SUPPLEMENT TO "THE EXPECTED NUMBER OF NASH EQUILIBRIA OF A NORMAL FORM GAME"

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THE EXPECTED NUMBER OF ROOTS OF A UNIVARIATE POLYNOMIAL

PROPOSITION 3.1 OF ENNE gives the formula for the mean number of roots of a multihomogeneous system of equations that was first published in McLennan (2002). A condensed version of the proof from that paper is given in the Appendix of ENNE. The purpose of this supplement is to illustrate the workings of this argument by following it from beginning to end in the simplest case, when there is only one equation and one unknown. Since Section 5 of ENNE follows a similar sequence of steps, this calculation is also illustrative of the argument presented there. To some extent the discussion here is modelled on Edelman and Kostlan (1995), which is a very readable exposition of a broad range of related ideas.

First of all, consider the problem, first studied by Kac (1943), of determining the mean number of real roots of a quadratic polynomial $at^2 + bt + c$, where (a, b, c) is a random point (various distributions will be considered) in $\mathbb{R}^3 \setminus \{0\}$. We wish to determine the probability of the set of coefficient vectors (a, b, c) at which the discriminant $b^2 - 4ac$ is positive, so it seems natural to integrate the quantity of interest, here the number of solutions, across the given measure on the parameter space. From a mathematical point of view, however, this sort of calculation is typically ill-behaved. The set of solutions of an economic model is typically the set of fixed points of a function or correspondence. For polynomials it is relatively hard to pass from a given polynomial to its roots. *In contrast, if, for a particular point in the solution space, we ask what parameters have that point as a solution, the set of such parameters is often very well behaved.* For general equilibrium theory this point of view is emphasized in Balasko (1988) where the equilibrium manifold of an exchange economy is displayed as a vector bundle in which the set of endowments that have a particular equilibrium allocation is the fiber.

For the quadratic polynomial the idea of integrating over the space of solutions leads us to define $\gamma : \mathbb{R} \to \mathbb{R}^3$ by $\gamma(t) = (t^2, t, 1)/\|(t^2, t, 1)\|$. Observing that t is a root of the polynomial $at^2 + bt + c$ if and only if $(a, b, c) \perp \gamma(t)$, we see that the probability that the polynomial will have a root in the interval $(t, t + \Delta t)$ is, for small Δt, approximately equal to the probability that (a, b, c) is in the cone lying between the planes orthogonal to $\gamma(t)$ and $\gamma(t + \Delta t)$ respectively. In particular, if (a, b, c) is uniformly distributed in the unit sphere in $\mathbb{R}^3$, this probability is approximately $1/\pi$ times the distance from $\gamma(t)$ to $\gamma(t + \Delta t)$.

This line of reasoning (with proper attention to detail) shows that for this distribution the mean number of real roots is the length of the curve γ divided by π. While this may not seem like an obvious increase in tractability, in fact the length of γ, and a host of related issues, have been studied extensively. (Cf., Edelman and Kostlan (1995).)

Naturally the most interesting distributions on the space of coefficient vectors are those for which the computation is nontrivial but tractable. To the extent that the distribution is intended as a model of a "typical" or "randomly chosen" coefficient vector, a distribution that satisfies some symmetry condition has some automatic claim to being "unbiased." When the coefficient vector (a, b, c) is uniformly distributed in the unit sphere, the three coefficients are treated symmetrically, but this is actually somewhat unnatural. An arguably more natural approach, which leads to more tractable computations, begins by homogenizing the polynomial: consider the bivariate quadratic $ax^2+bxy+cy^2$. The natural spaces in which to look for solutions are the unit circle and one dimensional projective space, which is obtained from the circle by identifying antipodal points.

Consider, for $0 \le \theta < 2\pi$, the transformation of variables corresponding to rotating these spaces θ radians:

$$(x, y) = (w\cos\theta + z\sin\theta, -w\sin\theta + z\cos\theta). \tag{1}$$

This substitution results in the quadratic polynomial $a_\theta w^2 + b_\theta wz + c_\theta z^2$, where

$$\begin{aligned}
a_\theta &:= a\cos^2\theta - b\cos\theta\sin\theta + c\sin^2\theta,\\
b_\theta &:= 2(a-c)\cos\theta\sin\theta + b(\cos^2\theta - \sin^2\theta),\\
c_\theta &:= a\sin^2\theta + b\cos\theta\sin\theta + c\cos^2\theta.
\end{aligned}$$

The transformation $(a, b, c) \mapsto (a_\theta, b_\theta, c_\theta)$ of coefficient vectors maps $p := (2^{-1/2}, 0, 2^{-1/2})$ to itself, and it maps $(0, 1, 0)$ to $q(\theta) := (-\cos\theta\sin\theta, \cos^2\theta - \sin^2\theta, \cos\theta\sin\theta)$, which is a unit vector orthogonal to $(2^{-1/2}, 0, 2^{-1/2})$. Let $C := \{q(\theta) : 0 \le \theta < 2\pi\}$. The uniform distribution on any circle in S^2 of the form $\alpha p + \beta C$ will be an invariant measure for these transformations. In particular, this symmetry alone does not determine a unique distribution of coefficient vectors.

A somewhat different perspective results from assuming that (a, b, c) is centrally multinormally distributed with covariance matrix C. Then $(a_\theta, b_\theta, c_\theta)$ will also be centrally multinormally distributed, say with covariance matrix C_θ. Insisting on invariance, in the sense that $C_\theta = C$ for all θ, does not determine C uniquely. But it turns out that there is a unique diagonal matrix with this property.

More generally, let C be the $(n+1) \times (n+1)$ diagonal matrix with $(i+1, i+1)$-entry $\binom{n}{i}$. If the coefficient vector $(a_0, \ldots, a_n)$ of the polynomial

$a_0x^n + a_1x^{n-1}y + \cdots + a_ny^n$ is centrally multinormally distributed with covariance matrix C, then the distribution of coefficient vectors is invariant under the transformations resulting from the substitutions (1). Moreover, C is the unique (up to multiplication by a scalar) diagonal covariance matrix with this property.[1] Our goal here is to compute the mean number of roots, in one dimensional projective space, of the polynomial

$$f(x, y) := a_0x^n + a_1x^{n-1}y + \cdots + a_ny^n$$

when the coefficient vector has this distribution.

Let $\mathcal{H}$ be the space of coefficient vectors $f = (a_0, \ldots, a_n)$ endowed with the inner product

$$\langle (a_0, \ldots, a_n), (b_0, \ldots, b_n) \rangle = \sum_{i=0}^{n} \binom{n}{i}^{-1} a_ib_i. \tag{2}$$

Relative to this inner product, the coefficient vector is multinormally distributed with covariance matrix equal to the identity matrix. Let $\mathcal{M}$ denote the unit sphere relative to this inner product. Let $F : \mathcal{H} \times \mathbb{R}^2 \to \mathbb{R}$ be the *evaluation map*:

$$F(f, (x, y)) := f(x, y) = a_0x^n + a_1x^{n-1}y + \cdots + a_ny^n.$$

From the point of view of the distribution of roots, the central multinormal distribution on the space of coefficient vectors and the uniform distribution on $\mathcal{M}$ are equivalent, and the latter is more convenient in certain respects.

Let N denote the unit sphere S^1 in $\mathbb{R}^2$. The *incidence variety* is

$$V := F^{-1}(0) \cap (\mathcal{M} \times N) = \{(f, (x, y)) \in \mathcal{M} \times N : f(x, y) = 0\}.$$

Note that F is invariant under the action

$$O(f, (x, y)) := (f \circ O^{-1}, O(x, y))$$

of $O(2)$ on $\mathcal{H} \times \mathbb{R}^2$, so V is invariant under this action, and there is an action on V defined by restriction.

We would like to show that each point $(f, (x, y)) \in V$ is a regular point of the restriction of F to $\mathcal{M} \times N$, since then the regular value theorem (e.g., Guillemin and Pollack (1965)) would imply that V is an n-dimensional C^∞ manifold. Since $(x, y) \neq (0, 0)$, not all of the monomials $x^{n-i}y^i$ vanish, so f is a regular point of $F(\cdot, (x, y)) : \mathcal{H} \to \mathbb{R}$. Interpreting f as an element of $T_f\mathcal{H}$

[1]These results, due to Kostlan (1993), are not particularly easy to prove. Perhaps the most accessible and self contained account is Edelman and Kostlan (1995, pp. 15–17). Kostlan (2002) gives a fuller characterization of the invariant multinormal distributions.

and abusing notation by identifying f and $(f, 0) \in T_{(f,(x,y))}(\mathcal{H} \times \mathbb{R}^2)$, we have $DF(f, (x, y))f = 0$. (To see this note that (x, y) is a root of αf for all $\alpha \in \mathbb{R}$.) Since f is a regular point of $F(\cdot, (x, y))$, there must be a vector $v \in T_f\mathcal{M}$ such that $DF(f, (x, y))v \neq 0$. It follows that f is a regular point of the restriction of $F(\cdot, (x, y))$ to $\mathcal{M}$, which implies that $(f, (x, y))$ is a regular point of $F|_{\mathcal{M}\times N}$, as desired.

Let

$$\pi_1 : V \to \mathcal{M} \quad \text{and} \quad \pi_2 : V \to N$$

be the natural projections. Sard's theorem implies that the critical values of π_1 constitute a set of measure zero in $\mathcal{M}$, so they can be ignored in computing the expected number of real roots of the equation. A measure μ on V may be defined by requiring that if $U \subset V$ is an open set containing only regular points of π_1 and the restriction of π_1 to U is injective, then $\mu(U) = \mathbf{U}_{\mathcal{M}}(\pi_1(U))$. For any open $Z \subset V$ the expected number of real roots corresponding to points $(x, y) \in Z$ is

$$\int_{\mathcal{M}} \#(\pi_1^{-1}(x) \cap Z)\, d\mathbf{U}_{\mathcal{M}}(x) = \mu(Z).$$

In turn there is a measure ν on N defined by requiring that, for each measurable $E \subset N$, $\nu(E) = \mu(\pi_2^{-1}(E))$, i.e., $\nu = \mu \circ \pi_2^{-1}$. If $Z = \pi_2^{-1}(\pi_2(Z))$, then $\mu(Z) = \nu(\pi_2(Z))$. In this sense ν is the distribution of roots.

It turns out that π_2 is the projection of a C^∞ sphere bundle. The *fiber* above $(x, y) \in N$ is

$$V_{(x,y)} := \{f \in \mathcal{M} : (f, (x, y)) \in V\}.$$

Consider a particular $(x_0, y_0) \in N$. As the set of coefficient vectors in $\mathcal{M}$ that are orthogonal to $(x_0^n, x_0^{n-1}y_0, \ldots, y_0^n)$, $V_{(x_0,y_0)}$ is an $(n-1)$-dimensional sphere in $\mathcal{M}$. Varying (x, y) in a neighborhood of (x_0, y_0) can be thought of as inducing a motion of the sphere $V_{(x,y)}$ in $\mathcal{M}$, and, roughly speaking, the probability of having a root in a small neighborhood of (x_0, y_0) will be proportional to the speed at which $V_{(x,y)}$ moves as we vary (x, y) near (x_0, y_0).

This intuition is made rigorous, at the natural level of generality, by an integral formula of Shub and Smale (1993, p. 273). (Cf. Blum et al. (1998, p. 240).) We now describe the consequence of this formula in the current context. For $(f, (x, y)) \in V$ let

$$A(f, (x, y)) : T_{(x,y)}N \to T_f\mathcal{M}$$

be the linear map whose graph is the orthogonal complement $\perp_{(f,(x,y))}$ of $T_{(f,(x,y))}V_{(x,y)}$ in $T_{(f,(x,y))}V$, and let $A^*(f, (x, y))$ be the adjoint of this map. In this setting the Shub–Smale integral formula states that for any open set $Z \subset V$,

$$\int_{\mathcal{M}} \#(\pi_1^{-1}(f) \cap Z)\, df = \int_N \int_{V_{(x,y)} \cap Z} \det\big(A^*(f,(x,y))A(f,(x,y))\big)^{1/2}\, df\, d(x,y). \tag{3}$$

(The idea expressed in this formula is geometric, insofar as the assumed measures on $\mathcal{M}$, N, and $V_{(x,y)}$ are the natural notions of volume derived from the inclusions of these spaces in $\mathcal{M}$, $\mathbb{R}^2$, and $\mathcal{M} \times \mathbb{R}^2$ respectively.) In particular, the expected number of roots is the integral of $\#(\pi_1^{-1}(f))$ with respect to the uniform distribution on $\mathcal{M}$, so

$$\int_{\mathcal{M}} \#(\pi_1^{-1}(f) \cap Z)\, d\mathbf{U}_{\mathcal{M}}(f) = \frac{1}{\operatorname{vol}(\mathcal{M})} \int_{\mathcal{M}} \#(\pi_1^{-1}(f) \cap Z)\, df. \tag{4}$$

When $Z = \pi_2^{-1}(Y)$ for some open $Y \subset N$, so that

$$\int_N \left(\int_{V_{(x,y)} \cap Z} \cdots\, df \right) d(x,y) = \int_Y \left(\int_{V_{(x,y)}} \cdots\, df \right) d(x,y),$$

the right-hand side of the formula above can be further simplified by exploiting the invariances arising out of the action

$$O(f,(x,y)) = (f \circ O^{-1}, O(x,y))$$

of $O(2)$ on $\mathcal{M} \times N$. Without going into any detail at this point (examples of this sort of argument occur in Section 5 and the Appendix of ENNE) we simply assert that

$$\int_{V_{(x,y)}} \det\big(A^*(f,(x,y))A(f,(x,y))\big)^{1/2}\, df$$

is a constant function of $(x,y) \in N$. So, since the action of $O(2)$ on N is transitive, for any open $Y \subset N$ and *any* $(x_0, y_0) \in N$ (including those not in Y)

$$\int_Y \int_{V_{(x,y)}} \det\big(A^*(f,(x,y))A(f,(x,y))\big)^{1/2}\, df\, d(x,y) = \operatorname{vol}(Y) \cdot \int_{V_{(x_0,y_0)}} \det\big(A^*(f,(x_0,y_0))A(f,(x_0,y_0))\big)^{1/2}\, df. \tag{5}$$

In evaluating the integral on the right-hand side we are now free to choose (x_0, y_0) as we please, and it turns out to be simplest to work with $(1,0)$. Note that for $f = (a_0, a_1, \ldots, a_n) \in \mathcal{M}$ the condition $f \in V_{(1,0)}$ amounts to $a_0 = 0$.

LEMMA A.1: *For all* $f = (0, a_1, \dots, a_n) \in V_{(1,0)}$,

$$\det\bigl(A^*(f,(1,0))A(f,(1,0))\bigr)^{1/2} = |a_1|.$$

PROOF: We have

$$\begin{aligned} T_f\mathcal{M} &:= \{\phi = (\phi_0, \phi_1, \dots, \phi_n) \in \mathcal{H} : \langle f, \phi\rangle = 0\} \\ &= \left\{\phi \in \mathcal{H} : \binom{n}{1}^{-1} a_1\phi_1 + \cdots + \binom{n}{n}^{-1} a_n\phi_n = 0\right\} \end{aligned}$$

and

$$T_{(1,0)}N = \{(\xi, \psi) \in \mathbb{R}^2 : (1,0)\cdot(\xi,\psi) = 0\} = \{(0,\psi) : \psi \in \mathbb{R}\}.$$

Observe that

$$\left.\frac{\partial(a_1x^{n-1}y + \cdots + a_ny^n)}{\partial y}\right|_{(x,y)=(1,0)} = a_1$$

and consequently

$$DF(f,(1,0))(\phi,(0,\psi)) = Df(1,0)(0,\psi) + \phi(1,0) = a_1\psi + \phi_0.$$

(Here $\phi(1,0)$ is the polynomial ϕ evaluated at $(1,0)$.) Therefore

$$T_{(f,(1,0))}V = \bigl\{(\phi,(0,\psi)) \in T_f\mathcal{M} \times T_{(1,0)}N : a_1\psi + \phi_0 = 0\bigr\}.$$

A vector $(\phi,(0,\psi)) \in T_{(f,(1,0))}V$ is in $T_{(f,(1,0))}V_{(1,0)}$ if and only if $\psi = 0$, in which case $\phi_0 = 0$. That is,

$$T_{(f,(1,0))}V_{(1,0)} = \bigl\{\bigl((0,\phi_1,\dots,\phi_n),(0,0)\bigr) \in T_f\mathcal{M} \times T_{(1,0)}N\bigr\}.$$

A vector $(\phi,(0,\psi)) \in T_{(f,(1,0))}V$ that is orthogonal to $T_{(f,(1,0))}V_{(1,0)}$ must satisfy

$$0 = \langle\phi,\phi'\rangle = \binom{n}{1}^{-1}\phi_1\phi_1' + \cdots + \binom{n}{n}^{-1}\phi_n\phi_n'$$

for all $\phi' \in T_f\mathcal{M}$ such that $\phi_0' = 0$, and $0 = \binom{n}{1}^{-1}a_1\phi_1 + \cdots + \binom{n}{n}^{-1}a_n\phi_n$ since $\phi \in T_f\mathcal{M}$, so $\phi_1 = \cdots = \phi_n = 0$. This means that

$$\perp_{(f,(1,0))} = \bigl\{\bigl((\phi_0,0,\dots,0),(0,\psi)\bigr) : a_1\psi + \phi_0 = 0\bigr\},$$

and $A(f,(1,0))$ is the linear map taking $(0,1) \in T_{(1,0)}N$ to $(-a_1,0,\dots,0) \in T_f\mathcal{M}$. The adjoint $A^*(f,(1,0))$ is the map taking $(\phi_0,\phi_1,\dots,\phi_n) \in T_f\mathcal{M}$ to

$(0, -a_1\phi_0) \in T_{(1,0)}N$. We conclude that $A^*(f, (1, 0))A(f, (1, 0))$ is the map taking $(0, 1) \in T_{(1,0)}N$ to $(0, a_1^2) \in T_{(1,0)}N$, and its determinant is a_1^2, as desired. *Q.E.D.*

We now endow $V_{(1,0)}$ with the geometrically natural coordinate system. Recalling that the inner product (2) is not the usual one, let

$$z_0 = \frac{a_0}{\sqrt{\binom{n}{0}}}, \quad z_1 = \frac{a_1}{\sqrt{\binom{n}{1}}}, \ldots, \quad z_n = \frac{a_n}{\sqrt{\binom{n}{n}}}$$

be the system of coordinates for $\mathcal{H}$ in which the ith standard unit basis vector is the vector of unit length, relative to (2), that is a positive multiple of the coefficient of the monomial $x^{n-i}y^i$. Then $V_{(1,0)}$ is the $(n-1)$-dimensional unit sphere in the coordinate subspace given by $z_0 = 0$, so

$$\text{(6)} \qquad \int_{V_{(1,0)}} |a_1|\, dz = \sqrt{n} \int_{V_{(1,0)}} |z_1|\, dz.$$

Summarizing the work to this point, (3), (4), (5), and (6) combine to imply that, for any open $Y \subset N$:

$$\text{(7)} \qquad \int_{\mathcal{M}} \#\big(\pi_1^{-1}(f) \cap \pi_2^{-1}(Y)\big)\, d\mathbf{U}_{\mathcal{M}}(f) = \frac{\operatorname{vol}(Y)}{\operatorname{vol}(\mathcal{M})} \cdot \sqrt{n} \int_{V_{(1,0)}} |z_1|\, df.$$

The calculation may be completed in numerous ways. We choose one that illustrates in simplified form the ideas underlying Lemmas 5.11 and 5.12 and the notation employed there. Let $D = (-1, 1)$ be the open unit disk in $\mathbb{R}^1$, and let $E = S^{n-2}$ and $F = S^{n-1}$ be the $(n-2)$-dimensional and $(n-1)$-dimensional unit spheres. The fiber $V_{(1,0)}$ corresponds to F, and we will arrive at a more tractable version of the right-hand side of (7) by means of the change of variables

$$\gamma : D \times E \to F \quad \text{defined by} \quad \gamma(p, r) = \big(p, (1 - p^2)^{1/2} r\big).$$

The determinant of the Jacobean of this function is computed as follows. The partial derivative of γ with respect to p is $(1, -p(1 - p^2)^{-1/2} r)$ and the norm of this vector is $(1 - p^2)^{-1/2}$. Evaluating $D\gamma(p, r)$ at the various elements of an orthonormal basis of $T_{(p,r)}(D \times E) = T_pD \times T_rE$ whose first element is in T_pD, one finds that the image is a pairwise orthogonal basis of $T_{\gamma(p,r)}F$ whose elements other than the first all have length $(1 - p^2)^{-1/2}$. Therefore

$$|\det D\gamma(p, r)| = (1 - p^2)^{-1/2}((1 - p^2)^{1/2})^{n-2} = (1 - p^2)^{(n-3)/2}.$$

Noting that the indefinite integral of $p(1 - p^2)^{(n-3)/2}$ is $-(1 - p^2)^{(n-1)/2}/(n - 1)$, we combine these calculations in the change of variables formula, computing that

$$\begin{aligned}\int_F |z_1|\,df &= \int_{D\times E} |p|\cdot|\det D\gamma(p,r)|\,d(p,r)\\ &= \mathrm{vol}(E)\cdot\int_D |p|\cdot(1-p^2)^{(n-3)/2}\,dp\\ &= 2\,\mathrm{vol}(E)\int_0^1 p(1-p^2)^{(n-3)/2}\,dp\\ &= \frac{2\,\mathrm{vol}(E)}{n-1}.\end{aligned}$$

Substituting this in (7) and applying the formula

$$\mathrm{vol}(S^{m-1}) = 2\frac{\pi^{m/2}}{\Gamma(\frac{m}{2})} \qquad (m \geq 1) \tag{8}$$

for the volume of the $(m-1)$-dimensional unit sphere (e.g., Federer (1969, p. 251)) we may finally conclude that, for any open $Y \subset N$,

$$\begin{aligned}&\int_{\mathcal{M}} \#\big(\pi_1^{-1}(f)\cap\pi_2^{-1}(Y)\big)\,d\mathbf{U}_{\mathcal{M}}(x)\\ &\quad = \mathbf{U}_N(Y)\cdot\frac{\mathrm{vol}(N)\,\mathrm{vol}(E)}{\mathrm{vol}(\mathcal{M})}\cdot\frac{2\sqrt{n}}{n-1}\\ &\quad = \mathbf{U}_N(Y)\cdot\frac{2\pi\cdot\Gamma(\frac{n+1}{2})\cdot 2\pi^{(n-1)/2}}{2\pi^{(n+1)/2}\cdot\Gamma(\frac{n-1}{2})}\cdot\frac{2\sqrt{n}}{n-1} = \mathbf{U}_N(Y)\cdot 2\sqrt{n},\end{aligned}$$

where the last equality derives from the formula $\Gamma(s+1) = s\Gamma(s)$. Since there are two roots in the circle corresponding to each point in one-dimensional projective space, the mean number of roots in one-dimensional projective space is $\sqrt{n}$.

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