Econometrica Supplementary Material

SUPPLEMENT TO "COMPETITION FOR A MAJORITY"

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THIS APPENDIX provides additional results that supplement those published in Barelli, Govindan, and Wilson (2014). Here we add the suffix "-BGW" to references to assumptions and results in that article.

S.1. SIMPLE-MAJORITY GAMES

For the case of simple-majority games, we specify a slightly different tie-breaking rule that implies the same result as Theorem 3.11-BGW even if the number of voters is even. We use the notation from Section 3.5-BGW except that the weight of each voter k is $w_k = 1/K$.

Obviously, Assumption 3.3-BGW cannot hold when the number of voters is even, so it is dropped. Assumption 3.4-BGW on diversity of preferences remains the same. Assumption 3.5-BGW relating strategy sets has to be changed.

ASSUMPTION S.1.1—Relationship Between Candidates' Strategy Sets—The Simple-Majority Version: *Fix* $x = (x_i, x_j) \in D$.

(1) *If* $\overline{L}_i(x)$ *is nonempty, then* $|L^0(x)| \geq 2$.

(2) *If* $L_i^*(x)$ *is nonempty and* $|L^0(x)| \geq 2$, *then*:

(a) *If* $\pi_i(y_i^k, x_j) = 0$ *for some* $k \in L_i^*(x)$, *then, for all* $k \in K^*(x_j)$, $\pi_j(y_j^k, x_i) \in \{0, 1\}$ *and, in fact, equals* $+1$ *if* $|L^0(x)| \geq 3$.

(b) *If* $\pi_i(y_i^k, x_j) = -1$ *for some* $k \in L_i^*(x)$, *then, for all* $k \in K^*(x_j)$, $\pi_j(y_j^k, x_i) = +1$.[1]

EXAMPLE S.1.2:

1. In the setting of Example 3.6-BGW, set the weights to $w_k = 1/4$ for every k. Condition (1) of Assumption S.1.1 holds because $\overline{L}_i(x)$ is nonempty iff $x_i = x_j$ and then $|L^0(x)| = K \geq 3$. Condition (2)(a) is illustrated by the policy pair (x_i, x_j) described in Example 3.6-BGW: in fact, $\pi_i(y_i^k, x_j) = 0$ for $k = 1, 3$, as y_i^1 (resp. y_i^3) wins voter 3 (resp. 1) and loses voter 1 (resp. 3), so

[1]Observe that if $\pi_i(y_i^k, x_j) = 0$ (resp. $\pi_i(y_i^k, x_j) = -1$) for some $k \in L_i^*(x)$, then $\pi_i(y_i^k, x_j) = 0$ (resp. $\pi_i(y_i^k, x_j) = -1$) for all $k \in L_i^*(x)$.

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each such policy gets $2/4$ votes against x_j. So we must show that $\pi_j(y_j^k, x_i) \geq 0$ for $k = 1, 3, 4$. And this is true, as it is equal to zero for $k = 1, 3$ (both y_j^1 and y_j^3 win one and lose one of the tied voters, so each gets $2/4$ votes against x_i) and it is equal to $+1$ for $k = 4$, as y_j^4 wins both tied voters 1 and 3 and retains voter 4, so j gets $3/4$ votes against x_i.

2. To illustrate the second part of condition (2)(a), modify Example 3.6-BGW by adding two voters and two dimensions, $K = 6$, $N = 5$, continuing with Euclidean preferences having ideal points $a^1 = (1, 0, 0, 0, 0)$, $a^2 = (0, 1, 0, 0, 0)$, $a^3 = (0, 0, 0, 0, 0)$, $a^4 = (0, 0, 1, 0, 0)$, $a^5 = (0, 0, 0, 1, 0)$, and $a^6 = (0, 0, 1, 0, 1)$. Again the strategy sets are the Pareto set, the convex hull of the ideal policies. For simple-majority rule, the weights are $w_k = 1/6$ for every k. Consider $x_i = (1/4, 1/4, 0, 0, 0)$ and $x_j = (1/4, 0, 1/4, 0, 0)$. Now $L^0(x) = \{1, 3, 5\}$, $L^i(x) = \{2\}$, and $L^j(x) = \{4, 6\}$. We have $\pi_i(y_i^k, x_j) = 0$ for $k \in \{1, 3\} = L_i^*(x)$, as y_i^1 (resp. y_i^3) wins voters 3 and 5 (resp. 1 and 5) and loses voter 1 (resp. 3), totaling $3/6$ votes from voters 2, 3, and 5 (resp. 1, 2, and 5). We must show that $\pi_j(y_j^k, x_i) = +1$ for $k = 1, 3, 4$, and this follows because y_j^k for $k = 1, 3, 4$, wins at least two of the tied voters, and retains voters 4 and 6 (relative to x_i), so j gets at least $4/6$ votes.

3. To illustrate condition (2)(b) of Assumption S.1.1, again modify Example 3.6-BGW, but now add only one voter and one dimension ($K = 5$, $N = 4$), with ideal policies $a^1 = (1, 0, 0, 0)$, $a^2 = (0, 1, 0, 0)$, $a^3 = (0, 0, 0, 0)$, $a^4 = (0, 0, 1, 0)$, and $a^5 = (0, 0, 1, 1)$, and $w_k = 1/5$ for all k. For the pair $x_i = (1/4, 1/4, 0, 0)$ and $x_j = (1/4, 0, 1/4, 0)$, we have $L^0(x) = \{1, 3\}$, $L^i(x) = \{2\}$, and $L^j(x) = \{4, 5\}$. Now $\pi_i(y_i^k, x_j) = -1$ for $k = 1, 2$, for the same reason as above, as x_j retains voters 4 and 5 and wins one more voter (voter 1 for $k = 1$ and voter 3 for $k = 3$), so it gets $3/5$ votes relative to y_i^k. So we have to verify that $\pi_j(y_j^k, x_i) = +1$ for $k = 1, 3, 4$. This follows, as y_j^k for $k = 1, 3, 4$ wins at least one voter, plus voters 4 and 5 that are already won (relative to x_i).

Again, the tie-breaking rule is specified in terms of the implied payoff function $\tilde{\pi} \in \Pi$.

DEFINITION S.1.3—Modified Tie-Breaking Rule $\mathcal{T}^S$: Suppose $x \in D$.

(T1) For each i, let $V(x_i)$ be as in Assumption S.1.1. Suppose, for some i, $L_i^*(x)$ is nonempty and $L^0(x)$ has at least two voters. For this i:

(a) If $\tilde{\pi}_i(y_i^k, x_j) = 0$ for some $k \in L_i^*(x)$, then $\tilde{\pi}_i(x_i, x_j)$ is zero if $|L^0(x)| = 2$ and -1 if $|L^0(x)| \geq 3$.

(b) If $\tilde{\pi}_i(y_i^k, x_j) = -1$ for some $k \in L_i^*(x)$, then $\tilde{\pi}_i(x_i, x_j) = -1$.

(T2) Suppose $L_i^*(x)$ is empty for each i or $L^0(x) = \{k\}$ for some k. If $\sum_{k' \in L^j(x)} w_{k'} = 1/2$, then $\tilde{\pi}_i(x_i, x_j) = -1/2$.

(T3) In all other cases, $\tilde{\pi}_i(x_i, x_j) = 0$ for each i.[2]

[2]Again, we could use fair coin tosses for each tied voter.

The rule $\mathcal{T}^S$ differs from the rule $\mathcal{T}$ used in Section 3.5-BGW only in that provisions (T1)(a) and (T2) are added—and the condition that $L_i^*(x)$ has at least two voters if $\overline{L}_i(x)$ is empty, when invoking (T1), is relaxed—to accommodate the fact that with an even number of voters, the game could end in a draw. Without these changes, $\mathcal{T}^S$ is the same as $\mathcal{T}$.

From Example S.1.2(3), we see that provision (T1)(b) is analogous to provision (T1) of tie-breaking rule $\mathcal{T}$: candidate j is in a very advantageous situation when $\tilde{\pi}_i(y_i^k, x_j) = -1$ for all $k \in L_i^*(x)$, as winning a single one of the tied voters guarantees a victory, whereas candidate i has to win all of the tied voters. In such a situation, $\mathcal{T}^S$ awards the election to j. Provision (T1)(a) handles draws: from Example S.1.2(1), we see that $\pi_i(y_i^k, x_j) = 0$ and $|L^0(x)| = 2$ for all $k \in L_i^*(x)$ is a symmetric situation, so the rule $\mathcal{T}^S$ declares it a draw; from Example S.1.2(2), we see that candidate j is in an advantageous situation when $\pi_i(y_i^k, x_j) = 0$ and $|L^0(x)| \geq 3$ for all $k \in L_i^*(x)$, as j has the upper hand in the non-tied battles, so $\mathcal{T}^S$ awards the election to j.

EXAMPLE S.1.4: Return to the setting of Example S.1.2(1). Consider the pair (x_i, x_j) with $x_i = (0, 0, 0)$ and x_j in the intersection of 1's indifference surface and the face spanned by voters 1, 2, and 4, in such a way that voter 4 prefers x_j to x_i (for instance, $x_j = (\frac{3-\sqrt{5}}{4}, \frac{1}{2}, \frac{\sqrt{5}-1}{4})$). Then $L^0(x) = \{1\}$ and $L^j(x) = \{2, 4\}$, so the premise of condition (T2) of the rule $\mathcal{T}^S$ applies, and the rule then says that $\tilde{\pi}_i(x_i, x_j) = -1/2$. We see that candidate j is in a stronger position because he has already secured 2/4 votes. But y_j^1 loses voter 1, so it fails to beat x_i. The relatively stronger position of candidate j is then captured by awarding the election to him with probability 3/4 rather than 1/2.

The payoff function $\tilde{\pi}$ induced by the tie-breaking rule $\mathcal{T}^S$ satisfies payoff approachability. As in the proof of Theorem 3.12-BGW, one shows that the payoff function satisfies condition 1 of Proposition 2.12-BGW, and then it is sufficient to show that payoff approachability is satisfied at each (x_i, σ_j) where σ_j has finite support in $D(x_i)$. This property is verified by Lemma S.2.1 in Section S.2 below, which then proves the existence theorem for simple-majority games.

THEOREM S.1.5—Existence and Invariance of the Value in Simple Majority Games: *The game $G(\tilde{\pi})$ has an equilibrium and its value is the value of every variant $G(\pi')$ with $\pi' \in \Pi$.*

S.2. PROOF OF THEOREM 5.4-BGW

We begin with a preliminary lemma about the payoff function $\tilde{\pi}$ that describes the tie-breaking rule $\mathcal{T}^S$, introduced in Definition S.1.3 for simple-majority games. In this game, fix (x_i, σ_j) such that the support of σ_j is finite and contained in $D(x_i)$. Choose $\bar{\varepsilon}$ as in the proof of Theorem 3.12-BGW and fix a neighborhood $V(x_i)$ also as there. The following lemma then proves payoff approachability for (x_i, σ_j) and, additionally, yields properties used to prove Theorem 5.4-BGW for simple-majority Colonel Blotto games.

LEMMA S.2.1: *There exists $k \in K^*(x_i)$ such that $\tilde{\pi}_i(x_i, \sigma_j) \leq \tilde{\pi}_i(y_i^k, \sigma_j)$. Moreover, the inequality is strict if one of the following conditions holds*:

(1) *$\overline{K}(x_i)$ is nonempty and there is a positive probability of* (T2) *or* (T3) *being used*.

(2) *$K^*(x_i)$ has at least three coordinates and there is a positive probability of* (T2) *or* (T3) *being used*.

(3) *$K^*(x_i)$ has two coordinates and* (T2) *or* (T3) *is used in resolving a tie (x_i, x_j) for which $L^0(x_i, x_j) \neq \{k\}$ for both k's in $K^*(x_i)$*.

(4) *$\overline{K}(x_i)$ is empty and* (T1) *is invoked for some (x_i, x_j) because i satisfies the conditions for the rule and either*: *$|L^0(x)| \geq 3$ and $\tilde{\pi}_i(y_i^{k'}, x_j) = 0$ for some $k' \in K^*(x_i)$; or $u_{k''}(x_i) \neq u_{k''}(x_j)$ for some $k'' \in K^*(x_i)$*.

PROOF: The proof becomes transparent once we compare the payoffs to x_i and y_i^k against x_j for each k and x_j, which we now do.

If (T1) is invoked and $\tilde{\pi}_i(y_i^k, x_j)$ is 0 (resp. -1) for some $k \in L^*(x)$, then $\tilde{\pi}_i(y_i^{k'}, x_j)$ is 0 (resp. -1) for all k' in $L_i^*(x)$, because of simple-majority scoring, and $\tilde{\pi}_i(y_i^{k'}, x_j)$ is 1 (resp. nonnegative) for $k' \in K^*(x_i) \setminus L_i^*(x)$. Thus, in this case, $\tilde{\pi}_i(x_i, x_j) \leq \tilde{\pi}_i(y_i^k, x_j)$ for all $k \in K^*(x_i)$, with strict inequality if $\overline{K}(x_i)$ is empty and either: (i) $|L^0(x)| \geq 3$ and $\tilde{\pi}_i(y_i^{k'}, x_j) = 0$ for some $k' \in K^*(x_i)$; or (ii) $u_k(x_i) \neq u_k(x_j)$.

If (T1) is invoked because $\tilde{\pi}_j(y_j^k, x_i)$ is 0, then $\tilde{\pi}_i(x_i, x_j)$ is zero if $|L^0(x)| = 2$ and $+1$ if $|L^0(x)| \geq 3$. By Assumption S.1.1, $\tilde{\pi}_i(y_i^k, x_j)$ is nonnegative in the former case and is $+1$ in the latter. Likewise, if (T1) is invoked because $\tilde{\pi}_j(y_j^k, x_i)$ is -1, then $\tilde{\pi}_i(y_i^k, x_j) = +1$ by Assumption S.1.1. In short, $\tilde{\pi}_i(y_i^k, x_j) \geq \tilde{\pi}_i(x_i, x_j)$ for all k. Thus, y_i^k does at least as well as x_i against every x_j for which (T1) is applied.

There remains to consider x_j's for which (T2) or (T3) is invoked.

Suppose $L_i^*(x)$ is empty for each i. If $|L^j(x)| = K/2$, then $\tilde{\pi}_i(x_i, x_j) = -1/2$ from (T2). For any $k \in K^*(x_i)$, because $k \notin L^0(x)$, $u_{k'}(y_i^k) > u_{k'}(x_j)$ for all $k' \in L^0(x)$, so $|L^i(y_i^k, x_j)| = K/2$ as well, and $\tilde{\pi}_i(y_i^k, x_j) = 0$. Likewise, if $|L^i(x)| = K/2$, then $\tilde{\pi}_i(x_i, x_j) = 1/2$ from (T2), and because $k \notin L^0(x)$, $\tilde{\pi}_i(y_i^k, x_j) = 1$. Summing up, $\tilde{\pi}_i(y_i^k, x_j) - \tilde{\pi}_i(x_i, x_j) = 1/2$ if either $|L^i(x)|$ or $|L^j(x)|$ equals $K/2$. This difference is equal to $+1$ otherwise (i.e., if neither of the candidates has half the votes outside of $L^0(x)$). Thus, all y_i^k's do strictly better against all these x_j's.

Suppose $L^0(x)$ contains just one voter, say k. If $k \notin K^*(x_i)$, then the payoff difference is as in the previous paragraph. If $k \in K^*(x_i)$, then $\overline{K}(x_i)$ is empty (by point (1) of Assumption S.1.1) and $\tilde{\pi}_i(y_i^k, x_j) - \tilde{\pi}_i(x_i, x_j) = -1 + 1/2 =$

$-1/2$ (resp. $0-1/2=-1/2$) if $|L^j(x)|=K/2$ (resp. $|L^i(x)|=K/2$). This difference is equal to -1 if neither of the candidates has half of the voters outside of $L^0(x)$. But observe that, for every $k'\neq k$ in $K^*(x_i)$, $\tilde{\pi}_i(y_i^{k'},x_j)-\tilde{\pi}_i(x_i,x_j)=\tilde{\pi}_i(x_i,x_j)-\tilde{\pi}_i(y_i^k,x_j)$, as $u_k(y_i^{k'})>u_k(x_j)>u_k(y_i^k)$. Thus, each $y_i^{k'}$ does strictly better against all these x_j's.

Finally, suppose $L^0(x)$ contains at least two voters, either $L_i^*(x)$ or $L_j^*(x)$ is nonempty, but each player for whom it is nonempty that he can achieve $+1$ rather than 0 or -1 specified there. Then, if $L_i^*(x)$ is nonempty, $\tilde{\pi}_i(y_i^k,x_j)=1$ for some $k\in L_i^*(x)$ (otherwise (T1) would apply) and it holds for all k while $\tilde{\pi}_i(x_i,x_j)=0$; on the other hand, if $L_i^*(x)$ is empty, then trivially each y_i^k achieves $+1$.

We now complete the proof of the lemma as follows. Obviously, if $\overline{K}(x_i)$ is nonempty, then y_i^k does at least as well as x_i against each x_j in the support of σ_j and strictly better against all x_j's for which (T1) is not invoked, proving the first statement and points (1)–(3) of the second, with point (4) being vacuously true. Assume from now on that $\overline{K}(x_i)$ is empty.

Each y_i^k does as well against all x_j to which (T1) applies and strictly better against those x_j's for which the condition of point (4) of the lemma holds. If (T2) or (T3) is not used with positive probability, then the first statement of the lemma holds as does point (4), while points (1)–(3) are vacuous.

Suppose (T2) or (T3) is invoked with positive probability. If there is one k for which no tie is just on this voter's utility, then y_i^k does strictly better than x_i, as the calculations above show. Thus, the inequality holds, regardless of the conditions of points (2)–(4), if there is such a k. Suppose then that, for each $k\in K^*(x_i)$, there is an x_j that ties with x_i just on k. It is clear that at least one of the y_i^k's would do as well as x_i against σ_j. Moreover, if there are at least three coordinates in $K^*(x_i)$, one of them would do strictly better, proving point (2). Also, if there are only two such k's, then one of them would do strictly better than x_i against σ_j unless each tie involves exactly one of these k's, which proves point (3). Observe that when there are two such k's, and x_i is not inferior to some y_i^k against σ_j, then x_i and each y_i^k give the same payoff against the conditional distribution over the x_j's for which (T2) or (T3) is used.

Coming to ties involving (T1), it is clear now that if there is a tie with an x_j where the rule is invoked because of i, then for x_i to do at least as well as all y_i^k, we must have $K^*(x_i)\subset L^0(x)$ and $\tilde{\pi}_i(y_i^k,x_j)=-1$ for each $k\in K^*(x_i)$ if $|L^0(x)|>2$. If this is violated for some x_i and if x_i is already not dominated by some y_i^k against the conditional over x_j's where (T1) is not used, then $K^*(x)$ has two coordinates and, as we argued at the end of the last paragraph, each k would do equally well against those not involving (T1), with the result that it would do strictly better against σ_j, proving point (4). *Q.E.D.*

We now recall and prove Theorem 5.4-BGW for simple-majority Colonel Blotto games.

THEOREM 5.4-BGW: *Let σ^* be an equilibrium that is invariant under all the symmetries of the game. If $R_1/R_2 < r^*$, then $(\sigma_1^* \otimes \sigma_2^*)(D) = 0$—that is, at the equilibrium σ^*, the tie-breaking rule $\mathcal{T}^S$ has zero probability of being invoked.*

We set up some notation and prove a number of preliminary claims before proving the theorem. Suppose x_i is a strategy in X_i such that $\sigma_j^*(D(x_i)) > 0$. We can decompose σ_j^* into σ_j^{c,x_i} and σ_j^{d,x_i}, where the former puts zero probability on $X_j \setminus D(x_i)$ and the latter puts probability 1 on it. Let $\mathcal{L}(x_i)$ be the set of quadruples $L = (L^0, L^i, L^j, Tn)$ such that there is a positive probability under σ_j^* of the set $D^L(x_i)$ consisting of x_j's such that $(L^0, L^i, L^j) = (L^0(x_i, x_j), L^i(x_i, x_j), L^j(x_i, x_j))$ and provision (Tn) of rule $\mathcal{T}^S$ is used, where $n \in \{1, 2, 3\}$. For simplicity, from here on we suppress Tn in the notation. For each L, choose a point $x_j(L) \in D^L(x_i)$ and consider the conditional distribution $\tilde{\sigma}_j^{x_i}$ over the $x_j(L)$'s given by $\tilde{\sigma}_j^{x_i}(x_j(L)) = (\sum_{L'} \sigma_j^*(D^{L'}(x_i)))^{-1} \sigma_j^*(D^L(x_i))$. Choose a neighborhood $V(x_i)$ such that, for each $y_i \in V(x_i)$ and L, $y_{i,k} > x_{j,k}(L)$ if $k \in L^i$, and $y_{i,k} < x_{j,k}(L)$ if $k \in L^j$.

CLAIM S.2.2: $\tilde{\pi}_i(x_i, \sigma_j^*) = \sigma_j^*(X_j \setminus D(x_i)) \tilde{\pi}_i(x_i, \sigma_j^{c,x_i}) + \sigma_j^*(D(x_i)) \tilde{\pi}_i(x_i, \tilde{\sigma}_j^{x_i})$.

PROOF: As the payoff $\tilde{\pi}_i(x_i, \cdot)$ is constant on each $D^L(x_i)$, $\tilde{\pi}_i(x_i, \sigma_j^{d,x_i}) = \tilde{\pi}_i(x_i, \tilde{\sigma}_j^{x_i})$ and the result follows. *Q.E.D.*

CLAIM S.2.3: *If x_i is a best reply to σ_j^*, then $\tilde{\pi}_i(x_i, \tilde{\sigma}_j^{x_i}) \geq \tilde{\pi}_i(y_i^k, \tilde{\sigma}_j^{x_i})$ for all $k \in K^*(x_i)$.*

PROOF: Assume to the contrary that $\tilde{\pi}_i(x_i, \tilde{\sigma}_j^{x_i}) < \tilde{\pi}_i(y_i^k, \tilde{\sigma}_j^{x_i})$ for some $k \in K(x_i)$. For each $\varepsilon > 0$, let $W^\varepsilon(x_i)$ be the set of y_i such that $|y_{i,k} - x_{i,k}| < \varepsilon$. For each L, let $D^{\varepsilon,L}(x_i)$ be the set of x_j in $D^L(x_j)$ such that $|x_{i,k} - x_{j,k}| > \varepsilon$ for $k \notin L^0$, and let $D^\varepsilon(x_i)$ be the union of the $D^{\varepsilon,L}(x_i)$'s. Choose ε small enough such that each $x_j(L)$ belongs to $D^\varepsilon(x_i)$. Define $\tilde{\sigma}_j^{\varepsilon,x_i}$ to be the distribution over $x_j(L)$ that assigns probability $\sigma_j^{d,x_i}(D^{\varepsilon,L}(x_i))/\sum_{L'} \sigma_j^{d,x_i}(D^{\varepsilon,L'}(x_i))$ to $x_j(L)$. By construction, $\tilde{\pi}_i(y_i(W^\varepsilon(x_i), k), \cdot)$ is constant on the set $D^{\varepsilon,L}(x_i)$ for each L and $\tilde{\pi}_i(y_i(W^\varepsilon(x_i), k), x_j) \in [-1, 1]$ for all x_j. Hence,

$$\begin{aligned}\tilde{\pi}_i\big(y_i\big(W^\varepsilon(x_i), \sigma_j^{d,x_i}\big)\big) \in \big(\sigma_j^{d,x_i}\big(D^\varepsilon(x_i)\big)\big) \tilde{\pi}_i\big(y_i\big(W^\varepsilon(x_i), k\big), \tilde{\sigma}_j^{\varepsilon,x_i}\big) \\ \pm \sigma_j^{d,x_i}\big(D(x_i) \setminus D^\varepsilon(x_i)\big).\end{aligned}$$

Obviously, $\tilde{\pi}_i(y_i(W^\varepsilon(x_i), k), x_j(L)) = \tilde{\pi}_i(y_i^k, x_j(L))$ for all $x_j(L)$. Moreover, $\tilde{\sigma}_j^{\varepsilon,x_i}$ converges to $\tilde{\sigma}_j^{x_i}$ and $D^\varepsilon(x_i)$ converges to $D(x_i)$. Therefore,

$$\lim_{\varepsilon \downarrow 0} \tilde{\pi}_i\big(y_i\big(W^\varepsilon(x_i), k\big), \sigma_j^{d,x_i}\big) = \tilde{\pi}_i\big(y_i^k, \tilde{\sigma}_j^{x_i}\big) > \tilde{\pi}_i\big(x_i, \sigma_j^{d,x_i}\big).$$

Since $\lim_{\varepsilon\downarrow 0}\tilde{\pi}_i(y_i(W^\varepsilon(x_i),k),\sigma_j^{c,x_i})=\tilde{\pi}_i(x_i,\sigma_j^{c,x_i})$, we then have that $\tilde{\pi}_i(x_i,\sigma_j^*)<\lim_{\varepsilon\downarrow 0}\tilde{\pi}_i(y_i(W^\varepsilon(x_i),k),\sigma_j^*)$ and σ_j^* is not a best reply to σ_j^*, a contradiction. *Q.E.D.*

The next three claims argue directly about points $(x_i,x_j)\in D$.

CLAIM S.2.4: *If x_i is a vertex, then there exists x'_j obtained by permuting the coordinates of x_j such that* (T1) *does not apply to (x_i,x'_j).*

PROOF: Let x_i be a strategy that assigns R_i to a battle, say $k=1$. Observe first that for (T1) to be used in deciding a tie between x_i and x_j's, this battle must belong to $L^0(x)$. If $R_1=R_2$, this means that $x_i=x_j$ and (T3) is operative. If $R_1>R_2$, then $i=2$ and $\tilde{\pi}_i(y_i^1,x_j)=-1$. Since $R_1<r^*R_2$, there exists some $k'\neq 1$ such that $0<x_{j,k'}<R_2$. There exists some x'_j that swaps these two coordinates and now (T3) applies to (x_i,x'_j). *Q.E.D.*

CLAIM S.2.5: *Suppose x_i is not a vertex, and* (T1) *applies to $(x_i,x_j)\in D$. If $\tilde{\pi}_i(y_i^k,x_j)$ is either 0 or -1 for some $k\in L^*(x)$, then either*: (i) *there exists $k'\in K^*(x_i)$ such that $x_{i,k'}\neq x'_{j,k'}$ for some x'_j obtained from permuting the coordinates of x_j; or* (ii) $|L^0(x)|\geq 3$ *and $\tilde{\pi}_i(y_i^k,x_j)\geq 0$ for some $k\in L_i^*(x)$.*

PROOF: If $R_1=R_2$, conclusion (i) is valid, since otherwise $x_i=x_j$ and (T3) would apply. If $R_1>R_2$ and $i=1$, then conclusion (i) is obvious.

Assume now that $i=2$, $R_1>R_2$, and conclusion (i) of the claim is violated. Then $x_{i,k}=x_{j,k}$ for each positive coordinate of x_i. If $\tilde{\pi}_i(x_i,x_j)=0$ for some k, then K is even, $|L^0(x)|=2$, and $|L^j(x)|=K/2-1$, while if $\tilde{\pi}_i(x_i,x_j)=-1$, then either $|L^j(x)|=\lfloor K/2\rfloor$ (K can be odd or even) or $|L^0(x)|\geq 3$, K is even and $|L^j(x)|=K/2-1$. If $|L^0(x)|=2$, then $|L^j(x)|=K-|L^0(x)|=K-2>\lfloor K/2\rfloor-1$. Thus, when $|L^j(x)|=K/2-1$, $|L^0(x)|\geq 3$.

If $|L^j(x)|=\lfloor K/2\rfloor$, then $|L^0(x)|=\lceil K/2\rceil$. Therefore, there exists k' such that $x_{i,k'}\geq R_2/\lceil K/2\rceil$. Moreover, since $|L^j(x)|=\lfloor K/2\rfloor$, and $R_1<r^*R_2$, there exists a coordinate k'' such that $x_{i,k''}=0<x_{j,k''}<R_1-R_2<R_2/\lceil K/2\rceil$. There exists x'_j that swaps these two coordinates and $(x_i,x'_j)\in D$. Now there is a coordinate, namely k', for which $x_{i,k'}>x'_{j,k'}$, a contradiction. So (i) must hold.

If $|L^j(x)|=K/2-1$, then, as we saw above, $|L^0(x)|\geq 3$. Therefore, $\tilde{\pi}_i(y_i^k,x_j)=0$ for each $k\in L_i^*(x)$, which proves (ii). *Q.E.D.*

CLAIM S.2.6: *Suppose $(x_i,x_j)\in D$, both x_i and x_j have two positive coordinates, $L^*(x_i)$ is nonempty, and* (T2) *or* (T3) *applies. There exists another x'_j obtained by a permutation of coordinates from x_j where* (T2) *or* (T3) *applies as well but where (x_i,x'_j) are either tied in two or more coordinates or in a zero coordinate.*

PROOF: Suppose x_i and x_j are tied in just one coordinate, say $k = 1$, and that this coordinate is positive for both players. Then $K = 3$ and i wins, say, $k = 2$ and j wins $k = 3$. Derive x'_j from x_j by permuting coordinates 2 and 3. x'_j ties with x_i in coordinates 1 and 3. *Q.E.D.*

PROOF OF THEOREM 5.4-BGW: Fix $x_1 \in D$ such that $\sigma_j^*(D(x_i)) > 0$. We show that x_i is not a best reply to σ_j^*, which proves the result.

Fix x_j in $D(x_i)$. Let $L = (L^0(x), L^i(x), L^j(x))$. Observe that if x'_j is obtained by permuting coordinates of x_j, then there exists $x'_j(L')$ in the support of $\tilde{\sigma}_j^{x_i}$, where $L' = (L^0(x_i, x'_j), L^i(x_i, x'_j), L^j(x_i, x'_j))$. Using this fact, the proof of the theorem follows quite easily. If x_i is a vertex, by Claim A.4, point (1) of Lemma S.2.1 holds for $\tilde{\sigma}_j^{x_i}$, and by Claim S.2.3, x_i is not a best reply to σ_j^*.

The other cases work similarly. If x_i is not a vertex, but (T1) applies to (x_i, x_j), then combining Claim A.5, point (4) of Lemma S.2.1, and Claim S.2.3 proves the result.

If (T2) or (T3) applies to (x_i, x_j), then by point (2) of Lemma S.2.1, x_i has only two nonzero coordinates. Claim A.6, point (3) of Lemma S.2.1, and Claim S.2.3 finish the proof. *Q.E.D.*

REFERENCE

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