Econometrica Supplementary Material

SUPPLEMENT TO "TESTING FOR STOCHASTIC MONOTONICITY"

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MATHEMATICAL APPENDIX: PROOFS

A.1. *Informal Discussion of the Proof Technique*

ALTHOUGH THE TEST IS EASY to implement, proving Theorem 3.1 involves several lengthy steps. Since establishing these steps requires techniques that are not commonly used in econometrics, we now give an informal description of our proof techniques and provide some discussions behind them. Specifically, our proof of Theorem 3.1 consists of the following three steps:

Step 1. The asymptotic approximation of $\widehat{U}_n(y,x)/c_n(x)$ by a Gaussian process (Appendix A.2).

Step 2. The asymptotic approximation of the excursion probability of the maximum of the Gaussian process on a fixed set (Appendix A.3).

Step 3. The asymptotic approximation of the excursion probability of the maximum of the Gaussian process on an increasing set (Appendix A.4).

In particular, in Step 1, we show that $\widehat{U}_n(y,x)/c_n(x)$ can be approximated uniformly over (y,x) by $\xi_n[F_Y(y), h_n^{-1}x]$, where $F_Y(\cdot)$ is the cumulative distribution function (c.d.f.) of Y and ξ_n is a sequence of Gaussian processes $\{\xi_n(u,s):(u,s)\in[0,1]\times[0,h_n^{-1}]\}$ with continuous sample paths such that

$$\text{(A1)}\qquad \begin{aligned} &E[\xi_n(u,s)]=0,\\ &E[\xi_n(u_1,s_1)\xi_n(u_2,s_2)]=[\min(u_1,u_2)-u_1u_2]\rho(s_1-s_2)\end{aligned}$$

for $u,u_1,u_2\in[0,1]$ and $s,s_1,s_2\in[0,h_n^{-1}]$, where $\rho(\cdot)$ is some known smooth function. See Appendix A.2 for the exact form of $\rho(\cdot)$.

First of all, note that by Step 1, taking the supremum of $\widehat{U}_n(y,x)/c_n(x)$ over (y,x) corresponds to taking the supremum of $\xi_n[F_Y(y), h_n^{-1}x]$ over (y,x) asymptotically. Since F_Y is the c.d.f. and $h_n\to 0$, this means that we need to take the supremum of the Gaussian process ξ_n over the product space of a fixed set (in the direction of y) and an increasing set (in the direction of x).

In general, it is expected that the asymptotic distribution of a suitably normalized version of the supremum of a Gaussian process over an increasing set converges to one of extreme value distributions. If the supremum is taken over Gaussian processes with a one-dimensional parameter, then the corresponding probability theory and applications on statistical problems are well understood. See, for example, Leadbetter, Lindgren, and Rootzén (1983). However, for Gaussian processes with multidimensional parameters (often called Gaussian fields), the probability theory is less developed and applications on statistical problems are rare. Unfortunately, we need to deal with $\xi_n(u,s)$ that has two

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parameters and approximate the distribution of its supremum over an increasing set. These tasks are Steps 2 and 3. The important reference we have used to carry out Steps 2 and 3 is Piterbarg (1996), who developed a general theory for approximations of the suprema of Gaussian fields.

Once Step 2 is established, then there is a general approximation method to achieve Step 3. Thus, Step 2 is the critical step in proving Theorem 3.1. Note that the covariance function of ξ_n in (A1) is the product of a Brownian bridge covariance function and a stationary covariance function. In this paper, we develop a new result for the excursion probability of the maximum of the Gaussian process ξ_n (Theorem A.2). To be specific, the approximating Gaussian process contains both a stationary and a nonstationary part, and therefore we need to extend existing results that only apply to either one or the other case. For example, see Section 7 of Piterbarg (1996) for the stationary case and Sections 8 and 9 of Piterbarg (1996) for the nonstationary case, but to our best knowledge, there is no known result regarding our case in the literature.

A.2. *Gaussian Process Approximation*

Let $f_X(\cdot)$, $F_X(\cdot)$, and $F_Y(\cdot)$, respectively, denote the probability density function (p.d.f.) the c.d.f. of X, and the c.d.f. of Y. Define

$$\rho(s) = \frac{\int q(z)q(z-s)K(z)K(z-s)\,dz}{\int q^2(z)K^2(z)\,dz},$$

where $q(u) = \int \operatorname{sgn}(u-w)K(w)\,dw$ was defined in the main text. Let $\xi(u,s)$ denote a Gaussian process $\{\xi(u,s):(u,s)\in[0,1]\times\mathbb{R}\}$ with continuous sample paths such that

$$E[\xi(u,s)] = 0,$$
$$E[\xi(u_1,s_1)\xi(u_2,s_2)] = [\min(u_1,u_2) - u_1u_2]\rho(s_1-s_2)$$

for $u, u_1, u_2 \in [0,1]$ and $s, s_1, s_2 \in \mathbb{R}$. Define $\mathcal{X}_n = [0, 1/h_n]$ and let ξ_n be the restriction of ξ to $[0,1]\times\mathcal{X}_n$.

THEOREM A.1: *Let Assumption* 3.1 *hold*. *Let* h_n *satisfy*

$$h_n(\log n)^{1/2} \to 0, \quad nh_n^3 \to \infty, \quad \text{and} \quad nh_n^2/(\log n)^2 \to \infty.$$

Then there exists a sequence of Gaussian processes $\{\xi_n(u,s):(u,s)\in[0,1]\times\mathcal{X}_n\}$ *with continuous sample paths such that*

$$E[\xi_n(u,s)] = 0,$$
$$E[\xi_n(u_1,s_1)\xi_n(u_2,s_2)] = [\min(u_1,u_2) - u_1u_2]\rho(s_1-s_2)$$

for $u, u_1, u_2 \in [0,1]$ *and* $s, s_1, s_2 \in \mathcal{X}_n$, *and that*

$$\sup_{(y,x)\in\mathcal{Y}\times\mathcal{X}} \left| n^{1/2}\frac{\widehat{U}_n(y,x)}{\widehat{\sigma}_n(x)} - \xi_n[F_Y(y), h_n^{-1}x] \right|$$
$$= O_p\bigl(n^{-1/2}h_n^{-3/2} + n^{-1/4}h_n^{-1/2}(\log n)^{1/2} + h_n(\log h_n^{-1})^{1/2}\bigr).$$

PROOF: The proof of the theorem follows closely Theorem 3.1 of Ghosal, Sen, and van der Vaart (2000). In particular, the theorem can be proved by combining arguments almost identical to those used in the proof of Theorem 3.1 of Ghosal, Sen, and van der Vaart (2000) with the lemmas proved in Section A.6. The only difference here is that because of the estimated X_i's, an additional term of order $O_p(n^{-1/2}h_n^{-3/2})$ appears. *Q.E.D.*

A.3. *Asymptotic Behavior of the Excursion Probability on the Fixed Set*

Since the distribution of $\xi_n(u,s)$ does not depend on n, for the purpose of deriving the distribution of the supremum statistic S_n, it suffices to consider the asymptotic behavior of the excursion probability of the maximum of the Gaussian process $\xi(u,s)$ that has the same covariance function as $\xi_n(u,s)$.

We first consider the asymptotic behavior of the tail probability of the maximum of $\xi(u,s)$ on a fixed set $[0,1]\times I$, where $I \equiv [0,L]$ is an interval with a fixed length L. Define

$$\Psi(a) = \frac{1}{\sqrt{2\pi}}\int_a^\infty \exp\left(-\frac{1}{2}x^2\right)dx.$$

THEOREM A.2: *Let* λ *denote the quantity defined in Theorem* 3.1. *In addition, let* $I = [0,L]$. *Then*

$$\Pr\Bigl(\max_{(u,s)\in[0,1]\times I}\xi(u,s) > a\Bigr) = L\left(\frac{8\lambda}{\pi}\right)^{1/2} a\exp(-2a^2)[1+o(1)]$$

as $a \to \infty$.

The following lemmas are useful to prove Theorem A.2.

LEMMA A.1: *Let* $\Pi_\delta = [1/2-\delta(a), 1/2+\delta(a)]$, *where* $\delta(a) = a^{-1}\log a$. *Then*

$$\Pr\Bigl(\max_{(u,s)\in[0,1]\times I}\xi(u,s) > a\Bigr) = \Pr\Bigl(\max_{(u,s)\in\Pi_\delta\times I}\xi(u,s) > a\Bigr)[1+o(1)]$$

as $a \to \infty$.

PROOF: For all sufficiently large a,

$$\begin{aligned}\text{(A2)}\qquad &\Pr\Big(\max_{(u,s)\in\Pi_\delta\times I}\xi(u,s)>a\Big)\\ &\le \Pr\Big(\max_{(u,s)\in[0,1]\times I}\xi(u,s)>a\Big)\\ &\le \Pr\Big(\max_{(u,s)\in\Pi_\delta\times I}\xi(u,s)>a\Big)+\Pr\Big(\max_{(u,s)\in\{[0,1]\setminus\Pi_\delta\}\times I}\xi(u,s)>a\Big).\end{aligned}$$

Note that

$$\begin{aligned}&E[\xi(u_1,s_1)-\xi(u_2,s_2)]^2\\ &\quad = u_1(1-u_1)+u_2(1-u_2)-2[\min(u_1,u_2)-u_1u_2]\rho(s_1-s_2).\end{aligned}$$

Furthermore, by some straightforward manipulation,

$$E[\xi(u_1,s_1)-\xi(u_2,s_2)]^2 \le C|u_1-u_2|+|s_1-s_2|$$

for some constant C. Thus, Assumption of Piterbarg (1996, p. 118) is satisfied. Then since

$$\max_{(u,s)\in\{[0,1]\setminus\Pi_\delta\}\times I}\sigma^2(u,s)\le 1/4-\delta(a)^2,$$

by Theorem 8.1 of Piterbarg (1996, p. 119), there exists a constant C such that

$$\begin{aligned}\text{(A3)}\qquad &\Pr\Big(\max_{(u,s)\in\{[0,1]\setminus\Pi_\delta\}\times I}\xi(u,s)>a\Big)\\ &\le C\,\mathrm{mes}\big(\{[0,1]\setminus\Pi_\delta\}\times I\big)a^4\Psi\bigg(\frac{a}{[1/4-\delta(a)^2]^{1/2}}\bigg),\end{aligned}$$

$\mathrm{mes}(A)$ denotes the Lebesgue measure of a set A. Note that by (D.8) of Piterbarg (1996, p. 15), as $a\to\infty$,

$$a^4\Psi\bigg(\frac{a}{[1/4-\delta(a)^2]^{1/2}}\bigg)\sim\frac{1}{\sqrt{2\pi}}a^3\exp\bigg(\frac{-a^2/2}{1/4-\delta(a)^2}\bigg),$$

where $A\sim B$ stands for $A/B\to 1$. Also, for some fixed interior point $\bar{s}\in I$, we have

$$\Pr\bigg(\xi\bigg(\frac{1}{2},\bar{s}\bigg)>a\bigg)=\Psi\bigg(\frac{a}{2}\bigg)\sim\frac{2}{\sqrt{2\pi}}a^{-1}\exp\bigg(\frac{-a^2/2}{4}\bigg).$$

Then it is easy to show that as $a\to\infty$, the probability on the left-hand side of (A3) converges to zero at a rate of $\exp[-2a^2+O(\log a)]$ and $\Pr(\xi(1/2,\bar{s})>a)$

converges to zero at a rate of $\exp[-a^2/8 - O(\log a)]$. Thus, the probability on the left-hand side of (A3) converges to zero faster than $\Pr(\xi(1/2, \bar{s}) > a)$. Since $\Pr(\xi(1/2, \bar{s}) > a) \leq \Pr(\max_{(u,s)\in\Pi_\delta\times I} \xi(u, s) > a)$,

$$\Pr\Big(\max_{(u,s)\in\{[0,1]\setminus\Pi_\delta\}\times I} \xi(u, s) > a\Big) = o\Big[\Pr\Big(\max_{(u,s)\in\Pi_\delta\times I} \xi(u, s) > a\Big)\Big].$$

Then the lemma follows immediately from (A2). *Q.E.D.*

Let $\sigma^2(u, s) = u(1-u)$ and $r[(u_1, s_1), r(u_2, s_2)] = [\min(u_1, u_2) - u_1u_2]\rho(s_1 - s_2)$, respectively, denote the variance and covariance functions of $\xi(u, s)$.

LEMMA A.2: *As* $u \to 1/2$,

$$\sigma^2(u, s) = \frac{1}{4} - \left(u - \frac{1}{2}\right)^2 [1 + o(1)]. \tag{A4}$$

Furthermore, *as* $(u_1, u_2) \to (1/2, 1/2)$ *and* $|s_1 - s_2| \to 0$,

$$\begin{aligned} &r[(u_1, s_1), r(u_2, s_2)] \\ &\quad = \frac{1}{4} - \frac{1}{2}|u_1 - u_2|[1 + o(1)] - \frac{\lambda}{8}(s_1 - s_2)^2[1 + o(1)] \\ &\qquad - \frac{1}{2}\left(u_1 - \frac{1}{2}\right)^2 [1 + o(1)] - \frac{1}{2}\left(u_2 - \frac{1}{2}\right)^2 [1 + o(1)]. \end{aligned} \tag{A5}$$

PROOF: The first result (A4) follows easily from a second-order Taylor series expansion of the variance of $\xi(u, s)$ with respect to u. We now consider the second result (A5). In view of the proof of Theorem 9.2 of Piterbarg (1996, p. 138), note that as $(u_1, u_2) \to (1/2, 1/2)$,

$$\frac{\min(u_1, u_2) - u_1u_2}{\sqrt{u_1(1 - u_1)u_2(1 - u_2)}} = 1 - \frac{1}{2}\frac{|u_1 - u_2|}{\sqrt{u_1(1 - u_1)u_2(1 - u_2)}} + o(|u_1 - u_2|). \tag{A6}$$

Note that by (4.9) of Ghosal, Sen, and van der Vaart (2000),

$$\rho(s_1 - s_2) = 1 - \frac{\lambda(s_1 - s_2)^2}{2} + o(|s_1 - s_2|^2) \tag{A7}$$

as $|s_1 - s_2| \to 0$. As in (A4), a Taylor series expansion of $\sigma(u, s)$ around $u = 1/2$ gives

$$\sigma(u, s) = \frac{1}{2} - \left(u - \frac{1}{2}\right)^2 [1 + o(1)] \quad \text{as} \quad u \to \frac{1}{2}$$

for any $s \in I$. Thus, we have

$$\text{(A8)} \qquad \sqrt{u_1(1-u_1)u_2(1-u_2)} = \frac{1}{4} - \frac{1}{2}\left(u_1 - \frac{1}{2}\right)^2 [1+o(1)] - \frac{1}{2}\left(u_2 - \frac{1}{2}\right)^2 [1+o(1)]$$

as $(u_1, u_2) \to (1/2, 1/2)$. Then the lemma follows from combining (A6) and (A7) with (A8). *Q.E.D.*

Let $\varepsilon > 0$ be a fixed constant. Define Gaussian processes $\psi_1^-(u)$ and $\psi_1^+(u)$ such that

$$\psi_1^-(u) = \frac{\zeta_1^-(u)}{2^{3/2}[1+4(1+\varepsilon)(u-0.5)^2]} \quad \text{and}$$

$$\psi_1^+(u) = \frac{\zeta_1^+(u)}{2^{3/2}[1+4(1-\varepsilon)(u-0.5)^2]},$$

where $\zeta_1^-(u)$ and $\zeta_1^+(u)$ are Gaussian stationary processes with zero means and the covariance functions $r_1^-(u) = \exp[-4(1-\varepsilon)|u|]$ and $r_1^+(u) = \exp[-4(1+\varepsilon)|u|]$. In addition, define mean-zero stationary Gaussian processes $\psi_2^-(s)$ and $\psi_2^+(s)$ such that they are independent of $\psi_1^-(u)$ and $\psi_1^+(u)$, and have the covariance functions of the form

$$r_2^-(s) = \frac{1}{8}[1 - \lambda(1-\varepsilon)s^2 + o(s^2)],$$

$$r_2^+(s) = \frac{1}{8}[1 - \lambda(1+\varepsilon)s^2 + o(s^2)],$$

respectively. Finally, define

$$\psi^-(u,s) = \psi_1^-(u) + \psi_2^-(s) \quad \text{and} \quad \psi^+(u,s) = \psi_1^+(u) + \psi_2^+(s).$$

LEMMA A.3: *Let $\varepsilon > 0$ be any fixed, arbitrarily small constant. Then for all sufficiently large a,*

$$\Pr\Big(\max_{(u,s)\in\Pi_\delta\times I} \psi^-(u,s) > a\Big) \le \Pr\Big(\max_{(u,s)\in\Pi_\delta\times I} \xi(u,s) > a\Big) \le \Pr\Big(\max_{(u,s)\in\Pi_\delta\times I} \psi^+(u,s) > a\Big).$$

PROOF: As noted in the proofs of Theorems D.4 and 8.2 of Piterbarg (1996, pp. 23 and 133), the lemma follows from Lemma A.2 and the fact that the distribution of the maximum is monotone with respect to the variance and the Slepian inequality (see, for example, Theorem C.1 of Piterbarg (1996, p. 6)). *Q.E.D.*

LEMMA A.4: *Let $\varepsilon > 0$ be any fixed, arbitrarily small constant. As $a \to \infty$,*

$$\Pr\Big(\max_{u\in\Pi_\delta} 2^{3/2}\psi_1^-(u) > a\Big) = 2^{1/2}\frac{(1-\varepsilon)}{(1+\varepsilon)^{1/2}}\exp\left(\frac{-a^2}{2}\right)[1+o(1)], \tag{A9}$$

$$\Pr\Big(\max_{u\in\Pi_\delta} 2^{3/2}\psi_1^+(u) > a\Big) = 2^{1/2}\frac{(1+\varepsilon)}{(1-\varepsilon)^{1/2}}\exp(\frac{-a^2}{2})[1+o(1)]. \tag{A10}$$

PROOF: This lemma can be proved by one of results given in the proof of Theorem D.4 of Piterbarg (1996, p. 21). In particular, using the notation used in the proof of Theorem D.4 of Piterbarg (1996), the excursion probability of $2^{3/2}\psi_1^-(u)$ can be obtained by the result of Case 1 with $\alpha = 1$, $\beta = 2$, $b = 4(1+\varepsilon)$, and $d = 4(1-\varepsilon)$. It follows from the second display on page 22 of Piterbarg (1996) that as $a \to \infty$,

$$\Pr\Big(\max_{u\in\Pi_\delta} 2^{3/2}\psi_1^-(u) > a\Big) = \frac{H_1\Gamma(1/2)[4(1-\varepsilon)]}{[4(1+\varepsilon)]^{1/2}}a\Psi(a)[1+o(1)],$$

where H_1 is the Pickands constant with $\alpha = 1$ (defined on pages 13 and 16 of Piterbarg (1996)) and $\Gamma(\cdot)$ is the Gamma function. Note that $\Gamma(1/2) = \sqrt{\pi}$. Furthermore, by (9.6) of Piterbarg (1996, p. 138), $H_1 = 1$ and by (D.8) of Piterbarg (1996, p. 15),

$$a\Psi(a) \sim (2\pi)^{-1/2}\exp(-a^2/2)$$

as $a \to \infty$. Therefore, (A9) follows immediately. The excursion probability of $2^{3/2}\psi_1^+(u)$ can be obtained analogously. *Q.E.D.*

LEMMA A.5: *Let $\varepsilon > 0$ be any fixed, arbitrarily small constant. As $a \to \infty$,*

$$\Pr\Big(\max_{s\in I} 2^{3/2}\psi_2^-(s) > a\Big) = \frac{[(\lambda/2)(1-\varepsilon)]^{1/2}L}{\pi}\exp\left(\frac{-a^2}{2}\right)[1+o(1)], \tag{A11}$$

$$\Pr\Big(\max_{s\in I} 2^{3/2}\psi_2^+(s) > a\Big) = \frac{[(\lambda/2)(1+\varepsilon)]^{1/2}L}{\pi}\exp\left(\frac{-a^2}{2}\right)[1+o(1)]. \tag{A12}$$

PROOF: Recall that $I = [0, L]$. By Theorem D.2 of Piterbarg (1996, p. 16) and a simple scaling of $\psi_2^-(u)$,

$$\Pr\Big(\max_{s\in[0,L]} 2^{3/2}\psi_2^-(s) > a\Big) = H_2L^*a\Psi(a)[1+o(1)],$$

where H_2 is the Pickands constant with $\alpha = 2$ and $L^* = [\lambda(1-\varepsilon)]^{1/2}L$. By (F.4) of Piterbarg (1996, p. 31), $H_2 = 1/\sqrt{\pi}$. Then (A11) follows immediately. The excursion probability of $2^{3/2}\psi_2^+(u)$ can be obtained similarly. *Q.E.D.*

PROOF OF THEOREM A.2: Let $\varepsilon > 0$ be any fixed, arbitrarily small constant. Note that $\psi^-(u,s)$ and $\psi^+(u,s)$ are convolutions of $\psi_1^-(u)$ and $\psi_2^-(s)$ and of $\psi_1^+(u)$ and $\psi_2^+(s)$, respectively. Then an application of Lemma 8.6 of Piterbarg (1996, p. 128) with Lemmas A.4 and A.5 gives

$$\text{(A13)}\quad \Pr\Big(\max_{(u,s)\in\Pi_\delta\times I} 2^{3/2}\psi^-(u,s) > a\Big) = L\frac{(1-\varepsilon)^{3/2}}{(1+\varepsilon)^{1/2}}\left(\frac{\lambda}{\pi}\right)^{1/2} a\exp\left(\frac{-a^2}{4}\right)[1+o(1)],$$

$$\text{(A14)}\quad \Pr\Big(\max_{(u,s)\in\Pi_\delta\times I} 2^{3/2}\psi^+(u,s) > a\Big) = L\frac{(1+\varepsilon)^{3/2}}{(1-\varepsilon)^{1/2}}\left(\frac{\lambda}{\pi}\right)^{1/2} a\exp\left(\frac{-a^2}{4}\right)[1+o(1)].$$

Then as $a\to\infty$, by Lemma A.1,

$$\Pr\Big(\max_{(u,s)\in[0,1]\times I} 2^{3/2}\xi(u,s) > a\Big) = L\left(\frac{\lambda}{\pi}\right)^{1/2} a\exp\left(\frac{-a^2}{4}\right)[1+o(1)]$$

since the choice of ε can be made arbitrarily small and the constants on the right-hand sides of (A13) and (A14) are continuous at $\varepsilon = 0$. Therefore, the theorem follows immediately. *Q.E.D.*

A.4. *Asymptotic Behavior of the Excursion Probability on the Increasing Set*

THEOREM A.3: *For any x,*

$$\Pr\Big(4\beta_n\Big\{\max_{(u,s)\in[0,1]\times\mathcal{X}_n} \xi(u,s) - \beta_n\Big\} < x\Big) = \exp\left\{-\exp\left(-x-\frac{x^2}{8\beta_n^2}\right)\left[1+\frac{x}{4\beta_n^2}\right]\right\} + o(1),$$

where β_n is defined in equation (7) *in the main text*.

PROOF: This theorem can be proved using arguments similar to those used in the proof of Theorem G.1 of Piterbarg (1996). Note that the covariance function of $\xi(u,s)$, that is, $r[(u_1,s_1),r(u_2,s_2)]$, has compact support and in particular it is zero when $|s_1-s_2|>2$. Define an increasing sequence m_n such that $m_n \to \infty$ but $m_n h_n \to 0$ as $n \to \infty$. That is, m_n converges to infinity slower than h_n^{-1}. Further, define sequences of sets

$$I_k = [k(m_n h_n)^{-1}, (k+1)(m_n h_n)^{-1} - 2],$$
$$J_k = [(k+1)(m_n h_n)^{-1} - 2, (k+1)(m_n h_n)^{-1}]$$

for $k = 0, 1, \dots, m_n - 1$. Then we have

$$\begin{aligned}\text{(A15)}\qquad &\Pr\Bigl(\max_{(u,s)\in[0,1]\times\mathcal{X}_n} \xi(u,s) < a\Bigr)\\ &\quad = \Pr\Bigl(\max_{(u,s)\in[0,1]\times[\bigcup_k I_k]} \xi(u,s) < a\Bigr)\\ &\qquad - \Pr\Bigl(\max_{(u,s)\in[0,1]\times[\bigcup_k I_k]} \xi(u,s) < a, \max_{(u,s)\in[0,1]\times[\bigcup_k J_k]} \xi(u,s) \ge a\Bigr).\end{aligned}$$

We first consider the first probability on the right-hand side of (A15). Let $c^* = (\frac{8\lambda}{\pi})^{1/2}$. For each x, choose $a_n = \beta_n + x/(4\beta_n)$, where β_n is the largest solution to the equation

$$\text{(A16)}\qquad h_n^{-1} c^* \beta_n \exp(-2\beta_n^2) = 1.$$

Since I_k's are separated by the diameter of the support and the distribution of $\xi(u,s)$ is stationary in the direction of s, it follows from Theorem A.2 that

$$\begin{aligned}&\Pr\Bigl(\max_{(u,s)\in[0,1]\times[\bigcup_k I_k]} \xi(u,s) < a_n\Bigr)\\ &\quad = \Bigl[1 - \Pr\Bigl(\max_{(u,s)\in[0,1]\times I_0} \xi(u,s) \ge a_n\Bigr)\Bigr]^{m_n}\\ &\quad = \exp\Bigl(m_n \log\Bigl[1 - \Pr\Bigl(\max_{(u,s)\in[0,1]\times I_0} \xi(u,s) \ge a_n\Bigr)\Bigr]\Bigr)\\ &\quad = \exp\Bigl(-m_n \Pr\Bigl(\max_{(u,s)\in[0,1]\times I_0} \xi(u,s) \ge a_n\Bigr)\Bigr)\\ &\qquad + O\Bigl(m_n\Bigl[\Pr\Bigl(\max_{(u,s)\in[0,1]\times I_0} \xi(u,s) \ge a_n\Bigr)\Bigr]^2\Bigr)\\ &\quad = \exp\bigl\{-m_n[(m_n h_n)^{-1} - 2] c^* a_n \exp(-2a_n^2)[1 + o(1)]\bigr\} + O(h_n),\end{aligned}$$

so that

$$\begin{aligned}\text{(A17)}\qquad &\Pr\Bigl(\max_{(u,s)\in[0,1]\times[\bigcup_k I_k]} \xi(u,s) < a_n\Bigr)\\ &\quad = \exp\Bigl\{-\exp\Bigl(-x - \frac{x^2}{8\beta_n^2}\Bigr)\Bigl[1 + \frac{x}{4\beta_n^2}\Bigr]\Bigr\} + o(1).\end{aligned}$$

Now consider the second probability on the right-hand side of (A15). Note that again using Theorem A.2 and the fact that the distribution of $\xi(u,s)$ is stationary in the direction of s,

$$\begin{aligned}
&\Pr\Big(\max_{(u,s)\in[0,1]\times[\bigcup_k I_k]} \xi(u,s) < a_n, \max_{(u,s)\in[0,1]\times[\bigcup_k J_k]} \xi(u,s) \geq a_n\Big) \\
&\quad \leq \Pr\Big(\max_{(u,s)\in[0,1]\times[\bigcup_k J_k]} \xi(u,s) \geq a_n\Big) \leq m_n \Pr\Big(\max_{(u,s)\in[0,1]\times J_1} \xi(u,s) \geq a_n\Big) \\
&\quad = m_n \Pr\Big(\max_{(u,s)\in[0,1]\times[0,2]} \xi(u,s) \geq a_n\Big) = O(m_n h_n) = o(1).
\end{aligned}$$

This and (A17) together prove the theorem. *Q.E.D.*

A.5. *Proofs of Theorems 3.1 and 3.2*

PROOF OF THEOREM 3.1: Since $\beta_n[n^{-1/2}h_n^{-3/2} + n^{-1/4}h_n^{-1/2}(\log n)^{1/2} + h_n(\log n)^{1/2}] \to 0$, the main theorem 3.1 is an immediate consequence of Theorems A.1 and A.3. *Q.E.D.*

PROOF OF THEOREM 3.2: The theorem can be proved by arguments similar to those used to prove Theorem 5.1 of Ghosal, Sen, and van der Vaart (2000). In fact, when $F_x(y|x) > 0$ for some (y,x), S_n is of order $O_p(n^{1/2}h_n^{3/2})$ and the consistency follows from the restriction that $nh_n^3/\log h_n^{-1} \to \infty$. *Q.E.D.*

A.6. *Lemmas for Proving Theorem A.1*

Define

$$\begin{aligned}
&V_n(y,x,\theta) \\
&\quad = \frac{2}{n(n-1)} \sum_{1\leq i<j\leq n} [1(Y_i \leq y) - 1(Y_j \leq y)]\,\mathrm{sgn}[\psi(W_i,\theta) - \psi(W_j,\theta)] \\
&\qquad \times K_{h_n}[\psi(W_i,\theta) - x]K_{h_n}[\psi(W_j,\theta) - x],
\end{aligned}$$

so that $\widehat{U}_n(y,x) = V_n(y,x,\widehat{\theta})$. Also, since $F_x(y|x) \equiv 0$, define the projection of $V_n(y,x,\theta)$ by

$$\begin{aligned}
&\widehat{V}_n(y,x,\theta) \\
&\quad = 2n^{-1}\sum_{i=1}^{n}[1(Y_i \leq y) - F(y)] \\
&\qquad \times \int \mathrm{sgn}[\psi(W_i,\theta) - \psi(\tilde{w},\theta)]K_{h_n}[\psi(\tilde{w},\theta) - x]\,dF_W(\tilde{w}) \\
&\qquad \times K_{h_n}[\psi(W_i,\theta) - x].
\end{aligned}$$

LEMMA A.6: *Let Θ denote a neighborhood of θ_0.*

$$\sup_{(y,x,\theta)\in\mathcal{Y}\times\mathcal{X}\times\Theta} |V_n(y,x,\theta) - \widehat{V}_n(y,x,\theta)| = O_p(n^{-1}h_n^{-2}).$$

PROOF: The proof is similar to that of Lemma 3.1 of Ghosal, Sen, and van der Vaart (2000). Hence, we will only indicate the differences. Consider a class of functions $\mathcal{M} = \{m_{(y,x,\theta)} : (y,x,\theta) \in \mathcal{Y}\times\mathcal{X}\times\Theta\}$, where

$$\begin{aligned} &m_{(y,x,\theta)}\big((y_1,w_1),(y_2,w_2)\big) \\ &\quad = [1(y_1 \le y) - 1(y_2 \le y)]\,\mathrm{sgn}[\psi(w_1,\theta) - \psi(w_2,\theta)] \\ &\qquad \times K_{h_n}[\psi(w_1,\theta) - x]K_{h_n}[\psi(w_2,\theta) - x]. \end{aligned}$$

This class is contained in the product of the classes

$$\begin{aligned} \mathcal{M}_1 &= \{1(y_1 \le y) - 1(y_2 \le y) : y \in \mathcal{Y}\}, \\ \mathcal{M}_2 &= \left\{K\left(\frac{\psi(w_1,\theta) - x}{h_n}\right) : (x,\theta) \in \mathcal{X}\times\Theta\right\}, \\ \mathcal{M}_3 &= \left\{K\left(\frac{\psi(w_2,\theta) - x}{h_n}\right) : (x,\theta) \in \mathcal{X}\times\Theta\right\}, \\ \mathcal{M}_4 &= \big\{h_n^{-2}\,\mathrm{sgn}[\psi(w_1,\theta) - \psi(w_2,\theta)] \\ &\qquad \times 1\big\{|\psi(w_1,\theta) - \psi(w_2,\theta)| \le 2h_n\big\} : \theta \in \Theta\big\}. \end{aligned}$$

Since θ is finite dimensional and K is of bounded variation, $\mathcal{M}$ is a Vapnik–Červonenkis (VC) class with the envelope function Ch_n^{-2} with some positive finite constant C, by Lemmas 2.6.15 and 2.6.18 of van der Vaart and Wellner (1996). Then using Theorem 2.6.7 of van der Vaart and Wellner (1996) and following the proof of Lemma 3.1 of Ghosal, Sen, and van der Vaart (2000), we have, for some finite constant C,

$$E\Big[\sup_{(y,x,\theta)\in\mathcal{Y}\times\mathcal{X}\times\Theta} |V_n(y,x,\theta) - \widehat{V}_n(y,x,\theta)|\Big] \le Cn^{-1}h_n^{-2},$$

which gives the conclusion of the lemma. *Q.E.D.*

LEMMA A.7:

$$\sup_{(y,x)\in\mathcal{Y}\times\mathcal{X}} |\widehat{U}_n(y,x) - \widehat{V}_n(y,x,\theta_0)| = O_p\big(n^{-1/2}\big).$$

PROOF: Note that by Assumption 3.1(h),

$$
\begin{aligned}
&|\widehat{V}_n(y,x,\widehat{\theta}) - \widehat{V}_n(y,x,\theta_0)| \\
&= \Bigg| 2n^{-1}\sum_{i=1}^{n}[1(Y_i \le y) - F(y)] \\
&\quad \times \left\{ \int \operatorname{sgn}[\psi(W_i,\widehat{\theta}) - \psi(\tilde{w},\widehat{\theta})] K_{h_n}[\psi(\tilde{w},\widehat{\theta}) - x]\, dF_W(\tilde{w}) \right. \\
&\quad \times K_{h_n}[\psi(W_i,\widehat{\theta}) - x] \\
&\quad - \int \operatorname{sgn}[\psi(W_i,\theta_0) - \psi(\tilde{w},\theta_0)] K_{h_n}[\psi(\tilde{w},\theta_0) - x]\, dF_W(\tilde{w}) \\
&\quad \left. \times K_{h_n}[\psi(W_i,\theta_0) - x] \right\} \Bigg| \\
&\le C\Bigg[\|\widehat{\theta} - \theta_0\| n^{-1}\sum_{i=1}^{n} K_{h_n}[\psi(W_i,\widehat{\theta}) - x] \\
&\quad + n^{-1}\sum_{i=1}^{n}\big\{K_{h_n}[\psi(W_i,\widehat{\theta}) - x] - K_{h_n}[\psi(W_i,\theta_0) - x]\big\} \Bigg]
\end{aligned}
$$

for some positive constant $C < \infty$, which is independent of (y, x). Also, note that using the standard empirical process method (for example, van der Vaart and Wellner (1996)), it is straightforward to show that for a $n^{-1/2}$ neighborhood Θ_n of θ_0,

$$
\begin{aligned}
&\sup_{(x,\theta)\in\mathcal{X}\times\Theta_n} n^{-1}\sum_{i=1}^{n} K_{h_n}[\psi(W_i,\theta) - x] = O_p(1), \\
&\sup_{(x,\theta)\in\mathcal{X}\times\Theta_n} n^{-1}\left|\sum_{i=1}^{n}\big\{K_{h_n}[\psi(W_i,\theta) - x] - K_{h_n}[\psi(W_i,\theta_0) - x]\big\}\right| \\
&= O_p\big(n^{-1/2}\big).
\end{aligned}
$$

Then the lemma follows from the root-n consistency of $\widehat{\theta}$ and Lemma A.6 since $\widehat{U}_n(y,x) = V_n(y,x,\widehat{\theta})$. *Q.E.D.*

Define

$$
\begin{aligned}
\phi_{n,y,x}(Y,X) &= 2[1(Y \le y) - F_Y(y)] \\
&\quad \times \int \operatorname{sgn}(X - \tilde{x}) K_{h_n}(\tilde{x} - x)\, dF_X(\tilde{x}) K_{h_n}(X - x).
\end{aligned}
$$

LEMMA A.8: *There exists a sequence of Gaussian processes $G_n(\cdot)$, indexed by $\mathcal{Y} \times \mathcal{X}$, with continuous sample paths and with*

$$E[G_n(y, x)] = 0 \quad \textit{for} \quad (y, x) \in \mathcal{Y} \times \mathcal{X},$$
$$E[G_n(y_1, x_1) G_n(y_2, x_2)] = E\big[\phi_{n,y_1,x_1}(Y, X)\phi_{n,y_2,x_2}(Y, X)\big]$$

for (y_1, x_1), *and* $(y_2, x_2) \in \mathcal{Y} \times \mathcal{X}$, *such that*

$$\sup_{(y,x)\in\mathcal{Y}\times\mathcal{X}} \big|n^{1/2}\widehat{V}_n(y, x, \theta_0) - G_n(y, x)\big| = O\big(n^{-1/4}h_n^{-1}(\log n)^{1/2}\big) \quad a.s.$$

PROOF: As in the proof of Lemma 3.2 of Ghosal, Sen, and van der Vaart (2000), we use Theorem 1.1 of Rio (1994). Since it can be proved using arguments identical to those used to prove Lemma 3.2 of Ghosal, Sen, and van der Vaart (2000), we will only highlight the differences. To apply Rio's theorem, we rewrite $\varphi_{n,y,x}(Y, X)$ as

$$\begin{aligned}\phi_{n,y,x}(Y, X) &= 2[1(U \le u) - u] \\ &\qquad \times \int \operatorname{sgn}(X - \tilde{x}) K_{h_n}(\tilde{x} - x)\, dF_X(\tilde{x}) K_{h_n}(X - x) \\ &\equiv \varphi_{n,u,x}(U, X),\end{aligned}$$

where $U = F_Y(Y)$ and $u = F_Y(y)$. Then U is uniformly distributed in $[0, 1] \equiv \mathcal{U}$. Thus, Theorem 1.1 of Rio (1994) can be applied to a normalized empirical process associated with $\varphi_{n,u,x}(U, X)$. First, we verify that the class of functions $(v, t) \mapsto h_n\varphi_{n,u,x}(v, t)$, indexed by $(u, x) \in \mathcal{U} \times \mathcal{X}$, is uniformly of bounded variation (UBV). By the definition of Rio (1994), it suffices to show that

$$\sup_{(u,x)\in\mathcal{U}\times\mathcal{X}} \sup_{g\in\mathcal{D}_2([0,1]^2)} \left(\int_{\mathbb{R}^2} h_n\varphi_{n,u,x}(v, t) \operatorname{div} g(v, t)\, dv\, dt \Big/ \|g\|_\infty\right) < \infty,$$

where $\mathcal{D}_2([0, 1]^2)$ denotes the space of C^∞ functions with values in $\mathbb{R}^2$ and with compact support included in $[0, 1]^2$, div denotes the divergence, and $\|g\|_\infty = \sup_{(v,t)\in\mathbb{R}^2} \|g(v, t)\|$ with $\|\cdot\|$ being the usual Euclidean norm. To do so, note that for any $g(v, t) \equiv (g_v(v, t), g_t(v, t))$,

$$\begin{aligned}&\int_{\mathbb{R}^2} \varphi_{n,u,x}(v, t) \operatorname{div} g(v, t)\, dv\, dt \\ &\quad = \int_{\mathbb{R}^2} 2[1(v \le u) - u] \int \operatorname{sgn}(t - \tilde{x}) K_{h_n}(\tilde{x} - x)\, dF_X(\tilde{x}) K_{h_n}(t - x) \\ &\qquad \times \left[\frac{\partial g_v(v, t)}{\partial v} + \frac{\partial g_t(v, t)}{\partial t}\right] dv\, dt\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}} \int_{\mathbb{R}} 2[1(v \le u) - u] \frac{\partial g_v(v,t)}{\partial v}\, dv \\
&\quad \times \int \operatorname{sgn}(t - \tilde{x}) K_{h_n}(\tilde{x} - x)\, dF_X(\tilde{x}) K_{h_n}(t - x)\, dt \\
&\quad + \int_{\mathbb{R}^2} 2[1(v \le u) - u] \\
&\quad \times \int \operatorname{sgn}(t - \tilde{x}) K_{h_n}(\tilde{x} - x)\, dF_X(\tilde{x}) K_{h_n}(t - x) \frac{\partial g_t(v,t)}{\partial t}\, dv\, dt.
\end{aligned}
$$

Then it is straightforward to verify that

$$
\sup_{g \in \mathcal{D}_2([0,1]^2)} \left(\int_{\mathbb{R}^2} \varphi_{n,u,x}(v,t) \operatorname{div} g(v,t)\, dv\, dt \Big/ \|g\|_\infty \right) = O(h_n^{-1})
$$

uniformly over $(u, x) \in \mathcal{U} \times \mathcal{X}$. This implies that the class of functions $\{h_n \varphi_{n,u,x} : (u, x) \in \mathcal{U} \times \mathcal{X}\}$ satisfies the UBV condition of Rio (1994). Furthermore, it is also straightforward to verify that

$$
\sup_{g \in \mathcal{D}_2([a,b]^2)} \left(\int_{\mathbb{R}^2} \varphi_{n,u,x}(v,t) \operatorname{div} g(v,t)\, dv\, dt \Big/ \|g\|_\infty \right) = O(h_n^{-1}[b-a])
$$

uniformly over $(u, x) \in \mathcal{U} \times \mathcal{X}$. This implies that the class of functions $\{h_n \varphi_{n,u,x} : (u, x) \in \mathcal{U} \times \mathcal{X}\}$ also satisfies the local UBV (LUBV) condition of Rio (1994). We now verify that the class of functions $\{h_n \varphi_{n,u,x} : (u, x) \in \mathcal{U} \times \mathcal{X}\}$ is a VC class. The function $h_n \varphi_{n,u,x}$ is bounded by a constant uniformly in $(u, z) \in \mathcal{U} \times \mathcal{X}$ and is obtained by taking an average of

$$
2h_n[1(v \le u) - 1(\tilde{u} \le u)] \operatorname{sgn}(\tilde{x} - t) K_{h_n}(\tilde{x} - t) K_{h_n}(t - x)
$$

over $(\tilde{u}, \tilde{x})$. Then it is easy to show that $\{h_n \varphi_{n,u,x} : (u, x) \in \mathcal{U} \times \mathcal{X}\}$ is a VC class by using arguments similar to those used in the proof of Lemma 3.2 of Ghosal, Sen, and van der Vaart (2000), in particular equation (8.5). Finally, by applying Theorem 1.1 of Rio (1994), there exists a sequence of centered Gaussian processes $G_n(u, x)$ with covariance

$$
E[G_n(u_1, x_1) G_n(u_2, x_2)] = E\big[\varphi_{n,u_1,x_1}(U, X) \varphi_{n,u_2,x_2}(U, X)\big].
$$

By switching back to the original variable Y and its corresponding index y, we obtain the desired result. *Q.E.D.*

Define

$$
\sigma_n^2(x) = 4 \int \left[\int \operatorname{sgn}(\bar{x} - \tilde{x}) K_{h_n}(\tilde{x} - x)\, dF_X(\tilde{x}) K_{h_n}(\bar{x} - x) \right]^2 dF_X(\bar{x})
$$

and

$$\sigma^2(x) = 4\left[\int q^2(u)K^2(u)\,du\right][f_X(x)]^3.$$

LEMMA A.9:

(a) $\sup_{x\in\mathcal{X}} |h_n\sigma_n^2(x) - \sigma^2(x)| = o(1)$,

(b) $\lim\inf_{n\to\infty} h_n \inf_{x\in\mathcal{X}} \sigma_n^2(x) > 0$,

(c) $\sup_{x\in\mathcal{X}} |\widehat{\sigma}_n^2(x) - \sigma_n^2(x)| = O_p\big(n^{-1/2}h_n^{-2}\big)$.

PROOF: Parts (a) and (b) of the lemma follow directly from Lemma 3.3(a) and (b) of Ghosal, Sen, and van der Vaart (2000). To prove part (c) of the lemma, note that $\widehat{\sigma}_n^2(x)$ depends on the estimated X_i. To deal with this, let $\tilde{\sigma}_n^2(x,\theta)$ be the same as $\widehat{\sigma}_n^2(x)$ except that $\widehat{X}_i$ is replaced by $\psi(W_i,\theta)$. As in the proof of Lemma A.6, modifying the proof of Lemma 3.3 of Ghosal, Sen, and van der Vaart (2000) gives

$$\sup_{x\in\mathcal{X}}\sup_{\theta\in\Theta} |\tilde{\sigma}_n^2(x,\theta) - E\tilde{\sigma}_n^2(x,\theta)| = O_p\big(n^{-1/2}h_n^{-2} + n^{-1}h_n^{-3} + n^{-3/2}h_n^{-4}\big),$$

where Θ is a neighborhood of θ_0. Then part (c) follows from the restriction on h_n and the fact that $E\tilde{\sigma}_n^2(x,\theta)$ is Lipschitz continuous with respect to θ. *Q.E.D.*

LEMMA A.10: *For the sequence of Gaussian processes* $\{G_n(y,x) : (y,x) \in \mathcal{Y}\times\mathcal{X}\}$ *obtained in Lemma* A.8, *there corresponds a sequence of Gaussian processes* $\{\xi_n(u,s) : (u,s) \in [0,1]\times\mathcal{X}_n\}$ *with continuous sample paths such that*

$$E[\xi_n(u,s)] = 0,$$
$$E[\xi_n(u_1,s_1)\xi_n(u_2,s_2)] = [\min(u_1,u_2) - u_1u_2]\rho(s_1 - s_2)$$

for $u, u_1, u_2 \in [0,1]$ *and* $s, s_1, s_2 \in \mathcal{X}_n$, *where*

$$\sup_{(y,x)\in\mathcal{Y}\times\mathcal{X}} \left|\frac{G_n(y,x)}{\sigma_n(x)} - \xi_n[F_Y(y), h_n^{-1}x]\right| = O_p\big(h_n\sqrt{\log h_n^{-1}}\big).$$

PROOF: Let $\mathcal{G}_n$ denote the class of functions $\{g_{n,u,x} : (u,x) \in \mathcal{U}\times\mathcal{X}\}$, where $g_{n,u,x}(U,X) = \varphi_{n,u,x}(U,X)/\sigma_n(x)$. Also, let $\tilde{\mathcal{G}}_n$ denote the class of functions $\{\tilde{g}_{n,u,x} : (u,x) \in \mathcal{U}\times\mathcal{X}\}$, where

$$\tilde{g}_{n,u,x}(U,X) = \tilde{\varphi}_{n,u,x}(U,X)/\tilde{\sigma}_{n,x}(X),$$
$$\tilde{\varphi}_{n,u,x}(U,X) = [1(U \le u) - u]$$

$$\times \int \operatorname{sgn}(X - \tilde{x}) K_{h_n}(\tilde{x} - x)\, d\tilde{x}\, K_{h_n}(X - x),$$

$$\tilde{\sigma}_{n,x}(X) = \left[\int \left(\int \operatorname{sgn}(\bar{x} - \tilde{x}) K_{h_n}(\tilde{x} - x)\, d\tilde{x}\right)^2 \left[K_{h_n}(\bar{x} - x)\right]^2 d\bar{x}\right]^{1/2} \times [f_X(X)]^{1/2}.$$

As explained in Remark 8.3 of Ghosal, Sen, and van der Vaart (2000), it is possible to extend Lemma A.8 in that there exists a sequence of Gaussian bridges, say $\{B_n(g) : g \in \mathcal{G}_n \cup \tilde{\mathcal{G}}_n\}$, with

$$E[B_n(g)] = 0, \quad E[B_n(g_1)B_n(g_2)] = \operatorname{cov}(g_1, g_2)$$

for all $g, g_1, g_2 \in \mathcal{G}_n \cup \tilde{\mathcal{G}}_n$ and with continuous sample paths with respect to the L_2 metric such that

$$G_n(u, x) = \sigma_n(x) B_n(\varphi_{n,u,x}),$$

where $G_n(u, x)$ is defined in the proof of Lemma A.8. Now let $\tilde{\xi}_n(u, x) = B_n(\tilde{g}_{n,u,x})$ and $\gamma_n(u, x) = G_n(u, x)/\sigma_n(x) - \tilde{\xi}_n(u, x)$. As in the proof of Lemma 3.4 of Ghosal, Sen, and van der Vaart (2000), note that $\gamma_n(u, x)$ is a mean-zero Gaussian process with

$$E[\gamma_n(u_1, x_1)\gamma_n(u_2, x_2)] = E\left[\left(g_{n,u_1,x_1} - \tilde{g}_{n,u_1,x_1}\right)\left(g_{n,u_2,x_2} - \tilde{g}_{n,u_2,x_2}\right)\right].$$

Then the lemma can be proved using identical arguments to those used in the proof of Lemma 3.4 of Ghosal, Sen, and van der Vaart (2000). *Q.E.D.*

A.7. *Proof of Theorem 6.1*

This theorem can be proved using arguments similar to those used in the proof of Theorem 3.1. In particular, the following lemmas can be proved (the whose proofs are omitted here for brevity) and then the desired result follows.

Recall that

$$\rho(s) = \frac{\int q(z)q(z - s)K(z)K(z - s)\, dz}{\int q^2(z)K^2(z)\, dz},$$

where $q(u) = \int \operatorname{sgn}(u - w)K(w)\, dw$. Let $\boldsymbol{\xi}(u, \mathbf{s})$ denote a Gaussian process $\{\boldsymbol{\xi}(u, \mathbf{s}) : (u, \mathbf{s}) \in [0, 1] \times \mathbb{R}^d\}$ with continuous sample paths such that

$$E[\boldsymbol{\xi}(u, \mathbf{s})] = 0,$$

$$E[\boldsymbol{\xi}(u_1, \mathbf{s}_1)\boldsymbol{\xi}(u_2, \mathbf{s}_2)] = [\min(u_1, u_2) - u_1 u_2] \prod_{j=1}^{d} \rho(s_{1j} - s_{2j})$$

for $u, u_1, u_2 \in [0,1]$ and $\mathbf{s}, \mathbf{s}_1 \equiv (s_{11}, \ldots, s_{1d}), \mathbf{s}_2 \equiv (s_{21}, \ldots, s_{2d}) \in \mathbb{R}^d$. Define $\mathcal{X}_n = [0, 1/h_n]^d$ and let $\boldsymbol{\xi}_n$ be the restriction of $\boldsymbol{\xi}$ to $[0,1] \times \mathcal{X}_n$.

LEMMA A.11: *Let Assumption* 6.1 *hold*. *Let h_n satisfy*

$$h_n(\log n)^{1/2} \to 0, \quad nh_n^{3d} \to \infty, \quad \textit{and} \quad nh_n^{d+1}/(\log n)^{d+1} \to \infty.$$

Then there exists a sequence of Gaussian processes $\{\boldsymbol{\xi}(u, \mathbf{s}) : (u, \mathbf{s}) \in [0,1] \times \mathcal{X}_n\}$ with continuous sample paths such that

$$E[\boldsymbol{\xi}_n(u, \mathbf{s})] = 0,$$

$$E[\boldsymbol{\xi}_n(u_1, \mathbf{s}_1)\boldsymbol{\xi}_n(u_2, \mathbf{s}_2)] = [\min(u_1, u_2) - u_1 u_2] \prod_{j=1}^{d} \rho(s_{1j} - s_{2j})$$

for $u, u_1, u_2 \in [0,1]$ and $\mathbf{s}, \mathbf{s}_1 \equiv (s_{11}, \ldots, s_{1d}), \mathbf{s}_2 \equiv (s_{21}, \ldots, s_{2d}) \in \mathcal{X}_n$, and that

$$\sup_{(y,\mathbf{x}) \in \mathcal{Y} \times \mathcal{X}} \left| n^{1/2} \frac{\widehat{\mathbf{U}}_n(y, \mathbf{x})}{\widehat{\mathbf{s}}_n(x)} - \boldsymbol{\xi}_n[F_Y(y), h_n^{-1}\mathbf{x}] \right|$$
$$= O_p\big[n^{-1/2} h_n^{-3d/2} + n^{-1/2(d+1)} h_n^{-1/2} (\log n)^{1/2} + h_n (\log n)^{1/2}\big].$$

LEMMA A.12: *Let λ denote the quantity defined in Theorem* 3.1 *and let $\mathbf{I} \equiv [0, L]^d$ be a rectangle with a fixed volume L^d. Then*

$$\Pr\Big(\max_{(u,\mathbf{s}) \in [0,1] \times \mathbf{I}} \boldsymbol{\xi}(u, \mathbf{s}) > a \Big) = L^d 2^{-(d-1)} \left(\frac{8\lambda}{\pi} \right)^{d/2} a^d \exp(-2a^2)[1 + o(1)]$$

as $a \to \infty$.

LEMMA A.13: *For any x,*

$$\Pr\Big(4\mathbf{b}_n \Big\{ \max_{(u,s) \in [0,1] \times \mathcal{X}_n} \boldsymbol{\xi}(u, \mathbf{s}) - \mathbf{b}_n \Big\} < x \Big)$$
$$= \exp\left\{ -\exp\left(-x - \frac{x^2}{8\mathbf{b}_n^2} \right) \left[1 + \frac{x}{4\mathbf{b}_n^2} \right]^d \right\} + o(1),$$

where $\mathbf{b}_n$ is defined in equation (16) *in the main text*.

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