Econometrica Supplementary Material

SUPPLEMENT TO "VECTOR EXPECTED UTILITY AND ATTITUDES TOWARD VARIATION"

(*Econometrica*, Vol. 77, No. 3, May 2009, 801–855)

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This document contains the following supplemental material: an omitted proof (Section S.1), formal statements and proofs of results characterizing complementary independence for other decision models (Section S.2), probabilistic sophistication for VEU preferences (Section S.3), and the analysis of the consumption–savings example of Section 4.5 (Section S.4).

S.1. OMITTED PROOFS

PROOF OF LEMMA 2: A binary relation $\succeq$ on a convex subset Φ of $B(\Sigma)$ is a *preorder* if it is reflexive and transitive; is *monotonic* if $a \geq b$ implies $a \succeq b$; is *conic* if $a \succeq b$ and $\alpha \in (0,1)$ imply $\alpha a + (1-\alpha)c \succeq \alpha b + (1-\alpha) \succeq c$; is *continuous* if $a^k \to a$, $b^k \to b$, and $a^k \succeq b^k$ for all k imply $a \succeq b$; is *nontrivial* if $a \succeq b$ and not $b \succeq a$ for some a, b.

Now, for $a, b \in B_0(\Sigma, u(X))$, let $a \succeq_0 b$ iff the left-hand side of Eq. (22) holds; also, for $a, b \in B(\Sigma, u(X))$, let $a \succeq b$ iff the left-hand side of Eq. (23) holds.

I closely mimic Proposition 4 in Ghirarduto, Maccheroni, and Marinacci (2004; GMM). Monotonicity, transitivity, and continuity of $\succeq_0$ and $\succeq$ follow directly from the definition and the properties of I. Reflexivity follows from monotonicity. To show that $\succeq_0$ and $\succeq$ are conic (i.e., independent), consider $\alpha \in (0,1)$ and $a, b, c \in B_0(\Sigma, u(X))$ or, respectively, $B(\Sigma, u(X))$. Then, for all $\beta \in (0,1]$, note that $\beta[\alpha a + (1-\alpha)c] + (1-\beta)d = \beta\alpha a + (1-\beta\alpha)[\frac{\beta(1-\alpha)}{1-\beta\alpha}c + \frac{1-\beta}{1-\beta\alpha}d]$ and similarly for b. Thus, $a \succeq_0 b$ or, respectively, $a \succeq b$ implies, in particular, that

$$\begin{aligned}
&I\big(\beta[\alpha a + (1-\alpha)c] + (1-\beta)d\big) \\
&\quad = I\left(\beta\alpha a + (1-\beta\alpha)\left[\frac{\beta(1-\alpha)}{1-\beta\alpha}c + \frac{1-\beta}{1-\beta\alpha}d\right]\right) \\
&\quad \geq I\left(\beta\alpha b + (1-\beta\alpha)\left[\frac{\beta(1-\alpha)}{1-\beta\alpha}c + \frac{1-\beta}{1-\beta\alpha}d\right]\right) \\
&\quad = I\big(\beta[\alpha b + (1-\alpha)c] + (1-\beta)d\big)
\end{aligned}$$

for all $\beta \in (0,1]$, so $\alpha a + (1-\alpha)c \succeq_0 \alpha b + (1-\alpha)c$ or, respectively, $\alpha a + (1-\alpha)c \succeq \alpha b + (1-\alpha)c$. The case $\alpha = 1$ is trivial.

Finally, if $\succeq_0$ is trivial, then in particular the conjunction "$\gamma \succeq_0 \gamma'$ and not $\gamma' \succeq_0 \gamma$" is false for all $\gamma, \gamma' \in u(X)$. Take $\gamma > \gamma'$: then $\gamma \succeq_0 \gamma'$ by monotonicity, and so it must be the case that also $\gamma' \succeq_0 \gamma$. By the definition of $\succeq_0$, taking

 DOI: 10.3982/ECTA7564

$\alpha = 1$, this implies that $I(\gamma) = I(\gamma')$, which contradicts the fact that I is normalized. The same argument applies to $\succeq$.

The first claim now follows by applying Proposition A.2 in GMM to $\succeq_0$.

For the second statement, note that continuity of I implies that the left-hand side of Eq. (23) holds iff $I(\alpha a + (1-\alpha)c) \geq I(\alpha b + (1-\alpha)c)$ for all $c \in B_0(\Sigma, u(X))$: that is, one can restrict attention to mixtures with simple functions. It then follows that $\succeq_0$ is the restriction of $\succeq$ to $B_0(\Sigma, u(X))$.

Define $\succeq'$ on $B(\Sigma, u(X))$ by stipulating that, for all $a, b \in B(\Sigma, u(X))$, $a \succeq' b$ iff $q(a) \geq q(b)$ for all $q \in \mathcal{C}$. Then $\succeq'$ is easily seen to be a nontrivial, monotonic, continuous, conic preorder, and clearly $a \succeq' b$ iff $a \succeq b$ for $a, b \in B_0(\Sigma, u(X))$: that is, $\succeq_0$ is also the restriction of $\succeq'$ to $B_0(\Sigma, u(X))$. Therefore, for all $a, b \in B_0(\Sigma, u(X))$, $a \succeq b$ iff $a \succeq' b$. It remains to be shown that this implies $\succeq = \succeq'$.

Thus, suppose $a \succeq b$ for some $a, b \in B(\Sigma, u(X))$. Then, for every $\alpha \in (0, 1)$, $\alpha a(\Omega), \alpha b(\Omega) \subset \operatorname{int} u(X)$ and $\alpha a \succeq \alpha b$ because $\succeq$ is conic. Hence, there exist sequences (a^k), (b^k) in $B_0(\Sigma, u(X))$ such that $a^k \geq \alpha a$, $b^k \leq \alpha b$, $a^k \to \alpha a$, and $b^k \to \alpha b$ in the supremum norm. Then $a^k \succeq \alpha a \succeq \alpha b \succeq b^k$ for all k, so also $a^k \succeq' b^k$. Since $\succeq'$ is continuous, taking limits as $k \to \infty$ yields $\alpha a \succeq' \alpha b$ and taking limits as $\alpha \to 1$ yields $a \succeq' b$. Exchanging the roles of $\succeq$ and $\succeq'$ yields the converse implication. *Q.E.D.*

S.2. CHARACTERIZATIONS OF COMPLEMENTARY INDEPENDENCE FOR OTHER MODELS

PROPOSITION 7—Complementary Independence for MEU and CEU Preferences:

1. *A MEU preference $\succcurlyeq$ satisfies Axiom 7 if and only if there is $p \in C$ such that, for all $q \in C$, $2p - q \in C$ (that is, p is the barycenter of C).*
2. *A CEU preference $\succcurlyeq$ satisfies Axiom 7 if and only if there is $p \in ba_1(\Sigma)$ such that, for all $E \in \Sigma$, $v(E) + [1 - v(\Omega \setminus E)] = 2p(E)$.*

In statements 1 and 2, $p \in ba_1(\Sigma)$ is the unique probability charge that satisfies $f \succcurlyeq \bar{f} \Leftrightarrow \int u \circ f \, dp \geq \int u \circ \bar{f} \, dp$ for all complementary pairs $(f, \bar{f})$, where u is the utility function in the MEU or CEU representation of $\succcurlyeq$.

PROOF: Part 1 follows from Lemma 3 and the observation that, for MEU preferences, the set $\mathcal{C}$ constructed in Lemma 2 coincides with C (cf. GMM, Section 5.1).

For part 2, notice that the Choquet integral is positively homogeneous; hence, I has a unique extension from $B_0(\Sigma, u(X))$ to $B_0(\Sigma)$, and $J(a) = \frac{1}{2}I(a) - \frac{1}{2}I(-a)$ for all $a \in B_0(\Sigma)$. If $\succcurlyeq$ satisfies complementary independence, then, using the VEU representation, $I(1_E) = p(E) + A(\mathrm{E}_p[\zeta 1_E])$ and $I(-1_E) = -p(E) + A(-\mathrm{E}_p[\zeta 1_E]) = -p(E) + A(\mathrm{E}_p[\zeta 1_E])$, so $I(1_E) - I(-1_E) = 2p(E)$. On the other hand, using the CEU representation, $I_v(E) = v(E)$ and $I_v(-1_E) = -[1 - v(\Omega \setminus E)]$; since $I = I_v$, the claim follows. In the opposite direction, suppose that $a=\sum_{k=1}^K \alpha_k 1_{E_k}$ for a partition $E_1,\dots,E_K$ of Ω and numbers $\alpha_1<\alpha_2<\cdots<\alpha_K$. Then $I_v(a)=\sum_{k=1}^K \alpha_k[v(\bigcup_{\ell=k}^K E_\ell)-v(\bigcup_{\ell=k+1}^K E_\ell)]$ and similarly, invoking the condition in the proposition,

$$\begin{aligned} I_v(-a) &= \sum_{k=1}^K(-\alpha_k)\left[v\left(\bigcup_{\ell=1}^k E_\ell\right)-v\left(\bigcup_{\ell=1}^{k-1} E_\ell\right)\right] \\ &= \sum_{k=1}^K(-\alpha_k)\left[2p\left(\bigcup_{\ell=1}^k E_\ell\right)-1+v\left(\bigcup_{\ell=k+1}^K E_\ell\right)-2p\left(\bigcup_{\ell=1}^{k-1} E_\ell\right)\right. \\ &\qquad \left. +1-v\left(\bigcup_{\ell=k}^K E_\ell\right)\right] \\ &= -2\sum_{k=1}^K \alpha_k p(E_k)+I_v(a), \end{aligned}$$

and so $\frac{1}{2}I(a)-\frac{1}{2}I(-a)=J(a)$, where J is the linear functional represented by p. The claim now follows from Lemma 1. *Q.E.D.*

PROPOSITION 8—Complementary Independence for Variational Preferences: *Let $\succcurlyeq$ be a variational preference and assume that the utility function u is unbounded either above or below. Then $\succcurlyeq$ satisfies Axiom 7 if and only if there exists $p\in ba_1(\Sigma)$ such that*

$$\forall q\in ba_1(\Sigma),\quad 2p-q\in ba_1(\Sigma)\quad\Rightarrow\quad c^*(q)=c^*(2p-q)$$

and

$$2p-q\notin ba_1(\Sigma)\quad\Rightarrow\quad c^*(q)=\infty.$$

In particular, $c^*(p)=0$. Finally, p is the unique probability charge such that, for all complementary pairs $(f,\bar f)$, $f\succcurlyeq \bar f \Leftrightarrow \int u\circ f\,dp\geq\int u\circ \bar f\,dp$.

The reader is referred to Maccheroni, Marinacci, and Rustichini (2006) for a discussion of the unboundedness assumption.

PROOF OF PROPOSITION 8: The preference $\succcurlyeq$ has a niveloidal representation I_{c^*}, u, where $I_c(a)=\min_{q\in ba_1(\Sigma)}\int a\,dq+c^*(q)$. For conciseness, say that c^* is *symmetric around* $p\in ba_1(\Sigma)$ iff it satisfies the condition in Proposition 8. By Lemma 1, Axiom 7 holds iff the functional J defined by $J(a)=\frac{1}{2}\gamma+\frac{1}{2}I_{c^*}(a)-\frac{1}{2}I_{c^*}(\gamma-a)$ is affine. Thus it suffices to show that J is affine iff c^* is symmetric around p.

Suppose that c^* is symmetric around p. Consider a complementary pair $(f, \bar{f})$ and let $z \in X$ be such that $\frac{1}{2}f(\omega) + \frac{1}{2}\bar{f}(\omega) \sim z$. Thus, $a \equiv u \circ f = 2u(z) - u \circ \bar{f} \equiv \gamma - u \circ \bar{f}$. Now let $q^* \in \arg\min_{q \in ba_1(\Sigma)} \int a\,dq + c^*(q)$; since clearly $c^*(q^*) < \infty$, $2p - q^* \in ba_1(\Sigma)$ and $c^*(q^*) = c^*(2p - q^*)$. Now, for all $q \in ba_1(\Sigma)$ such that $2p - q \in ba_1(\Sigma)$,

$$\begin{aligned}
&\int (\gamma - a)\,d(2p - q) + c^*(2p - q) \\
&\quad = \gamma - 2\int a\,dp + \int a\,dq + c^*(q) \\
&\quad \geq \gamma - 2\int a\,dp + \int a\,dq^* + c^*(q^*) \\
&\quad = \int (\gamma - a)\,d(2p - q^*) + c^*(2p - q^*).
\end{aligned}$$

Since any $q \in ba_1(\Sigma)$ such that $2p - q \in ba_1(\Sigma)$ can obviously be written as $q = 2p - [2p - q]$ and all other $q \in ba_1(\Sigma)$ have $c^*(q) = \infty$, it follows that $I_{c^*}(\gamma - a) = \gamma - 2\int a\,dp + \int a\,dq^* + c^*(2p - q^*) = \gamma - 2\int a\,dp + I_{c^*}(a)$. Therefore, $J(a) = \frac{1}{2}\gamma + \frac{1}{2}I_{c^*}(a) - \frac{1}{2}I_{c^*}(\gamma - a) = \int a\,dp$, that is, J is affine and represented by p.

In the opposite direction, suppose that $\gamma + \frac{1}{2}I_{c^*}(a) - \frac{1}{2}I_{c^*}(\gamma - a) = \int a\,dp$ for all $a, \gamma - a \in B_0(\Sigma)$; also, for every $f \in \mathcal{F}_0$, let $m_f \in X$ be such that $u(m_f) = \frac{1}{2}\min_{\omega \in \Omega} u(f(\omega)) + \frac{1}{2}\max_{\omega \in \Omega} u(f(\omega))$ and recall that $u(x_f) = I_{c^*}(u \circ f)$. For every $q \in ba_1(\Sigma)$ such that $2p - q \in ba_1(\Sigma)$,

$$\begin{aligned}
c^*(2p - q) &= \sup_{f \in \mathcal{F}_0} u(x_f) - \int u \circ f\,d(2p - q) \\
&= -2\int u \circ f\,dp + \sup_{f \in \mathcal{F}_0} I_{c^*}(u \circ f) - \int (-u \circ f)\,dq \\
&= -2\int u \circ f\,dp + \sup_{f \in \mathcal{F}_0} 2\int u \circ f\,dp \\
&\qquad + I_{c^*}(2u(m_f) - u \circ f) - 2u(m_f) - \int (-u \circ f)\,dq \\
&= \sup_{f \in \mathcal{F}_0} I_{c^*}(2u(m_f) - u \circ f) - \int [2u(m_f) - u \circ f]\,dq \\
&= \sup_{f \in \mathcal{F}_0} I_{c^*}(u \circ f) - \int u \circ f\,dq = c^*(q).
\end{aligned}$$

The last step follows because, for every $f \in \mathcal{F}_0$, there is $\bar{f} \in \mathcal{F}_0$ such that $u \circ \bar{f} = 2u(m_f) - u \circ f$; therefore, computing the supremum over $f \in \mathcal{F}_0$ is the same as computing it over the complementary acts $\bar{f}$ constructed from each $f \in \mathcal{F}_0$ in this way. If instead $2p - q \notin ba_1(\Sigma)$ but $c^*(q) < \infty$, the above calculations still show that

$$\sup_{f \in \mathcal{F}_0} u(x_f) - \int u \circ f \, d(2p - q) = c^*(q) < \infty.$$

Now $2p(\Omega) - q(\Omega) = 1$, so there must be $E \in \Sigma$ such that $2p(E) - q(E) < 0$. Therefore,

$$\begin{aligned}
&\sup_{f \in \mathcal{F}_0} u(x_f) - \int u \circ f \, d(2p - q) \\
&\quad = \sup_{f \in \mathcal{F}_0} I_{c^*}(u \circ f) - \int u \circ f \, d(2p - q) \\
&\quad \geq \sup_{\alpha, \beta \in u(X) : \alpha > \beta} I_{c^*}(\beta + (\alpha - \beta)1_E) - \int [\beta + (\alpha - \beta)1_E] \, d(2p - q) \\
&\quad = \sup_{\alpha, \beta \in u(X) : \alpha > \beta} I_{c^*}(\beta + (\alpha - \beta)1_E) - \beta - (\alpha - \beta)[2p(E) - q(E)] \\
&\quad \geq \sup_{\alpha, \beta \in u(X) : \alpha > \beta} \beta - \beta - (\alpha - \beta)[2p(E) - q(E)] = \infty,
\end{aligned}$$

which contradicts $c^*(q) < \infty$. The second equality follows from the fact that $2p(\Omega) - q(\Omega) = 1$, and the second inequality follows from monotonicity of I_{c^*}; the final equality uses the fact that $u(X)$ is unbounded and $2p(E) - q(E) < 0$. *Q.E.D.*

PROPOSITION 9: *Let $\succcurlyeq$ be a smooth-ambiguity preference (with finite support μ). If there exists $p \in ba_1(\Sigma)$ such that $\mu(q) = \mu(2p - q)$ for all $q \in ba_1(\Sigma)$, then Axiom 7 holds. Furthermore, if $0 \in \operatorname{int} u(X)$, p is the only probability charge such that, for all complementary pairs $(f, \bar{f})$, $f \succcurlyeq \bar{f}$ iff $\mathrm{E}_p[u \circ f] \geq \mathrm{E}_p[u \circ \bar{f}]$.*

PROOF: Let $(h, \bar{h})$ be complementary and write $a = u \circ h$, $\gamma - a = u \circ \bar{h}$. Then $h \succcurlyeq \bar{h}$ iff $\int \phi(\mathrm{E}_q[a]) \, d\mu \geq \int \phi(\mathrm{E}_q[\gamma - a]) \, d\mu$, that is, iff $\int \phi(\mathrm{E}_q[a]) \, d\mu \geq \int \phi(\gamma + \mathrm{E}_q[-a]) \, d\mu$. Under the assumption that $\mu(q) = \mu(2p - q)$, this can be rewritten as

$$\begin{aligned}
\int \phi(\mathrm{E}_q[a]) \, d\mu &\geq \int \phi(\gamma + \mathrm{E}_{2p-q}[-a]) \, d\mu \\
&= \int \phi(\gamma - 2\mathrm{E}_p[a] + \mathrm{E}_q[a]) \, d\mu.
\end{aligned}$$

Since ϕ is strictly increasing, this holds if and only if $\mathrm{E}_p[a] \geq \frac{\gamma}{2}$.

Now let f, $\bar{f}$, g, $\bar{g}$, and α be as in Axiom 7. Suppose that $f \succcurlyeq \bar{f}$ and $g \succcurlyeq \bar{g}$. Letting $u \circ \bar{f} = \gamma_f - u \circ f$ and $u \circ \bar{g} = \gamma_g - u \circ g$, the preceding argument implies that $\mathrm{E}_p[u \circ f] \geq \frac{1}{2}\gamma_f$ and $\mathrm{E}_p[u \circ g] \geq \frac{1}{2}\gamma_g$. Hence, $\mathrm{E}_p[u \circ (\alpha f + (1-\alpha)g)] \geq \gamma_\alpha \equiv \alpha\gamma_f + (1-\alpha)\gamma_g$. Since $u \circ (\alpha\bar{f} + (1-\alpha)\bar{g}) = \gamma_\alpha - u \circ (\alpha f + (1-\alpha)g)$, conclude that $\alpha f + (1-\alpha)g \succcurlyeq \alpha\bar{f} + (1-\alpha)\bar{g}$, that is, the axiom holds.

Finally, if $u \circ \bar{f} = \gamma - u \circ f$, then as noted above, $f \succcurlyeq \bar{f}$ iff $\mathrm{E}_p[u \circ f] \geq \frac{1}{2}\gamma$. Substituting for γ and simplifying, this is equivalent to $\frac{1}{2}\mathrm{E}_p[u \circ f] \geq \frac{1}{2}\mathrm{E}_p[u \circ \bar{f}]$, and the factor $\frac{1}{2}$ can be dropped. Now consider $q \neq p$, so there is $a \in B_0(\Sigma)$ with $\mathrm{E}_p[a] > E_q[a]$. Since by assumption $0 \in \operatorname{int} u(X)$, assume $[-1, 1] \subset u(X)$. Construct $f \in \mathcal{F}_0$ such that $u \circ f(\Omega) \subset [0, \frac{1}{2}]$ and $u \circ f = \alpha a + \beta$ with $\alpha > 0$. Then let $\bar{f} \in \mathcal{F}$ be such that $u \circ \bar{f} = -u \circ f$. Finally, construct g and $\bar{g}$ such that $u \circ g = u \circ f - \mathrm{E}_p[u \circ f]$ and $u \circ \bar{g} = u \circ \bar{f} - \mathrm{E}_p[u \circ \bar{f}]$: this is possible as $\frac{1}{2} \geq u \circ f(\omega) \geq 0 \geq u \circ \bar{f}(\omega) \geq -\frac{1}{2}$ and $[-1, 1] \subset u(X)$. Clearly, $\mathrm{E}_p[u \circ g] = 0 = \mathrm{E}_p[u \circ \bar{g}]$ and $u \circ \bar{g} = -u \circ f + \mathrm{E}_p[u \circ f] = -u \circ g$; hence, $g \sim \bar{g}$. However, $\mathrm{E}_q[u \circ g] = \mathrm{E}_q[u \circ f] - \mathrm{E}_p[u \circ f] < 0$ and $\mathrm{E}_q[u \circ \bar{g}] = \mathrm{E}_q[-u \circ g] > 0$, that is, $\mathrm{E}_q[u \circ \bar{g}] > \mathrm{E}_q[u \circ g]$, which is inconsistent with $g \sim \bar{g}$. *Q.E.D.*

S.3. PROBABILISTIC SOPHISTICATION FOR VEU PREFERENCES

An induced likelihood ordering $\succcurlyeq_\ell$ is *represented* by a probability $\mu \in ca_1(\Sigma)$ iff, for all $E, F \in \Sigma$, $E \succcurlyeq_\ell F$ iff $\mu(E) \geq \mu(F)$. Finally, a probability measure μ is *convex-ranged* iff, for every event $E \in \Sigma$ such that $\mu(E) > 0$ and for every $\alpha \in (0, 1)$, there exists $A \in \Sigma$ such that $A \subset E$ and $\mu(A) = \alpha\mu(E)$.

PROPOSITION 10: *Fix a VEU preference relation $\succcurlyeq$ and let $p \in ca_1(\Sigma)$ be the corresponding baseline probability. If the induced likelihood ordering $\succcurlyeq_\ell$ is represented by a convex-ranged probability measure $\mu \in ca_1(\Sigma)$, then $\mu = p_1$.*

PROOF: Fix $x, y \in X$ with $x \succ y$. Since the ranking of bets xEy is represented by μ and also by the map defined by $E \mapsto u(x)p(E) + u(y)p(E^c) + A(\mathrm{E}_p[\zeta \cdot xEy])$, there exists an increasing function $g : [0, 1] \to [u(y), u(x)]$ such that $u(x)p(E) + u(y)p(E^c) + A(\mathrm{E}_p[\zeta \cdot xEy]) = g(\mu(E))$ for all events E [this function g will in general depend upon x and y, but this is inconsequential]. Since $A(\mathrm{E}_p[\zeta \cdot yEx]) = A(\mathrm{E}_p[\zeta \cdot (x + y - xEy)]) = A(\mathrm{E}_p[\zeta \cdot xEy])$,

$$g(\mu(E)) - g(1 - \mu(E)) = [u(x) - u(y)](2p(E) - 1) \tag{28}$$

for all events $E \in \Sigma$. Since g is increasing, so is the map $\gamma \mapsto g(\gamma) - g(1 - \gamma)$; thus, $\mu(E) = \mu(F)$ if and only if $p(E) = p(F)$. Now, since μ is convex-ranged, for any integer n there exists a partition $\{E_1^n, \ldots, E_n^n\}$ of Ω such that $\mu(E_j^n) = \frac{1}{n}$ for all $j = 1, \ldots, n$; correspondingly, $p(E_j^n) = p(E_k^n)$ for all $j, k \in$

$\{1, \ldots, n\}$ and, therefore, $p(E_j^n) = \frac{1}{n}$ for all $j = 1, \ldots, n$. This implies that for every event E such that $\mu(E)$ is rational, $p(E) = \mu(E)$.

To extend this equality to arbitrary events, note that for every event E such that $\mu(E) > 0$ and number $r < \mu(E)$, since μ is convex-ranged, there exists $L \subset E$ such that $\mu(L) = \frac{r}{\mu(E)}\mu(E) = r$. Similarly, for every event E such that $\mu(E) < 1$ and number $r > \mu(E)$, there exists an event $U \supset E$ such that $\mu(U) = r$. To see this, note that $\mu(\Omega \setminus E) > 0$ and $1 - r < \mu(\Omega \setminus E)$, so there exists $L \subset \Omega \setminus E$ such that $\mu(L) = 1 - r$; hence, $U = \Omega \setminus L$ has the required properties.

Now consider sequences of rational numbers $\{\ell_n\}_{n\geq 0} \subset [0, 1]$ and $\{u_n\}_{n\geq 0} \subset [0, 1]$ such that $\ell_n \uparrow \mu(E)$ and $u_n \downarrow \mu(E)$. By the preceding argument, for every $n \geq 1$ there exist sets $L_n \subset E \subset U_n$ such that $\mu(L_n) = \ell_n$ and $\mu(U_n) = u_n$. It was shown above that $p(L_n) = \mu(L_n)$ and $p(U_n) = \mu(U_n)$; moreover, $L_n \subset E \subset U_n$ implies that $p(L_n) \leq p(E) \leq p(U_n)$. Therefore, $p(E) = \mu(E)$. *Q.E.D.*

S.4. CONSUMPTION–SAVINGS PROBLEM: FORMALITIES

As a preliminary step, consider a two-period version of the problem with EU preferences,

$$\max_{s\in[0,w]} v(w - s) + \delta[\pi v(Hs) + (1 - \pi)v(Ls)];$$

that is, find the optimal amount of savings s given wealth w, discount factor δ, and probability of high return π. It is easy to verify that the solution is linear: $s = \alpha w$, where $\alpha \in (0, 1)$ depends upon all parameters but not on w. This standard result will be used below to construct the solution to the multiperiod problem with VEU preferences.

Now verify Eqs. (10), (11), and (12). Fix $0 \leq \tau < T$ and $0 \leq t < T - 1$. If $t \geq \tau - 1$, then one easily verifies that $\mathrm{E}_p[\zeta_t|\Pi_\tau(\omega)] = \mathrm{E}_p[\zeta_t] = 0$ for all ω. If instead $t < \tau - 1$, then $\mathrm{E}_p[\zeta_t|\Pi_\tau(\omega)] = \zeta_t(\omega)$.

For $\tau = 0$, this implies that $(\zeta_t)_{0\leq t<T-1}$ satisfies the properties in Definition 1. For $\tau > 0$, together with Eq. (7), this implies that $\zeta_{t,\Pi_\tau(\omega)}(\omega) = p(\Pi_\tau(\omega))\zeta_t(\omega)$ for $t \geq \tau - 1$ and $\zeta_{t,\Pi_\tau(\omega)}(\omega) = 0$ otherwise. Equation (12) follows immediately.

This fact and Eq. (7) imply that, for all $F \in \Pi_\tau$,

$$\begin{aligned} V_F(f) &= \mathrm{E}_p[u \circ f|F] - \sum_{t=0}^{T-2} \bigl|\mathrm{E}_p[\zeta_{t,F} u \circ f|F]\bigr| \\ &= \sum_{t=0}^{T} \delta^t \mathrm{E}_p[v \circ f_t|F] - \sum_{t=\max(0,\tau-1)}^{T-2} \left| \mathrm{E}_p\left[\zeta_{t,F} \sum_{s=0}^{T} \delta^s v \circ f_s \,\middle|\, F \right] \right|. \end{aligned}$$

Now if $s \leq \tau$, then $\mathrm{E}_p[\zeta_t v \circ f_s|\Pi_\tau(\omega)] = v \circ f_s(\omega)\mathrm{E}_p[\zeta_t|\Pi_\tau(\omega)]$, which is 0 for $t \geq \tau - 1$. If $s > \tau$ and $t \geq s$, then f_s depends upon $r_0, \ldots, r_{s-1}$ and ζ_t

depends upon r_t, r_{t+1}, and these are independent (given $\Pi_\tau(\omega)$), so $\mathrm{E}_p[\zeta_t v \circ f_s|\Pi_\tau(\omega)] = \mathrm{E}_p[v \circ f_s|\Pi_\tau(\omega)]\mathrm{E}_p[\zeta_t|\Pi_\tau(\omega)]$, which again equals 0. Finally, if $s = t+1$, then $\mathrm{E}_p[\zeta_t v \circ f_{t+1}|\Pi_\tau(\omega)] = \mathrm{E}_p[\mathrm{E}_p[\zeta_t v \circ f_{t+1}|\Pi_{t+1}(\omega)]\big|\Pi_\tau] = \mathrm{E}_p[v \circ f_{t+1}\mathrm{E}_p[\zeta_t|\Pi_{t+1}]\big|\Pi_\tau(\omega)] = 0$, because $t \geq \max(0, \tau - 1)$ implies $t+1 \geq \tau$ and $\mathrm{E}_p[\zeta_t|F] = 0$ for all $F \in \Pi_{t+1}$. Taking $\tau = 0$, this argument yields Eq. (10) directly; for $\tau > 0$, note that since $t \geq \tau - 1$, $\zeta_{t,\Pi_\tau(\omega)}(\omega) = p(\Pi_\tau(\omega))\zeta_t(\omega)$, and so again Eq. (11) follows (cf. footnote 28).

Consistent planning can be formalized as follows. Let $B_T = \{f \in \mathcal{F}_A(w_0) : f_T = w_T^f\}$. Then, assuming that $B_{\tau+1}$ has been defined for $\tau < T$, let

$$B_\tau = \bigcap_{\omega\in\Omega}\bigcup_{w\geq 0} \arg \max_{f\in B_{\tau+1} : w_\tau^f(\omega)=w} V_\tau(f|\Pi_\tau(\omega)).$$

The following result implies the stated equivalence (see item 4 in the proposition for $\tau = 0$). For $a, b \in \{H, L\}$, let $\eta(a, b) = 1$ if $a = b$ and $\eta(a, b) = -1$ otherwise.

PROPOSITION 11: *For all $w \geq 0$, $\tau = 0, \ldots, T$, and $F \in \Pi_\tau$, the problem in Eq.* (13) *has a unique solution, which takes the form $s_{\tau,F}(w) = \alpha_\tau w$; for $\varepsilon > 0$ small, $\alpha_{\tau,F} \in [0, 1]$. Furthermore,*

$$\begin{aligned}
&V_\tau(w) = \beta_\tau^p v(w),\\
&\Phi_{\tau,t}(w|F) = \beta_{\tau,t} v(w) \qquad (t = \tau, \ldots T-2),\\
&\Phi_{\tau,\tau-1}(w|F) = \eta(r_{\tau-1}, H)\cdot\beta_{\tau,\tau-1} v(w),\\
&\Phi_{\tau,\tau-2}(w|F) = \eta(r_{\tau-2}, r_{\tau-1})\cdot\beta_{\tau,\tau-2} v(w),
\end{aligned}$$

where $\beta_{\tau,t} \to 0$ as $\varepsilon \to 0$. Finally, (for $\varepsilon > 0$ small) for all $\tau = 0, \ldots, T$, $\omega \in \Omega$, and $f \in B_\tau$, the following statements hold:

1. $f_\tau(\omega) = (1 - \alpha_{\tau,\Pi_\tau(\omega)})w_\tau^f(\omega)$.
2. $V_\tau(w_\tau^f(\omega)) = \sum_{t=\tau}^T \delta^{t-\tau}\mathrm{E}_p[v \circ f_t|\Pi_\tau(\omega)]$.
3. *For all* $t = \tau - 2, \ldots, T-2$, $\Phi_{\tau,t}(w_\tau^f(\omega)|\Pi_\tau(\omega)) = \mathrm{E}_p[\zeta_{t,\Pi_\tau(\omega)} \times \sum_{s=t+2}^T \delta^{s-\tau} v \circ f_s|\Pi_\tau(\omega)]$.
4. *If $f, g \in B_\tau$ and $w_\tau^f(\omega) = w_\tau^g(\omega)$, then $f_t(\omega') = g_t(\omega')$ for all $t = \tau, \ldots, T$ and $G \in \Pi_t$ with $G \subset \Pi_\tau(\omega)$.*

PROOF: For $\tau = T$, the objective function in Eq. (13) reduces to $v(w - s)$. Thus, the unique solution is $s_{T,F}^*(w) = 0$, that is, $\alpha_{T,F} = 0$. Clearly $V_T(w) = v(w)$, and $\Phi_{T,t}$ can only be defined for $t = T-2$, in which case $\Phi_{T,T-2}(w|F) = \zeta_{T-2,F}(\omega)V_T(w) = \eta(r_{T-2}(\omega), r_{T-1}(\omega))\cdot 2^{-T}\varepsilon v(w)$, where $\omega = F$ [actually, $F = \{\omega\}$]. Thus, $\beta_{T,T-2} = 2^{-T}\varepsilon$. Note that $\beta_{T,T-2} \to 0$ as $\varepsilon \to 0$.

Now assume the claim is true for $\tau + 1 \leq T$. Then the objective in Eq. (13) is equivalent to

$$\begin{aligned}
&v(w-s) + \delta\beta^{p}_{\tau+1}\left[\frac{1}{2}v(Hs) + \frac{1}{2}v(Ls)\right] \\
&\quad - \delta[\beta_{\tau+1,\tau-1} + \beta_{\tau+1,\tau}][v(Hs) - v(Ls)] \\
&\quad - \delta\sum_{t=\tau+1}^{T-2}\beta_{\tau+1,t}[v(Hs) + v(Ls)],
\end{aligned}$$

which is a two-period consumption–savings problem with EU preferences, probability of high output equal to

$$\pi = \frac{\frac{1}{2}\beta^{p}_{\tau+1} - \sum_{t=\tau-1}^{T-2}\beta_{\tau+1,t}}{\beta^{p}_{\tau+1} - 2\sum_{t=\tau+1}^{T-2}\beta_{\tau+1,t}}$$

and discount factor equal to

$$\delta_{\pi} \equiv \frac{\delta}{\beta^{p}_{\tau+1} - 2\sum_{t=\tau+1}^{T-2}\beta_{\tau+1,t}}.$$

Since $\beta_{\tau+1,t} \to 0$ as $\varepsilon \to 0$, for ε small, $\pi, \delta_{\pi} \in (0,1)$, so $\alpha_{\tau,F} \in [0,1]$. To complete the inductive step, the statement about $V_{\tau}(w)$ follows from standard arguments, so consider the functions $\Phi_{\tau,t}$. For $t > \tau$,

$$\begin{aligned}
\Phi_{\tau,t}(w|F) &= \delta\bigl\{\Phi_{\tau+1,t}(H\alpha_{\tau+1}w|F\cap H_{\tau}) + \Phi_{\tau+1,t}(L\alpha_{\tau+1}w|F\cap L_{\tau})\bigr\} \\
&= \delta\bigl\{\beta_{\tau+1,t}v(H\alpha_{\tau+1}w) + \beta_{\tau+1,t}v(L\alpha_{\tau+1}w)\bigr\}
\end{aligned}$$

and the claim follows from the properties of power utility; for $t = \tau$, we get

$$\begin{aligned}
\Phi_{\tau,\tau}(w|F) = \delta\bigl\{&\eta(H,H)\cdot\beta_{\tau+1,\tau}v(H\alpha_{\tau+1}w) \\
&+ \eta(L,H)\cdot\beta_{\tau+1,\tau}v(L\alpha_{\tau+1}w)\bigr\}
\end{aligned}$$

and again the claim follows; for $t = \tau - 1$,

$$\begin{aligned}
\Phi_{\tau,\tau-1}(w|F) = \delta\bigl\{&\eta(r_{\tau-1},H)\cdot\bigl[\beta_{\tau+1,\tau-1}v(H\alpha_{\tau+1}w) \\
&+ \eta(r_{\tau-1},L)\cdot\beta_{\tau+1,\tau-1}v(L\alpha_{\tau+1}w)\bigr]\bigr\}
\end{aligned}$$

$$= \delta\big\{\eta(r_{\tau-1},H)\cdot\big[\beta_{\tau+1,\tau-1}v(H\alpha_{\tau+1}w) - \beta_{\tau+1,\tau-1}v(L\alpha_{\tau+1}w)\big]\big\};$$

finally, for $t = \tau - 2$,

$$\Phi_{\tau,\tau-2}(w|F) = \mathrm{E}_p\big[\zeta_{\tau-2,F}V_\tau((1-\alpha_{\tau,F})w)|F\big] = \eta(r_{\tau-2},r_{\tau-1})\cdot\varepsilon 2^{-\tau}\beta_\tau^p v((1-\alpha_{\tau,F})w)$$

and the assertion follows. Note that $\beta_{\tau,\tau-2}\to 0$ as $\varepsilon\to 0$. Furthermore, if $\beta_{\tau+1,t}\to 0$ for $t=\tau-1,\ldots,T-2$ as $\varepsilon\to 0$, then also $\beta_{\tau,t}\to 0$.

Turn to the final claim. For $\tau = T$, by construction $f_T = w_T^f = (1-\alpha_{T,\Pi_T(\omega)})w_T^f$, as $\alpha_{T,\Pi_T(\omega)} = 0$; also, $V_T(w_T^f(\omega)) = v(w_T^f(\omega)) = v\circ f_T(\omega) = \mathrm{E}_p[v\circ f_T|\Pi_T(\omega)]$. The only continuation adjustment to be examined is

$$\Phi_{T,T-2}(w_T^f(\omega)|\Pi_T(\omega)) = \zeta_{T-2}(\omega)V_T(w_T^f(\omega)) = \zeta_{T-2}(\omega)v(w_T^f(\omega)) = \zeta_{T-2}(\omega)v(f_T(\omega)) = \mathrm{E}_p[\zeta_{T-2}v\circ f_T|\Pi_T(\omega)],$$

so item 3 holds. Finally, item 4 holds trivially.

Now assume the claim is true for $\tau+1\le T$ and consider $\tau<T$. Fix $\omega\in\Omega$ and $w\ge 0$ for which $C(w,\omega)\equiv\{f\in B_{\tau+1}: w_t^f(\omega)=w\}\ne\emptyset$. Clearly, for every $s\in[0,w]$ there is an act $f\in C(w,\omega)$ with $f_\tau(\omega)=w-s$. Furthermore, any two acts $f,g\in C(w,\omega)$ such that $f_t(\omega)=g_t(\omega)$ clearly also satisfy $w_{\tau+1}^f(\omega')=w_{\tau+1}^g(\omega')$ for all $\omega'\in\Pi_\tau(\omega)$, and item 4 of the inductive hypothesis implies that then $f_t(\omega')=g_t(\omega')$ as well for all $t=\tau+1,\ldots,T$. Therefore, $V_\tau(f|\Pi_\tau(\omega))=V_\tau(g|\Pi_\tau(\omega))$. Also, if $f\in C(w,\omega)$, then $f\in B_{\tau+1}\subset\mathcal{F}_A(w_0)$ and so $w-f_\tau(\omega)\in[0,w]$. Thus, one can identify each choice of $s\in[0,w]$ with a class of acts in $C(w,\omega)$ that deliver the same continuation payoff; conversely, these classes partition $C(w,\omega)$.

Now consider $f\in C(w,\omega)$ and let $s=w-f_\tau(\omega)$. By the induction hypothesis, since $f\in B_{\tau+1}$, for all $\omega'\in\Pi_\tau(\omega)$, $V_{\tau+1}(r_\tau(\omega')s)=\sum_{t=\tau+1}^T\delta^{t-\tau-1}\mathrm{E}_p[v\circ f_t|\Pi_{\tau+1}(\omega')]$, so by iterated expectations $\delta\mathrm{E}_p[V_{\tau+1}(r_\tau s)|\Pi_\tau(\omega)]=\sum_{t=\tau+1}^T\delta^{t-\tau}\times\mathrm{E}_p[v\circ f_t|\Pi_\tau(\omega)]$. Moreover, again for $\omega'\in\Pi_{\tau+1}(\omega)$, $\Phi_{\tau+1,t}(r_\tau(\omega')s|\Pi_{\tau+1}(\omega'))=\mathrm{E}_p[\zeta_{t,\Pi_{\tau+1}(\omega')}\sum_{s=t+2}^T\delta^{s-\tau-1}v\circ f_s|\Pi_{\tau+1}(\omega')]$ for all $t=\tau-1,\ldots,T-2$. Since, for $\omega'\in\Pi_\tau(\omega)$, $\Pi_{\tau+1}(\omega')$ equals either $\Pi_\tau(\omega)\cap H_\tau$ or $\Pi_\tau(\omega)\cap L_\tau$, Eq. (12) and the induction hypothesis imply that

$$\delta\big\{\Phi_{\tau+1,t}(Hs|\Pi_\tau(\omega)\cap H_\tau)+\Phi_{\tau+1,t}(Ls|\Pi_\tau(\omega)\cap L_\tau)\big\} = \mathrm{E}_p\left[\zeta_{t,\Pi_\tau(\omega)}\sum_{s=t+2}^T\delta^{s-\tau}v\circ f_s\,\middle|\,\Pi_\tau(\omega)\right].$$

Therefore, $V_\tau(f|\Pi_\tau(\omega)$ equals the value of the objective function in Eq. (13) at $s=w-f_\tau(\omega)$. It then follows that f maximizes $V_\tau(\cdot|\Pi_\tau(\omega)$ over $C(w,\omega)$ if and only if $w - f_\tau(\omega) = \alpha_{\tau,\Pi_\tau(\omega)} w$. A fortiori, this is the case for $f \in B_\tau$. This and the induction hypothesis immediately imply item 4. Finally, items 2 and 3 follow from the arguments given in the last paragraph (which apply to any act that prescribes the consistent planning choices from time $\tau + 1$ onward). *Q.E.D.*

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Manuscript received November, 2007; final revision received December, 2008.