SUPPLEMENT TO "EFFICIENCY IN REPEATED GAMES REVISITED: THE ROLE OF PRIVATE STRATEGIES." TECHNICAL DETAILS FOR EXAMPLE 2

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FIRST, WE SHOW how to derive the PE payoff in Figure 2 in the main paper. The private equilibrium in Section 3.2 of the main paper relies only on the assumption $p(Y|D,D) > 0 = p(Y|D,C) = p(Y|C,D)$; thus it also works in the present example, irrespective of the level of ε. The incentive constraints (6) and (7) in Section 3.2 reduce to a quadratic equation in q,

$$(1-\delta)\{(h-d)q+d\} = \delta q p(Y|D,D)(1-q-qh), \tag{1}$$

where q is the probability to play D. In Example 2, we have $h=6$, $d=1$, and $p(Y|D,D)=1/3$. Hence (1) becomes

$$7\delta q^2 + (15-16\delta)q + 3(1-\delta) = 0.$$

Whereas we are interested in the most efficient equilibrium, we choose the smaller root: $q(\delta) \equiv q = \frac{1}{14\delta}(-15+16\delta-\sqrt{225-564\delta+340\delta^2})$. Computation shows that this solution is real and lies in $[0,1]$ when $\delta \geq 0.992$. The associated symmetric private equilibrium payoff for each player is $v(\delta) = 1-7q(\delta)$, the graph of which is depicted by the solid line in Figure 2.

Next, we present the lemmas and the derivation of $\delta(\overline{v})$ that we cited when we derived the upper bound of the PPE payoffs. Throughout this supplement, $\overline{v}$ refers to the best symmetric payoff of the PPE payoffs.

LEMMA 1: *When $\overline{v} > 0$, there exists a positive constant L independent of ε such that $\Delta_1(\omega)+\Delta_2(\omega) \geq L$ is satisfied for all ω.*

PROOF: When $\overline{v} > 0$, the first-period action profile in the best symmetric PPE lies in the set

$$Q_+ = \{(q_1,q_2)|g_1(q_1,q_2)+g_2(q_1,q_2) > 0\},$$

where q_i is the probability that player i chooses action D. If this were not the case, so that $g_1'+g_2' \leq 0$ in the formula (16) in the paper, a continuation equilibrium would provide a better symmetric PPE with payoff $(v_1'(\omega)+v_2'(\omega))/2 > \overline{v}$, which contradicts our premise that $\overline{v}$ is the best symmetric PPE payoff. Whereas $F(q_1,q_2) \equiv g_1(q_1,q_2)+g_2(q_1,q_2) = 2-6q_2-6q_1+10q_1q_2$, we have

$$(q_1,q_2) \in Q_+ \quad \Rightarrow \quad q_i < 1/3 \quad \text{for} \quad i=1,2. \tag{2}$$

This is shown as follows. Note that $F(q_1, q_2)$ is linear in q_1 and that both $F(0, q_2) = 2 - 6q_2$ and $F(1, q_2) = 4(q_2 - 1)$ are nonpositive if $q_2 \geq 1/3$. Hence $F(q_1, q_2)$, which is a convex combination of those values, is nonpositive if $q_2 \geq 1/3$. A symmetric argument shows that F is nonpositive if $q_1 \geq 1/3$. Hence F is positive only if $q_1, q_2 < 1/3$.

Note that, for any (q_1, q_2), we have (i) $p(Y|q_1, q_2) \leq p(X_k|q_1, q_2)$, $k = 1, 2$, and (ii) $p(Y|q_1, q_2)$ does not depend on ε. Hence, for any $(q_1, q_2) \in Q_+$ and any ω, $p(\omega|q_1, q_2)$ is bounded below by $r \equiv \min_{q_1, q_2 \in [0,1/3]} p(Y|q_1, q_2)$ (we used (2) here), which is a positive number independent of ε. Now consider the dynamic programming equation (16) in the paper. Because $\overline{v} > 0$, $g_1'(q_1, q_2) + g_2'(q_1, q_2) \leq 2$, and $\sum_\omega (\Delta_1(\omega) + \Delta_2(\omega)) p(\omega|q_1, q_2) \leq r \min_\omega (\Delta_1(\omega) + \Delta_2(\omega))$ (this is implied by $\Delta_1(\omega) + \Delta_2(\omega) \leq 0$ (see the main paper) and $r \leq p(\omega|q_1, q_2)$), we have

$$\forall\, \omega, \quad -L \leq \Delta_1(\omega) + \Delta_2(\omega)$$

for $L \equiv 2/r$. *Q.E.D.*

LEMMA 2: *For any (large) constant $K > 0$, we can find a (small enough) $\underline{\varepsilon} > 0$ such that $\overline{v} > 0$ requires*

$$\begin{aligned}(\Delta_1(\omega), \Delta_2(\omega)) &\in D \\ &= \bigl\{(\Delta_1, \Delta_2) | -L \leq \Delta_1 + \Delta_2 \leq 0 \text{ and } \Delta_i > K \text{ for } i = 1 \text{ or } 2\bigr\}\end{aligned}$$

for some ω if $\varepsilon \leq \underline{\varepsilon}$.

PROOF: Suppose the claim does not hold. Then, for any $K > 0$ and any $\underline{\varepsilon} > 0$, there must be some $\varepsilon \leq \underline{\varepsilon}$ for which $\overline{v} > 0$ is sustained as a symmetric PPE by $(\Delta_1(\omega), \Delta_2(\omega))$, which lies for all ω in a compact set

$$D' = \bigl\{(\Delta_1, \Delta_2) | -L \leq \Delta_1 + \Delta_2 \leq 0 \text{ and } \Delta_i \leq K \text{ for } i = 1, 2\bigr\}.$$

Let (q_1, q_2) be the first-period action to sustain $\overline{v}$. Whereas $\overline{v} > 0$, the proof of Lemma 1 above shows that $(q_1, q_2) \in Q_+$. In addition, the incentive compatibility condition

$$g(D, q_j) - g(C, q_j) \leq \sum_{\omega = X_1, X_2, Y} \Delta_i(\omega)\bigl[p(\omega|C, q_j) - p(\omega|D, q_j)\bigr] \tag{3}$$

is satisfied for $i, j = 1, 2$ and $j \neq i$, which should hold with equality if player i mixes C and D.

Given that this is true for any $\underline{\varepsilon} > 0$, there is a sequence $\{\varepsilon^n, \Delta_1^n, \Delta_2^n, q_1^n, q_2^n\}$ such that $\varepsilon^n \to 0$ as $n \to \infty$, where (i) $\Delta_i^n \equiv (\Delta_i^n(Y), \Delta_i^n(X_1), \Delta_i^n(X_2))$, (ii) $(\Delta_1^n, \Delta_2^n, q_1^n, q_2^n)$ satisfies incentive constraint (3), and (iii) $(\Delta_1^n, \Delta_2^n, q_1^n, q_2^n)$

lies in compact set $(D')^3 \times [0, 1/3]^2$ (here we used (2)). By (iii), there is a converging subsequence; let $(\Delta_1^\#, \Delta_2^\#, q_1^\#, q_2^\#)$ be its limit. Whereas both sides of incentive constraint (3) are continuous in $(\varepsilon, \Delta_1, \Delta_2, q_1, q_2)$, the limit also satisfies (3).[1] In the limit where $\varepsilon = 0$, outcomes X_1 and X_2 *always* realize with an equal probability *for any action profile*. Hence, essentially we can regard $\{X_1, X_2\}$ as a single outcome X. This enables us to use our results in Section 3.1 of the main paper, which presumes two outcomes X and Y. To this end, define $\Delta_i^\#(X) \equiv \frac{1}{2}\Delta_i^\#(X_1) + \frac{1}{2}\Delta_i^\#(X_2)$. Whereas the limit satisfies (3), a simple calculation shows that $(\Delta_i^\#(X), \Delta_i^\#(Y))$ satisfies the incentive constraint for the game with two outcomes X and Y.

The limit also satisfies $q_1^\#, q_2^\# \le 1/3$, which implies that a unilateral deviation from $(q_1^\#, q_2^\#)$ makes X (i.e., $\{X_1, X_2\}$) more likely. Hence, $(q_1^\#, q_2^\#)$ is in set Q defined in Section 3.1 of the main paper. Then the upper bound in Lemma 1 in the main paper applies.[2] Therefore, the payoff associated with the limit is bounded above by

$$\max_{\mathbf{q}\in[0,1]} g(C, q) - \frac{d(q)}{L(q) - 1} = \max_{\mathbf{q}\in[0,1]} (1 - 7q) - \frac{1 + 5q}{\frac{3-q}{2+q} - 1}$$
$$< 1 - \frac{1}{\frac{3}{2} - 1} = 1 - 2 < 0.$$

However, whereas the payoffs along the sequence are strictly positive, their limits should be nonnegative. This constitutes a contradiction. *Q.E.D.*

Finally, we show how to derive $\delta(\overline{v})$, a lower bound of δ to satisfy

$$\left(\frac{1-\delta}{\delta} D + (v_1^0, v_2^0)\right) \cap V^F \neq \emptyset, \tag{4}$$

where (v_1^0, v_2^0) is an equilibrium payoff profile to obtain symmetric payoff $\overline{v}$ (possibly with public randomization). Note that if this condition (4) is satisfied for some δ', then it is also satisfied for all $\delta > \delta'$. Hence, any value of δ such that $(\frac{1-\delta}{\delta} D + (v_1^0, v_2^0)) \cap V^F = \emptyset$ is a lower bound of discount factor to satisfy (4).

A reasonably tight lower bound is obtained by the value of δ that is determined as in Figure S1. The two lines defined by $v_1 + 7v_2 = 8$ and $7v_1 + v_2 = 8$ lie on the Pareto frontier of the feasible payoff set V^F, so that V^F is contained in set W in the figure. The shaded areas correspond to set $\frac{1-\delta}{\delta} D + v'$. We pick the point v' (such that $2\overline{v} = v_1' + v_2'$) off the 45° line to deal with the possibility that (v_1^0, v_2^0) may not be a symmetric payoff profile. The particular choice of point v'

[1]Note that signal distribution p is a continuous function of ε.

[2]This follows from the fact that the upper bound in Lemma 1 in the main paper is derived by the incentive constraint and $\mathbf{q} \in Q$, both of which are satisfied by the limit point.

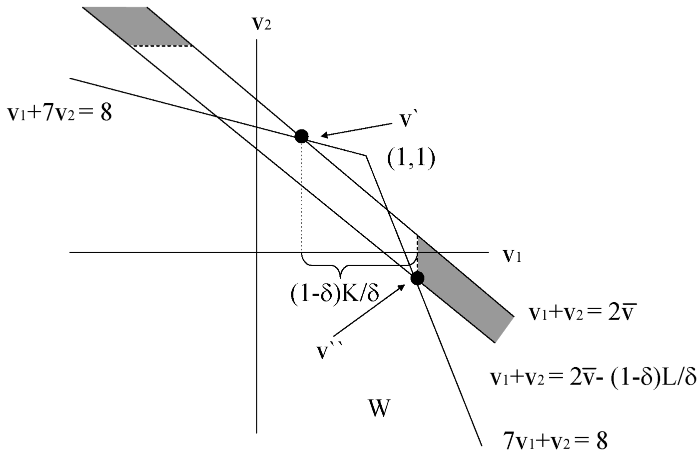

FIGURE S1.

makes sure that, if δ is determined as in Figure S1, then $\frac{1-\delta}{\delta}D + (v_1^0, v_2^0)$ always lies outside of W (hence outside of V^F) for *any* possible choice of (v_1^0, v_2^0) (i.e., for any (v_1^0, v_2^0) in W (hence in V^F) that satisfies $v_1^0 + v_2^0 = 2\overline{v}$). In summary, if δ is determined as in Figure S1, then we have $\frac{1-\delta}{\delta}D + (v_1^0, v_2^0) \cap V^F = \emptyset$.

Figure S1 shows that we have

$$v_1'' - v_1' = \frac{1-\delta}{\delta}K. \tag{5}$$

The value of v_1' is obtained by solving $v_1 + v_2 = 2\overline{v}$ and $v_1 + 7v_2 = 8$, and we find $v_1' = \frac{7\overline{v}-4}{3}$. Similarly, v_1'' is determined by $v_1 + v_2 = 2\overline{v} - (\frac{1-\delta}{\delta})L$ and $7v_1 + v_2 = 8$, and we find $v_1'' = (8 - 2\overline{v} + (\frac{1-\delta}{\delta})L)/6$. By plugging these results into equation (5), we obtain a lower bound of the discount factor to support $\overline{v}$:

$$\delta(\overline{v}) = \frac{3K - \frac{L}{2}}{3K - \frac{L}{2} + 8(1-\overline{v})}.$$

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