Econometrica Supplementary Material

SUPPLEMENT TO "COMMENT ON 'COMMITMENT VS. FLEXIBILITY'"

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S.1. PROOF OF PROPOSITION 3

IN THE APPENDIX of Ambrus and Egorov (2013), the proof of Proposition 3 contained only the idea of proof of the result that $w(\theta_p) < z(u(\theta_p))$ is possible, so money-burning for high types is possible. Here, we present the complete proof of this fact.

Our strategy is to build on Example 1, approximate it with a continuous distribution, and show that, for sufficiently close approximations, the optimal contract must have money-burning. Take $U(c) = \sqrt{c}$, $W(k) = \sqrt{k}$, $y = 1$ (then $z(u) = \sqrt{1 - u^2}$), $\beta = \frac{1}{20}$. Take $\varepsilon \in (0, \frac{1}{10})$, and let F_ε be the atomless distribution with finite support given by the following p.d.f.:

$$f_\varepsilon(\theta) = \begin{cases} 0, & \text{if } \theta < \dfrac{1}{10} - \varepsilon, \\[2ex] \dfrac{\dfrac{10}{11} - \dfrac{\varepsilon}{2}}{\varepsilon}, & \text{if } \dfrac{1}{10} - \varepsilon \leq \theta < \dfrac{1}{10}, \\[2ex] \dfrac{\varepsilon}{10 - \dfrac{1}{10}}, & \text{if } \dfrac{1}{10} \leq \theta < 10, \\[2ex] \dfrac{\dfrac{1}{11} - \dfrac{\varepsilon}{2}}{\varepsilon}, & \text{if } 10 \leq \theta < 10 + \varepsilon, \\[2ex] 0, & \text{if } 10 + \varepsilon \leq \theta. \end{cases}$$

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We have $G_\varepsilon(\theta) = F_\varepsilon(\theta) + \theta(1-\beta)f_\varepsilon(\theta)$ equal to

$$G_\varepsilon(\theta) = \begin{cases} 0, & \text{if } \theta < \dfrac{1}{10} - \varepsilon, \\ \left(\theta - \dfrac{1}{10} + \varepsilon\right) \dfrac{\frac{10}{11} - \frac{\varepsilon}{2}}{\varepsilon} & \\ \quad + \theta\left(1 - \dfrac{1}{20}\right) \dfrac{\frac{10}{11} - \frac{\varepsilon}{2}}{\varepsilon}, & \text{if } \dfrac{1}{10} - \varepsilon \le \theta < \dfrac{1}{10}, \\ \dfrac{10}{11} - \dfrac{\varepsilon}{2} + \left(\theta - \dfrac{1}{10}\right) \dfrac{\varepsilon}{10 - \frac{1}{10}} & \\ \quad + \theta\left(1 - \dfrac{1}{20}\right) \dfrac{\varepsilon}{10 - \frac{1}{10}}, & \text{if } \dfrac{1}{10} \le \theta < 10, \\ \dfrac{10}{11} + \dfrac{\varepsilon}{2} + (\theta - 10) \dfrac{\frac{1}{11} - \frac{\varepsilon}{2}}{\varepsilon} & \\ \quad + \theta\left(1 - \dfrac{1}{20}\right) \dfrac{\frac{1}{11} - \frac{\varepsilon}{2}}{\varepsilon}, & \text{if } 10 \le \theta < 10 + \varepsilon, \\ 1, & \text{if } 10 + \varepsilon \le \theta. \end{cases}$$

Direct computations give the threshold θ_p as a decreasing function of ε on $(0, \frac{1}{10})$, which monotonically increases from $\frac{5620-\sqrt{28754482}}{780} = 0.33$ to $\frac{1}{2}$ as ε decreases from $\frac{1}{10}$ to 0:

$$\theta_p(\varepsilon) = \frac{1}{390\varepsilon}\big(1010\varepsilon + 180 - \sqrt{5}\sqrt{3861\varepsilon^3 + 202538\varepsilon^2 + 58680\varepsilon + 6480}\big).$$

In particular, this implies that all individuals with $\theta \ge \frac{1}{2}$ are pooled.

Let us prove that this contract must involve money-burning for ε small enough for all individuals with $\theta \ge \frac{1}{2}$. Recall the values V and $\tilde{V}$ we defined in Example 1 as the ex ante payoff from the optimal contract and the optimal contract subject to no money-burning in state $\theta_h = 10$; we had $V > \tilde{V}$. In this example, for $\varepsilon \in (0, \frac{1}{10})$, let us define the ex ante payoff from the optimal contract as V_ε and that from the optimal contract with the constraint that types $\theta \geq \frac{1}{2}$ do not burn money (and thus types $\theta > \theta_p(\varepsilon)$ do not burn money) by $\tilde{V}_\varepsilon$. We now prove that $\liminf_{\varepsilon\to 0} V_\varepsilon \geq V$ and that $\limsup_{\varepsilon\to 0} \tilde{V}_\varepsilon \leq \tilde{V}$; this would establish that, for ε small enough, money-burning must be used for the types $\theta \geq \frac{1}{2}$.

We first prove that $\liminf_{\varepsilon\to 0} V_\varepsilon \geq V$. Let us take the optimal contract for the two-type case, $\Xi = (c_l, k_l, c_h, k_h)$, and provide these two options, (c_l, k_l) and (c_h, k_h), to all types from $\frac{1}{10} - \varepsilon$ to $10 + \varepsilon$. From Proposition 1, we know that type $\theta_l = \frac{1}{10}$ is indifferent between the two contracts; then single-crossing considerations will imply that types $\theta < \frac{1}{10}$ will choose (c_l, k_l), while types $\theta > \frac{1}{10}$ will choose (c_h, k_h). The ex ante utility from such contract equals

$$V'_\varepsilon = \left(\frac{10}{11} - \frac{\varepsilon}{2}\right)\left(\left(\frac{1}{10} - \frac{\varepsilon}{2}\right)u_l + w_l\right) + \varepsilon\left(\left(5 + \frac{1}{20}\right)u_h + w_h\right) + \left(\frac{1}{11} - \frac{\varepsilon}{2}\right)\left(\left(10 + \frac{\varepsilon}{2}\right)u_h + w_h\right).$$

Clearly, we have $\lim_{\varepsilon\to 0} V'_\varepsilon = V_\varepsilon$. But we have taken some contract, not necessarily optimal, so $V_\varepsilon \geq V'_\varepsilon$ for all ε. This implies $\liminf_{\varepsilon\to 0} V_\varepsilon \geq V$.

Let us now prove that $\limsup_{\varepsilon\to 0} \tilde{V}_\varepsilon \leq \tilde{V}$. Suppose this is not the case, and there exists $\delta > 0$ and a monotonically decreasing sequence $\varepsilon_1, \varepsilon_2, \ldots$ with $\lim_{n\to\infty} \varepsilon_n = 0$ such that $\tilde{V}_{\varepsilon_n} > \tilde{V} + \delta$ for all $n \in \mathbb{N}$. Suppose that $\Xi^{\varepsilon_n} = \{(c^{\varepsilon_n}(\theta), k^{\varepsilon_n}(\theta))\}_{\theta\in[1/10-\varepsilon_n, 10+\varepsilon_n]}$ is the optimal contract for ε_n, subject to no money-burning for types $\theta \geq \frac{1}{2}$. Let us construct a binary contract $(c_l^{\varepsilon_n}, k_l^{\varepsilon_n}, c_h^{\varepsilon_n}, k_h^{\varepsilon_n})$ in the following way. We let $(c_h^{\varepsilon_n}, k_h^{\varepsilon_n}) = (c^{\varepsilon_n}(10), k^{\varepsilon_n}(10))$ be the contract that type $\theta_h = 10$ chooses under Ξ^{ε_n} (as well as all types $\theta > \theta_p(\varepsilon)$). We let $(c_l^{\varepsilon_n}, k_l^{\varepsilon_n})$ be the contract that maximizes $\max_{\theta\in[1/10-\varepsilon_n, \varepsilon_n]} \theta U(c^{\varepsilon_n}(\theta)) + W(k^{\varepsilon_n}(\theta))$ (the reason we do not take $(c^{\varepsilon_n}(\frac{1}{10}), k^{\varepsilon_n}(\frac{1}{10}))$ is that, even in the optimal contract, the type θ_l may get a relatively low payoff, which is not a problem if this type has zero mass, but may be a problem if it has a mass of $\frac{10}{11}$); suppose that this maximum is reached at $\theta = \tilde{\theta}_{\varepsilon_n}$.

Let us compute the ex ante payoff from the following contract $\tilde{\Xi}^{\varepsilon_n}$: $(\tilde{c}^{\varepsilon_n}(\theta), \tilde{k}^{\varepsilon_n}(\theta)) = (c_l^{\varepsilon_n}, k_l^{\varepsilon_n})$ if $\theta \leq \frac{1}{10}$ and $(\tilde{c}^{\varepsilon_n}(\theta), \tilde{k}^{\varepsilon_n}(\theta)) = (c_h^{\varepsilon_n}, k_h^{\varepsilon_n})$ if $\theta > \frac{1}{10}$ for different distributions of θ. We first take f_{ε_n}; the payoff from this contract (note that this contract need not be incentive compatible!) is

$$\tilde{V}'_{\varepsilon_n} = \left(\frac{10}{11} - \frac{\varepsilon_n}{2}\right)\left(\left(\frac{1}{10} - \frac{\varepsilon_n}{2}\right)u_l^{\varepsilon_n} + w_l^{\varepsilon_n}\right) + \varepsilon_n\left(\left(5 + \frac{1}{20}\right)u_h^{\varepsilon_n} + w_h^{\varepsilon_n}\right) + \left(\frac{1}{11} - \frac{\varepsilon_n}{2}\right)\left(\left(10 + \frac{\varepsilon_n}{2}\right)u_h^{\varepsilon_n} + w_h^{\varepsilon_n}\right)$$

(where $u_l^{\varepsilon_n} = U(c_l^{\varepsilon_n}) = \sqrt{c_l^{\varepsilon_n}}$, etc. are defined as usual). But under the contract Ξ^{ε_n}, types $\theta > 10$ get exactly the same allocation as in $\tilde{\Xi}^{\varepsilon_n}$, and types $\theta < \frac{1}{10}$ get payoff

$$\theta u_l^{\varepsilon_n} + w_l^{\varepsilon_n} \geq |\theta - \theta_{\varepsilon_n}| + \tilde{\theta}_{\varepsilon_n} u_l^{\varepsilon_n} + w_l^{\varepsilon_n} \geq |\theta - \theta_{\varepsilon_n}| + \theta U\big(c^{\varepsilon_n}(\theta)\big) + W\big(k^{\varepsilon_n}(\theta)\big),$$

since $u_l^{\varepsilon_n} \in (0, 1)$. Consequently,

$$\tilde{V}'_{\varepsilon_n} - \tilde{V}_{\varepsilon_n} \geq -\left(\frac{10}{11} - \frac{\varepsilon_n}{2}\right)\varepsilon_n - \varepsilon_n\left(5 + \frac{1}{20} + 1\right),$$

where the second term certainly exceeds the possible difference between $\tilde{V}'_{\varepsilon}$ and $\tilde{V}_{\varepsilon}$ coming from $\theta \in (\frac{1}{10}, 10)$. But the right-hand side tends to 0 as $\varepsilon_n \to 0$, so for n high enough, $\tilde{V}'_{\varepsilon_n} > \tilde{V}_{\varepsilon_n} - \frac{\delta}{3}$.

Let us now take the binary distribution as in Example 1 and consider the payoff under $\tilde{\Xi}^{\varepsilon_n}$ (again, this contract need not be incentive compatible under this distribution). We have

$$\tilde{V}''_{\varepsilon_n} = \frac{10}{11}\left(\frac{1}{10}u_l^{\varepsilon_n} + w_l^{\varepsilon_n}\right) + \frac{1}{11}\left(10u_h^{\varepsilon_n} + w_h^{\varepsilon_n}\right).$$

Clearly,

$$\tilde{V}''_{\varepsilon_n} - \tilde{V}'_{\varepsilon_n} \geq -\varepsilon_n\left(5 + \frac{1}{20} + 1\right) - \left(\frac{1}{11} - \frac{\varepsilon_n}{2}\right)\frac{\varepsilon_n}{2},$$

so for n high enough, we have $\tilde{V}''_{\varepsilon_n} > \tilde{V}'_{\varepsilon_n} - \frac{\delta}{3}$.

Consider now the sequence of contracts $\tilde{\Xi}^{\varepsilon_n}$. It is characterized by two pairs $(c_l^{\varepsilon_n}, k_l^{\varepsilon_n})$ and $(c_h^{\varepsilon_n}, k_h^{\varepsilon_n})$; moreover, $c_h^{\varepsilon_n} + k_h^{\varepsilon_n} = y$ is satisfied for every n. Let us pick a subsequence $\{n_r\}$ such that $(c_l^{\varepsilon_{n_r}}, k_l^{\varepsilon_{n_r}})$ and $(c_h^{\varepsilon_{n_r}}, k_h^{\varepsilon_{n_r}})$ converge to some $(\hat{c}_l, \hat{k}_l)$ and $(\hat{c}_h, \hat{k}_h)$; this is possible since B is compact and, moreover, we have

$\hat{c}_h + \hat{k}_h = y$. Denote the ex ante payoff from this contract under the binary distribution by $\hat{V}$. We have

$$\hat{V} = \frac{10}{11}\left(\frac{1}{10}\hat{u}_l + \hat{w}_l\right) + \frac{1}{11}(10\hat{u}_h + \hat{w}_h);$$

here we used the fact that $U(\cdot)$ and $W(\cdot)$ are continuous. We have

$$\lim_{r\to\infty}\left(\hat{V} - \tilde{V}''_{\varepsilon_{n_r}}\right) = 0$$

by construction, and therefore, for r high enough, $\hat{V} > \tilde{V}''_{\varepsilon_{n_r}} - \frac{\delta}{3}$.

This shows that there is some n such that $\hat{V} > \tilde{V}_{\varepsilon_n} - \delta$. But we took the sequence such that $\tilde{V}_{\varepsilon_n} > \tilde{V} + \delta$ for all n, which implies that $\hat{V} > \tilde{V}$. Recall, however, that $\tilde{V}$ is the ex ante payoff in the optimal contract with no money-burning for the high type, and $\hat{V}$ is the ex ante payoff in one of such contracts. We would get a contradiction if we prove that the contract $(\hat{c}_l, \hat{k}_l)$ and $(\hat{c}_h, \hat{k}_h)$ is incentive compatible. To do so, let us write the following two incentive compatibility constraints that the contract $\tilde{\Xi}^{\varepsilon_{n_r}}$ satisfies:

$$\tilde{\theta}_{\varepsilon_{n_r}} u_l^{\varepsilon_{n_r}} + \frac{1}{20} w_l^{\varepsilon_{n_r}} \geq \tilde{\theta}_{\varepsilon_n} u_h^{\varepsilon_{n_r}} + \frac{1}{20} w_h^{\varepsilon_{n_r}};$$

$$10 u_h^{\varepsilon_{n_r}} + \frac{1}{20} w_h^{\varepsilon_{n_r}} \geq 10 u_l^{\varepsilon_{n_r}} + \frac{1}{20} w_l^{\varepsilon_{n_r}}.$$

Taking the limits as $r \to \infty$ and using the fact that $\tilde{\theta}_{\varepsilon_{n_r}} \in [\frac{1}{10} - \varepsilon_{n_r}, \frac{1}{10}]$ and thus tends to $\frac{1}{10}$, we get

$$\frac{1}{10}\hat{u}_l + \frac{1}{20}\hat{w}_l \geq \frac{1}{10}\hat{u}_h + \frac{1}{20}\hat{w}_h;$$

$$10\hat{u}_h + \frac{1}{20}\hat{w}_h \geq 10\hat{u}_l + \frac{1}{20}\hat{w}_l.$$

This proves that the contract $(\hat{c}_l, \hat{k}_l, \hat{c}_h, \hat{k}_h)$ is incentive compatible, and thus $\hat{V} \leq \tilde{V}$. We have reached a contradiction which proves that $\limsup_{\varepsilon\to 0} \tilde{V}_\varepsilon \leq \tilde{V}$.

Consequently, we have established both $\liminf_{\varepsilon\to 0} V_\varepsilon \geq V$ and $\limsup_{\varepsilon\to 0} \tilde{V}_\varepsilon \leq \tilde{V}$. But $V > \tilde{V}$; therefore, for ε close to 0, $V_\varepsilon > \tilde{V}_\varepsilon$. This means that there is $\varepsilon > 0$ for which the optimal contract must involve money-burning in the allocation that types $\theta > \theta_p(\varepsilon)$ get, and the mass of these agents is at least $\frac{1}{11}$ (as $\theta_p(\varepsilon) < \frac{1}{2}$). This completes the proof that $w(\theta_p) < z(u(\theta_p))$ is possible. *Q.E.D.*

S.2. ADDITIONAL FORMAL RESULTS

PROPOSITION 1: *Take any convex functions $U(\cdot)$ and $W(\cdot)$ such that the function $z(u)$ has at least one point $u_0 \in (0, y)$ with $|\frac{dz}{du}|_{u=u_0}| \geq 1$ (this would be the case, for example, if $W = U$, or if $W'(0) = \infty$ and $W(0) \neq -\infty$). Then there exists an open set of parameter values μ, θ_l, β (with θ_h found from $\mu\theta_l + (1-\mu)\theta_h = 1$) such that the optimal contract necessarily includes money-burning.*

PROOF: Given $U(\cdot)$ and $W(\cdot)$, the set A is fixed. Let $w = z(u)$ be the equation that determines the upper boundary of this set and let $k = |\frac{dz}{du}(u_0)| \geq 1$. By assumption that $W(0) \neq -\infty$ and convexity of A, the number $s = \frac{z(u_0)-W(0)}{U(y)-u_0} \in (k, \infty)$. For any $\beta \in (0, \frac{1}{s}) \subset (0, 1)$, let $\theta_l(\beta) = \beta s$. In this case, u_0 will be the u_0 from formulation of Proposition 2 in Ambrus and Egorov (2013). We have

$$\mu\left((1-\beta)\Big/\left(\frac{1}{\left|\dfrac{dz}{dx}(u_0)\right|} - \frac{\beta}{\theta_l(\beta)}\right)\right) = \mu\frac{1-\beta}{\dfrac{1}{k} - \dfrac{1}{s}}.$$

But $s \in (k, \infty)$ and $k \geq 1$ imply $\frac{1}{k} - \frac{1}{s} \in (0, 1)$, which means that inequality

$$\mu\left((1-\beta)\Big/\left(\frac{1}{\left|\dfrac{dz}{du}\right|_{u=u_0}\right|} - \frac{\beta}{\theta_l}\right)\right) > 1$$

must hold for β sufficiently close to 0 and μ sufficiently close to 1 (and θ_l, θ_h derived by $\theta_l = \beta s$ and $\theta_h = \frac{1-\mu\theta_l}{1-\mu}$). Moreover, for μ close to 1, we will have θ_h arbitrarily high; in particular, $\theta_h > s = \frac{\theta_l(\beta)}{\beta}$. The latter implies $\beta > \frac{\theta_l}{\theta_h}$, and we have $\beta < \beta^*$ by construction, so in this case, indeed, a separating contract is optimal by Proposition 1 in Ambrus and Egorov (2013). Finally, since varying u_0 would not change the inequalities above, then the set of parameters β, μ, θ_l for which money-burning is optimal contains an open set. *Q.E.D.*

REFERENCE

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