SUPPLEMENT TO "INCENTIVE PROBLEMS WITH UNIDIMENSIONAL HIDDEN CHARACTERISTICS: A UNIFIED APPROACH"

(*Econometrica*, Vol. 78, No. 4, July 2010, 1201–1237)

BY MARTIN F. HELLWIG

This Appendix provides formal proofs of Lemmas 2.7–2.9 in the text of the paper. The lemmas are repeated here for convenience.

S1. PROOF OF LEMMA 2.7

LEMMA 2.7: *A contract menu* $(w(\cdot), y(\cdot))$ *that is defined on* $T \subset [t_0, t_1]$ *is incentive-compatible and individually rational on* T *if and only if there exists an extension of* $(w(\cdot), y(\cdot))$ *to the interval* $[t_0, t_1]$ *that is incentive-compatible and individually rational on* $[t_0, t_1]$.

Before proving this lemma, I note some implications of the weak single-crossing condition, that is, the requirement that

$$\frac{\partial}{\partial t}\frac{|u_y(w, y, t)|}{u_w(w, y, t)} \leq 0 \tag{1}$$

for all w, y, and t.

LEMMA 1: *Let* $t, \bar{t}$, *and* $(w, y), (\bar{w}, \bar{y})$ *be such that* $\bar{t} > t$, $\bar{y} < y$, *and*

$$u(w, y, t) > u(\bar{w}, \bar{y}, t). \tag{2}$$

Then

$$u(w, y, \bar{t}) > u(\bar{w}, \bar{y}, \bar{t}). \tag{3}$$

The argument is practically identical to the argument for the analogous implication of strict single crossing in Milgrom and Shannon (1994). It is therefore left to the reader.

Given this result, the following lemma shows that under the weak single-crossing condition (1), incentive-compatible contract menus are monotone *unless* all the relevant types are indifferent between the contracts in question.

LEMMA 2: *Let* $t, \bar{t}$, *and* $(w, y), (\bar{w}, \bar{y})$ *be such that* $\bar{t} > t$, $\bar{y} < y$,

$$u(w, y, t) \geq u(\bar{w}, \bar{y}, t) \tag{4}$$

and

$$u(\bar{w}, \bar{y}, \bar{t}) \geq u(w, y, \bar{t}). \tag{5}$$

 DOI: 10.3982/ECTA7726

Then

$$(6)\qquad u(\bar{w}, \bar{y}, t') = u(w, y, t')$$

for all $t' \in [t, \bar{t}]$, *that is, all types between* t *and* $\bar{t}$ *are indifferent between the contracts* (w, y) *and* $(\bar{w}, \bar{y})$.

PROOF: By Lemma 1, $y > \bar{y}$ and (4) imply $u(w + \varepsilon, y, t') > u(\bar{w}, \bar{y}, t')$ for all $t' \in [t, \bar{t}]$ and all $\varepsilon > 0$, hence,

$$(7)\qquad u(w, y, t') \geq u(\bar{w}, \bar{y}, t')$$

for all $t' \in [t, \bar{t}]$. Similarly, $y > \bar{y}$ and (5) imply

$$(8)\qquad u(\bar{w}, \bar{y}, t') \geq u(w, y, t')$$

for all $t' \in [t, \bar{t}]$. (6) follows immediately. *Q.E.D.*

LEMMA 3: *A contract menu* $(w(\cdot), y(\cdot))$ *that is defined on a set* $X \subset [t_0, t_1]$ *is incentive-compatible on* X *if and only if, for any* $\underline{t} \in X$ *and* $\bar{t} \in X$ *such that* $(\underline{t}, \bar{t}) \cap X = \emptyset$, *there exists an extension of* $(w(\cdot), y(\cdot))$ *to* $X \cup (\underline{t}, \bar{t})$ *that is incentive-compatible on* $X \cup (\underline{t}, \bar{t})$.

PROOF: The if part of the lemma is trivial. To prove the only if part, let $X \subset [t_0, t_1]$ and suppose that $(w(\cdot), y(\cdot))$ is incentive-compatible on X. Let $\underline{t} \in X$ and $\bar{t} \in X$ be such that $(\underline{t}, \bar{t}) \cap X = \emptyset$. Incentive compatibility of $(w(\cdot), y(\cdot))$ on X implies that

$$(9)\qquad u(w(\underline{t}), y(\underline{t}), \underline{t}) \geq u(w(\bar{t}), y(\bar{t}), \underline{t})$$

and

$$(10)\qquad u(w(\bar{t}), y(\bar{t}), \bar{t}) \geq u(w(\underline{t}), y(\underline{t}), \bar{t}).$$

Because u is continuous, there exists $\hat{t} \in [\underline{t}, \bar{t}]$ such that

$$(11)\qquad u(w(\underline{t}), y(\underline{t}), \hat{t}) = u(w(\bar{t}), y(\bar{t}), \hat{t}).$$

Extend $w(\cdot)$ and $y(\cdot)$ to the interval $(\underline{t}, \bar{t})$ by setting

$$(12)\qquad (w(t), y(t)) = (w(\underline{t}), y(\underline{t})) \quad \text{for} \quad t \in (\underline{t}, \hat{t}),$$

$$(13)\qquad (w(t), y(t)) = (w(\bar{t}), y(\bar{t})) \quad \text{for} \quad t \in (\hat{t}, \bar{t}),$$

and

$$(14)\qquad (w(\hat{t}), y(\hat{t})) = (w(\bar{t}), y(\bar{t})) \quad \text{if} \quad \hat{t} > \underline{t}.$$

To prove that the extended contract menu is incentive-compatible, I first note that if $y(\bar{t}) \geq y(\underline{t})$, then Lemma 1 and (11) imply

$$u(w(\underline{t}), y(\underline{t}), t) \geq u(w(\bar{t}), y(\bar{t}), t) \tag{15}$$

for all $t \in (\underline{t}, \hat{t}]$ and

$$u(w(\bar{t}), y(\bar{t}), t) \geq u(w(\underline{t}), y(\underline{t}), t) \tag{16}$$

for all $t \in [\hat{t}, \bar{t}]$. If $y(\bar{t}) < y(\underline{t})$, these same inequalities follow from Lemma 2; indeed, in this case, (15) and (16) must hold as equations for all $t \in [\underline{t}, \bar{t}]$.

Consider the incentive compatibility condition

$$u(w(t), y(t), t) \geq u(w(t'), y(t'), t) \tag{17}$$

for $t \in [\underline{t}, \bar{t}]$ and $t' \in X$ satisfying $y(t') \leq y(\underline{t})$. Incentive compatibility of $(w(\cdot), y(\cdot))$ on X implies that

$$u(w(\underline{t}), y(\underline{t}), \underline{t}) \geq u(w(t'), y(t'), \underline{t}) \tag{18}$$

for all $t' \in X \cap [t_0, \underline{t})$. By Lemma 1, it follows that

$$u(w(\underline{t}), y(\underline{t}), t) \geq u(w(t'), y(t'), t) \tag{19}$$

for all $t \in [\underline{t}, \bar{t}]$. By (12)–(14) and (15), it follows that

$$u(w(t), y(t), t) \geq u(w(t'), y(t'), t). \tag{20}$$

A precisely symmetric argument shows that (20) must also hold for $t \in [\underline{t}, \bar{t}]$ and $t' \in X$ satisfying $y(t') \leq y(\bar{t})$.

For $t' \in X$ satisfying $y(\underline{t}) < y(t') < y(\bar{t})$, Lemma 2 implies that

$$u(w(\underline{t}), y(\underline{t}), \underline{t}) = u(w(t'), y(t'), \underline{t}) \tag{21}$$

if $t' < \underline{t}$ and

$$u(w(\bar{t}), y(\bar{t}), \bar{t}) = u(w(\underline{t}), y(\underline{t}), \bar{t}) \tag{22}$$

if $t' > \bar{t}$. In either case, one again obtains (20) for all $t \in [\underline{t}, \bar{t}]$.

For $t \in X$ and $t' \in X \cup (\underline{t}, \bar{t})$, the validity of (17) follows trivially from the incentive compatibility of $(w(\cdot), y(\cdot))$ on X and the observation that the extension of the domain of the contract menu to $X \cup (\underline{t}, \bar{t})$ has not changed its range. *Q.E.D.*

PROOF OF LEMMA 2.7: The if part of the lemma is trivial. To prove the only if part, observe that the set $[t_0, t_1] \setminus T$ can be represented as a countable union of open intervals $I_1, I_2, \ldots$. If one applies Lemma 3 successively, with $X_1 = T$,

$(\underline{t}_1, \bar{t}_1) = I_1$, $X_2 = T \cup I_1$, $(\underline{t}_2, \bar{t}_2) = I_2$, and so forth, then, in the limit, one obtains an extension of $(w(\cdot), y(\cdot))$ that is incentive-compatible on $[t_0, t_1]$.

To prove that this extended contract menu is also individually rational, one notes that, because $t_0 \in T$, the contract $(w(t_0), y(t_0))$ that is assigned to the lowest type has not been changed. By the individual rationality of the original contract menu, $u(w(t_0), y(t_0), t_0) \geq 0 = u(0, 0, t_0)$. By Lemma 1, it follows that $u(w(t_0), y(t_0), t) \geq u(0, 0, t) = 0$ for any $t > t_0$. By incentive compatibility, one also has $u(w(t), y(t), t) \geq u(w(t_0), y(t_0), t)$, hence, $u(w(t), y(t), t) \geq 0$. *Q.E.D.*

S2. PROOF OF LEMMA 2.8

LEMMA 2.8: *A nondecreasing contract menu* $(w(\cdot), y(\cdot))$ *is incentive-compatible and individually rational on* $[t_0, t_1]$ *if and only if the induced indirect utility function* $v(\cdot)$ *satisfies the integral equation*

$$v(t) = v(t_0) + \int_{t_0}^{t} u_t(w(\tau), y(\tau), \tau)\, d\tau \tag{23}$$

for $t \in [t_0, t_1]$ *and, moreover,*

$$v(t_0) \geq 0. \tag{24}$$

As mentioned in the text, Lemma 2.8 is little more than a slight generalization of the characterization result in Mirrlees (1976).

LEMMA 4: *If a contract menu* $(w(\cdot), y(\cdot))$ *is nondecreasing and incentive-compatible on* $[t_0, t_1]$, *then the induced indirect utility function* $v(\cdot)$ *satisfies*

$$v(t) = v(t_0) + \int_{t_0}^{t} u_t(w(\tau), y(\tau), \tau)\, d\tau \tag{25}$$

for all $t \in [t_0, t_1]$.

PROOF: The argument follows Baron and Myerson (1982). For any t and $\bar{t}$, incentive compatibility implies

$$v(t) = u(w(t), y(t), t) \geq u(w(\bar{t}), y(\bar{t}), t) \tag{26}$$

and

$$v(\bar{t}) = u(w(\bar{t}), y(\bar{t}), \bar{t}) \geq u(w(t), y(t), \bar{t}). \tag{27}$$

From (26) and (27), one obtains

$$
\begin{aligned}
(28)\qquad & u(w(t), y(t), t) - u(w(t), y(t), \bar{t}) \\
& \geq v(t) - v(\bar{t}) \\
& \geq u(w(\bar{t}), y(\bar{t}), t) - u(w(\bar{t}), y(\bar{t}), \bar{t}),
\end{aligned}
$$

hence

$$
(29)\qquad \frac{\int_{\bar{t}}^{t} u_t(w(t), y(t), \tau)\, d\tau}{t - \bar{t}} \geq \frac{v(t) - v(\bar{t})}{t - \bar{t}} \geq \frac{\int_{\bar{t}}^{t} u_t(w(\bar{t}), y(\bar{t}), \tau)\, d\tau}{t - \bar{t}}.
$$

Because $u_t(\cdot, \cdot, \cdot)$ is continuous and, by monotonicity, the triples $(w(t), y(t), \tau)$ and $(w(\bar{t}), y(\bar{t}), \tau)$ belong to the compact set $[0, w(t_1)] \times [0, y(t_1)] \times [t_0, t_1]$, the integrands on both sides of (29) are uniformly bounded. The function $v(\cdot)$ is therefore Lipschitz continuous, hence absolutely continuous on $[t_0, t_1]$.

If t is a continuity point of the contract menu $(w(\cdot), y(\cdot))$ and if $\bar{t}$ is close to t, then, by standard arguments, the right-hand side and the left-hand side of (29) are both approximately equal to $u_t(w(t), y(t), t)$. In this case, the (ordinary) derivative of $v(\cdot)$ at t exists and is equal to $u_t(w(t), y(t), t)$. Because the nondecreasing function $t \to (w(t), y(t))$ has at most countably many points of discontinuity, it follows that the function $t \to u_t(w(t), y(t), t)$ is a Radon–Nikodym derivative for the absolutely continuous function $v(\cdot)$. The validity of (25) follows immediately. *Q.E.D.*

LEMMA 5: *If a contract menu* $(w(\cdot), y(\cdot))$ *on an interval* $[t_0, t_1]$ *is nondecreasing and the induced indirect utility function* $v(\cdot)$ *satisfies the integral equation* (25), *then* $(w(\cdot), y(\cdot))$ *is incentive-compatible on* $[t_0, t_1]$.

PROOF: The argument follows Mirrlees (1976); see also Appendix B in the online version of Hellwig (2007). For any t and $\bar{t}$, (25) implies

$$
(30)\qquad u(w(t), y(t), t) - u(w(\bar{t}), y(\bar{t}), \bar{t}) = \int_{\bar{t}}^{t} u_t(w(\tau), y(\tau), \tau)\, d\tau.
$$

Because the left-hand side of (30) is equal to

$$
\begin{aligned}
& \int_{\bar{t}}^{t} u_w(w(\tau), y(\tau), \tau)\, dw(\tau) + \int_{\bar{t}}^{t} u_y(w(\tau), y(\tau), \tau)\, dy(\tau) \\
& \quad + \int_{\bar{t}}^{t} u_t(w(\tau), y(\tau), \tau)\, d\tau,
\end{aligned}
$$

it follows that

$$
(31)\qquad \int_{\bar{t}}^{t} u_w(w(\tau), y(\tau), \tau)\, dw(\tau) + \int_{\bar{t}}^{t} u_y(w(\tau), y(\tau), \tau)\, dy(\tau) = 0
$$

for all t and $\bar{t}$. By standard arguments, it follows that, for every measurable function f, one has

$$\text{(32)} \qquad \int_{t_0}^{t_1} f(\tau)u_w(w(\tau), y(\tau), \tau)\, dw(\tau) + \int_{t_0}^{t_1} f(\tau)u_y(w(\tau), y(\tau), \tau)\, dy(\tau) = 0.$$

In particular, therefore,

$$\text{(33)} \qquad \int_{\bar{t}}^{t} \chi(\tau, t)u_w(w(\tau), y(\tau), \tau)\, dw(\tau) + \int_{\bar{t}}^{t} \chi(\tau, t)u_y(w(\tau), y(\tau), \tau)\, dy(\tau) = 0$$

for every $\bar{t}$, t, and every measurable function χ. If one sets

$$\chi(\tau, t) = \frac{u_w(w(\tau), y(\tau), t)}{u_w(w(\tau), y(\tau), \tau)},$$

one infers that

$$\text{(34)} \qquad \int_{\bar{t}}^{t} u_w(w(\tau), y(\tau), t)\, dw(\tau) + \int_{\bar{t}}^{t} \frac{u_w(w(\tau), y(\tau), t)}{u_w(w(\tau), y(\tau), \tau)} u_y(w(\tau), y(\tau), \tau)\, dy(\tau) = 0$$

for all $\bar{t}$, t, and τ between $\bar{t}$ and t.

The single-crossing condition implies that

$$\frac{u_w(w(\tau), y(\tau), \tau)}{|u_y(w(\tau), y(\tau), \tau)|} \lesseqgtr \frac{u_w(w(\tau), y(\tau), t)}{|u_y(w(\tau), y(\tau), t)|} \quad \text{as} \quad \tau \lesseqgtr t,$$

hence, since u_y takes negative values,

$$\text{(35)} \qquad \frac{u_y(w(\tau), y(\tau), \tau)}{u_w(w(\tau), y(\tau), \tau)} \lesseqgtr \frac{u_y(w(\tau), y(\tau), t)}{u_w(w(\tau), y(\tau), t)} \quad \text{as} \quad \tau \lesseqgtr t.$$

Because u_w takes positive values and $y(\cdot)$ is nondecreasing, (34) and (35) imply that

$$\text{(36)} \qquad \int_{\bar{t}}^{t} u_w(w(\tau), y(\tau), t)\, dw(\tau) + \int_{\bar{t}}^{t} u_y(w(\tau), y(\tau), t)\, dy(\tau) \geq 0$$

for all t and $\bar{t}$. Therefore,

$$u(w(t), y(t), t) - u(w(\bar{t}), y(\bar{t}), t) \geq 0 \tag{37}$$

for all t and $\bar{t}$. *Q.E.D.*

LEMMA 6: *A contract menu $(w(\cdot), y(\cdot))$ that is incentive-compatible on an interval $[t_0, t_1]$ has induced utility satisfying $v(t) \geq 0$ if and only if $v(t_0) \geq 0$.*

PROOF: It suffices to observe that, by (25), $v(\cdot)$ is a nondecreasing function. *Q.E.D.*

Lemma 2.8 follows from Lemmas 4–6.

S3. PROOF OF LEMMA 2.9

LEMMA 2.9: *For any incentive-compatible contract menu $(w(\cdot), y(\cdot))$, there exists a nondecreasing incentive-compatible contract menu $(\bar{w}(\cdot), \bar{y}(\cdot))$ that provides the agent with the same payoff $v(t) = u(w(t), y(t), t)$ for all t and that satisfies*

$$\int [\bar{y}(t) - \bar{w}(t)]\, dF(t) \geq \int [y(t) - w(t)]\, dF(t); \tag{38}$$

moreover, the inequality in (38) is strict unless the contract menus $(w(\cdot), y(\cdot))$ and $(\bar{w}(\cdot), \bar{y}(\cdot))$ are equivalent.

To prove this lemma, I need a stronger version of Lemma 2. The following result implies that if an incentive-compatible contract menu violates monotonicity, then, in the relevant part of their domains, the indifference curves of the relevant types must coincide.

LEMMA 7: *If any two types t and $\bar{t}$ are both indifferent between two contracts (w, y) and $(\bar{w}, \bar{y}) \ll (w, y)$, then, between these two contracts, their indifference curves coincide, that is, for any (w', y') with $(\bar{w}, \bar{y}) \leq (w', y') \leq (w, y)$,*

$$u(w', y', t) = u(w, y, t) \quad \textit{if and only if} \quad u(w', y', \bar{t}) = u(\bar{w}, \bar{y}, \bar{t}). \tag{39}$$

PROOF: Without loss of generality, suppose that $t \leq \bar{t}$. If $(\bar{w}, \bar{y}) \leq (w', y') \leq (w, y)$, then, by Lemma 1,

$$u(w', y', t) = u(w, y, t) \tag{40}$$

implies $u(w', y' + \varepsilon, \bar{t}) < u(w, y, \bar{t})$ for all $\varepsilon > 0$, hence,

$$u(w', y', \bar{t}) \leq u(w, y, \bar{t}). \tag{41}$$

Because type t is indifferent between (w, y) and $(\bar{w}, \bar{y})$, and because $(\bar{w}, \bar{y}) \leq (w', y')$ by Lemma 1, (40) also implies $u(w' + \varepsilon, y', \bar{t}) > u(\bar{w}, \bar{y}, \bar{t})$ for all $\varepsilon > 0$, hence,

$$(42) \qquad u(w', y', \bar{t}) \geq u(\bar{w}, \bar{y}, \bar{t}).$$

Because type $\bar{t}$ is also indifferent between (w, y) and $(\bar{w}, \bar{y})$, (41) and (42) imply

$$(43) \qquad u(w', y', \bar{t}) = u(\bar{w}, \bar{y}, \bar{t}).$$

Thus, (40) implies (43). By a precisely symmetric argument, one also finds that (43) implies (40). *Q.E.D.*

To proceed with the proof of Lemma 2.9 itself, I need some additional notation. Given an incentive-compatible contract menu $(w(\cdot), y(\cdot))$ with associated indirect utility function $v(\cdot)$, for any t, let

$$(44) \qquad I(t) = \{\bar{t} \in T | u(w(\bar{t}), y(\bar{t}), t) = v(t) \text{ and } u(w(t), y(t), \bar{t}) = v(\bar{t})\}$$

be the set of types $\bar{t}$ such that both t and $\bar{t}$ are indifferent between the pairs $(w(t), y(t))$ and $(w(\bar{t}), y(\bar{t}))$. By Lemma 2, we know that, for any t, the set $I(t)$ contains any $\bar{t} > t$ for which $y(\bar{t}) < y(t)$; indeed, if such $\bar{t}$ exists, the set $I(t)$ has the entire interval $[t, \bar{t}]$ as a subset. By Lemmas 2 and 7 jointly, in this case, the set $I(t)$ also contains any $t' < \bar{t}$ for which $y(t') \geq y(\bar{t})$; indeed, it has the entire interval $[t', \bar{t}]$ as a subset.

Given the set $I(t)$, let

$$(45) \qquad \chi(t) := \{(w, y) | u(w, y, t) = v(t) \text{ and } y(t') \leq y \leq y(t'') \text{ for some } t', t'' \in I(t)\}$$

be the segment of type t's indifference curve through $(w(t), y(t))$ that lies between the contracts assigned to types in $I(t)$, and let $\bar{\chi}(t)$ be the closure of $\chi(t)$. Any contract (w, y) in $\bar{\chi}(t)$ provides type t with the same utility $v(t)$ as the contract $(w(t), y(t))$. It is therefore of interest to ask which of these contracts is most profitable for the principal.

LEMMA 8: *If $(w(\cdot), y(\cdot))$ is an incentive-compatible contract menu, then, for any $t \in [t_0, t_1]$, the problem*

$$(46) \qquad \max_{(w,y) \in \bar{\chi}(t)} [y - w]$$

has a unique solution $(\bar{w}(t), \bar{y}(t))$. The contract menu $(\bar{w}(\cdot), \bar{y}(\cdot))$ is nondecreasing and incentive-compatible.

PROOF: Uniqueness of the solution to the problem $\max_{(w,y)\in\bar{\chi}(t)}[y-w]$ follows from the strict quasiconcavity of u in w and y. To prove weak monotonicity, suppose that $t<\bar{t}$ and $\bar{y}(t)>\bar{y}(\bar{t})$. By Lemma 2, one has $\bar{t}\in I(t)$. By the definition of $I(\cdot)$, it follows that $I(t)=I(\bar{t})$. By Lemma 7, one then also has $\chi(t)=\chi(\bar{t})$, hence $\bar{\chi}(t)=\bar{\chi}(\bar{t})$. Because the solution to problem (46) is unique and depends on t only through the constraint set $\bar{\chi}(t)$, it follows that $(\bar{w}(t),\bar{y}(t))=(\bar{w}(\bar{t}),\bar{y}(\bar{t}))$. The assumption that $t<\bar{t}$ and $\bar{y}(t)>\bar{y}(\bar{t})$ has thus led to a contradiction and must be false. Thus $t<\bar{t}$ implies $\bar{y}(t)\geq\bar{y}(\bar{t})$. Weak monotonicity of $\bar{w}(\cdot)$ then follows from incentive compatibility.

By construction,

$$u(\bar{w}(t),\bar{y}(t),t)=v(t) \tag{47}$$

for all t. To prove incentive compatibility, it therefore suffices to show that

$$v(t)\geq u(\bar{w}(\bar{t}),\bar{y}(\bar{t}),t) \tag{48}$$

for all t and all $\bar{t}$. Since $(\bar{w}(\bar{t}),\bar{y}(\bar{t}))\in\bar{\chi}(\bar{t})$, there exists a sequence $\{(w^k(\bar{t}),y^k(\bar{t}))\}$ of elements of $\chi(\bar{t})$ that converges to $(\bar{w}(t),\bar{y}(t))$. To prove (48), it therefore suffices to show that

$$v(t)\geq u(w^k(\bar{t}),y^k(\bar{t}),t) \tag{49}$$

for all k.

By the definition of $\chi(\bar{t})$, there exist sequences $\{\bar{t}'_k\}$, $\{\bar{t}''_k\}$ of elements of $I(\bar{t})$ such that, for any k, one has

$$y(\bar{t}'_k)\leq y^k(\bar{t})\leq y(\bar{t}''_k); \tag{50}$$

moreover,

$$u(w^k(\bar{t}),y^k(\bar{t}),\bar{t})=u(w(\bar{t}'_k),y(\bar{t}'_k),\bar{t})=u(w(\bar{t}''_k),y(\bar{t}''_k),\bar{t}). \tag{51}$$

If $t<\bar{t}$, (50) and (51) in combination with Lemma 1 imply

$$u(w^k(\bar{t}),y^k(\bar{t}),t)\leq u(w(\bar{t}'_k),y(\bar{t}'_k),t). \tag{52}$$

Because incentive compatibility requires

$$v(t)\geq u(w(\bar{t}'_k),y(\bar{t}'_k),t), \tag{53}$$

(49) follows immediately. If $t>\bar{t}$, one similarly obtains

$$u(w^k(\bar{t}),y^k(\bar{t}),t)\leq u(w(\bar{t}''_k),y(\bar{t}''_k),t)\leq v(t), \tag{54}$$

which also yields (49). *Q.E.D.*

To establish Lemma 2.9, it now suffices to observe that, by construction, one has

$$\bar{y}(t) - \bar{w}(t) \geq y(t) - w(t) \tag{55}$$

for all t and that the inequality in (55) is strict unless $(w(t), y(t)) = (\bar{w}(t), \bar{y}(t))$.

REFERENCES

BARON, D. P., AND R. B. MYERSON (1982): "Regulating a Monopolist With Unknown Costs," *Econometrica*, 50, 911–930. [4]

HELLWIG, M. F. (2007): "A Contribution to the Theory of Optimal Utilitarian Income Taxation," *Journal of Public Economics*, 91, 1449–1477. Available at http://www.coll.mpg.de/pdf_dat/2007_02online.pdf. [5]

MILGROM, P., AND C. SHANNON (1994): "Monotone Comparative Statics," *Econometrica*, 62, 157–180. [1]

MIRRLEES, J. M. (1976): "Optimal Tax Theory: A Synthesis," *Journal of Public Economics*, 6, 327–358. [4,5]

Max Planck Institute for Research on Collective Goods, Kurt-Schumacher-Strasse 10, D-53113 Bonn, Germany; hellwig@coll.mpg.de.

Manuscript received February, 2008; final revision received April, 2010.