SUPPLEMENT TO "REPUTATIONAL BARGAINING WITH MINIMAL KNOWLEDGE OF RATIONALITY"

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This supplement shows that the characterization of the maxmin payoff and posture (Theorem 1) continues to apply when the solution concept is strengthened from first-order knowledge of rationality to iterated conditional dominance, or when the continuous-time bargaining protocol of the text is replaced by any discrete-time bargaining protocol with sufficiently frequent offers. However, the characterization does not apply with *both* iterated conditional dominance and discrete-time bargaining, as the fact that (complete-information) discrete-time bargaining is solvable by iterated conditional dominance implies that the predictions of the model with iterated conditional dominance and discrete-time bargaining depend on the order and relative frequency of offers.[1]

ITERATED CONDITIONAL DOMINANCE

THIS SECTION SHOWS that Theorem 1 continues to hold under a natural notion of iterated conditional dominance. Because the model has incomplete information and is not a multistage game with observed actions (as players do not observe each other's choice of demand paths on the integers), no off-the-shelf version of iterated conditional dominance is applicable, and even the simplest version that is applicable requires some new notation.

For integer t, let $\sigma_i(h^t)$ be the element of $\Delta(\mathcal{U}^t)$ prescribed by strategy σ_i at date $(t, 0)$ and history h^t. I first introduce the idea that a triple $(h^t, u_1^{\lfloor t \rfloor}, u_2^{\lfloor t \rfloor})$ is "σ_i-coherent" if $u_i^{\lfloor t \rfloor} \in \operatorname{supp} \sigma_i(h^{\lfloor t \rfloor})$ and at h^t the path of realized demands between $\lfloor t \rfloor$ and t coincides with $(u_1^{\lfloor t \rfloor}, u_2^{\lfloor t \rfloor})$.

DEFINITION 8: A triple $(h^t, u_1^{\lfloor t \rfloor}, u_2^{\lfloor t \rfloor})$ is σ_i-*coherent* if $u_i^{\lfloor t \rfloor} \in \operatorname{supp} \sigma_i(h^{\lfloor t \rfloor})$ and $(u_1^{\lfloor t \rfloor}(\tau), u_2^{\lfloor t \rfloor}(\tau)) = (u_1(\tau), u_2(\tau))$ for all $\tau \in [\lfloor t \rfloor, t]$, where $h^t = (u_1(\tau), u_2(\tau))_{\tau \le t}$. A history h^t is σ_i-*coherent* if there exist demand paths $(u_1^{\lfloor t \rfloor}, u_2^{\lfloor t \rfloor})$ such that $(h^t, u_1^{\lfloor t \rfloor}, u_2^{\lfloor t \rfloor})$ is σ_i-coherent.

For any strategy profile (σ_1, σ_2) and any triple $(h^t, u_1^{\lfloor t \rfloor}, u_2^{\lfloor t \rfloor})$ such that $(u_1^{\lfloor t \rfloor}(\tau), u_2^{\lfloor t \rfloor}(\tau)) = (u_1(\tau), u_2(\tau))$ for all $\tau \in [\lfloor t \rfloor, t]$, where $h^t = (u_1(\tau),$

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[1]Informally, there is a race between the number of rounds of iterated conditional dominance and the frequency of offers. I conjecture that for any number of rounds of iterated conditional dominance, the maxmin payoff and posture in discrete-time bargaining converge (in the sense of Proposition 6) to the maxmin payoff and posture in continuous-time bargaining as offers become frequent. This is consistent with Rubinstein bargaining, where the round at which any demand other than 0 or 1 is deleted goes to infinity as the time between offers vanishes (so that iterated conditional dominance has no "bite" in the continuous-time limit). I thank Jeff Ely for helpful comments on this point.

$u_2(\tau))_{\tau \leq t}$, each player i's expected payoff under strategy profile (σ_i, σ_j) conditional on reaching the triple $(h^t, u_1^{\lfloor t \rfloor}, u_2^{\lfloor t \rfloor})$ is well defined and is denoted

$$u_i(\sigma_i, \sigma_j | h^t, u_1^{\lfloor t \rfloor}, u_2^{\lfloor t \rfloor}).$$

I also write $u_i(\sigma_i, \sigma_j | h^t)$ for player i's expected payoff conditional on reaching h^t at date $(t, 0)$ for integer t.

I now define iterated conditional dominance. Informally, the idea is that a strategy is conditionally dominated if it is either strictly dominated or is "conditionally weakly dominated." The difference between the definition of iterated conditional dominance for the two players reflects the fact that player 1 is committed to strategy γ with probability ε and, therefore, that player 2 is restricted to assigning probability at least ε to strategy γ at histories that are consistent with γ. Note that the support of the γ-offsetting belief π_2^γ includes strategies that are iteratively conditionally dominated, as it is easy to verify that any strategy of player 1 that ever accepts a demand of 1 is iteratively conditionally dominated.

DEFINITION 9: For any posture γ and set of bargaining phase strategy profiles $\Omega = \Omega_1 \times \Omega_2 \subseteq \Sigma_1 \times \Sigma_2$, a strategy $\sigma_1 \in \Sigma_1$ is *conditionally dominated with respect to* (γ, Ω) if either of the following conditions hold:
- There exists a strategy $\sigma_1' \in \Sigma_1$ such that

$$u_1(\sigma_1', \pi_1) > u_1(\sigma_1, \pi_1)$$

for all beliefs $\pi_1 \in \Delta(\Omega_2)$.
- There exists a strategy $\sigma_1' \in \Sigma_1$ such that

$$u_1(\sigma_1', \pi_1 | h^t, u_1^{\lfloor t \rfloor}, u_2^{\lfloor t \rfloor}) \geq u_1(\sigma_1, \pi_1 | h^t, u_1^{\lfloor t \rfloor}, u_2^{\lfloor t \rfloor})$$

for all σ_1-coherent $(h^t, u_1^{\lfloor t \rfloor}, u_2^{\lfloor t \rfloor})$ and all beliefs $\pi_1 \in \Delta(\Omega_2)$, with strict inequality for some σ_1-coherent $(h^t, u_1^{\lfloor t \rfloor}, u_2^{\lfloor t \rfloor})$ and some belief $\pi_1 \in \Delta(\Omega_2)$.

A strategy $\sigma_2 \in \Sigma_2$ is *conditionally dominated with respect to* (γ, Ω) if either of the following conditions hold:
- There exists a strategy $\sigma_2' \in \Sigma_2$ such that

$$u_2(\sigma_2', \pi_2) > u_2(\sigma_2, \pi_2)$$

for all beliefs $\pi_2 \in \Delta(\Omega_1 \cup \{\gamma\})$ such that $\pi_2(\gamma) \geq \varepsilon$ with strict inequality only if $\gamma \in \Omega_1$.
- There exists a strategy $\sigma_2' \in \Sigma_2$ such that

$$u_2(\sigma_2', \pi_2 | h^t, u_1^{\lfloor t \rfloor}, u_2^{\lfloor t \rfloor}) \geq u_2(\sigma_2, \pi_2 | h^t, u_1^{\lfloor t \rfloor}, u_2^{\lfloor t \rfloor})$$

for all σ_2-coherent $(h^t, u_1^{\lfloor t \rfloor}, u_2^{\lfloor t \rfloor})$ that are inconsistent with γ and all beliefs $\pi_2 \in \Delta(\Omega_1)$, with strict inequality for some σ_2-coherent $(h^t, u_1^{\lfloor t \rfloor}, u_2^{\lfloor t \rfloor})$ that is inconsistent with γ and some belief $\pi_2 \in \Delta(\Omega_1)$.

A set of bargaining phase strategy profiles $\Omega = \Omega_1 \times \Omega_2 \subseteq \Sigma_1 \times \Sigma_2$ is *closed under conditional dominance given posture* γ if every $\sigma_i \in \Omega_i$ is conditionally undominated (i.e., not conditionally dominated) with respect to (γ, Ω). The set of *iteratively conditionally undominated strategies given posture* γ is

$$\Omega^{\mathrm{ICD}}(\gamma) \equiv \bigcup \{\Omega : \Omega \text{ is closed}$$

under conditional dominance given posture $\gamma\}$.

Player 1's *maxmin payoff under iterated conditional dominance given posture* γ is

$$u_1^{\mathrm{ICD}}(\gamma) \equiv \sup_{\sigma_1} \inf_{\sigma_2 \in \Omega_2^{\mathrm{ICD}}(\gamma)} u_1(\sigma_1, \sigma_2).$$

Player 1's *maxmin payoff under iterated conditional dominance* is

$$u_1^{\mathrm{ICD}} \equiv \sup_{\gamma} u_1^{\mathrm{ICD}}(\gamma).$$

A posture γ^{ICD} is a *maxmin posture under iterated conditional dominance* if there exists a sequence of postures $\{\gamma_n\}$ such that $\gamma_n(t) \to \gamma^{\mathrm{ICD}}(t)$ for all $t \in \mathbb{R}_+$ and $u_1^{\mathrm{ICD}}(\gamma_n) \to u_1^{\mathrm{ICD}}$.

This version of iterated conditional dominance is stronger than rationalizability in that $\Omega^{\mathrm{ICD}}(\gamma) \subseteq \Omega^{\mathrm{RAT}}(\gamma)$ for any posture γ. This can be seen by noting that every set Ω that is closed under conditional dominance is also closed under rationalizability, because rationalizability is equivalent to imposing only the first of the two conditions in the definition of conditional dominance (for both player 1 and player 2). An immediate consequence of this observation is that the maxmin payoff under iterated conditional dominance is weakly greater than the maxmin payoff (under first-order knowledge of rationality), that is, $u_1^{\mathrm{ICD}} \geq u_1^*$. In fact, the two payoffs are equal, as are the corresponding maxmin postures.

PROPOSITION 5: *Player 1's maxmin payoff under iterated conditional dominance equals her maxmin payoff, and the unique maxmin posture under iterated conditional dominance is the unique maxmin posture; that is, $u_1^{\mathrm{ICD}} = u_1^*$ and the unique maxmin posture under iterated conditional dominance is $\gamma^{\mathrm{ICD}} = \gamma^*$.*

The rest of this section is devoted to proving Proposition 5. The proof builds on that of Proposition 4. This is because it can be shown that the set of iteratively conditionally undominated strategies and the set of rationalizable strategies are identical up to strategies that are "exceptional" in the following sense.

DEFINITION 10: A strategy $\sigma_i \in \Sigma_i$ is *exceptional given posture* γ if either of the following conditions hold:

- $i \in \{1, 2\}$ and σ_i ever accepts a demand of 1, rejects a demand of 0, makes a demand of 0 or a path of demands converging to 0 (i.e., $\lim_{\tau \uparrow t} u_i(\tau) = 0$), or makes a demand of 1 at every successor of some history h^t.
- $i = 1$ and σ_i ever accepts a demand $u_2(t) \geq 1 - e^{-r(t^*-t)} \underline{\gamma}(t^*) > 0$ at any history h^t consistent with γ with $t \leq t^*$, rejects a demand $\overline{u_2}(t) \leq 1 - e^{-r(t^*-t)} \gamma(t^*) < 1$ at any history h^t consistent with γ with $t \leq t^*$, or demands $\gamma(t^*)$ at any history h^{t^*} consistent with γ.

The relationship between iterated conditional dominance and rationalizability is formalized in the following lemma, which is a key step in the proof of Proposition 5.

LEMMA 5: *For any posture γ, every strategy that is rationalizable and nonexceptional given posture γ is also iteratively conditionally undominated given posture γ.*

The proof of the lemma uses the concept of a unique optimal action: an action (accepting, rejecting, or choosing a demand path for the next integer) is the *unique optimal action* at a triple $(h^t, u_1^{\lfloor t \rfloor}, u_2^{\lfloor t \rfloor})$ under a belief π_i if every strategy σ_i that maximizes $u_i(\sigma_i, \pi_i | h^t, u_1^{\lfloor t \rfloor}, u_2^{\lfloor t \rfloor})$ prescribes that action at history h^t (where the arguments $(u_1^{\lfloor t \rfloor}, u_2^{\lfloor t \rfloor})$ are omitted in the case of choosing a demand path for the next integer).

PROOF OF LEMMA 5: Fix a posture γ. For the duration of the proof, I omit the modifier "given posture γ." To prove the lemma, I show that for every nonexceptional strategy σ_i and every σ_i-coherent history h^t that is inconsistent with γ, there exist demand paths $(u_1^{\lfloor t \rfloor}, u_2^{\lfloor t \rfloor})$ and belief π_i with support on strategies that are rationalizable and nonexceptional such that $(h^t, u_1^{\lfloor t \rfloor}, u_2^{\lfloor t \rfloor})$ is σ_i-coherent and σ_i prescribes the unique optimal action at $(h^t, u_1^{\lfloor t \rfloor}, u_2^{\lfloor t \rfloor})$ under belief π_2. If $i = 1$, this conclusion also holds at σ_i-coherent histories that are consistent with γ. This implies that the second of the two conditions in the definition of conditional dominance can never hold if σ_i is nonexceptional for $i = 1, 2$. Therefore, every nonexceptional strategy that is conditionally dominated is also strictly dominated, and hence every nonexceptional strategy that is rationalizable is also iteratively conditionally undominated.

I start by establishing a statement with the important implication that, starting from a history that is inconsistent with γ, any continuation strategy is part of a rationalizable strategy.

STEP 1: *Any strategy $\sigma_1 \in \Sigma_1$ that demands $\gamma(t)$ and rejects player 2's demand at every history h^t that is consistent with γ is rationalizable. Any strategy $\sigma_2 \in \Sigma_2$ that demands 1 and accepts at (and not before) the more favorable of dates $(t^*, -1)$ and $(t^*, 1)$ if player 1 follows γ until time t^* is rationalizable.*

PROOF: By the proof of Lemma 4, strategy $\dot{\pi}_2^\gamma$ is rationalizable for player 1 and strategy σ_2^γ is rationalizable for player 2. Now if strategies σ_1 and σ_2 are as in the statement, then $\sigma_1 \in \Sigma_1^*(\sigma_2^\gamma)$ and $\sigma_2 \in \Sigma_2^*(\dot{\pi}_2^\gamma)$, so σ_1 and σ_2 are rationalizable as well. *Q.E.D.*

STEP 2: *For $i = 1, 2$, if a strategy σ_i is nonexceptional and a history h^t is σ_i-coherent and inconsistent with γ, then there exist demand paths $(u_1^{\lfloor t \rfloor}, u_2^{\lfloor t \rfloor})$ and a belief π_i with support on strategies that are rationalizable and nonexceptional such that $(h^t, u_1^{\lfloor t \rfloor}, u_2^{\lfloor t \rfloor})$ is σ_i-coherent and σ_i prescribes the unique optimal action at $(h^t, u_1^{\lfloor t \rfloor}, u_2^{\lfloor t \rfloor})$ under belief π_i.*

PROOF: Fix a nonexceptional strategy σ_i and a history h^t that is σ_i-coherent and inconsistent with γ. Step 1 implies that any continuation strategy of player j is part of a rationalizable strategy. Hence, the restriction that π_i has support on strategies that are rationalizable and nonexceptional implies only that continuation strategies are nonexceptional.

Suppose that σ_i accepts at h^t. Then the fact that σ_i is nonexceptional and h^t is σ_i-coherent imply that $u_i(t) > 0$ and $u_j(t) < 1$. Let $(u_1^{\lfloor t \rfloor}, u_2^{\lfloor t \rfloor})$ specify that the players continue to demand $u_i(t)$ and $u_j(t)$ until $\lceil t \rceil$, and let π_i assign probability 1 to a rationalizable strategy under which at every successor history of h^t player j demands $\frac{1 + u_j(t)}{2}$ (after time $\lceil t \rceil$) and rejects any strictly positive demand; such a strategy exists by the previous paragraph, and is clearly nonexceptional (in particular, player j always chooses demand paths that always make demands $u_2(\tau) \in (0, 1)$). Then it is clear that accepting at h^t is the optimal action at $(h^t, u_1^{\lfloor t \rfloor}, u_2^{\lfloor t \rfloor})$ under belief π_i.[2]

Suppose that σ_i rejects at h^t. Then the facts that σ_i is nonexceptional and h^t is σ_i-coherent imply that $u_i(t) > 0$ and $u_j(t) > 0$. Let π_i assign probability 1 to a rationalizable and nonexceptional strategy under which player j reduces his demand to $u_j(t)/2$ by some time τ such that

$$e^{-r(\tau - t)}\left(1 - \frac{u_j(t)}{2}\right) > 1 - u_j(t);$$

subsequently demands $u_j(t)/2$ forever; and rejects player i's demand at every successor history of h^t unless player i demands 0 (or let $(u_1^{\lfloor t \rfloor}, u_2^{\lfloor t \rfloor})$ specify that player j's demand follows such a path, in case there is no integer between t and τ). Choose any $(u_1^{\lfloor t \rfloor}, u_2^{\lfloor t \rfloor})$ such that $(h^t, u_1^{\lfloor t \rfloor}, u_2^{\lfloor t \rfloor})$ is σ_i-coherent and player j's demands follow such a path. Now rejecting until time τ and then accepting (while never demanding 0) is strictly better for player i under belief π_i than is

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[2]Note that the possibility that player i could reject at h^t but accept "immediately" after h^t is ruled out by the assumption that the probability that a player accepts by date $(t, 1)$ is right-continuous in t.

accepting at h^t, so rejecting is the unique optimal action at $(h^t, u_1^{\lfloor t \rfloor}, u_2^{\lfloor t \rfloor})$ under belief π_i.

Finally, suppose that t is an integer and that σ_i chooses demand path u_i^t at h^t. The fact that σ_i is nonexceptional implies that $u_i^t(\tau) > 0$ for all $\tau \in [t, t+1)$ and that $\lim_{\tau \uparrow t+1} u_i^t(\tau) > 0$. Let π_i assign probability 1 to a rationalizable and nonexceptional strategy under which player j demands $1 - \frac{e^{-r} \lim_{\tau \uparrow t+1} u_i^t(\tau)}{2}$ at h^t and at every successor history of h^t; accepts at date $(t+1, -1)$ if $\dot{u}_i(\tau) = u_i^t(\tau)$ for all $\tau \in [t, t+1)$; and otherwise rejects any strictly positive demand at every successor history of h^t. Now choosing demand path u_i^t at h^t and rejecting player j's demand until time $t+1$ yields payoff $e^{-r} \lim_{\tau \uparrow t+1} u_i^t(\tau)$ under belief π_i, while every other continuation strategy yields payoff at most $\frac{e^{-r} \lim_{\tau \uparrow t+1} u_i^t(\tau)}{2}$ under belief π_i, so choosing demand path u_i^t is the unique optimal action at h^t under belief π_i.   Q.E.D.

STEP 3: *If strategy $\sigma_1 \in \Sigma_1$ is nonexceptional and a history h^t is σ_1-coherent and consistent with γ, then there exist demand paths $(u_1^{\lfloor t \rfloor}, u_2^{\lfloor t \rfloor})$ and a belief π_1 with support on strategies that are rationalizable and nonexceptional such that $(h^t, u_1^{\lfloor t \rfloor}, u_2^{\lfloor t \rfloor})$ is σ_1-coherent and σ_1 prescribes the unique optimal action at $(h^t, u_1^{\lfloor t \rfloor}, u_2^{\lfloor t \rfloor})$ under belief π_1.*

PROOF: If $t > t^*$, then if player 2 plays a rationalizable and nonexceptional strategy σ_2 that accepts at time t^* under strategy profile (γ, σ_2) (which exists), then player 2's continuation play starting from h^t is restricted only by the requirement that it is nonexceptional. Hence, the proof in this case is just like the proof of Step 2. I therefore assume that $t \leq t^*$.

Suppose that σ_1 accepts at h^t. Then the fact that σ_1 is nonexceptional, h^t is σ_1-coherent and consistent with γ, and $t \leq t^*$ implies that $u_1(t) > 0$ and $u_2(t) < 1 - e^{-r(t^*-t)}\underline{\gamma}(t^*)$. Define the strategy $\ddot{\sigma}_2 \in \Sigma_2$ as follows:

• If h^τ is consistent with γ, then demand 1 until time $\lceil t^* + 1 \rceil$, subsequently demand $\frac{1}{2}$, reject all positive demands until the more favorable of dates $(t^*, -1)$ and $(t^*, 1)$, and subsequently accept all demands of less than 1.

• If h^τ is inconsistent with γ, then demand $\frac{1+u_2(t)}{2}$ and reject all positive demands.

Note that $\ddot{\sigma}_2 \in \Sigma_2^*(\gamma)$, so $\ddot{\sigma}_2$ is rationalizable. In addition, $\ddot{\sigma}_2$ is clearly nonexceptional. Let π_1 assign probability 1 to $\ddot{\sigma}_2$, and let $(u_1^{\lfloor t \rfloor}, u_2^{\lfloor t \rfloor})$ specify that player 1 demands $u_1(\tau) = \gamma(\tau)$ for all $\tau \in [t, \lceil t \rceil)$ and that player 2 continues to demand $u_2(t)$ until $\lceil t \rceil$. Then accepting at $(h^t, u_1^{\lfloor t \rfloor}, u_2^{\lfloor t \rfloor})$ yields payoff $1 - u_2(t)$ under belief π_1, while any strategy that rejects at $(h^t, u_1^{\lfloor t \rfloor}, u_2^{\lfloor t \rfloor})$ yields strictly less. So accepting is the unique optimal action at $(h^t, u_1^{\lfloor t \rfloor}, u_2^{\lfloor t \rfloor})$ under belief π_1.

Suppose that σ_1 rejects at h^t. Then the fact that σ_1 is nonexceptional, h^t is σ_1-coherent and consistent with γ, and $t \leq t^*$ implies that $u_1(t) > 0$ and

$u_2(t) > 1 - e^{-r(t^*-t)}\underline{\gamma}(t^*)$. Let $\ddot{\sigma}_2$, π_1, and $(u_1^{\lfloor t \rfloor}, u_2^{\lfloor t \rfloor})$ be as above, with the modification that $\ddot{\sigma}_2$ demands $1 - \frac{e^{-r(t^*-t)}\underline{\gamma}(t^*)}{2}$ rather than $\frac{1+u_2(t)}{2}$ at histories h^τ that are inconsistent with γ.[3] Then rejecting and following strategy γ at $(h^t, u_1^{\lfloor t \rfloor}, u_2^{\lfloor t \rfloor})$ yields payoff $e^{-r(t^*-t)}\underline{\gamma}(t^*)$ under belief π_1, while any strategy that rejects at $(h^t, u_1^{\lfloor t \rfloor}, u_2^{\lfloor t \rfloor})$ yields strictly less. So rejecting is the unique optimal action at $(h^t, u_1^{\lfloor t \rfloor}, u_2^{\lfloor t \rfloor})$ under belief π_1.

Finally, suppose that t is an integer and that σ_1 chooses demand path u_1^t at h^t. Since u_1^t and γ are continuous on $[t, t+1)$, there are three cases.

Case 1. $u_1^t(\tau) = \gamma(\tau)$ for all $\tau \in [t, t+1)$.

Case 2. $u_1^t(t) = \gamma(t)$ but $u_1^t(\tau) \neq \gamma(\tau)$ for some $\tau \in [t, \min\{t+1, t^*\})$.

Case 3. $u_1^t(t) \neq \gamma(t)$.

Start with Case 1. Here, the fact that σ_1 is nonexceptional, h^t is consistent with γ, and $t \leq t^*$ implies that, in fact, $t+1 \leq t^*$, as σ_1 never demands $\gamma(t^*)$ at a history h^{t^*} consistent with γ. Now for all $\eta > 0$, Step 1 implies that there exists a rationalizable and nonexceptional strategy σ_2 that demands 1 at all times τ such that $e^{-r(\tau-(t+1))} \geq \eta$, accepts at (but not before) time t^* under strategy profile (γ, σ_2), rejects all strictly positive demands at dates $(t+1, -1)$ and earlier, rejects all strictly positive demands at dates after $(t+1, 0)$ such that $u_1(\tau) \neq \gamma(\tau)$ for some $\tau < t+1$, and accepts all demands less than 1 at dates after $(t+1, 0)$ such that $u_1(\tau) = \gamma(\tau)$ for all $\tau < t+1$ but $u_1(t+1) \neq \gamma(t+1)$. Since $e^{-r(t^*-(t+1))}\gamma(t^*) < 1$ (which follows from the definition of t^*), choosing demand path u_1^t and then deviating from γ to a demand close to 1 at time $t+1$ yields a strictly higher payoff under any belief that assigns probability 1 to such a strategy than does choosing any other demand path, for η sufficiently small.

For Case 2, the fact that σ_1 is nonexceptional, h^t is consistent with γ, and $t \leq t^*$ implies that $t < t^*$. Let τ_0 be the infimum over times $\tau \in [t, \min\{t+1, t^*\})$ such that $u_1^t(\tau) \neq \gamma(\tau)$. Now for all $\eta > 0$, Step 1 implies that there exists a rationalizable and nonexceptional strategy σ_2 that demands 1 at all times τ such that $e^{-r(\tau-\max\{t+1,t^*\})} \geq \eta$, accepts at (but not before) time t^* under strategy profile (γ, σ_2), rejects all strictly positive demands at all histories that are inconsistent with either γ or u_1^t, and, for all $k \in \{0, 1, \ldots, \lfloor \frac{t+1-\tau_0}{\eta} \rfloor - 1\}$, accepts at time $\tau_0 + k\eta$ with probability $\eta^k(1-\eta)$ if $u_1(\tau) = u_1^t(\tau)$ for all $\tau \in [t, \tau_0 + k\eta]$, and accepts at date $(t+1, -1)$ with probability $\eta^{\lfloor (t+1-\tau_0)/\eta \rfloor}$ if player 1's demands are consistent with u_1^t on $[t, t+1)$. Since u_1^t is continuous, player 1 receives a strictly higher payoff from choosing u_1^t and then rejecting until time $t+1$ than from mimicking γ, under the belief that player 2 plays such a strategy for sufficiently small η (as player 1 strictly prefers to have her demand accepted at any time prior to t^* than at t^*, by definition of t^*). In addition, player 1 receives a strictly higher payoff from choosing u_1^t than from

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[3]This modification serves only to ensure that $\ddot{\sigma}_2$ does not demand 1 forever after some history and, thus, that $\ddot{\sigma}_2$ is nonexceptional.

choosing any demand path that coincides with u_1^t until some time $\tau \in [t, t+1)$ and then diverges from u_1^t. Therefore, choosing demand path u_1^t is player 1's unique optimal action at h^t under the belief that player 2 plays such a strategy for sufficiently small η.

For Case 3, Step 1 implies that for all $\eta > 0$, there exists a rationalizable and nonexceptional strategy σ_2 that accepts at (but not before) time t^* under strategy profile (γ, t^*), rejects all strictly positive demands at all histories that are inconsistent with γ, and, for all $k \in \{0, 1, \ldots, \lfloor \frac{1}{\eta} \rfloor - 2\}$, with probability $\eta^k(1-\eta)$ demands 1 until time $t + k\eta$ and reduces its demand to η by time $t + (k+1)\eta$, and with probability $\eta^{\lfloor 1/\eta \rfloor}$ demands 1 until time $t + 1 - \eta$ and reduces its demand to η by time $t + 1$. Step 1 also implies that there exists a rationalizable and nonexceptional strategy that accepts at (but not before) time t^* under strategy profile (γ, t^*), demands $1 - (\frac{e^{-r} \lim_{\tau \uparrow t+1} u_i^t(\tau)}{2})\eta$ at h^t and at every successor history of h^t; accepts at date $(t+1, -1)$ if $u_i(\tau) = u_i^t(\tau)$ for all $\tau \in [t, t+1)$; and otherwise rejects any strictly positive demand at every successor history of h^t (as in the last part of the proof of Step 2). Let π_1 assign probability $1 - \eta$ to player 2's playing a strategy of the first kind and assign probability η to player 2's playing a strategy of the second kind. I claim that for η sufficiently small, u_1^t is player 1's unique optimal action at h^t under belief π_1. To see this, first note that $e^{-r(t^*-t)}\underline{\gamma}(t^*) < 1$ implies that choosing u_1^t is strictly better than choosing any demand path that coincides with γ until time t^*, for η sufficiently small. Finally, any strategy that chooses a demand path other than u_1^t that diverges from γ before time t^* does no better than choosing demand path u_1^t and rejecting until time $t+1$ in the event that player 2 plays a strategy of the first kind, and does strictly worse in the event that player 2 plays a strategy of the second kind. Therefore, choosing demand path u_1^t is player 1's unique optimal action at h^t under belief π_1, for $\eta > 0$ sufficiently small.   Q.E.D.

I now complete the proof of the lemma. By the definition of conditional dominance and Step 2, strategy σ_2 can be conditionally dominated (with respect to some (γ, Ω)) by strategy σ_2' only if either σ_2' strictly dominates σ_2 (with respect to (γ, Ω); i.e., if the first condition in the definition of conditionally dominance with respect to (γ, Ω) holds) or σ_2' agrees with σ_2 at all σ_2-coherent histories that are inconsistent with γ. But, again by the definition of conditional dominance, if σ_2' conditionally dominates σ_2 and agrees with σ_2 at all σ_2-coherent histories that are inconsistent with γ, then σ_2' must strictly dominate σ_2. The same argument applies for player 1, noting that Steps 2 and 3 imply that a strategy σ_1 can be conditionally dominated by a strategy σ_1' only if σ_1' strictly dominates σ_1 or if σ_1' agrees with σ_1 at all σ_1-coherent histories h^t, whether or not h^t is consistent with γ (which is needed for the argument given the difference in the definitions of conditional dominance for players 1 and 2). Therefore, for $i = 1, 2$, if σ_i is nonexceptional, then it cannot be conditionally dominated unless it is also strictly dominated. Finally, if σ_i is rationalizable

and nonexceptional, then it is not strictly dominated with respect to $(\gamma, \Omega^{\mathrm{RAT}})$, hence, it is not conditionally dominated with respect to $(\gamma, \Omega^{\mathrm{RAT}})$, and, hence, it also is not conditionally dominated with respect to the smaller set $(\gamma, \Omega^{\mathrm{ICD}})$. This proves that every rationalizable and nonexceptional strategy is iteratively conditionally undominated. *Q.E.D.*

I now prove Proposition 5.

PROOF OF PROPOSITION 5: As was the case for Proposition 4, it suffices to show that $u_1^{\mathrm{ICD}}(\gamma) = \underline{\gamma}(t^*)$ for every posture γ. The proof proceeds by approximating the γ-offsetting belief π_2^γ with beliefs $\{\pi_2^\gamma(\eta)\}_{\eta > 0}$ that have support on rationalizable and nonexceptional strategies only (unlike the offsetting belief π_2^γ itself, which assigns positive probability to the exceptional strategy $\tilde{\gamma}$), and by approximating the γ-offsetting strategy σ_2^γ with nonexceptional strategies $\{\sigma_2^\gamma(\eta)\}_{\eta > 0}$ such that $\sigma_2^\gamma(\eta) \in \Sigma_2^*(\pi_2^\gamma(\eta))$ for all $\eta > 0$. Lemma 5 then implies that the strategy $\sigma_2^\gamma(\eta)$ is iteratively conditionally undominated for all $\eta > 0$. Finally, as $\eta \to 0$,

$$\sup_{\sigma_1} u_1\big(\sigma_1, \sigma_2^\gamma(\eta)\big) \to u_1^*(\gamma),$$

which implies that $u_1^{\mathrm{ICD}}(\gamma) \le u_1^*(\gamma)$. Since $u_1^{\mathrm{ICD}}(\gamma) \ge u_1^*(\gamma)$ is immediate because $\Omega^{\mathrm{ICD}}(\gamma) \subseteq \Omega^{\mathrm{RAT}}(\gamma)$, this shows that $u_1^{\mathrm{ICD}}(\gamma) = u_1^*(\gamma) = \underline{\gamma}(t^*)$, completing the proof of the proposition.

I now present an argument that leads to the construction of the beliefs $\{\pi_2^\gamma(\eta)\}_{\eta > 0}$ and strategies $\{\sigma_2^\gamma(\eta)\}_{\eta > 0}$. I start by defining the strategies of player 1 that receive positive weight under belief $\pi_2^\gamma(\eta)$. Fix $t \in (0, t^*)$ and $\eta \in (0, \frac{1}{2})$. Let

$$\eta' \equiv \begin{cases} \min\left\{\eta, \dfrac{r(t^* - t)}{3}, \dfrac{\gamma(t^*)}{2}\right\}, & \text{if } \gamma(t^*) > 0, \\[2ex] \min\left\{\eta, \dfrac{r(t^* - t)}{3}\right\}, & \text{if } \gamma(t^*) = 0, \end{cases}$$

and let $\tilde{\gamma}(t, \eta)$ be the strategy that demands $u_1(\tau) = \gamma(\tau)$ for all $\tau \in [0, t)$; demands

$$u_1(\tau) = \frac{t + \eta'/r - \tau}{\eta'/r}\gamma(t) + \left(1 - \frac{t + \eta'/r - \tau}{\eta'/r}\right)(1 - \eta')$$

for all $\tau \in [t, t + \eta'/r]$; demands

$$u_1(\tau) = \frac{t + 2\eta'/r - \tau}{\eta'/r}(1 - \eta') + \left(1 - \frac{t + 2\eta'/r - \tau}{\eta'/r}\right)\eta'$$

for all $\tau \in [t + \eta'/r, t + 2\eta'/r]$; demands $u_1(\tau) = \eta'$ if $\tau > t + 2\eta'/r$; and accepts a demand of player 2 if and only if it equals 0. Intuitively, $\tilde{\gamma}(t, \eta)$ mimics γ until time t and then quickly rises to almost 1 before quickly falling to almost 0, where "quickly" and "almost 0" are both measured by η (the point of having η' rather than η in the formulas will become clear shortly).

I claim that $\tilde{\gamma}(t, \eta)$ is iteratively conditionally undominated. To see this, observe that $\tilde{\gamma}(t, \eta)$ is a best response to any strategy σ_2 with the following properties:

• σ_2 demands 1 and accepts at (and not before) date $(t^*, -1)$ if player 1 follows γ until time t^*.

• If h^τ is inconsistent with γ but consistent with $\tilde{\gamma}(t, \eta)$, then σ_2 demands 1 and accepts if and only if $\tau \geq t + \eta'/r$.

• If h^τ is inconsistent with both γ and $\tilde{\gamma}(t, \eta)$, then σ_2 demands 1 and rejects player 1's demand.

This follows because playing $\tilde{\gamma}(t, \eta)$ against such a strategy σ_2 yields payoff

$$e^{-rt - \eta'}\left(1 - \eta'\right),$$

while the only other positive payoff that can be obtained against strategy σ_2 is

$$e^{-rt^*}\gamma\left(t^*\right) \leq e^{-rt^*},$$

and $e^{-rt - \eta'}(1 - \eta') \geq e^{-rt^*}$ because $\eta' \leq \min\{\frac{1}{2}, \frac{r(t^* - t)}{3}\}$ (as can be easily checked). Now, by Step 1 of the proof of Lemma 5, there exists a rationalizable strategy σ_2 of this form, so $\tilde{\gamma}(t, \eta)$ is rationalizable. In addition, $\tilde{\gamma}(t, \eta)$ is nonexceptional, because $\gamma(t) > 0$ for all $t \in [0, t^*)$ (recalling the definition of γ^*) and $\tilde{\gamma}(t, \eta)$ always demands $\eta' \neq \gamma(t^*)$ at time t^*, so Lemma 5 implies that $\tilde{\gamma}(t, \eta)$ is iteratively conditionally undominated.

I now introduce versions of some of the key objects of Section 3.2, indexed by η. Let

$$\lambda(t, \eta) = \frac{rv(t) - v'(t)}{e^{-2\eta'}(1 - \eta') - v(t)}$$

if v is differentiable at t and $v(t) < e^{-2\eta'}(1 - \eta')$, and let $\lambda(t, \eta) = 0$ otherwise; and let

$$p(t, \eta) = \frac{v(t, -1) - v(t)}{e^{-2\eta'}(1 - \eta') - v(t)}$$

if $v(t) < v(t, -1) \leq e^{-2\eta'}(1 - \eta')$, and let $p(t, \eta) = 0$ otherwise. Define $\tilde{T}(\eta)$, $T(\eta)$, $t^*(\eta)$, $\hat{\lambda}(t, \eta)$, and $\hat{p}(t, \eta)$ as in Section 3.2, with $\lambda(t, \eta)$ and $p(t, \eta)$ replacing $\lambda(t)$ and $p(t)$ in the definitions. Note that as $\eta \to 0$, $\lambda(t, \eta) \downarrow \lambda(t)$ and $p(t, \eta) \downarrow p(t)$ for all $t \in \mathbb{R}_+$. Hence, $\hat{\lambda}(t, \eta) \downarrow \lambda(t)$, $\hat{p}(t, \eta) \downarrow p(t)$, $\tilde{T}(\eta) \uparrow \tilde{T}$, $T(\eta) \uparrow T$, and $t^*(\eta) \uparrow t^*$.

Let $\mu^\gamma(\eta)$ be the belief that player 1 rejects all nonzero demands of player 2 and that her path of demands begins by following $\gamma(t)$ and then switches to following $\tilde{\gamma}(t, \eta)$ at time t with hazard rate $\hat{\lambda}(t, \eta)$ and discrete probability $\hat{p}(t, \eta)$, for all $t < t^*(\eta)$.[4] Let $\mu^{\gamma,t}(\eta)$ be the belief that coincides with $\mu^\gamma(\eta)$ until date $(t, -1)$ and subsequently coincides with the belief that player 1 follows γ. Let $\pi_2^\gamma(\eta)$ put probability ε on strategy γ and put probability $1 - \varepsilon$ on strategy $\mu^{\gamma,t^*}(\eta)$. Let $\sigma_2^\gamma(\eta)$ be some best response to $\pi_2^\gamma(\eta)$ with the following properties:

- $\sigma_2^\gamma(\eta)$ demands 1 at all times t such that $t \leq t^*$ and $e^{-rt} \geq \eta$.
- $\sigma_2^\gamma(\eta)$ rejects player 1's demand at those histories that are consistent with $\pi_2^\gamma(\eta)$, where accepting and rejecting are both optimal actions.
- $\sigma_2^\gamma(\eta)$ rejects all positive demands at histories that are inconsistent with γ.
- $\sigma_2^\gamma(\eta)$ is nonexceptional.

It is clear that such a strategy exists. Furthermore, any such strategy is rationalizable, by Step 1 of the proof of Lemma 5, and hence any such strategy is iteratively conditionally undominated, by Lemma 5.

I claim that as $\eta \to 0$, the time at which agreement is reached under strategy profile $(\gamma, \sigma_2^\gamma(\eta))$ converges to t^*, uniformly over possible choices of $\sigma_2^\gamma(\eta)$ satisfying the above properties. To see this, observe that, as in Section 3.2, if $v(t) < e^{-2\eta'}(1 - \eta')$, then $\lambda(t, \eta)$ and $p(t, \eta)$ are the rate and probability of player 1 switching to $\tilde{\gamma}(t, \eta)$ that make player 2 indifferent between accepting and rejecting γ, and, under belief $\pi_2^\gamma(\eta)$, player 2 believes that player 1 switches to $\tilde{\gamma}(t, \eta)$ with rate and probability $\lambda(t, \eta)$ and $p(t, \eta)$ if $v(t) < e^{-2\eta'}(1 - \eta')$ and $t < t^*(\eta)$. Furthermore, since $\gamma(t)$ is positive and continuous on $[0, t^*)$, it follows that

$$\liminf_{\eta \to 0}\{t : v(t) \geq e^{-2\eta'}(1 - \eta')\} = t^*.$$

Hence, for small η, player 2 is indifferent between accepting and rejecting γ until close to time $\min\{t^*, t^*(\eta)\}$, and, therefore, $\sigma_2^\gamma(\eta)$ specifies that he rejects until close to time $\min\{t^*, t^*(\eta)\}$. Since $t^*(\eta) \to t^*$, this shows that the time at which agreement is reached under strategy profile $(\gamma, \sigma_2^\gamma(\eta))$ converges to t^*.

The proof is nearly complete. Strategy $\sigma_2^\gamma(\eta)$ is iteratively conditionally undominated for all $\eta > 0$. When facing strategy $\sigma_2^\gamma(\eta)$, the highest payoff that player 1 can receive when player 2 accepts at a history that is consistent with γ converges to $\underline{\gamma(t^*)}$ as $\eta \to 0$. Furthermore, for any $\eta > 0$, the most player 2 accepts at a history that is inconsistent with γ is η'; and the highest payoff player 1 can receive by accepting a demand of player 2 is η (since $\sigma_2^\gamma(\eta)$ demands 1 at

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[4]There is a technical problem here because it is not clear that $\mu^\gamma(\eta)$ can be written as a *finite-dimensional* distribution over Σ_1, that is, as an element of $\Delta(\Sigma_1)$. However, it should be clear that $\mu^\gamma(\eta)$ can in turn be approximated by a finite-dimensional distribution over Σ_1 in a way that suffices for the proof.

all times t such that $e^{-rt} \geq \eta$). It follows that

$$u_1^{\text{ICD}}(\gamma) \leq \lim_{\eta \to 0} \sup_{\sigma_1} u_1\big(\sigma_1, \sigma_2^\gamma(\eta)\big) = \max\Big\{\underline{\gamma}(t^*), \lim_{\eta \to 0} \eta', \lim_{\eta \to 0} \eta\Big\}$$

$$= \max\{\underline{\gamma}(t^*), 0, 0\} = \underline{\gamma}(t^*),$$

completing the proof.   *Q.E.D.*

DISCRETE-TIME BARGAINING WITH FREQUENT OFFERS

This section shows that Theorem 1 continues to hold when the continuous-time bargaining protocol of the text is replaced by any discrete-time bargaining protocol with sufficiently frequent offers. More precisely, for any sequence of discrete-time bargaining games that converges to continuous time (in that each player may make an offer close to any given time), the corresponding sequence of maxmin payoffs and postures converges to the continuous-time maxmin payoff and posture given by Theorem 1. Abreu and Gul (2000) provided a similar independence-of-procedures result for sequential equilibrium outcomes of reputational bargaining. Because my result concerns maxmin payoffs and postures rather than equilibria, my proof is very different from Abreu and Gul's.

Formally, replace the (continuous time) bargaining phase of Section 2 with the following procedure: There is a (commonly known) function $g : \mathbb{R}_+ \to \{0, 1, 2\}$ that specifies who makes an offer at each time. If $g(t) = 0$, no player takes an action at time t. If $g(t) = i \in \{1, 2\}$, then player i makes a demand $u_i(t) \in [0, 1]$ at time t and player j immediately accepts or rejects. If player j accepts, the game ends with payoffs $(e^{-rt}u_i(t), e^{-rt}(1 - u_i(t)))$; if player j rejects, the game continues. Let $I_i^g = \{t : g(t) = i\}$, and assume that $I_i^g \cap [0, t]$ is finite for all t and that I_i^g is infinite. The announcement phase is correspondingly modified so that player 1 announces a posture $\gamma : I_i^g \to [0, 1]$, and if player 1 becomes committed to posture γ (which continues to occur with probability ε), she demands $\gamma(t)$ at time t and rejects all of player 2's demands. I refer to the function g as a *discrete-time bargaining game*.

I now define convergence to continuous time. This definition is very similar to that of Abreu and Gul (2000), as is the above model of discrete-time bargaining and the corresponding notation.

DEFINITION 11: A sequence of discrete-time bargaining games $\{g_n\}$ *converges to continuous time* if for all $\Delta > 0$, there exists N such that for all $n \geq N$, $t \in \mathbb{R}_+$, and $i \in \{1, 2\}$, $I_i^{g_n} \cap [t, t + \Delta] \neq \emptyset$.

The maxmin payoff and posture in a discrete-time bargaining game are defined exactly as in Section 2. Let $u_1^{*,g}$ be player 1's maxmin payoff in discrete-time bargaining game g and let $u_1^{*,g}(\gamma)$ be player 1's maxmin payoff given

posture γ in g. The independence-of-procedures result states that for any sequence of discrete-time bargaining games converging to continuous time, the corresponding sequence of maxmin payoffs $\{u_1^{*,g_n}\}$ converges to u_1^* and any corresponding sequence of postures $\{\gamma^{g_n}\}$ such that $u_1^{*,g_n}(\gamma^{g_n}) \to u_1^*$ "converges" to γ^*, where u_1^* and γ^* are the maxmin payoff and posture identified in Theorem 1. The nature of the convergence of the sequence $\{\gamma^{g_n}\}$ to γ^* is slightly delicate. For example, there may be (infinitely many) times $t \in \mathbb{R}_+$ such that $\lim_{n\to\infty} \gamma^{g_n}(t)$ exists and is greater than $\gamma^*(t)$, because these demands may be "nonserious" (in that they are followed immediately by lower demands).[5] Thus, rather than stating the convergence in terms of $\{\gamma^{g_n}\}$ and γ^*, I state it in terms of the corresponding continuation values of player 2, which are the economically more important variables. Formally, given a posture γ^{g_n} in discrete-time bargaining game g^n, let

$$v^{g_n}(t) \equiv \max_{\tau \geq t:\tau \in I_1^{g_n}} e^{-r(\tau-t)}\big(1 - \gamma^{g_n}(\tau)\big).$$

Let $v^*(t) = \max\{1 - e^{rt}/(1 - \log \varepsilon), 0\}$, the continuation value corresponding to γ^* in the continuous-time model of Section 2. The independence-of-procedures result is as follows:

PROPOSITION 6: *Let $\{g_n\}$ be a sequence of discrete-time bargaining games converging to continuous time. Then $u_1^{*,g_n} \to u_1^*$, and if $\{\gamma^{g_n}\}$ is a sequence of postures with γ^{g_n} a posture in g_n and $u_1^{*,g_n}(\gamma^{g_n}) \to u_1^*$, then $v^{g_n}(t) \to v^*(t)$ for all $t \in \mathbb{R}_+$.*

The key fact behind the proof of Proposition 6 is that for any sequence of discrete-time postures $\{\gamma^{g_n}\}$ converging to some continuous-time posture γ, $\lim_{n\to\infty} u_1^{*,g_n}(\gamma^{g_n}) = \lim_{n\to\infty} u_1^*(\gamma^{g_n})$ (where $u_1^*(\gamma^{g_n})$ is the maxmin payoff given a natural embedding of γ^{g_n} in continuous time, defined formally in the proof). This fact is proved by constructing a belief that is similar to the γ^{g_n}-offsetting belief in each discrete-time game g_n and then showing that these beliefs converge to the γ-offsetting belief in the limiting continuous-time game.

PROOF OF PROPOSITION 6: Observe that a posture γ in discrete-time bargaining game g induces a "continuous-time posture" $\hat{\gamma}$ (i.e., a map from $\mathbb{R}_+ \to [0, 1]$) according to $\hat{\gamma}(t) = \gamma(\min\{\tau \geq t : \tau \in I_i^g\})$; that is, $\hat{\gamma}$'s time-t demand is simply γ's next demand in g. I henceforth refer to a posture γ in g as also being a continuous-time posture, with the understanding that I mean the posture $\hat{\gamma}$ defined above.

However, γ may *not* be a posture in the continuous-time bargaining game of Section 2, because it may be discontinuous at a noninteger time. To

avoid this problem, I now introduce a modified version of the continuous-time bargaining game of Section 2. Formally, let the continuous-time bargaining game g^{cts} be defined as in Section 2, with the following modifications: Most importantly, omit the requirement that player i's demand path $u_i^t : [t, t+1) \to [0, 1]$ (which is still chosen at integer times t) is continuous. Second, specify that the payoffs if player i accepts player j's offer at date $(t, -1)$ are $(e^{-rt}(1 - \liminf_{\tau \uparrow t} u_j(\tau)), e^{-rt} \liminf_{\tau \uparrow t} u_j(\tau))$ (because $\lim_{\tau \uparrow t} u_j(\tau)$ may now fail to exist). Third, add a fourth date $(t, 2)$ to each instant of time t. At date $(t, 2)$, each player i announces *accept* or *reject*, and, if player i accepts player j's offer at date $(t, 2)$, the game ends with payoffs $(e^{-rt}(1 - \liminf_{\tau \downarrow t} u_j(\tau)), e^{-rt} \liminf_{\tau \downarrow t} u_j(\tau))$. Adding the date $(t, 2)$ ensures that each player has a well defined best response to her belief, even though $u_j(t)$ may now fail to be right-continuous. One can check that the analysis of Sections 3 and 4, including Lemmas 1–3 and Theorem 1, continue to apply to the game g^{cts}, with the exception that in g^{cts}, the maxmin posture γ^* is not, in fact, unique; however, every maxmin posture corresponds to the continuation value function v^* (by the same argument as in Step 1 of the proof of Theorem 1).[6] Because of this, for the remainder of the proof I slightly abuse notation by writing $u_1^*(\gamma)$ for player 1's maxmin payoff given posture γ *in the game* g^{cts}, rather than in the model of Section 2. Importantly, $u_1^*(\gamma)$ equals player 1's maxmin payoff given γ in both g^{cts} and in the model of Section 2 when γ is a posture in the model of Section 2, but $u_1^*(\gamma)$ is well defined for all $\gamma : \mathbb{R}_+ \to [0, 1]$. Similarly, I write $u_1^*(v)$ for player 1's maxmin payoff given continuation value function $v : \mathbb{R}_+ \to [0, 1]$. This is well defined because $u_1^*(\gamma) = \min_{t \leq T} e^{-rt} \underline{\gamma}(t)$ by Lemma 3, T depends on γ only through v (by Lemma 1), and it can be easily verified that $\min_{t \leq T} e^{-rt} \underline{\gamma}(t) = \min_{t \leq T} e^{-rt}(1 - v(t))$ (and thus depends on γ only through v). A similar argument, which I omit, implies that one may write $u_1^{*, g_n}(v^{g_n})$ for player 1's maxmin payoff given continuation value function v^{g_n} in discrete-time bargaining game g_n.

I now establish two lemmas, from which Proposition 6 follows. Their proofs require some additional notation. Let Σ_i^g be the set of player i's strategies in g^{cts} with the property that player i's demand only changes at times $t \in I_i^g$, player i only accepts player j's offer at times $t \in I_j^g$, and player i's action at time t only depends on past play at times $\tau \in I_i^g \cup I_j^g$. One can equivalently view Σ_i^g as player i's strategy set in g itself. Thus, any belief π_2 in g may also be viewed as a belief in g^{cts} (with $\text{supp}(\pi_2) \subseteq \Sigma_i^g$).

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[6]The reason I did not use the game g^{cts} in Sections 3 and 4 is that it is difficult to interpret the assumption that player i can accept the demand $\liminf_{\tau \downarrow t} u_j(\tau)$ at time t, since the demand $u_j(\tau)$ has not yet been made at time t for all $\tau > t$. Thus, I view the game g^{cts} as a technical construct for analyzing the limit of discrete-time games and not as an appealing model of continuous-time bargaining in its own right.

LEMMA 6: *For any sequence of discrete-time bargaining games converging to continuous time, $\{g_n\}$, there exists a sequence of postures $\{\gamma^{g_{n'}}\}$ with $\gamma^{g_{n'}}$ a posture in g_n and $\lim_{n\to\infty} u_1^{*,g_n}(\gamma^{g_{n'}}) \geq u_1^*$.*

PROOF: Let $\gamma^{g_{n'}}$ be given by $\gamma^{g_{n'}}(t) = (\frac{n}{n+1})\gamma^*(\max\{\tau \leq t : \tau \in I_1^{g_n}\})$ for all $t \in \mathbb{R}_+$, with the convention that $\max\{\tau \leq t : \tau \in I_1^{g_n}\} \equiv 0$ if the set $\{\tau \leq t : \tau \in I_1^{g_n}\}$ is empty. I first claim that $\lim_{n\to\infty} u_1^*(\gamma^{g_{n'}}) \geq u_1^*$.[7] To show this, I first establish that $\tilde{T}(\gamma^{g_{n'}}) \leq \min\{\tau > T^1 : \tau \in I_1^{g_n}\}$ for all n, where T^1 is defined as in the proof of Theorem 1. Since γ^* (and thus $\gamma^{g_{n'}}$) are nondecreasing, $\sup_{\tau \geq t} e^{-r(\tau-t)}(1 - \gamma^{g_{n'}}(\tau)) = 1 - \gamma^{g_{n'}}(t)$. Therefore, by Lemma 1, $\tilde{T}(\gamma^{g_{n'}})$ satisfies

(S1)
$$\exp\left(-\int_0^{\tilde{T}(\gamma^{g_{n'}})} \frac{r\left(\frac{n+1}{n} - \gamma^*(\max\{\tau \leq t : \tau \in I_1^{g_n}\})\right)}{\gamma^*(\max\{\tau \leq t : \tau \in I_1^{g_n}\})} dt\right)$$
$$\times \prod_{t \in I_1^{g_n} \cap [0, \tilde{T}(\gamma^{g_{n'}}))} \frac{\gamma^*(\max\{\tau < t : \tau \in I_1^{g_n}\})}{\gamma^*(t)} \geq \varepsilon.$$

Now

(S2)
$$\exp\left(-\int_0^{\tilde{T}(\gamma^{g_{n'}})} \frac{r\left(\frac{n+1}{n} - \gamma^*(\max\{\tau \leq t : \tau \in I_1^{g_n}\})\right)}{\gamma^*(\max\{\tau \leq t : \tau \in I_1^{g_n}\})} dt\right)$$
$$\times \prod_{t \in I_1^{g_n} \cap [0, \tilde{T}(\gamma^{g_{n'}}))} \frac{\gamma^*(\max\{\tau < t : \tau \in I_1^{g_n}\})}{\gamma^*(t)}$$
$$\leq \exp\left(-\int_0^{\tilde{T}(\gamma^{g_{n'}})} \frac{r(1 - \gamma^*(t))}{\gamma^*(t)} dt\right) \frac{\gamma^*(0)}{\gamma^*(\max\{\tau < \tilde{T}(\gamma^{g_{n'}}) : \tau \in I_1^{g_n}\})}$$
$$\leq \exp\left(-\int_0^{\max\{\tau < \tilde{T}(\gamma^{g_{n'}}) : \tau \in I_1^{g_n}\}} \frac{r(1 - \gamma^*(t)) + \gamma^{*\prime}(t)}{\gamma^*(t)} dt\right).$$

Observe that if $\tilde{T}(\gamma^{g_{n'}}) > \min\{\tau > T^1 : \tau \in I_1^{g_n}\}$, then $\max\{\tau < \tilde{T}(\gamma^{g_{n'}}) : \tau \in I_1^{g_n}\} > T^1$ and, therefore, (S2) is less than ε, which contradicts (S1). Hence, $\tilde{T}(\gamma^{g_{n'}}) \leq \min\{\tau > T^1 : \tau \in I_1^{g_n}\}$ for all n. In addition, $\gamma^{g_{n'}}(t)$ is nondecreasing and $\gamma^{g_{n'}}(t) < 1$ for all t, which implies that $T(\gamma^{g_{n'}}) = \tilde{T}(\gamma^{g_{n'}})$. Therefore, by Lemma 3, $u_1^*(\gamma^{g_{n'}}) = \min_{t \leq \tilde{T}(\gamma^{g_{n'}})} e^{-rt}\gamma^{g_{n'}}(t)$. Since $\tilde{T}(\gamma^{g_{n'}}) \leq \min\{\tau > T^1 : \tau \in$

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[7]Theorem 1 implies that $\lim_{n\to\infty} u_1^*(\gamma^{g_{n'}}) \leq u_1^*$, so this inequality must hold with equality. But only the inequality is needed for the proof.

$I_1^{g_n}\}$ for all n and $\{g_n\}$ converges to continuous time, $\lim_{n\to\infty}\tilde{T}(\gamma^{g_n\prime}) \le T^1$. In addition, $\lim_{n\to\infty}\sup_{t\in\mathbb{R}_+} |\gamma^{g_n\prime}(t) - \gamma^*(t)| = 0$, so it follows that

$$\lim_{n\to\infty} u_1^*\big(\gamma^{g_n\prime}\big) = \lim_{n\to\infty} \min_{t\le\tilde{T}(\gamma^{g_n\prime})} e^{-rt}\gamma^{g_n\prime}(t)$$

$$\ge \lim_{n\to\infty} \min_{t\le\tilde{T}(\gamma^{g_n\prime})} e^{-rt}\gamma^*(t) \ge \min_{t\le T^1} e^{-rt}\gamma^*(t) = u_1^*.$$

Next, I claim that $u_1^{*,g_n}(\gamma^{g_n}) \ge u_1^*(\gamma^{g_n})$ for *any* posture γ^{g_n} in discrete-time bargaining game g^n. To see this, note that if $\mathrm{supp}(\pi_2) \subseteq \Sigma_1^{g_n}$ and $\sigma_2 \in \Sigma_2^{*,g_n}(\pi_2)$, then $\sigma_2 \in \Sigma_2^*(\pi_2)$ as well (i.e., there is no benefit to responding to a strategy in $\Delta(\Sigma_1^{g_n})$ with a strategy outside of $\Sigma_2^{g_n}$). Therefore, if $\pi_1 \in \Pi_1^{\gamma^{g_n},g_n}$ (i.e., if π_1 is consistent with knowledge of rationality in g_n), then $\pi_1 \in \Pi_1^{\gamma^{g_n},g^{\mathrm{cts}}}$; that is, $\Pi_1^{\gamma^{g_n},g_n} \subseteq \Pi_1^{\gamma^{g_n},g^{\mathrm{cts}}}$. Now

$$u_1^{*,g_n}\big(\gamma^{g_n}\big) = \sup_{\sigma_1\in\Sigma_1^{g_n}} \inf_{\pi_1\in\Pi_1^{\gamma^{g_n},g_n}} u_1(\sigma_1,\pi_1)$$

$$\ge \sup_{\sigma_1\in\Sigma_1^{g_n}} \inf_{\pi_1\in\Pi_1^{\gamma^{g_n},g^{\mathrm{cts}}}} u_1(\sigma_1,\pi_1)$$

$$= u_1\big(\gamma^{g_n},\sigma^{\gamma^{g_n}}\big)$$

$$= u_1^*\big(\gamma^{g_n}\big),$$

where $\sigma^{\gamma^{g_n}}$ is as in Definition 5, and the second line follows because $\Pi_1^{\gamma^{g_n},g_n} \subseteq \Pi_1^{\gamma^{g_n},g^{\mathrm{cts}}}$; the third line follows because $u_1(\gamma^{g_n},\sigma^{\gamma^{g_n}}) = \sup_{\sigma_1\in\Sigma_1^{\mathrm{cts}}} \inf_{\pi_1\in\Pi_1^{\gamma^{g_n},g^{\mathrm{cts}}}} u_1(\sigma_1,\pi_1)$ by Lemma 3, and $\gamma^{g_n} \in \Sigma_1^{g_n} \subseteq \Sigma_1^{\mathrm{cts}}$; and the fourth line follows by Lemma 3.

Combining the above claims, it follows that $\lim_{n\to\infty} u_1^{*,g_n}(\gamma^{g_n\prime}) \ge \lim_{n\to\infty} u_1^*(\gamma^{g_n\prime}) \ge u_1^*$.   *Q.E.D.*

LEMMA 7: *For any sequence of discrete-time bargaining games converging to continuous time $\{g_n\}$ and any sequence of functions $\{v^{g_n}\}$ such that v^{g_n} is a continuation value function in g_n and $\lim_{n\to\infty} v^{g_n}(t)$ exists for all $t \in \mathbb{R}_+$, it follows that $\lim_{n\to\infty} u_1^{*,g_n}(v^{g_n})$ exists and equals $\lim_{n\to\infty} u_1^*(v^{g_n})$.*

PROOF: Fix a sequence of continuation value functions $\{v^{g_n}\}$ (with v^{g_n} a continuation value function in discrete-time game g_n) converging pointwise to some function $v:\mathbb{R}_+ \to [0,1]$. I have already shown that $u_1^{*,g_n}(\gamma^{g_n}) \ge u_1^*(\gamma^{g_n})$ for any posture γ^{g_n} in game g_n or, equivalently, $u_1^{*,g_n}(v^{g_n}) \ge u_1^*(v^{g_n})$. This immediately implies that $\lim_{n\to\infty} u_1^{*,g_n}(v^{g_n}) \ge \limsup_{n\to\infty} u_1^*(v^{g_n})$ for every convergent subsequence of $\{u_1^{*,g_n}(v^{g_n})\}$. Hence, I must show that $\lim_{n\to\infty} u_1^{*,g_n}(v^{g_n}) \le \liminf_{n\to\infty} u_1^*(v^{g_n})$ for every convergent subsequence of $\{u_1^{*,g_n}(v^{g_n})\}$. I establish this inequality by assuming that there exists $\eta > 0$ such that $\lim_{n\to\infty} u_1^{*,g_n}(v^{g_n}) >$

$\liminf_{n \to \infty} u_1^*(v^{g_n}) + \eta$ for some convergent subsequence of $\{u_1^{*,g_n}(v^{g_n})\}$ and then deriving a contradiction. The approach is first to define analogs of the continuous-time γ-offsetting belief and the time $\tilde{T}$ (defined in Section 3.2) for game g_n, denoted $\pi_2^n \in \Sigma_2^{g_n}$ and $\tilde{T}^n \in \mathbb{R}_+$, and then show that $\tilde{T}^n \to \tilde{T}$.

I must introduce some additional notation before defining the belief π_2^n. Let $t_{g_n}^{\text{next}}(i) = \min\{\tau > t : \tau \in I_i^{g_n}\}$ be the time of player i's next demand at t. Given continuation value function v^{g_n} and any corresponding posture γ^{g_n}, let $\widetilde{\gamma^{g_n}}^n$ be defined as follows: First, $\widetilde{\gamma^{g_n}}^n$ demands $\widetilde{\gamma^{g_n}}^n(h^t) = \gamma^{g_n}(h^t)$ for all $t \in I_1^{g_n}$. Second, $\widetilde{\gamma^{g_n}}^n$ accepts player 2's demand at time $t \in I_2^{g_n}$ with probability

$$\hat{p}^n(t) \equiv \min\left\{ \frac{p^n(t)}{\chi^n(t)}, 1 \right\},$$

where

$$p^n(t) \equiv \max_{\substack{\tau < t: \\ \tau \in I_1^{g_n}, \tau_{g_n}^{\text{next}}(2) = t}} \frac{e^{r(t-\tau)} v^{g_n}(\tau) - v^{g_n}(t)}{1 - v^{g_n}(t)}$$

if $\{\tau < t : \tau \in I_1^{g_n}, \tau_{g_n}^{\text{next}}(2) = t\}$ is nonempty and $v^{g_n}(\tau) < 1$ for all time τ in this set, and $p^n(t) \equiv 0$ otherwise; and

$$\chi^n(t) \equiv \max\left\{ \frac{\prod_{\tau < t: \tau \in I_2^{g_n}} (1 - p^n(\tau)) - \varepsilon}{\prod_{\tau < t: \tau \in I_2^{g_n}} (1 - p^n(\tau))}, 0 \right\}.$$

Let $\tilde{T}^n$ be the supremum over times t at which $\chi^n(t_{g_n}^{\text{next}}(2)) \hat{p}^n(t_{g_n}^{\text{next}}(2)) = p^n(t_{g_n}^{\text{next}}(2))$ and let

$$T^n \equiv \sup_{\substack{t \geq \tilde{T}^n: \\ t \in I_1^{g_n}}} \arg \max e^{-rt} v^{g_n}(t).$$

By an argument similar to the proof of Lemma 2, if $\gamma^{g_n}(t) < \eta$ for some $t \leq T^n$, then there exists a belief $\pi_2 \in \Delta(\Sigma_1^{g_n})$ and a strategy $\sigma_2 \in \Sigma_2^{g_n}$ such that $\pi_2(\gamma^{g_n}) \geq \varepsilon$, $\sigma_2 \in \Sigma_2^{*,g_n}(\pi_2)$, and the demand $\gamma^{g_n}(t)$ is accepted under strategy profile (γ^{g_n}, σ_2). In particular, $u_1^{g_n}(\gamma^{g_n}, \sigma_2) < \eta$. Thus, by the hypothesis that $\lim_{n \to \infty} u_1^{*,g_n}(v^{g_n}) > \liminf_{n \to \infty} u_1^*(v^{g_n}) + \eta$, there must exist $N > 0$ such that $\gamma^{g_n}(t) \geq \eta$ for all $t \leq T^n$ and all $n > N$, and hence $v^{g_n}(t) \leq 1 - \eta$ for all $t \leq T^n$ and all $n > N$.

Let π_2^n assign probability ε to γ^{g_n} and probability $1 - \varepsilon$ to $\widetilde{\gamma^{g_n}}^n$, and fix $\sigma_2^n \in \Sigma_2^{*,g_n}(\pi_2^n)$ with the property that σ_2^n always demands 1 and rejects player 1's demand at any history at which player 1 has deviated from γ^{g_n} (which is

possible because π_2^n assigns probability 0 to such histories, except for terminal histories), as well as at any history at which player 2 is indifferent between accepting and rejecting player 1's demand under belief π_2^n. Note that γ^{g_n} is a best response to σ_2^n in g_n. This implies that $u_1^{*,g_n}(\gamma^{g_n}) \leq u_1^{g_n}(\gamma^{g_n}, \sigma_2^n)$ for all n. Thus, to show that $\lim_{n\to\infty} u_1^{*,g_n}(v^{g_n}) \leq \liminf_{n\to\infty} u_1^*(v^{g_n}) + \eta$ (the desired contradiction), it suffices to show that $\lim_{n\to\infty} u_1^{g_n}(\gamma^{g_n}, \sigma_2^n) \leq \liminf_{n\to\infty} u_1^*(v^{g_n}) + \eta$.

Observe that $p^n(t)$ satisfies

$$\exp\big(-r(t-\tau)\big)\big(p^n(t)(1) + \big(1 - p^n(t)\big)v^{g_n}(t)\big) \geq v^{g_n}(\tau)$$

for all $\tau \leq t$ such that $\tau \in I_1^{g_n}$ and $\tau_{g_n}^{\text{next}}(2) = t$. Hence, it is optimal for player 2 to reject player 1's demand γ at any time τ at which $\chi^n(\tau_{g_n}^{\text{next}}(2))\hat{p}^n(\tau_{g_n}^{\text{next}}(2)) = p^n(\tau_{g_n}^{\text{next}}(2))$ (under belief π_2^n). Therefore, $u_1^{g_n}(\gamma^{g_n}, \sigma_2^n) \leq \min_{t \leq T^n} e^{-rt}(1 - v^{g_n}(t))$. Now $u_1^*(v^{g_n}) = \min_{t \leq T(v^{g_n})} e^{-rt}(1 - v^{g_n}(t))$ and $\lim_{n\to\infty} \tilde{T}(v^{g_n}) = \tilde{T}(v)$. Hence, showing that $\lim_{n\to\infty} \tilde{T}^n = \tilde{T}(v) \equiv \tilde{T}$ would imply that $\lim_{n\to\infty} u_1^{g_n}(\gamma^{g_n}, \sigma_2^n) \leq \liminf_{n\to\infty} u_1^*(v^{g_n})$, yielding the desired contradiction. The remainder of the proof shows that $\lim_{n\to\infty} \tilde{T}^n = \tilde{T}$.

To see that $\lim_{n\to\infty} \tilde{T}^n = \tilde{T}$, first fix $t_0 \leq \tilde{T}$ and note that for all $\delta > 0$, there exists $N' > 0$ such that for all $t \leq t_0$ and all $n \geq N'$, if $g_n(t) = 2$, then $\min\{\tau \leq t : \tau \in I_1^{g_n}, \tau_{g_n}^{\text{next}}(2) = t\} \geq t - \delta$ (if this set is nonempty). Next, since both $e^{-rt}v(t)$ and $e^{-rt}v^{g_n}(t)$ are nonincreasing (as is easily checked) and $v^{g_n}(t) \to v(t)$ for all $t \in \mathbb{R}_+$, it follows that for all $\delta' > 0$, there exists $\delta > 0$ such that $t \leq t_0$ and $\tau \in [t - \delta, t]$ implies that $|e^{r(\tau-t)}v^{g_n}(\tau) - v(t, -1)| < \delta'$. Since $1 - v(t) \geq \eta$ for all $t \leq \tilde{T}$, combining these observations and letting S be the (countable) set of discontinuity points of $v(t)$, for all $\delta' > 0$, there exists N'' such that if $t = s_{g_n}^{\text{next}}(2)$ for some $s \in S \cap [0, t_0]$ and $n \geq N''$, then $|p^n(t) - \frac{v(t,-1)-v(t)}{1-v(t)}| < \delta'$.[8] Hence,

$$(S3) \qquad \lim_{n\to\infty} \prod_{s \in S \cap [0, t_0]} \big(1 - p^n\big(s_{g_n}^{\text{next}}(2)\big)\big) = \prod_{s \in S \cap [0, t_0]} \big(1 - p(s)\big)$$

for all $t_0 \leq \tilde{T}$, where p is as in Section 3.2.

Finally, I establish that whenever v is continuous on an interval $[t_0, t_\infty]$ with $t_\infty \leq \tilde{T}$, then

$$(S4) \qquad \lim_{n\to\infty} \prod_{t \in I_2^{g_n} \cap [t_0, t_\infty]} \big(1 - p^n(t)\big) = \exp\bigg(-\int_{t_0}^{t_\infty} \frac{rv(t) - v'(t)}{1 - v(t)}\, dt\bigg)$$

$$= \exp\bigg(-\int_{t_0}^{t_\infty} \lambda(t)\, dt\bigg),$$

---

[8]S is countable because $e^{-rt}v(t)$ is nonincreasing, and monotone functions have at most countably many discontinuities. Unlike in Section 3, S need not be a subset of $\mathbb{N}$ here.

where λ is as in Section 3.2. I will prove this fact by showing that the limit as $n \to \infty$ of a first-order approximation of the logarithm of $\prod_{t \in I_2^{g_n} \cap [t_0, t_\infty]} (1 - p^n(t))$ equals $-\int_{t_0}^{t_\infty} \frac{rv(t) - v'(t)}{1 - v(t)}$.

Let $\{t_{1,g_n}, t_{2,g_n}, \ldots, t_{K(n),g_n}\} = \{t \in [t_0, t_\infty] : p^n(t) > 0\}$, with $t_{k,g_n} < t_{k+1,g_n}$ for all $k \in \{1, \ldots, K(n) - 1\}$ and all $n \in \mathbb{N}$, and let $t_{0,g_n} = \max\{\tau : \tau \in I_1^{g_n}, \tau_{g_n}^{\text{next}}(2) = t_{1,g_n}\}$. Note that $K(n)$ is finite because $I_2^{g_n} \cap [t_0, t_\infty]$ is finite, and that, in addition, $t_{k,g_n}^{\text{next}}(1) < t_{k+1,g_n}$ for all k (where $t_{k,g_n}^{\text{next}}(1) \equiv t_{k,g_n,g_n}^{\text{next}}(1)$ to avoid redundant notation). Furthermore, since $e^{-r\tau} v^{g_n}(\tau)$ is nonincreasing, then

$$t_{k,g_n}^{\text{next}}(1) \in \underset{\substack{\tau < t_{k+1,g_n} : \\ \tau \in I_1^{g_n}, \tau_{g_n}^{\text{next}}(2) = t_{k+1,g_n}}}{\arg\max} \; e^{r(t_{k+1,g_n} - \tau)} v^{g_n}(\tau)$$

for all $k \in \{0, 1, \ldots, K(n) - 1\}$. Therefore,

$$(S5) \qquad \prod_{k=1}^{K(n)} \left(1 - p^n(t_{k,g_n})\right) = \prod_{k=1}^{K(n)} \underset{\substack{\tau < t_{k,g_n} : \\ \tau \in I_1^{g_n}, \tau_{g_n}^{\text{next}}(2) = t_{k,g_n}}}{\min} \frac{1 - e^{r(t_{k,g_n} - \tau)} v^{g_n}(\tau)}{1 - v^{g_n}(t_{k,g_n})}$$

$$= \prod_{k=1}^{K(n)} \underset{\substack{\tau < t_{k,g_n} : \\ \tau \in I_1^{g_n}, \tau_{g_n}^{\text{next}}(2) = t_{k,g_n}}}{\min} \frac{1 - e^{r(t_{k,g_n} - \tau)} v^{g_n}(\tau)}{1 - e^{-r(t_{k,g_n}^{\text{next}}(1) - t_k)} v^{g_n}(t_{k,g_n}^{\text{next}}(1))}$$

$$= \left(\prod_{k=1}^{K(n)-1} \frac{1 - e^{r(t_{k+1,g_n} - t_{k,g_n}^{\text{next}}(1))} v^{g_n}(t_{k,g_n}^{\text{next}}(1))}{1 - e^{-r(t_{k,g_n}^{\text{next}}(1) - t_{k,g_n})} v^{g_n}(t_{k,g_n}^{\text{next}}(1))} \right)$$

$$\times \frac{1 - e^{r(t_{1,g_n} - t_{0,g_n}^{\text{next}}(1))} v^{g_n}(t_{0,g_n}^{\text{next}}(1))}{1 - e^{-r(t_{K(n),g_n}^{\text{next}}(1) - t_{K(n),g_n})} v^{g_n}(t_{K(n),g_n}^{\text{next}}(1))}.$$

Next, taking a first-order Taylor approximation of $\log(1 - e^{rx} v^{g_n}(t))$ at $x = 0$ yields

$$\log\left(1 - e^{rx} v^{g_n}(t)\right) = \log\left(1 - v^{g_n}(t)\right) - \frac{rx v^{g_n}(t)}{1 - v^{g_n}(t)} + O(x^2).$$

Therefore, a first-order approximation of the logarithm of (S5) equals

$$\left(\sum_{k=1}^{K(n)-1} -(t_{k+1,g_n} - t_{k,g_n}) \frac{r v^{g_n}(t_{k,g_n}^{\text{next}}(1))}{1 - v^{g_n}(t_{k,g_n}^{\text{next}}(1))} \right)$$

$$+ \log\left(1 - e^{r(t_{1,g_n} - t_{0,g_n}^{\text{next}}(1))} v^{g_n}\left(t_{0,g_n}^{\text{next}}(1)\right)\right)$$

$$- \log\left(1 - e^{-r(t_{K(n),g_n}^{\text{next}}(1) - t_{K(n),g_n})} v^{g_n}\left(t_{K(n),g_n}^{\text{next}}(1)\right)\right).$$

I now show that

$$(\text{S6}) \qquad \lim_{n \to \infty} \sum_{k=1}^{K(n)-1} -(t_{k+1,g_n} - t_{k,g_n}) \frac{r v^{g_n}(t_{k,g_n}^{\text{next}}(1))}{1 - v^{g_n}(t_{k,g_n}^{\text{next}}(1))} = -\int_{t_0}^{t_\infty} \frac{r v(t)}{1 - v(t)} \, dt$$

and

$$(\text{S7}) \qquad \lim_{n \to \infty} \left(\log\left(1 - e^{r(t_{1,g_n} - t_{0,g_n}^{\text{next}}(1))} v^{g_n}\left(t_{0,g_n}^{\text{next}}(1) \right) \right) \right.$$

$$\left. - \log\left(1 - e^{-r(t_{K(n),g_n}^{\text{next}}(1) - t_{K(n),g_n})} v^{g_n}\left(t_{K(n),g_n}^{\text{next}}(1) \right) \right) \right)$$

$$= \int_{t_0}^{t_\infty} \frac{v'(t)}{1 - v(t)} \, dt,$$

which completes the proof of (S4). Equation (S7) is immediate, because, since v is continuous on $[t_0, t_\infty]$, both the left- and right-hand sides equal

$$\log\left(1 - v(t_0) \right) - \log\left(1 - v(t_\infty) \right).$$

To establish (S6), let

$$f^n(t) \equiv \exp\left(-r\left(\frac{1 + \eta}{\eta} \right) t \right) \frac{r v^{g_n}(t)}{1 - v^{g_n}(t)}$$

and let

$$f(t) \equiv \exp\left(-r\left(\frac{1 + \eta}{\eta} \right) t \right) \frac{r v(t)}{1 - v(t)}.$$

For all $n > N$, it can be verified that both $f^n(t)$ and $f(t)$ are nonincreasing on the interval $[t_0, t_\infty]$, using the facts that $e^{-rt} v^{g_n}(t)$ and $e^{-rt} v(t)$ are nonincreasing, and that $v^{g_n}(t) \le 1 - \eta$ for all $n > N$ and $t \le t_\infty \le \tilde{T}$. Fix $\zeta > 0$ and $m \in \mathbb{N}$. Because $v^{g_n}(t) \to v(t)$ for all $t \in \mathbb{R}_+$, there exists $N''' \ge N$ such that for all $n > N'''$, $|f^n(t) - f(t)| < \zeta$ for all t in the set

$$\left\{ t_0, \frac{(m-1)t_0 + t_\infty}{m}, \frac{(m-2)t_0 + 2t_\infty}{m}, \ldots, t_\infty \right\}.$$

Since both f^n and f are nonincreasing on $[t_0, t_\infty]$, this implies that

$$\left| f^n(t) - f(t) \right| < \zeta + \max_{k \in \{1, \ldots, K(n)-1\}} \left(f\left(\frac{(m-k)t_0 + kt_\infty}{m} \right) \right.$$

$$\left. - f\left(\frac{(m-k-1)t_0 + (k+1)t_\infty}{m} \right) \right)$$

for all $t \in [t_0, t_\infty]$. Since f is continuous on $[t_0, t_\infty]$, taking $m \to \infty$ implies that $|f^n(t) - f(t)| < 2\zeta$ for all $t \in [t_0, t_\infty]$. Therefore, $|\frac{rv^{g_n}(t)}{1 - v^{g_n}(t)} - \frac{rv(t)}{1 - v(t)}| \leq 2\zeta \exp(r(\frac{1+\eta}{\eta})t_\infty)$ for all $t \in [t_0, t_\infty]$. Hence,

$$
\begin{aligned}
\lim_{n \to \infty} & \sum_{k=1}^{K(n)-1} -(t_{k+1,g_n} - t_{k,g_n}) \frac{rv^{g_n}(t_{k,g_n}^{\text{next}}(1))}{1 - v^{g_n}(t_{k,g_n}^{\text{next}}(1))} \\
&= \lim_{n \to \infty} \sum_{k=1}^{K(n)-1} -(t_{k+1,g_n} - t_{k,g_n}) \frac{rv(t_{k,g_n}^{\text{next}}(1))}{1 - v(t_{k,g_n}^{\text{next}}(1))} \\
&= \lim_{n \to \infty} \sum_{k=1}^{K(n)-1} -(t_{k+1,g_n} - t_{k,g_n}) \frac{rv(t_{k,g_n})}{1 - v(t_{k,g_n})} \\
&= -\int_{t_0}^{t_\infty} \frac{rv(t)}{1 - v(t)} \, dt,
\end{aligned}
$$

where the first equality follows because $\sum_{k=1}^{K(n)-1}(t_{k+1,g_n} - t_{k,g_n}) \leq t_\infty - t_0$ for all $n \in \mathbb{N}$, the second equality follows because $t_{k,g_n}^{\text{next}}(1) \in [t_{k,g_n}, t_{k+1,g_n}]$ and v is continuous on $[t_0, t_\infty]$, and the third equality follows by definition of the (Riemann) integral.

Combining (S3) and (S4), it follows that

$$
\lim_{n \to \infty} \prod_{s \in I_2^{g_n} \cap [0,t]} \left(1 - p^n(s)\right) = \exp\left(-\int_0^t \lambda(s) \, ds\right) \prod_{s \in S \cap [0,t]} \left(1 - p(s)\right)
$$

for all $t \leq \tilde{T}$. This implies that $\lim_{n \to \infty} \tilde{T}^n = \tilde{T}$, completing the proof of the lemma. *Q.E.D.*

I now complete the proof of Proposition 6.

Let $\{g_n\}$ be a sequence of discrete-time bargaining games converging to continuous time. Recall that $u_1^{*,g_n} = \sup_{\gamma^{g_n}} u_1^{*,g_n}(\gamma^{g_n})$. Thus, there exists a sequence of postures $\{\gamma^{g_n}\}$, with γ^{g_n} a posture in g_n, such that $\lim_{n \to \infty} |u_1^{*,g_n} - u_1^{*,g_n}(\gamma^{g_n})| = 0$. Let $\{v^{g_n}\}$ be the corresponding sequence of continuation value functions. Because $e^{-rt}v^{g_n}(t)$ is nonincreasing and the space of monotone functions from $\mathbb{R}_+ \to [0, 1]$ is sequentially compact (by Helly's selection theorem or footnote 27), this sequence has a convergent subsequence $\{v^{g_k}\}$ converging to some v on $\mathbb{R}_+$.

I claim that $v = v^*$. Toward a contradiction, suppose not. Since v^* is the unique maxmin continuation value function in g^{cts}, there exists $\eta > 0$ such that $u_1^* > \lim_{k \to \infty} u_1^*(v^{g_k}) + \eta$. By Lemma 7, $\lim_{k \to \infty} u_1^{*,g_k}(v^{g_k}) = \lim_{k \to \infty} u_1^*(v^{g_k})$. Finally, by Lemma 6, there exists an alternative sequence of postures $\{\gamma^{g_{k'}}\}$ such

that $\lim_{k \to \infty} u_1^{*, g_k}(\gamma^{g_{k'}}) \geq u_1^*$. Combining these observations implies that there exists $K > 0$ such that for all $k \geq K$,

$$u_1^{*, g_k}(\gamma^{g_{k'}}) > u_1^* - \eta/3 > u_1^*(v^{g_k}) + 2\eta/3 > u_1^{*, g_k}(v^{g_k}) + \eta/3,$$

which contradicts the fact that $\lim_{k \to \infty} |u_1^{*, g_k} - u_1^{*, g_k}(\gamma^{g_k})| = 0$. Therefore, $v = v^*$. In addition, since this argument applies to any convergent subsequence of $\{v^{g_n}\}$ and since every subsequence of $\{v^{g_n}\}$ has a convergent subsubsequence, this implies that $v^{g_n} \to v^*$ pointwise.

A similar contradiction argument shows that $\lim_{k \to \infty} u_1^{*, g_k}(v^{g_k}) = u_1^*$ for any convergent subsequence $\{v^{g_k}\} \subseteq \{v^{g_n}\}$. Since $\lim_{k \to \infty} |u_1^{*, g_k} - u_1^{*, g_k}(\gamma^{g_k})| = 0$, it follows that $u_1^{*, g_k} \to u_1^*$, and since this argument applies to any convergent subsequence of $\{v^{g_n}\}$, this implies that $u_1^{*, g_n} \to u_1^*$.   *Q.E.D.*

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