SUPPLEMENT TO "NONPARAMETRIC IDENTIFICATION OF RISK AVERSION IN FIRST-PRICE AUCTIONS UNDER EXCLUSION RESTRICTIONS"
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The supplementary material contains the proof of Theorem 1.

DETERMINING THE SMOOTHNESS of the equilibrium strategy is difficult when the differential equation (2) in the text does not have an explicit solution, which is the case for general utility functions $U(\cdot)$. This is more so as this differential equation is known to have a singularity at $\underline{v}(I)$ when the reserve price is non-binding. To address these difficulties, we rewrite it as a differential equation in the bid quantile function $b(\alpha, I) = s[v(\alpha, I)]$, where $\alpha \in [0, 1]$ and $v(\alpha, I)$ is the α-quantile of $F(\cdot|I)$. We then view the latter differential equation as a member of a set (also called flow) of differential equations $E(B; t) = 0$ parameterized by $t \in [0, 1]$ in an unknown function $B(\cdot)$, where $E(B; 1) = 0$ corresponds to the general utility function $U(\cdot)$, while $E(B; 0) = 0$ corresponds to an appropriate constant relative risk aversion (CRRA) utility function; see (S.1)–(S.3). Next, we adopt a functional approach which exploits the existence, uniqueness, and smoothness of the equilibrium strategy in the CRRA case, where the solution of the differential equation (2) is known explicitly. In particular, our functional approach delivers the existence and uniqueness of the equilibrium strategy for a general utility function $U(\cdot)$ by a continuation argument theorem, thereby providing an alternative proof to those used in the economics literature. Moreover, our framework establishes the smoothness of the equilibrium strategy by an implicit functional theorem.

PROOF OF THEOREM 1

As our argument is done for every $I \in \mathcal{I}$, hereafter we omit the dependence on I. Theorem 1 follows from Theorems S1 and S2, where $b(\alpha) = s[v(\alpha)]$ is the α-bid quantile function with $\alpha \in [0, 1]$, $v(\alpha)$ is the α-quantile of $F(\cdot)$, and $s(\cdot)$ is a solution of (2) which must be strictly increasing by Lemma S1 below. Since $F[v(\alpha)] = \alpha$ implies $v'(\cdot) = 1/f[v(\cdot)]$, we have $b'(\alpha) = s'(v(\alpha))/f(v(\alpha))$. Hence, from (2), the bid quantile function $b(\cdot)$ must solve

$$\text{(S.1)} \qquad b'(\alpha) = \frac{I-1}{\alpha} \lambda[v(\alpha) - b(\alpha)] \quad \text{for} \quad \alpha \in (0, 1] \quad \text{with} \quad b(0) = v(0),$$

where $\lambda(\cdot)$ is $R + 1$ continuously differentiable with $\lambda'(\cdot) \geq 1$ on $[0, \infty)$ and $v(\cdot)$ is $R + 1$ continuously differentiable with $v'(\cdot) > 0$ on $[0, 1]$ as $U(\cdot) \in \mathcal{U}_R$ and $F(\cdot) \in \mathcal{F}_R$. Note that (S.1) is ill-conditioned at $\alpha = 0$. As for (2) in the text,

the solutions of (S.1) are not explicit except for simple utility functions such as CRRA. Specifically, when $U(x) = x^{1-c}/[1-c]$ for $0 \le c < 1$, it is well known that the equilibrium strategy exists and is unique so that the solution of (S.1) exists and is unique, namely

$$(S.2) \qquad b(\alpha) = \frac{I-1}{(1-c)\alpha^{(I-1)/(1-c)}} \int_0^\alpha r^{(I-1)/(1-c)-1} v(r) \, dr.$$

Moreover, following the proof of Lemma A2 in Guerre, Perrigne, and Vuong (2000), the equilibrium strategy in the CRRA case and hence the bid quantile function $b(\cdot) = s[v(\cdot)]$ are $R+1$ continuously differentiable on $[v(0), v(1)] = [\underline{v}, \overline{v}]$ and $[0, 1]$, respectively.

We now define our flow of differential equations $\{E(B; t) = 0; t \in [0, 1]\}$. For $t \in (0, 1]$, let

$$\Lambda(x; t) = \begin{cases} \dfrac{\lambda(tx)}{t} & \text{for } x \in \mathbb{R}_+, \\ \lambda'(0)x & \text{for } x \in \mathbb{R}_-, \end{cases}$$

$$V(\alpha; t) = v(0) + \frac{v(\alpha t) - v(0)}{t} \quad \text{for} \quad \alpha \in [0, 1].$$

These two functions are extended at $t = 0$ by considering their limits as $t \downarrow 0$, namely, $\Lambda(x; 0) = \lambda'(0)x$ for $x \in \mathbb{R}$ and $V(\alpha; 0) = v(0) + v'(0)\alpha$ for $\alpha \in [0, 1]$. For every $t \in [0, 1]$, note that $\Lambda(\cdot; t)$ and $V(\cdot; t)$ correspond to a utility function $U(x; t) = \exp(\int_0^x [1/\Lambda(u; t)] \, du) \in \mathcal{U}_R$ and a private value distribution $F(\cdot|\cdot; t) \in \mathcal{F}_R$, respectively. The flow of differential equations $\{E(B; t) = 0; t \in [0, 1]\}$ is then defined by

$$(S.3) \qquad B'(\alpha; t) = \frac{I-1}{\alpha} \Lambda(V(\alpha; t) - B(\alpha; t); t)$$

$$\text{for} \quad \alpha \in (0, 1] \quad \text{with} \quad B(0; t) = v(0),$$

which is analogous to (S.1). Note that $E(B; 0) = 0$ is

$$B'(\alpha; 0) = \frac{(I-1)\lambda'(0)}{\alpha} [v(0) + v'(0)\alpha - B(\alpha; 0)]$$

$$\text{for} \quad \alpha \in (0, 1] \quad \text{with} \quad B(0; 0) = v(0),$$

which corresponds to (S.1) for a CRRA utility function with parameter $0 \le c = 1 - 1/\lambda'(0) < 1$ as $\lambda'(\cdot) \ge 1$ and with a uniform private value distribution on $[v(0), v(0) + v'(0)]$. In particular, a key property is that $E(B; 0) = 0$ is known to admit a unique solution, namely

$$B(\alpha; 0) = v(0) + \frac{(I-1)\lambda'(0)}{(I-1)\lambda'(0) + 1} v'(0)\alpha$$

from (S.2). On the other hand, solving $E(B; 1) = 0$ is equivalent to solving (S.1) since $\Lambda(x; 1) = \lambda(x)$ and $V(\alpha; 1) = v(\alpha)$. Thus, the flow of differential equations $\{E_I(B; t) = 0; t \in [0, 1]\}$ is a path between $E(B; 0) = 0$ and $E(B; 1) = 0$.

The existence and uniqueness of the solution to $E(B; 1) = 0$ can be inferred from the existence and uniqueness of the solution to $E(B; 0)$ by a continuation argument given by Proposition 6.10 in Zeidler (1985) and reproduced below as Theorem Z1. Roughly this argument says that $E(B; 1) = 0$ admits a unique solution if $E(B; 0) = 0$ does under some regularity conditions on the functional operator associated with the differential equation $E(B; t) = 0$ and a so-called a priori condition defining the set of functions containing the potential solutions of $E(B; t) = 0$. This gives us the following theorem.

THEOREM S1: *If* $[U, F] \in \mathcal{U}_R \times \mathcal{F}_R$, *then for every* $I \in \mathcal{I}$, *statements* (i) *and* (ii) *hold*:

(i) *The differential equation* (S.1) *has a unique solution* $b(\cdot)$, *which is strictly increasing and continuously differentiable over* $[0, 1]$ *with* $b(\alpha) < v(\alpha)$ *for all* $\alpha \in (0, 1]$.

(ii) $s(\cdot) = b(F(\cdot))$ *is the unique solution of the differential equation* (2) *with initial condition* $s(v(0)) = v(0)$. *Moreover, this solution is strictly increasing and continuously differentiable on* $[v(0), v(1)]$ *with* $s(v) < v$ *for all* $v \in (v(0), v(1)]$, $s'(v) > 0$ *for all* $v \in [v(0), v(1)]$, *and* $s'(v(0)) = (I - 1)\lambda'(0)/[(I - 1) \times \lambda'(0) + 1] < 1$.

A main advantage of our functional approach is that it also delivers the smoothness of the equilibrium strategy. As above, we first study the differentiability of the bid quantile function $b(\cdot)$ on $[0, 1]$, building on an implicit functional theorem in Zeidler (1985, Theorem 4.B) and reproduced below as Theorem Z2. This theorem is applied to the flow of differential equations $\{E(B; t) = 0; t \in [0, 1]\}$.

THEOREM S2: *If* $[U, F] \in \mathcal{U}_R \times \mathcal{F}_R$, *then for every* $I \in \mathcal{I}$, *the following statements hold*:

(i) *The unique solution* $b(\cdot)$ *of* (S.1) *admits* $R + 1$ *continuous partial derivatives on* $[0, 1]$, *while* $b'(\alpha)$ *has* $R + 1$ *continuous partial derivatives on* $(0, 1]$.

(ii) *The unique solution* $s(\cdot)$ *of the differential equation* (2) *with initial condition* $s(v(0)) = v(0)$ *admits* $R + 1$ *continuous partial derivatives on* $[v(0), v(1)]$.

To prove Theorems S1 and S2 requires the establishment of some properties so as to satisfy the conditions of the continuation argument theorem and the implicit functional theorem. These properties follow from the next series of lemmas and corollaries, most of which are used to check the conditions of either theorem. In what follows, $\pi^{(k)}(\alpha; t)$, $V^{(k)}(\alpha; t)$, and $\Lambda^{(k)}(x; t)$ denote the kth derivatives of $\pi(\alpha; t)$, $V(\alpha; t)$, and $\Lambda(x; t)$ with respect to α, α, and x, respectively.

We first establish some properties that potential solutions to (2) must satisfy.

LEMMA S1: *If* $[U, F] \in \mathcal{U}_R \times \mathcal{F}_R$, *then for every* $I \in \mathcal{I}$, *solutions* $s(\cdot)$ *of* (2) *with boundary condition* $s(v(0)) = v(0)$ (*if any*), *are such that* (i) *and* (ii) *hold*:

(i) $s(\cdot)$ *is continuously differentiable on* $[v(0), v(1)]$.

(ii) $s(v) < v$ *for all* $v \in (v(0), v(1)]$ *and* $s'(v) > 0$ *for all* $v \in [v(0), v(1)]$ *with* $s'(v(0)) = (I - 1)\lambda'(0)/[(I - 1)\lambda'(0) + 1] < 1$.

PROOF: Fix $I \in \mathcal{I}$. Let $\tilde{\lambda}(x) = \lambda(x)$ for $x \geq 0$ and let $\tilde{\lambda}(x) = \lambda'(0)x$ for $x < 0$. Note that $\tilde{\lambda}(\cdot)$ is strictly increasing and continuously differentiable over $\mathbb{R}$ because $\lambda'(\cdot) \geq 1$ on $\mathbb{R}_+$. We establish (i) and (ii) for the potential solutions of the "extended" differential equation

$$(S.4) \qquad s'(v) = (I - 1)\frac{f(v)}{F(v)}\tilde{\lambda}(v - s(v))$$

$$\text{for} \quad v \in (v(0), v(1)] \quad \text{with} \quad s(v(0)) = v(0).$$

Since $\lambda(\cdot)$ and $\tilde{\lambda}(\cdot)$ coincide over $\mathbb{R}_+$, a solution of (2) with $s(v(0)) = v(0)$ is also a solution of (S.4). Conversely, a solution of (S.4) satisfying $s(v) < v$ for all $v \in (v(0), v(1)]$ is a solution of (2) with $s(v(0)) = v(0)$. In Step 2 we show that potential solutions of (S.4) must satisfy $s(v) < v$ for all $v \in (v(0), v(1)]$. Hence, $s(\cdot)$ is a solution of (2) with $s(v(0)) = v(0)$ if and only if it is a solution of (S.4). The desired result then follows.

STEP 1—Proof of (i) for Solutions of (S.4): Solutions $s(\cdot)$ of (S.4) are continuous on $[v(0), v(1)]$ and continuously differentiable on $(v(0), v(1)]$. Thus, it suffices to show the existence of $s'(v(0))$ with $\lim_{v \downarrow v(0)} s'(v) = s'(v(0))$. For $v \in (v(0), v(1)]$, let

$$\Psi(v) = (I - 1)\frac{f(v)(v - v(0))}{F(v)}\frac{\tilde{\lambda}(v - s(v))}{v - s(v)},$$

$$r(v) = \exp\left(-\int_v^{v(1)} \frac{\Psi(u)}{u - v(0)}\,du\right).$$

Also, let $\Psi(v(0)) = (I - 1)\lambda'(0)$ and $r(v(0)) = 0$. Thus, $\Psi(\cdot)$ is continuous and strictly positive on $[v(0), v(1)]$ since $[U, F] \in \mathcal{U}_R \times \mathcal{F}_R$. Hence, $0 < r(\cdot) < 1$ on $(v(0), v(1)]$. Moreover, $\frac{\Psi(v)}{v - v(0)} = \frac{\Psi(0) + o(1)}{v - v(0)}$ when $v \downarrow v(0)$. Thus, $\lim_{v \downarrow v(0)} r(v) = 0$ and $r(\cdot)$ is continuous on $[v(0), v(1)]$.

Now, (S.4) can be written as $s'(v) = \Psi(v)\frac{v - s(v)}{v - v(0)}$ for $v \in (v(0), v(1)]$ with $s(v(0)) = v(0)$, that is,

$$(S.5) \qquad (v - v(0))s'(v) + \Psi(v)(s(v) - v(0)) = \Psi(v)(v - v(0))$$

$$\text{for} \quad v \in (v(0), v(1)] \quad \text{with} \quad s(v(0)) = v(0).$$

Letting $C(v) = r(v)[s(v) - v(0)]$ yields, for $v \in (v(0), v(1)]$, $C'(v) = r(v) \times \Psi(v)\frac{s(v) - v(0)}{v - v(0)} + r(v)s'(v)$, so that (S.5) gives $C'(v) = r(v)\Psi(v)$. Thus, $C(v) =$

$C_0 + \int_{v(0)}^v r(u)\Psi(u)\,du$, where $C_0 = 0$ because $C(v(0)) = 0$. Hence, the potential solutions of (S.4) satisfy

$$s(v) = v(0) + \int_{v(0)}^v \frac{r(u)}{r(v)}\Psi(u)\,du.$$

But for $v(0) < u \le v \le v(1)$,

$$\frac{r(u)}{r(v)} = \exp\left(-\int_u^v \frac{\Psi(v(0)) + o(1)}{x - v(0)}\,dx\right)$$

$$= \exp\left(-[\Psi(v(0)) + o(1)]\log\frac{v - v(0)}{u - v(0)}\right)$$

$$= \left(\frac{u - v(0)}{v - v(0)}\right)^{[\Psi(v(0))+o(1)]}.$$

It follows that

$$s(v) - v(0) = \int_{v(0)}^v \left(\frac{u - v(0)}{v - v(0)}\right)^{(\Psi(v(0))+o(1))}[\Psi(v(0)) + o(1)]\,du$$

$$= \frac{\Psi(v(0))}{\Psi(v(0)) + 1}(v - v(0))(1 + o(1)),$$

showing that $s(v(0)) = v(0)$ as desired. Moreover, $s(\cdot)$ is differentiable at $v(0)$ with $s'(v(0)) = (I - 1)\lambda'(0)/[(I - 1)\lambda'(0) + 1]$ using $\Psi(v(0)) = (I - 1)\lambda'(0)$. On the other hand, $s'(v) = \Psi(v)\frac{v - s(v)}{v - v(0)}$ for $v > v(0)$ gives

$$\lim_{v\downarrow v(0)} s'(v) = \lim_{v\downarrow v(0)} \Psi(v)\left(1 - \frac{s(v) - v(0)}{v - v(0)}\right)$$

$$= \Psi(v(0))\big(1 - s'(v(0))\big) = s'(v(0))$$

as desired.

STEP 2—Proof of (ii) for Solutions of (S.4): We first prove that $s(v) < v$ for $v \in (v(0), v(1)]$ by contradiction. Observe that $0 < s'(v(0)) < 1$. It follows that $s(v) < v$ for $v > v(0)$ close enough to $v(0)$. Suppose that there is a v^* in $(v(0), v(1)]$ such that $s(v) < v$ for $v \in (v(0), v^*)$ and $s(v^*) = v^*$, so that $s'(v^*) = 0$ by (S.4). Since $R \ge 1$, differentiating (S.4) at v^* yields

$$\frac{s''(v^*)}{I - 1} = \frac{\partial}{\partial v}\left(\frac{f(v^*)}{F(v^*)}\right)\lambda(v^* - s(v^*)) + \frac{f(v^*)}{F(v^*)}\lambda'(v^* - s(v^*))(1 - s'(v^*))$$

$$= \frac{f(v^*)}{F(v^*)}\lambda'(0) > 0.$$

Hence, a second-order Taylor expansion for $\varepsilon > 0$ small enough yields $s(v^* - \varepsilon) = v^* + [s''(v^*) + o(1)]\varepsilon^2/2$. Thus, $s(v^* - \varepsilon) > v^* > v^* - \varepsilon$ for $\varepsilon > 0$ small enough, contradicting $s(v^* - \varepsilon) < v^* - \varepsilon$.

We next show that $s'(v) > 0$ for $v \in [v(0), v(1)]$. This follows immediately from $s'(v(0)) = (I - 1)\lambda'(0)/[(I - 1)\lambda'(0) + 1]$ and (S.4) using $s(v) < v$ for $v \in (v(0), v(1))$.   *Q.E.D.*

The next result, which follows from Lemma S1, relates the potential solutions of (2) to those of (S.1). It also provides some properties of the bid quantile function $b(\cdot)$.

COROLLARY S1: *If* $[U, F] \in \mathcal{U}_R \times \mathcal{F}_R$, *then for every* $I \in \mathcal{I}$, *the following statements hold*:

(i) $b(\cdot)$ *solves* (S.1) *if and only if* $b(\alpha) = s(v(\alpha))$, *where* $s(\cdot)$ *is a solution of* (2) *with* $s(v(0)) = v(0)$. *Equivalently*, $s(\cdot)$ *solves* (2) *with* $s(v(0)) = v(0)$ *if and only if* $s(v) = b(F(v))$, *where* $b(\cdot)$ *is a solution of* (S.1).

(ii) *Solutions* $b(\cdot)$ *of* (S.1), *if any, are continuously differentiable on* $[0, 1]$, *with* $b'(v) > 0$ *for all* $\alpha \in [0, 1]$ *and* $b(\alpha) < v(\alpha)$ *for all* $\alpha \in (0, 1]$.

PROOF: Note that $v(\alpha)$ is continuously differentiable on $[0, 1]$ with $v'(\alpha) = 1/f(v(\alpha)) > 0$ as $v(\alpha) = F^{-1}(\alpha)$. For part (i), setting $b(\alpha) = s(v(\alpha))$ yields $b'(\alpha) = s'(v(\alpha))/f(v(\alpha))$, so if $b(\cdot)$ solves (S.1), then the change of variable $\alpha = F(v)$ yields that $s(\cdot)$ solves (2) with the desired initial condition. Conversely, if $s(\cdot)$ solves (2) with $s(v(0)) = v(0)$, then elementary algebra yields that $b(\cdot)$ solves (S.1). The second assertion of (i) follows similarly. Part (ii) follows from Lemma S1 with $b(\alpha) = s(v(\alpha))$.   *Q.E.D.*

Instead of working with $B(\cdot; t)$, it is more convenient to make the change of variable $\pi(\cdot; t) = V(\cdot; t) - B(\cdot; t)$, where $V(\cdot; t)$ is continuously differentiable on $[0, 1]$. This gives the companion flow of differential equations $\{\tilde{E}(\pi; t) = 0; t \in [0, 1]\}$ defined by

$$(S.6) \qquad \pi'(\alpha; t) = V'(\alpha; t) - \frac{I - 1}{\alpha}\Lambda(\pi(\alpha; t); t)$$

$$\text{for} \quad \alpha \in (0, 1] \quad \text{with} \quad \pi(0; t) = 0.$$

The next result, which also follows from Lemma S1, provides a set Σ in which the potential solutions of (S.6) lie. Hereafter, we let $\mathbf{C}_1^0$ be the set of functions $\pi(\cdot)$ from $[0, 1]$ to $\mathbb{R}$ that are continuously differentiable on $[0, 1]$ and satisfy $\pi(0) = 0$.

COROLLARY S2: *Let* $[U, F] \in \mathcal{U}_R \times \mathcal{F}_R$. *For every* $I \in \mathcal{I}$, *define* $\bar{v}' = \max_{\alpha \in [0, 1]} v'(\alpha)$, *where* $0 < \bar{v}' < \infty$, *and let* $\Sigma = \{\pi(\cdot) \in \mathbf{C}_1^0; 0 < \pi(\alpha) < \bar{v}'$ *for* $\alpha \in (0, 1], \pi'(0) > 0\}$. *Then, for any* t *in* $[0, 1]$, *solutions* $\pi(\cdot; t)$ *of the differential equation* $\tilde{E}(\pi; t) = 0$, *if any, are in* Σ.

PROOF: Fix $t \in [0, 1]$. For $\alpha \in [0, 1]$, note that $V'(\alpha; t) = v'(\alpha t)$ and $V(0, t) = v(0)$. Hence, $0 \leq V(\alpha; t) = v(0) + \int_0^\alpha v'(ut) \, du \leq v(0) + \sup_{x \in [0,1]} v'(x) = v(0) + \bar{v}'$. Moreover, $V(\cdot; t)$ is $R + 1$ continuously differentiable on $[0, 1]$, while $\Lambda(\cdot; t)$ has the same properties as $\lambda(\cdot)$. Thus, (S.3) is similar to (S.1), thereby yielding that $B(\cdot; t)$ is continuously differentiable on $[0, 1]$ with $v(0) < B(\alpha; t) < V(\alpha; t)$ for all $\alpha \in (0, 1]$ by Corollary S1(ii). Now, $\pi(\cdot; t) = V(\cdot; t) - B(\cdot; t)$ solves (S.6) if and only if $B(\cdot; t)$ solves (S.3). Thus, $\pi(\cdot; t) \in C_1^0$ and $0 < \pi(\alpha; t) = V(\alpha; t) - B(\alpha; t) < V(\alpha; t) - v(0) \leq \bar{v}'$ for $\alpha \in (0, 1]$. Moreover, $\pi'(0; t) = V'(0; t) - B'(0; t) = v'(0) - s'(v(0); t)v'(0) > 0$ since $v'(0) > 0$ and $s'(v(0); t) = (I - 1)\Lambda'(0; t)/[(I - 1)\Lambda'(0; t) + 1] < 1$ by Lemma S1(ii), where $\Lambda'(0; t) = \lambda'(0) > 0$.   Q.E.D.

Next, we establish the smoothness of the auxiliary functions $\Lambda(x; t)$ and $V'(\alpha; t)$.

LEMMA S2: *If* $[U, F] \in \mathcal{U}_R \times \mathcal{F}_R$, *then for every* $I \in \mathcal{I}$, *statements* (i) *and* (ii) *are valid*:
   (i) $\Lambda(x; t)$ *is* R *continuously differentiable in* $(x, t) \in \mathbb{R}_+ \times [0, 1]$, *Moreover,* $(1/x)\partial^r \Lambda(x; t)/\partial t^r$ *is continuous in* $(x, t) \in \mathbb{R}_+ \times [0, 1]$ *for* $r = 0, \ldots, R$.
   (ii) $V'(\alpha; t)$ *is* R *continuously differentiable in* $(\alpha, t) \in [0, 1]^2$.

PROOF: Let $0 < t \leq 1$. For $x > 0$, the Leibnitz–Newton formula yields

$$\frac{\partial^r \Lambda(x; t)}{\partial t^r} = \frac{\partial^r}{\partial t^r} \left(\frac{\lambda(tx)}{t} \right)$$

$$= \sum_{j=0}^r \frac{r!}{j!(r - j)!} \frac{\partial^j \lambda(tx)}{\partial t^j} \frac{\partial^{r-j}}{\partial t^{r-j}} \left(\frac{1}{t} \right)$$

$$= \frac{(-1)^r r!}{t^{r+1}} \sum_{j=0}^r \frac{\lambda^{(j)}(tx)}{j!} (-tx)^j$$

for $0 \leq r \leq R$. On the other hand, a Taylor expansion of $\lambda(0) = \lambda(tx - tx) = 0$ around tx with integral remainder (see, e.g., Zeidler (1985, p. 77)) shows that

$$0 = \sum_{j=0}^r \frac{\lambda^{(j)}(tx)}{j!} (-tx)^j$$

$$+ \frac{(-tx)^{r+1}}{r!} \int_0^1 (1 - u)^r \lambda^{(r+1)}(tx - utx) \, du.$$

Hence, using the change of variable $\nu = 1 - u$, we obtain for $(x, t) \in (\infty) \times (0, 1]$,

$$\frac{1}{x} \frac{\partial^r \Lambda(x; t)}{\partial t^r} = x^r \int_0^1 \nu^r \lambda^{(r+1)}(\nu t x) \, d\nu,$$

$$\frac{\partial^{r_1 + r_2} \Lambda(x; t)}{\partial x^{r_1} \partial t^{r_2}} = \frac{\partial^{r_1}}{\partial x^{r_1}} \left(x^{r_2+1} \int_0^1 \nu^{r_2} \lambda^{(r_2+1)}(\nu t x) \, d\nu \right),$$

where $0 \leq r_1 + r_2 \leq R$. Using the Lebesgue dominated convergence theorem and the $R + 1$ continuous differentiability of $\lambda(\cdot)$ on $\mathbb{R}_+$, it can be checked that the above two functions are continuous on $\mathbb{R}_+ \times [0, 1]$, thereby establishing part (i). Part (ii) follows from $V'(\alpha; t) = v'(\alpha t)$ for $(\alpha, t) \in [0, 1]^2$, where $v'(\cdot)$ is R continuously differentiable on $[0, 1]$ because $F(\cdot | \cdot) \in \mathcal{F}_R$.   Q.E.D.

We now introduce some functional operators associated with the differential equation (S.6). Let $\mathbf{C}_0$ be the set of functions $\pi(\cdot)$ from $[0, 1]$ to $\mathbb{R}$ that are continuous on $[0, 1]$. As is well known, $\mathbf{C}_0$ is a Banach space equipped with the norm $\|\pi\|_0 = \sup_{\alpha \in [0,1]} |\pi(\alpha)|$. Similarly, $\mathbf{C}_1^0$ as defined earlier is a Banach space equipped with the norm $\|\pi\|_1 = \max_{r=0,1} \sup_{\alpha \in [0,1]} |\pi^{(r)}(\alpha)| = \sup_{\alpha \in [0,1]} |\pi^{(1)}(\alpha)|$.[1] In particular, $\boldsymbol{\Sigma}$ is an open subset of $\mathbf{C}_1^0$ since the open ball $\mathcal{V}(\pi; \varepsilon) = \{\zeta \in \mathbf{C}_1^0; \|\zeta - \pi\|_1 < \varepsilon\} \subset \boldsymbol{\Sigma}$ for any $\pi \in \boldsymbol{\Sigma}$ and $\varepsilon = \varepsilon_\pi$ small enough. Moreover, for every $t \in [0, 1]$, it can be checked that $\Lambda(\pi(\alpha); t)/\alpha$ and $V'(\alpha; t)$ are continuous in $\alpha \in [0, 1]$ whenever $\pi(\cdot) \in \mathbf{C}_1^0$. Thus, for every $t \in [0, 1]$, we can view the solutions of the differential equation (S.6) as the zeros of the functional operator $\mathbf{E}(\cdot; t)$ from $\mathbf{C}_1^0$ to $\mathbf{C}_0$ (see Lemma S3(i) below), where

$$\mathbf{E}(\cdot; t) : \pi(\cdot) \rightarrow \mathbf{E}(\pi; t)(\alpha) = \pi^{(1)}(\alpha) + \frac{I - 1}{\alpha} \Lambda(\pi(\alpha); t) - V'(\alpha; t),$$

$$\alpha \in [0, 1].$$

In what follows $\mathbf{E}^{r_1 r_2}(\pi; t) = \partial^{r_1 + r_2} \mathbf{E}(\pi; t) / \partial \pi^{r_1} \partial t^{r_2}$ denotes the Fréchet partial derivatives of $\mathbf{E}(\pi; \alpha)$ (see, e.g., Zeidler (1985)), which are linear operators from $(\mathbf{C}_1^0)^{r_1} \times \mathbb{R}^{r_2}$. For a linear operator $L : \mathcal{C}_1 \mapsto \mathcal{C}_0$ with Banach spaces $\mathcal{C}_i$ equipped with norms N_i, $\rho(L) = \sup_{x \in \mathcal{C}_1, N_1(x)=1} N_0(L(x))$ is the operator norm of L.

LEMMA S3: *If $[U, F] \in \mathcal{U}_R \times \mathcal{F}_R$, then for every $I \in \mathcal{I}$, the following statements hold* (i)–(iii):
    (i) $\mathbf{E}(\pi; t) \in \mathbf{C}_0$ *for all* $(\pi, t) \in \mathbf{C}_1^0 \times [0, 1]$.

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[1]To see that $\|\pi\|_1 = \|\pi^{(1)}\|_0$, note that $|\pi(\alpha)| = |\int_0^\alpha \pi^{(1)}(u) \, du| \leq \sup_{u \in [0,1]} |\pi^{(1)}(u)|$ for all $\alpha \in [0, 1]$.

(ii) $\mathbf{E}(\pi; t)$ *is R Fréchet differentiable in $(\pi, t) \in \Sigma \times [0, 1]$ with Fréchet partial derivatives $\mathbf{E}^{r_1 r_2}(\pi; t)$, $0 \le r_1 + r_2 \le R$, that are uniformly continuous over $\Sigma \times [0, 1]$.*

(iii) *The Fréchet partial derivative $\mathbf{E}^{10}(\pi; t)$ at $(\pi, t) \in \Sigma \times [0, 1]$ maps $\eta \in \mathbf{C}_1^0$ to $\mathbf{E}^{10}(\pi; t)(\eta) \in \mathbf{C}_0$ defined as $\mathbf{E}^{10}(\pi; t)(\eta)(\alpha) = \eta^{(1)}(\alpha) + \frac{I-1}{\alpha} \Lambda^{(1)}(\pi(\alpha); t) \times \eta(\alpha)$ for $\alpha \in [0, 1]$. Moreover, $\mathbf{E}^{10}(\pi; t)$ is one-to-one (bijective) from $\mathbf{C}_1^0$ to $\mathbf{C}_0$ with an inverse of bounded operator norm uniformly in $(\pi, t) \in \Sigma \times [0, 1]$.*

PROOF: Throughout, fix $I \in \mathcal{I}$.

(i) Fix $(\pi, t) \in \mathbf{C}_1^0 \times [0, 1]$. It is sufficient to study $\Lambda(\pi(\cdot); t)$, which is clearly continuous on $(0, 1]$. As $\alpha \downarrow 0$, $\pi(\alpha) = \pi'(0)\alpha + o(\alpha)$ since $\pi(0) = 0$ by definition of $\mathbf{C}_1^0$. For $t > 0$, it follows that $\Lambda(\pi(\alpha); t)/\alpha = \lambda(t\pi(\alpha))/(\alpha t) = \lambda'(0)\pi'(0) + o(1)$, the last expansion being also true for $t = 0$. Thus, $\mathbf{E}(\pi; t) \in \mathbf{C}_0$.

(ii) We first consider the Gâteaux derivatives of $\mathbf{E}(\pi; t)$. From Zeidler (1985), for example, these are obtained in two steps: In the first step, $\partial^{r_1 + r_2} \mathbf{E}(\pi + u\eta; t)/\partial u^{r_1} \partial t^{r_2}$ is computed, where $\eta \in \mathbf{C}_1^0$; in the second step, the term η^{r_1} arising in this expression is changed into $\eta_1 \times \cdots \times \eta_{r_1}$, where the η_r are in $\mathbf{C}_1^0$. For $1 \le r_1 + r_2 \le R$ and $\eta_1, \ldots, \eta_R$ in $\mathbf{C}_1^0$, the Gâteaux derivatives are

$$(S.7) \qquad \mathbf{E}^{10}(\pi; t)(\eta_1)(\alpha) = \eta_1^{(1)}(\alpha) + (I - 1)\Lambda^{(1)}(\pi(\alpha); t)\frac{\eta_1(\alpha)}{\alpha},$$

$$\mathbf{E}^{0r_2}(\pi; t)(\alpha) = \frac{I - 1}{\alpha}\frac{\partial^{r_2}\Lambda(\pi(\alpha); t)}{\partial t^{r_2}} - \frac{\partial^{r_2}V'(\alpha; t)}{\partial t^{r_2}} \quad \text{for} \quad r_2 \ge 1,$$

$$\mathbf{E}^{r_1 r_2}(\pi; t)(\eta_1, \ldots, \eta_{r_1})(\alpha)$$
$$= (I - 1)\frac{\partial^{r_1 + r_2}\Lambda(\pi(\alpha); t)}{\partial x^{r_1} \partial t^{r_2}}\frac{\eta_1(\alpha)}{\alpha}\eta_2(\alpha) \times \cdots \times \eta_{r_1}(\alpha), \quad \text{otherwise.}$$

Note that $\eta_1(\alpha)/\alpha$ belongs to $\mathbf{C}_0$ since $\eta_1 \in \mathbf{C}_1^0$. It follows that $\mathbf{E}^{r_1 r_2}(\pi; t) \times (\eta_1, \ldots, \eta_{r_1}) \in \mathbf{C}_0$ for $1 \le r_1 + r_2 \le R$ by Lemma S2.

We now show that $\mathbf{E}(\pi; t)$ is R Fréchet continuously differentiable over $\Sigma \times [0, 1]$ with Fréchet partial derivatives $\mathbf{E}^{r_1 r_2}(\pi; t)$. Note that $\mathbf{E}^{0r_2}(\pi; t)$ is uniformly continuous over $\Sigma \times [0, 1]$ by Lemma S2. Thus, from Proposition 4.8 in Zeidler (1985), part (ii) is proven if, for $r_1 \ge 1$, the following conditions hold:

(ii)(a) The map $(\eta_1, \ldots, \eta_{r_1}) \to \mathbf{E}^{r_1 r_2}(\pi; t)(\eta_1, \ldots, \eta_{r_1})$ is a continuous multilinear operator.

(ii)(b) The map $(\pi, t) \to \mathbf{E}^{r_1 r_2}(\pi; t)$ is uniformly continuous over $\Sigma \times [0, 1]$.

We show these for $(r_1, r_2) = (1, 0)$ only, the other cases being similar. Recall that if $\pi \in \Sigma$, then $\pi(\cdot)$ takes its values in $[0, \bar{v}']$, which is compact. For (ii)(a), we have to show that the operator norm $\rho_1(\mathbf{E}^{10}(\pi; t)) \equiv$

$\sup_{\eta_1 \in \mathbf{C}_1^0, \|\eta_1\|_1 = 1} \|\mathbf{E}^{10}(\pi; t)(\eta_1)\|_0 < \infty$ for all $(\pi, t) \in \boldsymbol{\Sigma} \times [0, 1]$. Since $\|\eta_1(\alpha)/\alpha\|_0 \le \|\eta_1\|_1$, it follows from (S.7) and Taylor inequality that

$$\|\mathbf{E}^{10}(\pi; t)(\eta_1)\|_0 = \left\| \eta_1^{(1)}(\alpha) + (I - 1)\Lambda^{(1)}(\pi(\alpha); t)\frac{\eta_1(\alpha)}{\alpha} \right\|_0$$

$$\le \|\eta_1\|_1 \left(1 + (I - 1) \sup_{(x,t) \in [0,\tilde{v}'] \times [0,1]} \left| \Lambda^{(1)}(x; t) \right| \right),$$

so that $\rho_1(\mathbf{E}^{10}(\pi; t)) < \infty$ by Lemma S2. For (ii)(b), we have for any (π_0, t_0) and (π, t) in $\boldsymbol{\Sigma} \times [0, 1]$,

$$\|\mathbf{E}^{10}(\pi_0; t_0)(\eta_1) - \mathbf{E}^{10}(\pi; t)(\eta_1)\|_0$$

$$= (I - 1)\left\| \left(\Lambda^{(1)}(\pi_0(\alpha); t_0) - \Lambda^{(1)}(\pi(\alpha); t)\right)\frac{\eta_1(\alpha)}{\alpha} \right\|_0$$

$$\le (I - 1)\|\eta_1\|_1 \left\| \Lambda^{(1)}(\pi_0(\alpha); t_0) - \Lambda^{(1)}(\pi(\alpha); t) \right\|_0.$$

It follows from Lemma S2 that $\rho_1(\mathbf{E}^{10}(\pi_0; t_0) - \mathbf{E}^{10}(\pi; t))$ can be made arbitrarily small by choosing $\|\pi_0 - \pi\|_1$ and $|t_0 - t|$ small enough.

(iii) Fix (π, t) in $\boldsymbol{\Sigma} \times [0, 1]$ and abbreviate $\mathbf{E}^{10}(\pi; t)$ into $\mathbf{E}^1$. The first part of (iii) has been established in (S.7). To show that this operator is one-to-one from $\mathbf{C}_1^0$ to $\mathbf{C}_0$, consider ζ in $\mathbf{C}_0$. Finding an $\eta \in \mathbf{C}_1^0$ with $\mathbf{E}^1(\eta) = \zeta$ amounts to solving the linear differential equation

$$(S.8) \qquad E_\zeta^1 : \eta^{(1)}(\alpha) + (I - 1)\Lambda^{(1)}(\pi(\alpha); t)\frac{\eta(\alpha)}{\alpha} = \zeta(\alpha) \quad \text{with} \quad \eta(0) = 0.$$

Proceeding as in Step 1 of the proof of Lemma S1 yields that the unique candidate solution is

$$\eta_\zeta(\alpha) = \int_0^\alpha \zeta(u)\frac{R(u)}{R(\alpha)}\,du \quad \text{where}$$

$$R(\alpha) = \exp\left(-(I - 1)\int_\alpha^1 \frac{\Lambda^{(1)}(\pi(u); t)}{u}\,du\right).$$

Note that $\Lambda^{(1)}(x; t) = \lambda'(tx) \ge 1$ and $\Lambda^{(1)}(\pi(u); t)/u = \lambda'(0)/u + O(1)$ when $u \downarrow 0$. Thus,

$$0 \le \frac{R(u)}{R(\alpha)} = \exp\left(-(I - 1)\int_u^\alpha \frac{\Lambda^{(1)}(\pi(\tau); t)}{\tau}\,d\tau\right) \le 1$$

$$\text{for} \quad 0 \le u \le \alpha \quad \text{and} \quad \lim_{\alpha \to 0} R(\alpha) = 0.$$

It follows that η_ζ is defined and continuously differentiable over $(0, 1]$. Observe now that $|\eta_\zeta(\alpha)| \leq \alpha\|\zeta\|_0$ so that setting $\eta_\zeta(0) = 0$ gives a continuous function over $[0, 1]$. For differentiability at 0, note that for $0 \leq u \leq \alpha$, we have $(\log\alpha - \log u)/(\alpha - u) \to +\infty$ as $\alpha \downarrow 0$. Thus, as $\alpha \downarrow 0$ we have

$$\frac{R(u)}{R(\alpha)} = \exp\left[-(I-1)\int_u^\alpha \left(\frac{\lambda'(0)}{\tau} + O(1)\right)d\tau\right]$$

$$= \exp\left(-(I-1)\lambda'(0)\log\frac{\alpha}{u} + O(\alpha - u)\right)$$

$$= \exp\left(-(I-1)\lambda'(0)\log\frac{\alpha}{u} + \frac{\alpha - u}{\log\alpha - \log u}O(1)\right)$$

$$= \left(\frac{u}{\alpha}\right)^{(I-1)\lambda'(0)}(1 + o(1)),$$

$$\eta_\zeta(\alpha) = \int_0^\alpha (\zeta(0) + o(1))\left(\frac{u}{\alpha}\right)^{(I-1)\lambda'(0)}(1 + o(1))\, du$$

$$= \frac{\zeta(0)}{(I-1)\lambda'(0) + 1}\alpha + o(\alpha).$$

Hence, η_ζ is differentiable at 0 with $\eta_\zeta^{(1)}(0) = \zeta(0)/[(I-1)\lambda'(0)+1]$. To check that $\eta_\zeta^{(1)}$ is continuous at 0, observe that (S.8) gives for $\alpha \downarrow 0$,

$$\eta_\zeta^{(1)}(\alpha) = \zeta(0) - (I-1)(\lambda'(0) + o(1))\left(\frac{\zeta(0)}{(I-1)\lambda'(0)+1} + o(1)\right)$$

$$= \eta_\zeta^{(1)}(0) + o(1).$$

Hence, $\eta_\zeta \in C_1^0$. Thus, $\mathbf{E}^1 : C_1^0 \mapsto \mathbf{C}_0$ is one-to-one with $[\mathbf{E}^1]^{-1}(\zeta) = \eta_\zeta$ for any $\zeta \in \mathbf{C}_0$.

Last, recall that $|\eta_\zeta(\alpha)| \leq \alpha\|\zeta\|_0$ and $0 \leq \pi(\alpha) \leq \bar{v}'$ for any $\pi \in \Sigma$. This gives

$$\|[\mathbf{E}^1]^{-1}(\zeta)\|_1 = \|\eta_\zeta^{(1)}\|_0 = \left\|\zeta(\alpha) - (I-1)\Lambda^{(1)}(\pi(\alpha); t)\frac{\eta(\alpha)}{\alpha}\right\|_0$$

$$\leq \|\zeta\|_0 + (I-1)\sup_{(x,t)\in[0,\bar{v}']\times[0,1]}\lambda'(tx)\|\zeta\|_0$$

$$= \left(1 + (I-1)\sup_{x\in[0,\bar{v}']}\lambda'(x)\right)\|\zeta\|_0.$$

Hence the operator norm $\rho([\mathbf{E}^1(\pi; t)]^{-1})$ is bounded uniformly in $(\pi, t) \in \Sigma \times [0, 1]$. *Q.E.D.*

We now prove Theorems S1 and S2. To prove Theorem S1, we use the following continuation argument from Zeidler (1985, Proposition 6.10).

THEOREM Z1—Continuation Argument: *Let $\mathcal{C}_1$ and $\mathcal{C}_0$ be some Banach spaces. For $\pi \in \mathcal{C}_1$, let $\mathcal{V}(\pi; \varepsilon)$ denote the ε-neighborhood of π in $\mathcal{C}_1$. Suppose the following conditions*:

(i) *The map $(\pi, t) \in \mathcal{C}_1 \times [0, 1] \mapsto \mathbf{E}(\pi; t) \in \mathcal{C}_0$ is continuous.*

(ii) *(A priori condition) There exists an open subset $\mathcal{S}$ of $\mathcal{C}_1$ and a number $\varepsilon > 0$ such that if (π_t, t) verifies $\mathbf{E}(\pi_t; t) = 0$, then $\mathcal{V}(\pi_t; \varepsilon) \subset \mathcal{S}$ for all $t \in [0, 1]$.*

(iii) *For any $t \in [0, 1]$, the operator $\mathbf{E}$ has a Fréchet derivative $\mathbf{E}_\pi$ with respect to $\pi \in \mathcal{S}$. The operators $(\pi, t) \mapsto \mathbf{E}(\pi; t)$ and $(\pi, t) \mapsto \mathbf{E}_\pi(\pi; t)$ are uniformly continuous over $\mathcal{S} \times [0, 1]$.*

(iv) *The linear operator $\eta \in \mathcal{C}_1 \mapsto \mathbf{E}_\pi(\pi; t)(\eta) \in \mathcal{C}_0$ is one-to-one and for some constant C, $\rho(\mathbf{E}_\pi(\pi; t)^{-1}) \leq C$ for all $(\pi, t) \in \mathcal{S} \times [0, 1]$.*
If $\mathbf{E}(\pi; 0) = 0$ has a unique solution π_0, then $\mathbf{E}(\pi; 1) = 0$ has a unique solution π_1.

PROOF: Fix $I \in \mathcal{I}$. Part (ii) follows from Lemma S1, Corollary S1, and part (i). Thus, it suffices to show the latter. In view of Corollary S1(ii), it remains to show the existence and uniqueness of the solution of (S.1), that is, $E(B; 1) = 0$ or equivalently $\tilde{E}(\pi; 1) = 0$. We apply Theorem Z1, where $\mathcal{C}_1 = \mathbf{C}_1^0$, $\mathcal{C}_0 = \mathbf{C}_0$, and $\mathcal{S} = \boldsymbol{\Sigma}$. Lemma S3 shows that conditions (i), (iii), and (iv) of Theorem Z1 hold. Hence, it remains to check condition (ii).

We begin with some inequalities. Let $\bar{v}'$ be as in Corollary S2 and define $\underline{v}' = \inf_{\alpha \in [0,1]} v'(\alpha)$, and $\underline{\lambda}' = \inf_{x \in [0, \bar{v}']} \frac{\lambda(x)}{x}$, and $\bar{\lambda}' = \sup_{x \in [0, \bar{v}']} \frac{\lambda(x)}{x}$, where $0 < \underline{v}' \leq \bar{v}' < \infty$ and $1 \leq \underline{\lambda}' \leq \bar{\lambda}' < \infty$ because $[U, F] \in \mathcal{U}_R \times \mathcal{F}_R$. Recall that $V'(\alpha; t) = v'(\alpha t)$ for $(\alpha, t) \in [0, 1]^2$, while $\pi(\cdot) \in \boldsymbol{\Sigma}$ takes its values in $[0, \bar{v}')$. Thus, for any $(\alpha, t) \in [0, 1]^2$ and $\pi(\cdot) \in \boldsymbol{\Sigma}$, we have $\underline{v}' \leq V'(\alpha; t) \leq \bar{v}'$ and $\underline{\lambda}' \pi(\alpha) \leq \Lambda(\pi(\alpha); t) \leq \bar{\lambda}' \pi(\alpha)$ since $\Lambda(x; t) = \lambda(xt)/t$ for $(x, t) \in \mathbb{R}_+ \times (0, 1]$, $\Lambda(x; 0) = \lambda'(0)x$ for $x \in \mathbb{R}_+$, and $\underline{\lambda}' \leq \lambda'(0) \leq \bar{\lambda}'$. For $t \in [0, 1]$, let $\pi(\cdot; t)$ be a solution of $\tilde{E}(\pi; t) = 0$ so that $\pi(\cdot) \in \boldsymbol{\Sigma}$ by Corollary S2. Since $\tilde{E}(\pi; t) = 0$ writes $\pi'(\alpha) + (I - 1)\Lambda(\pi(\alpha); t)/\alpha = V'(\alpha; t)$, the above inequalities yield, for all $(\alpha, t) \in [0, 1]^2$,

$$\underline{v}' \leq \pi'(\alpha; t) + (I - 1)\bar{\lambda}' \frac{\pi(\alpha; t)}{\alpha} \quad \text{and}$$

$$\pi'(\alpha; t) + (I - 1)\underline{\lambda}' \frac{\pi(\alpha; t)}{\alpha} \leq \bar{v}' \quad \text{with} \quad \pi(0; t) = 0.$$

Setting $\bar{C}(\alpha; t) = \pi(\alpha; t)\alpha^{(I-1)\bar{\lambda}'}$ so that $\bar{C}'(\alpha; t) = \alpha^{(I-1)\bar{\lambda}'}[\pi'(\alpha; t) + (I - 1)\bar{\lambda}'\pi(\alpha; t)/\alpha]$ yields $\underline{v}'\alpha^{(I-1)\bar{\lambda}'} \leq \bar{C}'(\alpha; t)$ from the first differential inequality. Thus, integrating and using $\bar{C}(0; t) = 0$ gives $\underline{v}'\alpha/[(I - 1)\underline{\lambda}' + 1] \leq \pi(\alpha)$ for all $(\alpha, t) \in [0, 1]^2$. Setting $\underline{C}(\alpha; t) = \pi(\alpha; t)\alpha^{(I-1)\underline{\lambda}'}$, proceeding similarly with the

second differential inequality, and combining yield

$$\text{(S.9)} \qquad \frac{\underline{v}'\alpha}{(I-1)\bar{\lambda}'+1} \leq \pi(\alpha;t) \leq \frac{\bar{v}'\alpha}{(I-1)\underline{\lambda}'+1} < \bar{v}'$$

for all $(\alpha, t) \in [0,1]^2$.

We now check condition (ii) of Theorem Z1. We have to show that $\mathcal{V}(\pi(\cdot;t);\varepsilon) \subset \Sigma$ for $\varepsilon > 0$ small enough and all $t \in [0,1]$. Recall that the neighborhood $\mathcal{V}(\pi(\cdot;t);\varepsilon)$ of $\pi(\cdot;t)$ in $\mathbf{C}_1^0$ consists of functions $\zeta(\cdot) \in \mathbf{C}_1^0$ with $\sup_{\alpha \in [0,1]} |\zeta'(\alpha) - \pi'(\alpha;t)| < \varepsilon$. In particular, $\zeta'(0) > \pi'(0;t) - \varepsilon$, where $\pi'(0;t) = V'(0;t) - B'(0;t) = v'(0)[1 - s'(v(0);t)] = v'(0)/[(I-1)\lambda'(0)+1] > 0$ by Lemma B1. Moreover, integrating and using $\pi(0;t) = \zeta(0) = 0$ give $\pi(\alpha;t) - \varepsilon\alpha < \zeta(\alpha) < \pi(\alpha;t) + \varepsilon\alpha$ for all $\alpha \in [0,1]$. Hence, for $\varepsilon > 0$ small enough, $\zeta'(0) > \pi'(0;t) - \varepsilon > 0$, while (S.9) yields

$$0 < \left(\frac{\underline{v}'}{(I-1)\bar{\lambda}'+1} - \varepsilon\right)\alpha < \zeta(\alpha) < \left(\frac{\bar{v}'}{(I-1)\underline{\lambda}'+1} + \varepsilon\right)\alpha < \bar{v}'$$

for all $\alpha \in (0,1]$,

for all $t \in [0,1]$. That is, there exists $\varepsilon > 0$ such that $\mathcal{V}(\pi(\cdot;t);\varepsilon)$ is a subset of Σ for all $t \in [0,1]$ and the a priori condition (ii) of Theorem Z1 is proven.

Q.E.D.

To prove Theorem S1, we use the following implicit functional theorem from Zeidler (1985, Theorem 4.B).

THEOREM Z2—*Implicit Functional Theorem: Let (π_0, t_0) be in $\mathcal{C}_1 \times [0,1]$, where $\mathcal{C}_1$ is a Banach space, and consider an R continuously Fréchet differentiable operator $\mathbf{E}(\cdot, \cdot)$ defined on a neighborhood of (π_0, t_0) with values in a Banach space $\mathcal{C}_0$ such that $\mathbf{E}(\pi_0, t_0) = 0$. If the Fréchet derivative $\mathbf{E}_\pi(\pi, t)$ of $\mathbf{E}(\pi, t)$ with respect to π is such that $\mathbf{E}_\pi(\pi_0, t_0)$ is one-to-one, then there exists a neighborhood $\mathcal{O}(t_0)$ of t_0 such that, for $t \in \mathcal{O}(t_0)$, the equation $\mathbf{E}(\pi, t) = 0$ has a unique solution $\pi(t)$, which is R continuously differentiable on $\mathcal{O}(t_0)$.*

PROOF: Fix $I \in \mathcal{I}$. Part (ii) follows from part (i) since $s(v) = b(F(v))$ by Corollary S1(ii) and $F \in \mathcal{F}_R$. Thus, it suffices to show part (i). Let $\mathcal{C}_1 = \mathbf{C}_1^0$ and $\mathcal{C}_0 = \mathbf{C}_0$. For any $t_0 \in [0,1]$, note that $\tilde{E}(\pi;t_0) = 0$ has a unique solution $\pi_0(\cdot) = \pi(\cdot;t_0)$ as it suffices to consider the flow of differential equations $\{\tilde{E}_0(\pi;u) = 0; u \in [0,1]\}$, where $\tilde{E}_0(\pi;u) \equiv \tilde{E}(\pi;ut_0)$ and to follow the proof of Theorem S1(i) with $\mathcal{S} = \Sigma$. As $\pi_0 \in \Sigma$ by Corollary S2, while $\Sigma \times [0,1]$ is a neighborhood of (π_0, t_0), Lemma S3 and Theorem Z2 yield that $\pi(t) = \pi(\cdot;t)$ is R continuously differentiable with respect to t in a neighborhood $\mathcal{O}(t_0)$ of t_0, and hence at t_0. As t_0 is arbitrary in $[0,1]$, then $\pi(\cdot;t)$ is R continuously differentiable in $t \in [0,1]$.

For $t \in (0, 1]$, note that $\pi(\alpha; t) = [v(\alpha t) - b(\alpha t)]/t$ for $\alpha \in [0, 1]$, where $b(\cdot)$ is the solution of (S.1). To see this, it suffices to verify that such a $\pi(\cdot; t)$ verifies (S.6) using $\pi'(\alpha; t) = v'(\alpha t) - b'(\alpha t)$, $V'(\alpha; t) = v'(\alpha t)$, $\Lambda(x; t) = \lambda(tx)/t$ for $x \geq 0$, and (S.1). Similarly, for $t = 0$, let $\pi(\alpha; 0) = [v'(0) - b'(0)]\alpha$ for $\alpha \in [0, 1]$, which can be seen to verify (S.6). In particular, $\pi'(1; t) = v'(t) - b'(t)$ for $t \in [0, 1]$. Thus, using $v'(t) = V'(1; t)$ and (S.6) at $\alpha = 1$ gives $b'(t) = (I - 1)\Lambda(\pi(1; t); t)$ for $t \in [0, 1]$, where $\Lambda(\cdot; \cdot)$ is R continuously differentiable on $\mathbb{R}_+ \times [0, 1]$ by Lemma S2(i) and $\pi(1; \cdot)$ is R continuously differentiable on $[0, 1]$. Hence, $b'(\cdot)$ is R continuously differentiable on $[0, 1]$, implying that $b(\cdot)$ is $R + 1$ continuously differentiable on $[0, 1]$ as desired. Last, using (S.1) shows that $b'(\cdot)$ is $R + 1$ continuously differentiable on $(0, 1]$. *Q.E.D.*

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