Econometrica Supplementary Material

SUPPLEMENT TO "DIRECTED SEARCH FOR EQUILIBRIUM WAGE–TENURE CONTRACTS"

(*Econometrica*, Vol. 77, No. 2, March 2009, 561–584)

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This supplementary appendix provides the proofs of Lemmas 3.2, B.1, B.3, and B.4, and Theorem 6.1. Lemmas B.1, B.2, B.3, and B.4 are used in the proof of Theorem 4.1 in the paper. All cross references to equations and sections use the numbering in the paper.

APPENDIX C

PROOF OF LEMMA 3.2: Consider the firm's optimization problem ($\mathcal{P}$). The state variable is V that obeys (3.3). Treat γ, defined in (3.6), as an auxiliary state variable whose law of motion is

$$\frac{d}{dt}\gamma(t, t_a) = -\big[r + p\big(F(V(t))\big)\big]\gamma(t, t_a). \tag{C.1}$$

Denote the shadow price of V as Λ_V and of γ as Λ_γ. Then the Hamiltonian of ($\mathcal{P}$) is

$$\begin{aligned}\mathcal{H}(t) &= (y - \tilde{w})\gamma(t, 0) + \Lambda_V[rV - S(V) - u(\tilde{w})] \\ &\quad - \Lambda_\gamma\big[r + p(F(V))\big]\gamma(t, 0),\end{aligned}$$

where I have suppressed the dependence of the variables on t, except that of γ. Denote $\Lambda_c(t) = \Lambda_V(t)/\gamma(t, 0)$, where the subscript c indicates the "current value." The optimality conditions of $\tilde{w}$, V, and γ are

$$-u'(\tilde{w})\Lambda_c - 1 \le 0 \quad \text{and} \quad \tilde{w} \ge 0, \quad \text{with complementary slackness,} \tag{C.2}$$

$$\dot{\Lambda}_c = \Lambda_\gamma\, dp(F(V))/dV, \tag{C.3}$$

$$\dot{\Lambda}_\gamma = -(y - \tilde{w}) + \Lambda_\gamma\big[r + p(F(V))\big]. \tag{C.4}$$

To derive (C.3), I have used the fact that $S'(V) = -p(F(V))$ (see Lemma 3.1).

Using (C.1), I can rewrite (C.4) as $\frac{d}{dt}[\gamma(t, 0)\Lambda_\gamma(t)] = -[y - \tilde{w}(t)]\gamma(t, 0)$. Integrating this equation under the transversality condition, $\lim_{t\to\infty}\gamma(t, 0) \times \Lambda_\gamma(t) = 0$, I get $\Lambda_\gamma(t) = \tilde{J}(t)$ for all t, where $\tilde{J}(\cdot)$ is given by (3.7). Substituting $\Lambda_\gamma = \tilde{J}$ into (C.3) and the Hamiltonian yields

$$\dot{\Lambda}_c = \tilde{J}\,\frac{dp(F(V))}{dV}, \tag{C.5}$$

$$\mathcal{H}(t) = \gamma(t, 0)\left[\frac{-d\tilde{J}(t)}{dt} + \Lambda_c(t)\dot{V}(t)\right]. \tag{C.6}$$

 DOI: 10.3982/ECTA7870

Because $p(F(V))$ strictly decreases in V for all $t < \infty$, $\dot{\Lambda}_c(t) < 0$ for all $t < \infty$.

Define t_0 by $\Lambda_c(t_0) = 0$. There is at most one such t_0, because $\dot{\Lambda}_c < 0$. Moreover, $\Lambda_c(t) > 0$ for all $t < t_0$ and $\Lambda_c(t) < 0$ for all $t > t_0$. For all $t \leq t_0$, $-u'(\tilde{w}(t))\Lambda_c(t) \leq 0$, in which case (C.2) implies $\tilde{w}(t) = 0$. For all $t > t_0$, the assumption $u'(0) = \infty$ ensures $\tilde{w}(t) > 0$: if $\tilde{w}(t) = 0$, then $-u'(\tilde{w}(t))\Lambda_c(t) - 1 = \infty > 0$, which contradicts (C.2).

The remainder of the proof establishes a sequence of results. First, $d\tilde{w}(t)/dt > 0$ for all $t > t_0$. Suppose, to the contrary, that $d\tilde{w}(t)/dt \leq 0$ at $t = t_1$ for some $t_1 \in (t_0, \infty)$. Because $\dot{\Lambda}_c < 0$, then

$$\frac{d}{dt}\big[-u'(\tilde{w}(t))\Lambda_c(t)\big] > -u''(\tilde{w}(t))\Lambda_c(t)\frac{d\tilde{w}(t)}{dt}, \quad \text{all } t < \infty.$$

Because $d\tilde{w}(t)/dt \leq 0$ at $t = t_1$ and $\Lambda_c(t) < 0$ for $t > t_0$, the derivative above on the right-hand side is strictly positive for t near t_1. As a result, there exists $\varepsilon > 0$ such that $-u'(\tilde{w}(t))\Lambda_c(t) > -u'(\tilde{w}(t_1))\Lambda_c(t_1) = 1$ for $t \in (t_1, t_1 + \varepsilon]$, where the equality follows from (C.2) and $\tilde{w}(t_1) > 0$. This result contradicts (C.2). Thus, I have shown that the wage path has the form

$$\text{(C.7)} \qquad \begin{aligned} &\tilde{w}(t) = 0, \quad \text{for} \quad t < t_0; \\ &\tilde{w}(t) > 0 \quad \text{and} \quad d\tilde{w}(t)/dt > 0, \quad \text{for} \quad t \in (t_0, \infty). \end{aligned}$$

Because $\tilde{w}(t)$ is bounded for all t and increasing, then $\tilde{w}(t) \nearrow \bar{w}$ as $t \to \infty$.

Second, $\mathcal{H}(t) = 0$ for all t. Differentiating (C.6) with respect to t and substituting (C.5) yields

$$\frac{d\mathcal{H}(t)}{dt} = -\gamma(t, 0)\big[1 + u'(\tilde{w}(t))\Lambda_c(t)\big]\frac{d\tilde{w}(t)}{dt} = 0,$$

where the second equality uses the results that $d\tilde{w}(t)/dt = 0$ for $t < t_0$, and $1 + u'(\tilde{w}(t))\Lambda_c(t) = 0$ for $t \geq t_0$. Because $\lim_{t\to\infty} \mathcal{H}(t) = 0$, then $\mathcal{H}(t) = 0$ for all t, which can be rewritten as

$$\text{(C.8)} \qquad d\tilde{J}(t)/dt = \Lambda_c(t)\dot{V}(t), \quad \text{all } t.$$

Third, $\dot{V}(t) > 0$ for all $t < \infty$, and $\tilde{J}(t)$ is maximized at $t = t_0$. Suppose, to the contrary, that $\dot{V}(t_1) \leq 0$ for some $t_1 < \infty$. If $t_1 > t_0$, then $d\tilde{w}(t)/dt > 0$ for all $t \in [t_1, \infty)$ (see (C.7)). Differentiating (3.3) yields

$$\frac{d\dot{V}(t)}{dt} = \big[r + p\big(F(V(t))\big)\big]\dot{V}(t) - u'(\tilde{w}(t))\frac{d\tilde{w}(t)}{dt}.$$

$\dot{V}(t_1) \leq 0$ implies $d\dot{V}(t)/dt < 0$ at $t = t_1$. By induction, $d\dot{V}(t)/dt < 0$ for all $t \in [t_1, \infty)$. Thus, $V(t)$ strictly decreases toward $\bar{V}$ as t increases from t_1 to ∞, contradicting the fact that $V(t) \le \bar{V}$ for all $t < \infty$. If $t_1 \le t_0$, then $\tilde{w}(t_1) = 0$ by (C.7), and so (3.3) implies $rV(t_1) - S(V(t_1)) \le u(0)$. This result and (3.4) yield

$$rV_u - S(V_u) - u(b) - \left[rV(t_1) - S(V(t_1))\right] + u(0) \ge 0.$$

Because $S'(V) < 0$, the left-hand side of the equation is strictly decreasing in $V(t_1)$. Because the left-hand side is negative at $V(t_1) = V_u$, then $V(t_1) < V_u$. In this case, the worker will quit into unemployment, which will be suboptimal to the firm—a contradiction.

Recall that $\Lambda_c(t) > 0$ for all $t < t_0$ and $\Lambda_c(t) < 0$ for all $t > t_0$. Equation (C.8) and $\dot{V} > 0$ imply that $d\tilde{J}(t)/dt > 0$ for all $t < t_0$ and $d\tilde{J}(t)/dt < 0$ for all $t > t_0$. That is, $\tilde{J}(t)$ is maximized at $t = t_0$.

Fourth, $t_0 \le 0$; thus, $\tilde{w}(t) > 0$ for all $t > 0$ and $d\tilde{w}(t)/dt > 0$ for all $t < \infty$ (see (C.7)). Suppose $t_0 > 0$, to the contrary. Then $\tilde{J}(t_0) > \tilde{J}(0)$ by the previous result. Let $\{\tilde{w}(t)\}_{t=0}^{\infty}$ be the optimal contract that generates $\tilde{J}(0)$ to the firm. Consider an alternative contract $\{\hat{w}(t)\}_{t=0}^{\infty}$, where $\hat{w}(t) = \tilde{w}(t + t_0)$ for all t. This alternative contract is feasible and generates a higher value to the firm, $\tilde{J}(t_0)$, than the optimal contract—a contradiction.

Finally, (3.9) and (3.10) hold. Because $\tilde{w}(t) > 0$ for all t, then $\Lambda_c(t) = -1/u'(\tilde{w}(t))$ for all t. Differentiating this equation with respect to t and substituting (C.5), I get (3.9). Substituting Λ_c into (C.8) yields (3.10). Because $\dot{V}(t) > 0$ and $\tilde{w}(t) > 0$ for all $t < \infty$, then $d\tilde{J}(t)/dt < 0$ for all $t < \infty$. *Q.E.D.*

PROOF OF LEMMA B.1: Let $w \in \Omega$. Part (i) of the lemma was established in the analysis immediately following (4.8). It is easy to verify from (4.3) that $J_w(V)$ is strictly decreasing and continuously differentiable, with $J'_w(V) = -1/u'(w(V)) < 0$. Since $w(V)$ is increasing, then $J'_w(V)$ is decreasing, and so $J_w(V)$ is (weakly) concave. Because $q_w(V) = k/J_w(V)$ and $p_w(V) = M(q_w(V))$, I have

$$p'_w(V) = \frac{M'(q_w(V))[q_w(V)]^2}{u'(w(V))k} < 0,$$

where $M'(q) < 0$ by Assumption 1. This shows that $p_w(V)$ is strictly decreasing and continuously differentiable. Moreover, parts (iii) and (iv) of Assumption 1 imply that $[M'(q)q^2]$ is decreasing in q. Because $q_w(V)$ is increasing in V, $M'(q_w(V))[q_w(V)]^2$ is decreasing in V. Because $1/u'(w(V))$ is increasing in V and $M' < 0$, then $p'_w(V)$ is decreasing. That is, $p_w(V)$ is (weakly) concave, and so part (ii) of the lemma holds.

If $w \in \Omega'$, that is, if $w(V)$ is strictly increasing for all $V < \bar{V}$, then it is straightforward to strengthen the argument for part (ii) to show that $J_w(V)$ and $p_w(V)$ are strictly concave, as stated in part (iii). *Q.E.D.*

PROOF OF LEMMA B.2: To prove part (i) of the lemma, pick two arbitrary functions $w_1, w_2 \in \Omega$, with $w_2(V) \ge w_1(V)$ for all V. Simplify the notation

J_{w_i} to J_i, F_{w_i} to F_i, and p_{w_i} to p_i, where $i = 1, 2$. Because $w_2(V) \geq w_1(V)$ for all V, (4.3) implies $J_2(V) \geq J_1(V)$, and the assumption $M' < 0$ implies $p_2(V) \geq p_1(V)$ for all V. Suppose, contrary to part (i) of the lemma, that $p_1(F_1(V)) > p_2(F_2(V))$ for some V. Let $q_i = k/J_i(F_i(V))$, $i = 1, 2$. Because $p_i(F_i(V)) = M(q_i)$ and $M(q)$ is strictly decreasing in q, the supposition implies $q_1 < q_2$ and, hence, $J_1(F_1(V)) > J_2(F_2(V))$. Monotonicity of J_w in w implies $J_2(F_2(V)) \geq J_1(F_2(V))$. In this case, $J_1(F_1(V)) > J_1(F_2(V))$ and so $F_1(V) < F_2(V)$. With these results, I can derive

$$\begin{aligned}
0 &< p_1(F_1(V)) - p_2(F_2(V)) \\
&= p_2'(F_2(V))[F_2(V) - V] - p_1'(F_1(V))[F_1(V) - V] \\
&< \big[p_2'(F_2(V)) - p_1'(F_1(V))\big][F_1(V) - V] \\
&= \frac{F_1(V) - V}{k}\left[\frac{M'(q_2)(q_2)^2}{u'(w_2(F_2(V)))} - \frac{M'(q_1)(q_1)^2}{u'(w_1(F_1(V)))}\right] \\
&\leq \frac{F_1(V) - V}{u'(w_1(F_1(V)))k}[M'(q_2)(q_2)^2 - M'(q_1)(q_1)^2].
\end{aligned}$$

The first inequality comes from the supposition, the first equality comes from (3.2), the second inequality comes from $F_2(V) > F_1(V)$ and $p_2'(F_2) < 0$, the second equality comes from computing $p_i'(F_i)$, and the last inequality comes from $M'(q_2) < 0$ and $w_2(F_2(V)) \geq w_1(F_2(V)) \geq w_1(F_1(V))$. Parts (iii) and (iv) of Assumption 1 imply that $M'(q)q^2$ is decreasing in q. Because $q_2 > q_1$, as shown above, the expression in the last line above is nonpositive—a contradiction.

To prove part (ii) of the lemma, let $w \in \Omega$, and $V_2 \geq V_1$. Note that $w(V_2) \geq w(V_1)$, because $w \in \Omega$. Moreover, because $[rV - S_w(V)]$ is strictly increasing in V, $rV_2 - S_w(V_2) \geq rV_1 - S_w(V_1)$. Hence, the following inequality holds:

$$\Delta \leq \Delta_1 \equiv \frac{1}{u'(w(V_1))}\begin{bmatrix} \max\{0, rV_1 - S_w(V_1) - u(w(V_1))\} \\ -\max\{0, rV_1 - S_w(V_1) - u(w(V_2))\} \end{bmatrix}.$$

Consider all possible cases: (a) $rV_1 - S_w(V_1) \geq u(w(V_2))$, (b) $rV_1 - S_w(V_1) \leq u(w(V_1))$, and (c) $u(w(V_1)) < rV_1 - S_w(V_1) < u(w(V_2))$. In each case, it can be verified that

$$\Delta_1 \leq \frac{u(w(V_2)) - u(w(V_1))}{u'(w(V_1))}.$$

Thus, the first inequality in (B.1) holds.

To establish the second inequality in (B.1), I first show that

$$\begin{aligned}
&\max\big\{0, rV_2 - S_w(V_2) - u(w(V_2))\big\} \\
&\quad \leq \frac{u'(w(V_2))}{u'(w(V_1))}\max\big\{0, rV_2 - S_w(V_2) - u(w(V_1))\big\}.
\end{aligned}$$

This inequality is evident when $rV_2 - S_w(V_2) \le u(w(V_2))$, because the left-hand side is 0 in that case. If $rV_2 - S_w(V_2) > u(w(V_2))$, the above inequality becomes

$$\frac{rV_2 - S_w(V_2) - u(w(V_2))}{u'(w(V_2))} \le \frac{rV_2 - S_w(V_2) - u(w(V_1))}{u'(w(V_1))}.$$

Because $[rV - S_w(V)]$ is strictly increasing in V, $rV_2 - S_w(V_2) \le r\bar{V} - S_w(\bar{V}) = u(\bar{w})$. In this case, (4.11) implies that $[rV - S_w(V) - u(w)]/u'(w)$ is decreasing in w for any given V and $S_w(V)$. Since $w(V_2) \ge w(V_1)$, the above inequality holds.

Using the above result, I obtain

$$\Delta \ge \frac{1}{u'(w(V_1))}\begin{bmatrix} \max\{0, rV_1 - S_w(V_1) - u(w(V_1))\} \\ -\max\{0, rV_2 - S_w(V_2) - u(w(V_1))\} \end{bmatrix}.$$

Consider all of the possible cases: (a) $u(w(V_1)) \ge rV_2 - S_w(V_2)$, (b) $u(w(V_1)) \le rV_1 - S_w(V_1)$, and (c) $rV_1 - S_w(V_1) < u(w(V_1)) < rV_2 - S_w(V_2)$. In each case, it is straightforward to deduce the second inequality in (B.1) from the above relation. *Q.E.D.*

PROOF OF LEMMA B.3: Let $w \in \Omega$, and consider the function $\psi w(V)$. With Lemma B.1, $\psi w(V)$ is a continuous and bounded function of V. Next, I prove that $\psi w(V)$ is an increasing function. To do so, let V_1 and V_2 be arbitrary values in $[\underline{V}, \bar{V}]$, with $V_2 \ge V_1$. Simplify the notation $f(V_i)$ to f_i, where f includes the functions w, J_w, F_w, S_w, and ψw. I show that $\psi w_2 \ge \psi w_1$. To do so, use the second inequality in (B.1) to obtain

$$\begin{aligned} \psi w_2 - \psi w_1 \ge{} & [r + p_w(F_{w1})]J_{w1} - [r + p_w(F_{w2})]J_{w2} \\ & + [rV_1 - S_{w1} - (rV_2 - S_{w2})]/u'(w_1) \\ ={} & [r + p_w(F_{w1})](J_{w1} - J_{w2}) + J_{w2}[p_w(F_{w1}) - p_w(F_{w2})] \\ & + [rV_1 - S_{w1} - (rV_2 - S_{w2})]/u'(w_1). \end{aligned}$$

Because $[rV - S_w(V)]' = r + p_w(F_w)$ and $[r\bar{V} - S_w(\bar{V})] = u(\bar{w})$, then

$$rV - S_w(V) = u(\bar{w}) - \int_V^{\bar{V}} \big[r + p_w(F_w(z))\big]\,dz.$$

Using this result and expressing $J_w(V)$ as in (4.3), I get

$$\begin{aligned} &[r + p_w(F_{w1})](J_{w1} - J_{w2}) + [rV_1 - S_{w1} - (rV_2 - S_{w2})]/u'(w_1) \\ &\quad = \int_{V_1}^{V_2} \left[\frac{r + p_w(F_{w1})}{u'(w(z))} - \frac{r + p_w(F_w(z))}{u'(w_1)}\right] dz \ge 0. \end{aligned} \tag{C.9}$$

The inequality follows from $p_w(F_{w1}) \geq p_w(F_w(z))$ and $u'(w(z)) \leq u'(w_1)$ for all $z \in [V_1, V_2]$. Because $p_w(F)$ is a decreasing function of F, I have established

$$\psi w_2 - \psi w_1 \geq J_{w2}[p_w(F_{w1}) - p_w(F_{w2})] \geq 0. \tag{C.10}$$

Now I verify $\psi w(V) \in [\underline{w}, \bar{w}]$ for all V, with $\psi w(\bar{V}) = \bar{w}$. Because $w(\bar{V}) = \bar{w}$, $J_w(\bar{V}) = k/\bar{q}$, and $p_w(\bar{V}) = 0$, it is clear that $\psi w(\bar{V}) = \bar{w}$. Since $\psi w(V)$ is increasing, $\psi w(V) \leq \psi w(\bar{V}) = \bar{w}$ for all V. Similarly, $\psi w(V) \geq \underline{w}$ for all V if and only if $\psi w(\underline{V}) \geq \underline{w}$. To establish the latter inequality, note that $w(\underline{V}) \geq \underline{w}$, because $w \in \Omega$. Using (4.11) and the fact that $r\underline{V} = u(b)$, I have

$$\frac{1}{u'(w(\underline{V}))}\left[r\underline{V} - S_w(\underline{V}) - u(w(\underline{V}))\right] \leq \frac{1}{u'(\underline{w})}[u(b) - S_w(\underline{V}) - u(\underline{w})].$$

The right-hand side of the inequality is nonnegative, because $\underline{w}$ is set to be small. Thus,

$$\begin{aligned}\psi w(\underline{V}) &\geq y - \left[r + p_w(F_w(\underline{V}))\right]J_w(\underline{V}) \\ &\qquad - \frac{1}{u'(\underline{w})}[u(b) - S_w(\underline{V}) - u(\underline{w})] \\ &\geq y - \left[r + p_{\bar{w}}(F_{\bar{w}}(\underline{V}))\right]\bar{J} - \frac{1}{u'(\underline{w})}[u(b) - S_{\underline{w}}(\underline{V}) - u(\underline{w})].\end{aligned}$$

The first inequality comes from the preceding result. The second inequality uses part (i) of Lemma B.2, the upper bound on J (defined in (4.8)), and the fact that $S_w(V)$ is increasing in w for any given V. With the above result, (4.10) implies $\psi w(\underline{V}) \geq \underline{w}$. Therefore, ψ maps functions in Ω back into functions in Ω.

Finally, if $V_2 > V_1$, the inequalities in (C.9) and (C.10) are strict, because $F_w(V)$ is strictly increasing and $p_w(F_w(V))$ is strictly decreasing in V for all $V < \bar{V}$ (see Lemma B.2). In this case, $\psi w \in \Omega' \subset \Omega$. This completes the proof of Lemma B.3. *Q.E.D.*

PROOF OF LEMMA B.4: I prove that the following inequality holds for all $w_1, w_2 \in \Omega$, and all V,

$$|\psi w_2(V) - \psi w_1(V)| \leq A\|w_2 - w_1\|, \tag{C.11}$$

where the norm is the sup norm and A is a finite constant. Once this is done, Lipschitz continuity of ψ is evident from the inequality

$$\|\psi w_2 - \psi w_1\| = \max_V |\psi w_2(V) - \psi w_1(V)| \leq A\|w_2 - w_1\|.$$

To show (C.11), take two arbitrary functions, $w_1, w_2 \in \Omega$, and fix V at an arbitrary value in $[\underline{V}, \bar{V}]$. Without loss of generality, assume $\psi w_2(V) \geq \psi w_1(V)$

for this given V. Since V is fixed, I suppress it from the functions if this does not cause confusion. Also, shorten the subscript w_i on J, p, F, and S to i, where $i = 1, 2$. I have

$$\begin{aligned}
0 &\leq \psi w_2(V) - \psi w_1(V) \\
&= [r + p_1(F_1)](J_1 - J_2) + J_2[p_1(F_1) - p_2(F_2)] + \Delta_2,
\end{aligned}$$

where

$$\Delta_2 = \max\left\{0, \frac{rV - S_1 - u(w_1)}{u'(w_1)}\right\} - \max\left\{0, \frac{rV - S_2 - u(w_2)}{u'(w_2)}\right\}.$$

To proceed, note that the following inequalities hold for all a_1 and a_2:

$$\max\{0, a_1\} - \max\{0, a_2\} \leq \max\{0, a_1 - a_2\} \leq |a_1 - a_2|.$$

Using these results, it is easy to verify that

$$\Delta_2 \leq \left|\frac{rV - S_1 - u(w_1)}{u'(w_1)} - \frac{rV - S_1 - u(w_2)}{u'(w_2)}\right| + \frac{|S_2 - S_1|}{u'(w_2)}.$$

Denote the first term on the right-hand side above as Δ_3. Define

$$\mu_1 = \min_{w \in [\underline{w}, \bar{w}]} |u''(w)|, \quad \mu_2 = \max_{w \in [\underline{w}, \bar{w}]} |u''(w)|, \tag{C.12}$$

where μ_1 and μ_2 are positive and finite. Because $(rV - S_1)$ is strictly increasing in V, $rV - S_1 \leq u(\bar{w})$. Also, concavity of u implies $u(\bar{w}) \leq u(w) + u'(w)(\bar{w} - w)$. Then

$$\left|\frac{d}{dw}\left(\frac{rV - S_1 - u(w)}{u'(w)}\right)\right| \leq 1 + \frac{\mu_2}{u'(\bar{w})}(\bar{w} - \underline{w}) \equiv A_1. \tag{C.13}$$

Hence,

$$\Delta_3 \leq A_1|w_2 - w_1|, \quad \Delta_2 \leq A_1|w_2 - w_1| + |S_2 - S_1|/u'(\bar{w}).$$

Substituting these results into the earlier expression for $[\psi w_2(V) - \psi w_1(V)]$ and using the bounds in (4.8), I obtain

$$\begin{aligned}
0 &\leq \psi w_2(V) - \psi w_1(V) \\
&\leq (r + \bar{p})|J_1 - J_2| + \bar{J}|p_2(F_2) - p_1(F_1)| \\
&\quad + |S_2 - S_1|/u'(\bar{w}) + A_1|w_2 - w_1|.
\end{aligned} \tag{C.14}$$

Let me examine the first three terms on the right-hand side above. With μ_2 defined in (C.12), the following inequality holds for all $w_1, w_2 \in [\underline{w}, \bar{w}]$:

$$\left|\frac{1}{u'(w_1(z))} - \frac{1}{u'(w_2(z))}\right| \le A_2\|w_2 - w_1\|, \quad \text{where} \quad A_2 \equiv \frac{\mu_2}{[u'(\bar{w})]^2}. \tag{C.15}$$

Using this result and (4.3), I have

$$|J_1 - J_2| \le \int_{\underline{V}}^{\bar{V}} \left|\frac{1}{u'(w_1(z))} - \frac{1}{u'(w_2(z))}\right| dz \le A_2(\bar{V} - \underline{V})\|w_2 - w_1\|. \tag{C.16}$$

To put a bound on the difference, $|p_2(F_2) - p_1(F_1)|$, define

$$B_1 \equiv m_1\bar{q}^2/k, \quad B_2 \equiv (\bar{q}m_2 + 2m_1)\bar{q}^3/k^2, \tag{C.17}$$

where m_1 and m_2 are the bounds specified in Assumption 1. Clearly, B_1 and B_2 are finite. Because $k/J_w = q_w \le \bar{q}$, it is straightforward to verify that

$$\left|\frac{dM(k/J_w)}{dJ_w}\right| \le B_1, \quad \left|\frac{d^2M(k/J_w)}{dJ_w^2}\right| \le B_2. \tag{C.18}$$

Using these bounds, (C.15), and (C.16), I can derive the following results for all $z \in [\underline{V}, \bar{V}]$:

$$|p_2(z) - p_1(z)| \le B_1|J_2(z) - J_1(z)| \le B_1A_2(\bar{V} - \underline{V})\|w_2 - w_1\|, \tag{C.19}$$

$$\begin{aligned} |p_2'(z) - p_1'(z)| &= \left|\frac{1}{u'(w_2(z))}\frac{d}{dJ_2}M\left(\frac{k}{J_2(z)}\right) \right. \\ &\qquad \left. - \frac{1}{u'(w_1(z))}\frac{d}{dJ_1}M\left(\frac{k}{J_1(z)}\right)\right| \\ &\le B_1\left|\frac{1}{u'(w_2(z))} - \frac{1}{u'(w_1(z))}\right| \\ &\qquad + \frac{1}{u'(w_2(z))}\left|\frac{d}{dJ_2}M\left(\frac{k}{J_2(z)}\right) - \frac{d}{dJ_1}M\left(\frac{k}{J_1(z)}\right)\right| \\ &\le B_1A_2\|w_2 - w_1\| + \frac{B_2}{u'(\bar{w})}|J_2(z) - J_1(z)| \\ &\le \left[B_1 + \frac{B_2}{u'(\bar{w})}(\bar{V} - \underline{V})\right]A_2\|w_2 - w_1\|. \end{aligned} \tag{C.20}$$

Now examine the difference $|p_2(F_2) - p_1(F_1)|$. Assume $F_2 \ge F_1$ without loss of generality. (If $F_2 \le F_1$, switch the roles of F_1 and F_2 in the proof, and the resulting bound is the same.) In the case where $p_2(F_2) < p_1(F_1)$, I have the inequalities

$$\begin{aligned} 0 &< p_1(F_1) - p_2(F_2) = -p_1'(F_1)(F_1 - V) + p_2'(F_2)(F_2 - V) \\ &\le (F_1 - V)[p_2'(F_1) - p_1'(F_1)]. \end{aligned}$$

The equality follows from (3.2), and the last inequality follows from the fact that $p_2'(F)(F - V)$ is decreasing in F. Because $0 \le F_1 - V \le \bar{V} - \underline{V}$, the above result and (C.20) imply

$$|p_2(F_2) - p_1(F_1)| \le \left[B_1 + \frac{B_2}{u'(\bar{w})}(\bar{V} - \underline{V})\right] A_2(\bar{V} - \underline{V})\|w_2 - w_1\|. \tag{C.21}$$

In the case where $p_2(F_2) \ge p_1(F_1)$, the following inequalities hold:

$$0 \le p_2(F_2) - p_1(F_1) \le p_2(F_1) - p_1(F_1) \le B_1 A_2(\bar{V} - \underline{V})\|w_1 - w_2\|.$$

The second inequality comes from the fact that p is decreasing and the last inequality comes from (C.19). Thus, (C.21) holds in this case too.

Next, turn to the difference, $|S_2 - S_1|$. Because S_1 is the maximum of $p_1(F)(F - V)$ over F, then $S_1 \ge p_1(F_2)(F_2 - V)$. Using the inequality and (C.19), I have

$$\begin{aligned} S_2 - S_1 &\le p_2(F_2)(F_2 - V) - p_1(F_2)(F_2 - V) \\ &= (F_2 - V)[p_2(F_2) - p_1(F_2)] \le B_1 A_2(\bar{V} - \underline{V})^2\|w_2 - w_1\|. \end{aligned}$$

Similarly, using the inequality, $S_2 \ge p_2(F_1)(F_1 - V)$, I can show that $(S_1 - S_2)$ is bounded by the same upper bound as above. Hence,

$$|S_2 - S_1| \le B_1 A_2(\bar{V} - \underline{V})^2\|w_2 - w_1\|. \tag{C.22}$$

Assembling (C.16), (C.21), and (C.22) into (C.14), I obtain (C.11), where A is given as

$$A = A_1 + A_2(\bar{V} - \underline{V})\left\{(r + \bar{p}) + \left[B_1\bar{J} + \frac{B_1 + B_2\bar{J}}{u'(\bar{w})}(\bar{V} - \underline{V})\right]\right\}.$$

Clearly, A is finite. Moreover, A is independent of the particular functions w_1 and w_2 with which the functions $(J_i, q_i, p_i, F_i, S_i)$ are constructed. *Q.E.D.*

PROOF OF THEOREM 6.1: First, I derive (6.2). Set $V = \bar{V}$ in (6.1). Because $\dot{V} = 0$ at $V = \bar{V}$, the left-hand side of (6.1) is equal to 0 at $V = \bar{V}$. Moreover, the integral in (6.1) is equal to zero, because $F^{-1}(\bar{V}) = \bar{V}$. Thus, at $V = \bar{V}$, (6.1) yields (6.2).

Second, I show that G is continuous; that is, G does not have any mass point. Suppose, to the contrary, that G has a mass $a > 0$ at some value $V \in [v_1, \bar{V}]$. Then $G(V) - G(V - \dot{V}\,dt) \geq a$ for all $dt > 0$ and so the left-hand side of (6.1) is equal to ∞. This is a contradiction, because the right-hand side of (6.1) is bounded.

Third, to establish (6.3) and continuity of g, denote the left-hand side derivative of G as $g(V_-)$. The left-hand side of (6.1) is equal to $g(V_-)\dot{V}$. Because G, F, F^{-1}, and $p(\cdot)$ are continuous, the right-hand side of (6.1) is continuous in V. Thus, $g(V_-)\dot{V}$ must be continuous. Because $\dot{V}$ is continuous, g must be continuous. Then I can express the left-hand side of (6.1) as $g(V)\dot{V}$. After substituting $p(v_1)$ from (6.2), (6.1) becomes (6.3).

Fourth, g is continuously differentiable for all $V \neq v_2$. To see this, note that F, F^{-1}, and $p(\cdot)$ are continuously differentiable. Since g is continuous, G is continuously differentiable and so the right-hand side of (6.3) is continuously differentiable for all $V \neq v_2$. Thus, the left-hand side of the equation, $g(V)\dot{V}$, must be continuously differentiable for all $V \neq v_2$. Because $\dot{V}$ is continuously differentiable, $g(V)$ is continuously differentiable for all $V \neq v_2$.

Fifth, I derive (6.5). For $V \in (v_1, v_2)$, $F^{-1}(V) < v_1$ and so (6.3) becomes

$$g_1(V)\dot{V} = \delta[1 - G_1(V)] - \int_{v_1}^{V} p(F(z))g_1(z)\,dz. \tag{C.23}$$

Note that $T'(V) = 1/\dot{V}$ from (4.1). Differentiating the function Γ in (6.4) yields

$$d\Gamma(V, v_1)/dV = -\big[\delta + p(F(V))\big]\Gamma(V, v_1)/\dot{V}. \tag{C.24}$$

With (C.24) and (C.23), it is straightforward to verify

$$\frac{d}{dV}\left[\frac{\dot{V}g_1(V)}{\Gamma(V, v_1)}\right] = 0. \tag{C.25}$$

Recall that $G_1(v_1) = 0$, because $G(V)$ is continuous for all V. Taking the limit $V \downarrow v_1$ in (C.23) leads to $g_1(v_1)\dot{v}_1 = \delta$. With this initial condition, integrating (C.25) from v_1 to V yields (6.5). Since g is continuous, taking the limit $V \uparrow v_2$ in (6.5) gives $g(v_2)$.

Finally, I derive (6.6) by examining the case $V \in [v_j, v_{j+1})$, where $j \geq 2$. In this case, $F^{-1}(V) \geq v_1$ and so (6.3) becomes

$$g_j(V)\dot{V} = \delta[1 - G(V)] - \int_{F^{-1}(V)}^{v_j} p(F(z))g_{j-1}(z)\,dz - \int_{v_j}^{V} p(F(z))g_j(z)\,dz. \tag{C.26}$$

I have separated the two groups of applicants who obtain jobs with values above V: one group comes from $(F^{-1}(V), v_j]$ and the other from $[v_j, V]$. With (C.26) and (C.24), I can derive

$$\frac{d}{dV}\left[\frac{\dot{V} g_j(V)}{\Gamma(V, v_1)}\right] = \frac{p(V)}{\Gamma(V, v_1)} g_{j-1}(F^{-1}(V)) \frac{dF^{-1}(V)}{dV}. \tag{C.27}$$

Integrating this equation from v_j to V yields (6.6). Because g is continuous, then $g_j(v_j) = \lim_{V \uparrow v_j} g_{j-1}(V)$, all j. This completes the proof of Theorem 6.1. *Q.E.D.*

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Manuscript received April, 2008; final revision received August, 2008.