Econometrica Supplementary Material

SUPPLEMENT TO "WEALTH INEQUALITY IN A LOW RATE ENVIRONMENT"

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APPENDIX B: APPENDIX FOR SECTION 3

B.1. *Sufficient Statistic in the Rentier Regime*

SO FAR, WE HAVE DERIVED OUR SUFFICIENT STATISTIC under the assumption that we are in the entrepreneur regime. We now show that that the sufficient statistic in words (i.e., equation (8)) also holds in the rentier regime. Rentiers own a diversified portfolio. Their portfolio can be seen as a "representative" tree with payout τ (the cash flow due to the fraction of trees that blossom every period) minus i (the negative cash flow due to the investment in existing trees) that grows at rate $g - \tau$ (the growth rate of trees that keep growing minus the fraction of trees that blossom). This implies a payout yield $(\tau - i)/q$ and a growth rate $g - \tau$. Note that the return on wealth r can be written as

$$r = \underbrace{\frac{\tau - i}{q}}_{\text{Payout yield}} + \underbrace{g - \tau.}_{\text{Growth rate of cash flows}}$$

Plugging this expression for r in the expression for Pareto inequality (see Proposition 1), we obtain

$$\theta = \max\left(\frac{g - \frac{i}{q} - \rho}{\eta + \tau}, \frac{g - \tau + \frac{\tau - i}{q} - \rho}{\eta}\right).$$

Differentiating with respect to r gives us

$$\partial_r \log\theta = \begin{cases} \dfrac{-\frac{i}{q}(-\partial_r \log q)}{g - \frac{i}{q} - \rho} & \text{for } r < r^* \text{ (entrepreneur regime),} \\ \dfrac{\frac{\tau - i}{q}(-\partial_r \log q)}{r - \rho} & \text{for } r > r^* \text{ (rentier regime).} \end{cases}$$

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The key takeaway is that the sufficient statistic (8) holds regardless of whether we are in the entrepreneur or rentier regime. In other words, the effect of r on Pareto inequality can be written as

$$\partial_r \log \theta = \frac{\text{Payout yield} \times \text{Duration}}{\text{Growth rate of wealth}} \tag{36}$$

for households reaching the right tail of the wealth distribution (see Appendix B.3 for a generalization of this insight). The only special thing about rentiers is that, because the "representative tree" they own has a constant growth rate, the numerator in the sufficient statistic is equal to one. Mathematically, this comes from the fact that the payout yield of the representative tree is exactly equal to the inverse of the duration (see footnote 33).

B.2. *Return on Wealth Versus Return on Capital*

Following the notation in the main text, the definition of the net return on capital in the endogenous investment extension is

$$\text{rok} \equiv \underbrace{a}_{\text{Production efficiency}} + \underbrace{g - \iota(g)}_{\text{Investment efficiency}}. \tag{37}$$

The net return on capital is the sum of production efficiency a (i.e., how much gross output is produced per unit of capital) and investment efficiency $g - \iota(g)$ (i.e., the difference between the growth rate of capital and the investment rate). This definition of the net return on capital fully summarizes the technological contribution of a firm to aggregate net value-added, and is consistent with the System of National Accounts.[51]

Plugging this definition into the expression for q given in (9), we can write

$$q = 1 + \frac{\text{rok} - r}{r + \tau - g}. \tag{38}$$

The second term in (38) represents the present value of rents. Indeed, from the perspective of a firm owner, the *average* return on investment is rok while the *marginal* return on investment is r, which means that investment generates Ricardian rents that accrue to firm owners.[52] Note that the presence of a convex investment adjustment cost function allows rok $> r$ to be sustained in equilibrium. The reason is that convex adjustment costs generate decreasing returns to scale in investment. Hence, rents accrue to firm owners

[51]The net return on capital is defined as net capital income over capital. In the National Accounts, net capital income is the sum of gross profits minus capital depreciation. Using our notation, gross profits are aK and the identity that implicitly defines the depreciation rate is

$$K_{t+1} - K_t = -\text{depreciation rate} \times K_t + I_t,$$

where K_t is capital and I_t is investment. Hence, the "depreciation rate" in our model is $\iota(g) - g$. Putting this together, we have that the net return on capital in our model is $a - (\iota(g) - g)$, which coincides with (37).

[52]See Cochrane (1991) for a formal proof that the first-order condition for investment in q-theory models (i.e., $\iota'(g) = q$ using our notation) implies that the marginal return on investment is equal to the discount rate (i.e., r using our notation). Note that, at the aggregate level, $(\text{rok} - r)K$ is the total value of "pure profits" that accrue to firm owners (e.g., Barkai (2020); Karabarbounis and Neiman (2019), and Gouin-Bonenfant (2022)). In our model, the pure profits are due to Ricardian rents, but, in general, they could also be due to market power.

due to the fact that their technology cannot be costlessly scaled to accommodate the supply of savings. We make this point quantitatively in Section 5, where we simulate a rise in savings in a general equilibrium model.

Note that we can write the return on wealth for entrepreneurs, while their firms keep growing, as

$$\frac{\mathrm{d}R_t}{R_t} = \mathrm{rok}\,\mathrm{d}t + (g - \mathrm{rok})\left(1 - \frac{1}{q}\right)\mathrm{d}t. \tag{39}$$

When entrepreneurs do not use external financing (i.e., $g = \mathrm{rok}$, or equivalently $a = \iota(g)$), their return on wealth is equal to the return on capital. However, when they use external financing (i.e., $g > \mathrm{rok}$, or equivalently $a < \iota(g)$), their return on wealth differs from the return on capital. In particular, their return on wealth *exceeds* the return on capital whenever $q > 1$. This comes from the fact that part of an entrepreneur's return comes from selling shares to outsiders. This is the key insight of our paper: for a given return on capital, a low cost of capital r benefits entrepreneurs who raise external financing.

B.3. *Extension: Heterogeneous Firm Dynamics*

PROOF OF PROPOSITION 2: Denote ω log wealth, $\boldsymbol{p}_{Et}(\omega)$ the joint density of log wealth and productivity state for entrepreneurs (an $S \times 1$ vector), and $p_{Rt}(\omega)$ the density of log wealth for rentiers. Moreover, denote $\boldsymbol{m}_{Et}(\xi)$, $m_{Rt}(\xi)$ the corresponding moment generating function for wealth for each type of agent:

$$\boldsymbol{m}_{Et}(\xi) \equiv \int_{\mathbb{R}} e^{\xi\omega} \boldsymbol{p}_{Et}(\omega)\,\mathrm{d}\omega, \qquad m_{Rt}(\xi) \equiv \int_{\mathbb{R}} e^{\xi\omega} p_{Rt}(\omega)\,\mathrm{d}\omega.$$

Applying the Laplace transform on the Kolmogorov forward equation gives the dynamics of these functions over time:

$$\partial_t \boldsymbol{m}_{Et}(\xi) = \mathcal{D}(\boldsymbol{q})^{\xi}\big(\xi\mathcal{D}(\boldsymbol{\mu}) + \mathcal{T}' - (\eta+\tau)\mathcal{I}\big)\mathcal{D}(\boldsymbol{q})^{-\xi}\boldsymbol{m}_{Et}(\xi) + \eta\mathcal{D}(\boldsymbol{q})^{\xi}\boldsymbol{\psi}, \tag{40}$$

$$\partial_t m_{Rt}(\xi) = \big(\xi(r-\rho) - \eta\big)m_{Rt}(\xi) + \tau\mathbf{1}'\boldsymbol{m}_{Et}(\xi), \tag{41}$$

where $\boldsymbol{q} \equiv (q_1, \dots, q_S)'$ is the vector of prices (i.e., the solution to the HJB, see equation (10)), $\boldsymbol{\psi} = (\psi_1, \dots, \psi_S)'$ is the distribution of firm types at birth, $\mathcal{D}(\boldsymbol{\mu})$ is the diagonal matrix with diagonal elements given by the vector $\boldsymbol{\mu}$, and $\mathcal{I}$ is the identity matrix.

Denote $p_t(\omega)$ the overall density of log wealth; that is, $p_t(\omega) = \mathbf{1}'\boldsymbol{p}_{Et}(\omega) + p_{Rt}(\omega)$. Hence, the moment generating function for wealth is given by

$$m_t(\xi) = \mathbf{1}'\boldsymbol{m}_{Et}(\xi) + m_{Rt}(\xi).$$

We are interested in characterizing the limit of $m_t(\xi)$ as $t \to \infty$ (stationary economy). Our assumption that there exists at least one state s such that the rate of capital accumulation is positive (i.e., $\mu_s > 0$) implies the existence of a unique $\theta_E > 0$ such that $\varrho(\frac{1}{\theta_E}\mathcal{D}(\boldsymbol{\mu}) + \mathcal{T}') = \eta + \tau$ (see Proposition 2 in Beare and Akira Toda (2022)). This allows us to characterize the limit $m_t(\xi)$ as time tends to infinity:

$$m(\xi) \equiv \lim_{t\to+\infty} m_t(\xi) = \left(1 + \frac{\tau}{\eta - \xi(r-\rho)}\right)\mathbf{1}'\mathcal{D}(\boldsymbol{q})^{\xi}\big((\eta+\tau)\mathcal{I} - \xi\mathcal{D}(\boldsymbol{\mu}) - \mathcal{T}'\big)^{-1}\eta\boldsymbol{\psi}$$

if $0 \leq \xi < \min(\frac{\eta}{r-\rho}, \frac{1}{\theta_E})$, and infinity if $\xi \geq \min(\frac{\eta}{r-\rho}, \frac{1}{\theta_E})$. That is, m has a pole at $\min(\frac{\eta}{r-\rho}, \frac{1}{\theta_E})$. Using Theorem 3.1 in Beare, Seo, and Akira Toda (2022), this implies that the long-run wealth distribution has a right Pareto tail with Pareto inequality $\theta = \max(\theta_E, \frac{r-\rho}{\eta})$. *Q.E.D.*

PROOF FOR PROPOSITION 3: We prove the proposition in three steps. In the first step, we prove that

$$\partial_r \log \theta = \frac{(\boldsymbol{u} \circ \boldsymbol{v})' \partial_r \boldsymbol{\mu}}{(\boldsymbol{u} \circ \boldsymbol{v})' \boldsymbol{\mu}}, \tag{42}$$

where $\boldsymbol{u}$ and $\boldsymbol{v}$ denote the left and right eigenvectors associated with the dominant eigenvalue of the matrix $\frac{1}{\theta}\mathcal{D}(\boldsymbol{\mu}) + \mathcal{T}$ and $\boldsymbol{u} \circ \boldsymbol{v}$ denotes the multiplication elementwise. This equation says that the semielasticity of Pareto inequality with respect to r can be written as the ratio between the average derivative of the growth rate of wealth $\partial_r \mu$ and the average growth rate of wealth μ, where the average is taken with respect to some density across productivity states $\boldsymbol{u} \circ \boldsymbol{v}$.[53] In the second step of the proof, we show that $\boldsymbol{u} \circ \boldsymbol{v}$ can be interpreted as the density of past states for individuals in the right tail of the wealth distribution. The third step concludes.

Step 1. Denote $\boldsymbol{u}(\theta, r)$ and $\boldsymbol{v}(\theta, r)$ the left and right eigenvectors associated with the dominant eigenvalue of the matrix $\frac{1}{\theta}\mathcal{D}(\boldsymbol{\mu}) + \mathcal{T}$, normalized so that $\boldsymbol{u}'\mathbf{1} = \boldsymbol{u}'\boldsymbol{v} = 1$. As seen in Proposition 2, Pareto inequality θ satisfies the equation

$$\left(\frac{1}{\theta}\mathcal{D}(\boldsymbol{\mu}) + \mathcal{T}\right)\boldsymbol{v}(\theta, r) = (\eta + \tau)\boldsymbol{v}(\theta, r).$$

Differentiating with respect to r gives

$$\begin{aligned} &\left(\frac{1}{\theta}\mathcal{D}(\partial_r \boldsymbol{\mu}) - \frac{1}{\theta^2}\mathcal{D}(\boldsymbol{\mu})\partial_r \theta\right)\boldsymbol{v} + \left(\frac{1}{\theta}\mathcal{D}(\boldsymbol{\mu}) + \mathcal{T}\right)(\partial_r \boldsymbol{v} + \partial_\theta \boldsymbol{v} \partial_r \theta) \\ &\quad = (\eta + \tau)(\partial_r \boldsymbol{v} + \partial_\theta \boldsymbol{v} \partial_r \theta). \end{aligned}$$

Left multiplying by $\boldsymbol{u}'$ gives

$$\boldsymbol{u}'\left(\frac{1}{\theta}\mathcal{D}(\partial_r \boldsymbol{\mu}) - \frac{1}{\theta^2}\mathcal{D}(\boldsymbol{\mu})\partial_r \theta\right)\boldsymbol{v} = 0,$$

since, by definition of $\boldsymbol{u}$, we have $\boldsymbol{u}'(\frac{1}{\theta}\mathcal{D}(\boldsymbol{\mu}) + \mathcal{T}) = \boldsymbol{u}'(\eta + \tau)$. Rearranging and dividing by $\boldsymbol{u}'\mathcal{D}(\boldsymbol{\mu})\boldsymbol{v}$ gives

$$\partial_r \log \theta = \frac{\boldsymbol{u}'\mathcal{D}(\partial_r \boldsymbol{\mu})\boldsymbol{v}}{\boldsymbol{u}'\mathcal{D}(\boldsymbol{\mu})\boldsymbol{v}}, \tag{43}$$

which is (42). Note that we can divide by $\boldsymbol{u}'\mathcal{D}(\boldsymbol{\mu})\boldsymbol{v}$ since this quantity is always positive: indeed, it corresponds to the derivative of $\xi \to \varrho(\xi\mathcal{D}(\boldsymbol{\mu}) + \mathcal{T})$ at $\xi = 1/\theta$, which is a strictly convex function (see Beare, Seo, and Akira Toda (2022)).

[53]Note that $\boldsymbol{u} \circ \boldsymbol{v}$ corresponds to a density as $\boldsymbol{u}$ and $\boldsymbol{v}$ are positive elementwise (they correspond to the eigenvectors associated with the dominant eigenvalue).

Step 2. We now show that $\boldsymbol{u} \circ \boldsymbol{v}$ can be interpreted as the density of past states for individuals in the right tail of the wealth distribution. We refer the reader to Lecomte (2007) and Chetrite and Touchette (2015) for similar derivations in the context of large deviation theory.

Consider a function f defined on the set of states $\{1, \ldots, S\}$. For an individual i in the wealth distribution with age a_i, denote $F_i = \int_0^{a_i} f(s_{ia})\,\mathrm{d}a$ the cumulative sum of $f(s_{ia})$ since birth. Denote $\boldsymbol{p}_E(\omega, F)$ the cross-sectional density of productivity state s, log wealth ω, and F for entrepreneurs. Denote $\boldsymbol{m}_E(\omega, \beta) \equiv \int_{\mathbb{R}} e^{\beta F} \boldsymbol{p}_E(\omega, F)\,\mathrm{d}F$ the moment generating function of F and $\tilde{\boldsymbol{m}}_E(\xi, \beta) \equiv \int_{\mathbb{R}^2} e^{\xi\omega+\beta F} \boldsymbol{p}_E(\omega, F)\,\mathrm{d}\omega\,\mathrm{d}F$ the joint moment generating function of F and ω. Applying the Laplace transform on the Kolmogorov-forward equation for $\boldsymbol{p}_E(\omega, F)$ gives a closed-form solution for $\tilde{\boldsymbol{m}}_E$:

$$0 = \mathcal{D}(\boldsymbol{q})^{\xi}\big(\beta\mathcal{D}(\boldsymbol{f}) + \xi\mathcal{D}(\boldsymbol{\mu}) + \mathcal{T}' - (\eta+\tau)\mathcal{I}\big)\mathcal{D}(\boldsymbol{q})^{-\xi}\tilde{\boldsymbol{m}}_E(\xi, \beta) + \eta\mathcal{D}(\boldsymbol{q})^{\xi}\boldsymbol{\psi}, \tag{44}$$

where $\boldsymbol{f} \equiv (f(s_1), \ldots, f(s_S))$. We know that $\varrho(\xi\mathcal{D}(\boldsymbol{\mu}) + \mathcal{T}) = \eta + \tau$ has a unique positive solution, given by $\xi = 1/\theta$. Hence, for β close enough to zero, $\varrho(\beta\mathcal{D}(\boldsymbol{f}) + \xi\mathcal{D}(\boldsymbol{\mu}) + \mathcal{T}) = \eta + \tau$ also has a unique positive solution, which we denote $\xi^*(\beta)$. Given (44), this implies that $\xi \to \tilde{\boldsymbol{m}}_E(\xi, \beta)$ has a pole in $\xi^*(\beta)$. Beare, Seo, and Akira Toda (2022) shows that this implies

$$\lim_{\omega\to\infty} \frac{1}{\omega} \log m_{Es}(\omega, \beta) = -\xi^*(\beta), \tag{45}$$

where m_{Es} denotes the s^{th} element of the vector $\boldsymbol{m}_E$ and $s \in \{1, \ldots, S\}$.

Taking the derivative of $\log m_{Es}$ with respect to β at zero gives

$$\partial_{\beta=0} \log m_{Es}(\omega, \beta) = \frac{\displaystyle\int_{\mathbb{R}} F p_{Es}(\omega, F)\,\mathrm{d}F}{\displaystyle\int_{\mathbb{R}} p_{Es}(\omega, F)\,\mathrm{d}F} = \mathbb{E}[F_i | \omega_i = \omega, s_i = s],$$

where p_{Es} denotes the s^{th} element of the vectors $\boldsymbol{p}_E$. Because the convergence (45) is locally uniform in β around zero, we get

$$\lim_{\omega\to\infty} \frac{1}{\omega} \mathbb{E}[F_i | \omega_i = \omega] = -\xi^{*\prime}(0).$$

Using a similar derivation as in step 1, one can show that $\xi^{*\prime}(0) = -((\boldsymbol{u}\circ\boldsymbol{v})'\boldsymbol{f})/((\boldsymbol{u}\circ\boldsymbol{v})'\boldsymbol{\mu})$, which gives

$$\lim_{\omega\to\infty} \frac{1}{\omega} \mathbb{E}[F_i | \omega_i = \omega] = \frac{(\boldsymbol{u}\circ\boldsymbol{v})'\boldsymbol{f}}{(\boldsymbol{u}\circ\boldsymbol{v})'\boldsymbol{\mu}}.$$

This is the sense in which $\boldsymbol{u} \circ \boldsymbol{v}$ can be interpreted as the density of states for individuals in the right tail of the wealth distribution.

Step 3. Combining the previous formula in the special case $f(s) = \partial_r \mu_s$ with (42) gives

$$\lim_{\omega\to\infty} \frac{1}{\omega} \mathbb{E}\left[\int_0^{a_i} \partial_r \mu_{s_{ia}}\,\mathrm{d}a \,\Big|\, \omega_i = \omega\right] = \partial_r \log\theta.$$

This implies

$$\lim_{\omega\to\infty}\frac{1}{\omega}\mathbb{E}\left[\int_0^{a_i}\partial_r\mu_{s_{ia}}\,\mathrm{d}a\Big|\omega_i=\omega\right]=\partial_r\log\theta$$

$$\Longrightarrow\quad\lim_{W\to\infty}\mathbb{E}\left[\frac{\dfrac{1}{a_i}\displaystyle\int_0^{a_i}\partial_r\mu_{s_{ia}}\,\mathrm{d}a}{\dfrac{1}{a_i}\displaystyle\int_0^{a_i}\mu_{s_{ia}}\,\mathrm{d}a}\Bigg|W_i=W\right]=\partial_r\log\theta.$$

Q.E.D.

B.4. *Extension: Debt Issuance*

We now provide a derivation of equation (15). Similar to the stylized model, we have, in the entrepreneur regime,

$$\theta=\frac{-\dfrac{i_\lambda}{q_\lambda}+g-\rho}{\eta+\tau}.$$

Hence, the effect of a small change in the required return on debt $\mathrm{d}r_f$ and in the required return on unlevered equity $\mathrm{d}r$ on θ is

$$\mathrm{d}\log\theta=\mathrm{d}\log\left(-\frac{i_\lambda}{q_\lambda}+g-\rho\right)=\frac{\mathrm{d}\left(-\dfrac{i_\lambda}{q_\lambda}\right)}{-\dfrac{i_\lambda}{q_\lambda}+g-\rho},\tag{46}$$

where the second equality uses the fact that g and ρ are exogenous parameters. Differentiating (13) and (14) give

$$\mathrm{d}i_\lambda=(\lambda-1)\,\mathrm{d}r_f,$$
$$\mathrm{d}\log q_\lambda=\lambda_M\,\mathrm{d}\log q,$$

where $\lambda_M\equiv\lambda q/q_\lambda$ denotes market leverage. Finally, equation (2) gives $\mathrm{d}\log q=\partial_r\log q\,\mathrm{d}r$; that is, the value of q only depends on r, not r_f. Combining these equations give

$$\begin{aligned}\mathrm{d}\left(-\frac{i_\lambda}{q_\lambda}\right)&=-\frac{\mathrm{d}i_\lambda}{q_\lambda}+\frac{i_\lambda}{q_\lambda}\,\mathrm{d}\log q_\lambda\\&=-\frac{\lambda-1}{q_\lambda}\,\mathrm{d}r_f-\frac{i_\lambda}{q_\lambda}\lambda_M(-\partial_r\log q)\,\mathrm{d}r\\&=-(\lambda_M-1)\,\mathrm{d}r_f-\frac{i_\lambda}{q_\lambda}\lambda_M(-\partial_r\log q)\,\mathrm{d}r.\end{aligned}$$

Combining with (46) and rearranging give (15).

B.5. *Constraints on External Financing*

We now consider an extension of the investment model described in Endogenous Investment with constraints on the amount of external financing. Formally, we assume that the book leverage of the firm must be λ (a constraint on debt issuance) and that the equity payout yield must be higher than a certain threshold $-B$ (a constraint on equity issuance). Formally, the investment problem faced by the entrepreneur is now

$$\begin{aligned} rq &= \max_g \{a - \iota(g) + gq + \tau(1-q)\} \\ \text{s.t.} \quad & i_\lambda \leq B q_\lambda, \end{aligned} \tag{47}$$

where, similar to the leverage extension, λ denotes book leverage, $i_\lambda = g - r_f + \lambda(\iota(g) - a - (g - r_f))$ denotes the flow of equity financing as a share of book equity, and $q_\lambda = 1 + \lambda(q-1)$ denotes the market value of equity divided by its book value.

The baseline model can be seen as a special case where $B = +\infty$ (i.e., no constraint on equity issuance). Another special case often considered in the literature on entrepreneurship is $B = 0$ (i.e., no equity issuance). More generally, this model allows us to consider the intermediate case $0 < B < +\infty$.

Denoting by $\upsilon/\lambda \geq 0$ the Lagrange multiplier on the financial constraint, the first-order condition for g in (47) gives

$$(1+\upsilon)\big(1 + \lambda\big(\iota'(g) - 1\big)\big) = q_\lambda. \tag{48}$$

When the constraint does not bind (i.e., $\upsilon = 0$), we obtain $\iota'(g) = q$, as in the model without constraints on external financing. In contrast, when the constraint binds (i.e., $\upsilon > 0$), investment is inefficiently low.

We now assume that the constraint binds (otherwise, this reverts to the model without constraints on external financing). As in the stylized model (see equation (7)), the effect of a change in the required return on debt r_f and in the required return on (unlevered) equity r on Pareto inequality is given by the relative change in the growth rate of entrepreneurs:

$$\begin{aligned} \mathrm{d}\log\theta &= \mathrm{d}\log\left(-\frac{i_\lambda}{q_\lambda} + g - \rho\right) \\ &= \frac{\mathrm{d}g}{-\frac{i_\lambda}{q_\lambda} + g - \rho}, \end{aligned}$$

where the second equality uses the fact that the constraint binds. To compute the change $\mathrm{d}g$, we differentiate the constraint on external financing

$$\begin{aligned} & \mathrm{d}i_\lambda = Bq_\lambda \,\mathrm{d}\log q_\lambda \\ \implies \quad & (\lambda - 1)\,\mathrm{d}r_f + \big(1 + \lambda\big(\iota'(g) - 1\big)\big)\,\mathrm{d}g = Bq_\lambda \lambda_M \,\mathrm{d}\log q \\ \implies \quad & (\lambda - 1)\,\mathrm{d}r_f + \frac{q_\lambda}{1+\upsilon}\,\mathrm{d}g = i_\lambda \lambda_M \,\mathrm{d}\log q \\ \implies \quad & \mathrm{d}g = (1+\upsilon)\left(\mathrm{d}r_f + \lambda_M\left(\frac{i_\lambda}{q_\lambda}\partial_r \log q \,\mathrm{d}r - \mathrm{d}r_f\right)\right). \end{aligned}$$

Plugging this into the expression for $\mathrm{d}\log\theta$ gives

$$\mathrm{d}\log\theta = (1+\upsilon)\frac{\mathrm{d}r_f + \lambda_M\left(\frac{i_\lambda}{q_\lambda}\partial_r \log q\,\mathrm{d}r - \mathrm{d}r_f\right)}{-\frac{i_\lambda}{q_\lambda} + g - \rho}.$$

This is the same as the sufficient statistic with leverage (Appendix B.4) with one key difference: the formula for the effect of r on $\log\theta$ is multiplied by $(1+\upsilon)$.

How important is this multiplier? To answer this question, note that the multiplier can be rewritten as $1+\upsilon = (1+\lambda(q-1))/(1+\lambda(\iota'(g)-1))$ using the first-order condition for investment (48). Since $1 < \iota'(g) \leq q$, we obtain that the multiplier is bounded below by 1 and bounded above by q_λ; that is, $1 < 1+\upsilon \leq q_\lambda$. The lower bound is attained in the limit $B \to \infty$ (i.e., there are no financial frictions) while the upper bound is attained in the limit $\iota'(g) \to 1$ (i.e., there are no adjustment costs). In the latter case, the constraint on external financing is the only force that keeps the growth rate of the firm from being infinite, as in Cagetti and De Nardi (2006) and Moll (2014).

APPENDIX C: APPENDIX FOR SECTION 4

C.1. *Estimating the Sufficient Statistic*

C.1.1. *Methodology*

We use annual data from Compustat (SP Global Market Intelligence (2023)) to estimate the equity payout yield and market leverage of the firms owned by top individuals in the U.S.

Equity Payout Yield. We start by showing that the equity payout yield can be written as the sum of a dividend yield and a buyback yield. Denote by $CF_t\,\mathrm{d}t$ the amount of cash distributed to equity holders during a time period $\mathrm{d}t$. This cash can be distributed through dividends or through share repurchases. Denoting D_t the dividend per share, P_t the price per share, and N_t the number of outstanding shares, the following accounting identity holds:

$$CF_t\,\mathrm{d}t = N_t D_t\,\mathrm{d}t - P_t\,\mathrm{d}N_t.$$

Dividing by the market value of the firm equity $N_t P_t$, we obtain

$$\underbrace{\frac{CF_t}{N_t P_t}\mathrm{d}t}_{\text{Equity payout yield}} = \underbrace{\frac{D_t}{P_t}\mathrm{d}t}_{\text{Dividend yield}} \underbrace{-\frac{\mathrm{d}N_t}{N_t}}_{\text{Buyback yield}}.$$

This says that the equity payout yield is the sum of the dividend yield and the buyback yield, where the buyback yield is defined as the opposite of the growth of the number of shares. Note that the buyback yield is positive when firms repurchase shares and negative when firms issue shares.

We first describe the construction of the dividend yield. In years in which a company is public, we observe in Compustat the amount of dividends during the year, $\texttt{dv}_t$, and the market value of equity at the end of the year $\texttt{mkvalt}_t$ (or the number of common shares outstanding $\texttt{csho}_t$ times the price per share $\texttt{prcc\_f}_t$ if it is missing) in Compustat. We then construct the dividend yield during the year as

$$\text{Dividend yield}_t = \frac{\texttt{dv}_t}{(\texttt{mkvalt}_{t-1} + \texttt{mkvalt}_t)/2}.$$

We winsorize this variable at the 1rst and 99th percentile to decreases the effect of measurement errors. Finally, we set the dividend yield to zero in years in which a company is private.

We now describe the construction of the buyback yield in years in which a company is public. We observe in Compustat the number of common shares outstanding $\texttt{chso}_t \times \texttt{adj\_f}_t$, where the adjustment accounts for the effect of stock splits.[54] We construct the buyback issuance yield in years in which a company is public as the opposite of the logarithmic growth in its number of common shares outstanding.

Our baseline model assumes that firms redistribute to equity holders the component of earnings that they do not invest in their production technology. In reality, firms may also use this cash to acquire other firms. To account for these additional financial transactions, we adjust our measure of buyback yield by the net cash used for acquisition, $\texttt{acq}_t - \texttt{sppe}_t$, divided by firm market equity. Intuitively, this means that we treat similarly a firm repurchasing its own shares and a firm purchasing the shares of another firm's share. Overall, our final measure for the buyback yield in years post-IPO is

$$\text{Buyback yield}_t = \log\left(\frac{\texttt{chso}_{t-1} \times \texttt{adj\_f}_{t-1}}{\texttt{chso}_t \times \texttt{adj\_f}_t}\right) + \frac{\texttt{acq} - \texttt{sppe}}{(\texttt{mkvalt}_{t-1} + \texttt{mkvalt}_t)/2}.$$

We winsorize this variable at the 1rst and 99th percentile to decreases the effect of measurement errors.

We estimate the buyback yield in years leading (and including) the IPO using hand-collected data. We rely on the ownership share of founders (immediately post-IPO) as reported on their S-1 forms, denoted Ω. Assuming that founders neither sold or purchased shares or received shares as part of their compensation, this ownership share corresponds to the inverse of the cumulative growth in the number of shares since founding date.[55] Overall, our measure for the buyback yield in years leading (and including) the IPO is

$$\text{Buyback yield}_t = \frac{\log(\Omega)}{t_{\text{IPO}} - t_0},$$

where t_{IPO} denotes the year of the IPO and t_0 denotes the year in which the firm was incorporated. Finally, we set the buyback yield of firms that are private in 2015 to zero.

Figure C.1 plots the average annual equity payout yield of firms that are public in 2015 as a function of their age. One can see that the equity payout yield gradually increases, from -10% in early years to 5% in later years. This reflects the fact that, similar to the trees in the stylized model, firms initially raise cash from equity holders (through equity issuance) and then start paying positive cash flows as they age (through dividend payments and/or equity repurchases).

[54]See Fama and French (2001) or Boudoukh, Michaely, Richardson, and Roberts (2007) for similar measurements of the buyback yield post-IPO.

[55]Note that this assumption leads us to underestimate (in magnitude) the buyback yield in case founders purchased or received shares before the IPO.

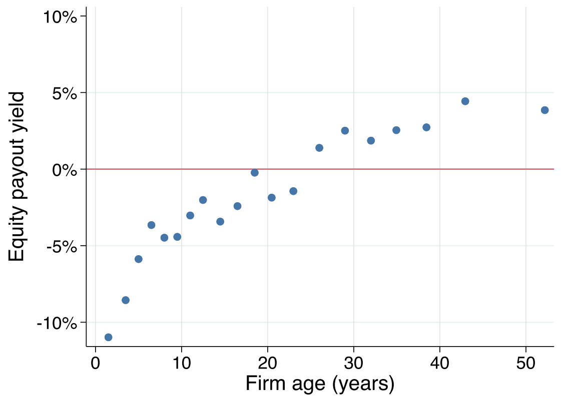


FIGURE C.1.—Annual equity payout yield as a function of firm age. *Notes.* This figure plots the average annual equity payout yield in 20 bins of firm age for the set of firms that are public in 2015.

Market Leverage. In years in which a company is public, we compute market leverage as the ratio between the market value of assets and the market value of equity. The market value of assets is computed as the market value of equity plus the value of debt, where the value of debt is constructed using Compustat asset `at` minus cash `che` minus shareholder equity `seq`. Overall, our measure of market leverage in years in which a firm is public is

$$\text{Market leverage} = \frac{\texttt{mkvalt} + (\texttt{at} - \texttt{che} - \texttt{seq})}{\texttt{mkvalt}}.$$

We winsorize this variable at the 1rst and 99th percentile to decrease the effect of measurement errors. We then construct the market leverage of firms before the IPO as their market leverage in the year following the IPO. Finally, we set the market leverage of firms that are private in 2015 as the average market leverage of firms that are public in 2015.

Growth Rate of Wealth. We measure normalized wealth in 2015, W_{2015}, by dividing individual wealth reported in the 2015 Forbes list by the aggregate household net worth of households from the Financial Accounts of the United States in 2015 (Board of Governors of the Federal Reserve System (US) (2023b)), divided by the number of households from the Census (U.S. Census Bureau (2023)). We then compute the lifetime average growth rate of each entrepreneur using equation (20).

C.1.2. *Sensitivity Analysis and Sampling Uncertainty*

We now assess the robustness of our estimated sufficient statistic along two dimensions.

Sensitivity Analysis. Our individual statistics (i.e., lifetime average equity payout yield, market leverage, and growth rate of wealth) might be measured with a bias. To assess the sensitivity of our estimated sufficient statistic to these potential biases, we reestimate the sufficient statistic after shifting uniformly all individual statistics by a given number. We report the results in Table C.I, with bootstrapped 95% confidence intervals.

The first two rows reports the sufficient statistic after shifting the equity payout yield of all individuals in our sample by ± 0.5 pp. The third and fourth rows report the sufficient statistic after shifting the market leverage by ± 0.1. Unsurprisingly, decreasing the equity

TABLE C.I
SENSITIVITY ANALYSIS FOR $\widehat{\partial_r \log \theta}$.

	Estimate	95% Confidence interval	
		Lower bound	Upper bound
Equity payout yield −0.5pp	−5.0	−6.3	−4.0
Equity payout yield +0.5pp	−3.4	−4.6	−2.5
Market leverage −0.1	−3.7	−4.8	−2.8
Market leverage +0.1	−4.8	−6.0	−3.8
Growth rate of wealth −5pp	−5.0	−6.4	−3.9
Growth rate of wealth +5pp	−3.7	−4.7	−2.8
Growth rate of wealth assuming initial wealth $W_{t_0} = 10$	−5.1	−6.5	−3.9
Growth rate of wealth assuming initial wealth $W_{t_0} = 0.1$	−3.6	−4.7	−2.8

payout yield or increasing the market leverage both tend to decrease the sufficient statistic $\partial_r \log \theta$, as they correspond to an increased reliance on external sources of financing.

Finally, the fifth and sixth rows report the sufficient statistic after shifting the growth rate of wealth by ±5 pp. Alternatively, we explore the sensitivity of our estimated sufficient statistic to the lifetime average growth rate of wealth by setting the entrepreneur's wealth the year of incorporation, W_0, to ten times more or less than the average wealth in the economy in (20) (as opposed to one in the baseline). We find that this does not change the sufficient statistic much. The reason is that the terminal wealth of the individuals in our sample is so high that small changes in their initial wealth do not matter much for their lifetime average growth rate.

Note that, in all specifications, the lower bound of the confidence interval remains well below zero, which suggests that the sign of the sufficient statistic (if not its magnitude) are robust to potential biases and sampling uncertainty.

Alternative Estimator. For the sake of simplicity, we estimated the sufficient statistic in the main text as a ratio of two averages: the average effect of required returns on the growth rate of wealth, in the numerator, and the average growth rate of wealth, in the denominator (see equation (19)). In the presence of firm heterogeneity, however, theory instructs us to compute the sufficient statistic as the average of a ratio computed at the individual level (see equation (12)). To examine the difference between these two methods, we now consider an alternative estimator for our sufficient statistic:

$$\widehat{\partial_r \log \theta}_{\text{alt}} = \frac{1}{N}\sum_{i=1}^{N} \frac{1 + \text{Market leverage}_{i,T} \times (\text{Equity payout yield}_{i,T} \times \text{Duration} - 1)}{\text{Growth rate}_{i,T}}; \quad (49)$$

that is, an average of ratios rather than a ratio of averages. We report this estimate, as well as the bootstrapped 95% confidence interval, in Table C.II. We find that this alternative estimate is very close to the original one.

TABLE C.II

ESTIMATES FOR $\widehat{\partial_r \log \theta}_{\text{ALT}}$.

	Estimate	95% Confidence interval	
		Lower bound	Upper bound
Duration = 35 years (baseline)	−4.1	−5.4	−3.1
Duration = 20 years	−4.0	−5.2	−3.1
Duration = 50 years	−4.2	−5.6	−3.1

Note: The alternative sufficient statistic is constructed using equation (49). The 95% confidence interval is constructed as a percentile bootstrap confidence interval using 1000 replications. Data are from Forbes, Compustat, and S-1 filings.

C.2. *Estimating Required Returns*

C.2.1. *The Required Return on Business Liabilities*

We now describe our methodology to estimate the required return on business liabilities (i.e., corporate equities and debts). We use publicly available annual data from the *Integrated Macroeconomic Accounts* (Bureau of Economic Activity (2023)), which combines sectoral data on income and expenditure from the National Accounts with data on financial transactions and holdings from the Financial Accounts. We focus on the corporate nonfinancial sector (i.e., Table S5) and deflate all variables using the Consumer Price Index for All Urban Consumers (U.S. Bureau of Labor Statistics (2023)).

Return Definition. Consider the return associated with a trading strategy that consists of holding all liabilities issued by the corporate sectors and purchasing all new issuances in every year. The realized return of owning the corporate sector between year t and $t+1$ is given by

$$r_{\text{corp},t+1} = \frac{\text{net operating surplus}_{t+1} - \text{net capital formation}_{t+1}}{\text{net liabilities}_t} + \frac{\text{net liabilities}_{t+1} - \text{net liabilities}_t}{\text{net liabilities}_t}. \tag{50}$$

Net operating surplus (line item 8) is a measure of net corporate profit (i.e., value-added minus worker compensation and capital depreciation). Net capital formation (line 28) measures capital formation (which includes investments in real estate, equipment, and intellectual property products) net of depreciation. Net operating surplus minus net capital formation thus accounts for all of the cash flows generated by the corporate sector. Finally, net liabilities measures the market value of debts and equities issued by the corporate sector minus the financial assets held by the corporate sector.

The first term in (50) corresponds to the payout yield. Corporate cash flows can be used to pay interests, dividends, stock buybacks, or debt repurchases. Given the trading strategy that we consider, all of these uses of corporate cash flows have the same economic implication: they represent flows of cash from corporations to households (see Abel, Mankiw, Summers, and Zeckhauser (1989) for an early discussion of this idea). The second term accounts for the contribution of the growth in the market value of liabilities.

To map (50) more closely to the model, we define the following variables:

$$\text{rok}_{t+1} \equiv \frac{\text{net operating surplus}_{t+1}}{\text{capital}_t}, \quad \text{(Return on capital)}$$

$$g_{t+1} \equiv \frac{\text{net capital formation}_{t+1}}{\text{capital}_t}, \quad \text{(Net capital formation rate)}$$

$$Q_t \equiv \frac{\text{net liabilities}_t}{\text{capital}_t}. \quad \text{(Tobin's Q)}$$

Given these definitions, the realized return defined in (50) can thus be rewritten as

$$\begin{aligned} r_{\text{corp},t+1} &= \frac{\text{rok}_{t+1} - g_{t+1}}{Q_t} + \frac{q_{t+1} \times \text{capital}_{t+1} - Q_t \times \text{capital}_t}{Q_t \times \text{capital}_t} \\ &= \frac{\text{rok}_{t+1} - g_{t+1}}{Q_t} + g_{t+1} \\ &\quad + \underbrace{\frac{\text{capital}_{t+1} - (1+g_{t+1})\text{capital}_t}{\text{capital}_t} + \frac{\text{capital}_{t+1}}{\text{capital}_t} \times \frac{Q_{t+1} - Q_t}{Q_t}}_{\text{revaluation gain}}. \end{aligned}$$

We call the sum of the last two terms the "revaluation gains." This term combines the growth in the replacement value of capital and the growth in Tobin's Q (revaluation of net financial assets liabilities relative to the replacement value of capital). In practice, the first term in this sum is mainly driven by the revaluation of real estate prices (real estate capital is reported using market values), and it averages to roughly zero in our sample.

Required Returns. To obtain a measure of expected returns, we make two assumptions. First, the investment rate and the return on capital are known one period in advance (i.e., $\mathbb{E}_t[g_{t+1}] = g_{t+1}$ and $\mathbb{E}_t[\text{rok}_{t+1}] = \text{rok}_{t+1}$). Second, expected revaluation gains are zero. See Campbell (2017) Chapter 5.5.2 for an analogous set of assumptions in the context of stock market returns. Combining the definition of realized returns (50) with these two assumptions, we obtain that expected returns can be written as

$$\mathbb{E}_t[r_{\text{corp},t+1}] = \frac{\text{rok}_{t+1} - g_{t+1}}{Q_t} + g_{t+1}, \tag{51}$$

which is directly observable. From now on, we refer to $\mathbb{E}_t[r_{\text{corp},t+1}]$ as the *required* return on wealth. The idea is that, the value of net liabilities Q_t is such that the expected return on investor's wealth is equal to their required return.

Aggregate Per Capita Growth. What matters in our sufficient statistic approach is the decline in *r net of aggregate growth per capita*. One simple method is to deflate our measure of required returns by the growth rate of capital per capita. This deflated measure of required returns simply corresponds to the payout yield, $(\text{rok}_{t+1} - g_{t+1})/Q_t$, plus the rate of population growth (Board of Governors of the Federal Reserve System (US) (2023c)). A second method is to deflate our measure of returns by TFP growth, as constructed in Feenstra, Inklaar, and Timmer (2015) (University of Groningen and University of California, Davis (2023)).

Results. We plot the our required returns on wealth series in Figure C.2. The key observation is that, for both deflators, there is a substantial decline in the required return net of per capita growth.

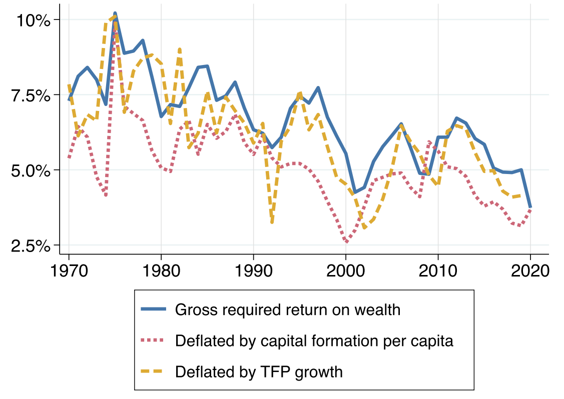


FIGURE C.2.—Required return on business liabilities net of aggregate growth. *Notes.* Data are from Bureau of Economic Activity (2023), Board of Governors of the Federal Reserve System (US) (2023c), U.S. Bureau of Labor Statistics (2023), and University of Groningen and University of California, Davis (2023).

Table C.III contains summary statistics on the return on capital, the required return on wealth (deflated or not), and Tobin's Q from 1985 through 2015. Computing the decline as the change in the average in 2015–2020 compared to the average in 1980–1985, we obtain that the change in the required return on corporate sector liabilities is −2.7 pp. Deflating by the growth rate of capital per capita gives a change of −2.3 pp. To be conservative, we use a change in required returns of −2 pp. in the main text.

C.2.2. *Required Returns on Corporate Liabilities Versus Required Returns on Corporate Debt*

To implement our sufficient statistic, we have assumed that the change in the required return on corporate liabilities $\mathrm{d}r$ was equal to the change in the required return on corporate debt $\mathrm{d}r_f$. To test this assumption, we now separately estimate the change in the required return on corporate debt. We then discuss the effect of this estimate on the change in Pareto inequality due to the change in required returns.

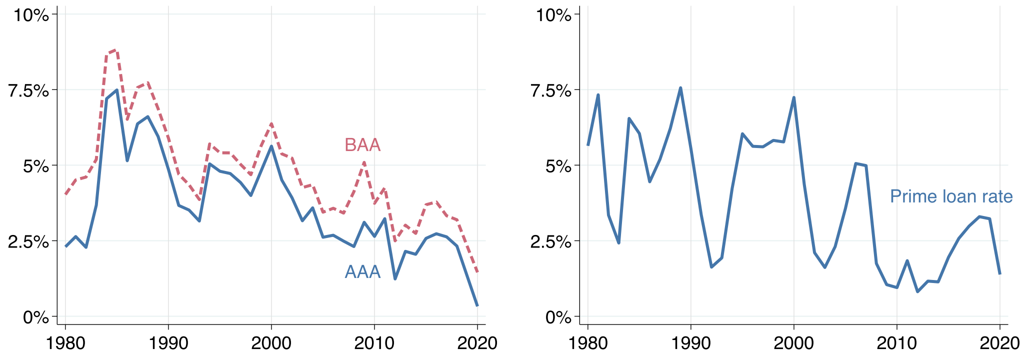
Debt issued by the corporate sector can take the form of bonds or bank loans. Assuming away the probability of default does not change over time, we can directly estimate this

TABLE C.III

REQUIRED RETURNS AND VALUATIONS OF THE U.S. CORPORATE SECTOR.

Moment	1980–1985	1980–2020	2015–2020
Return on capital (%)	6.5	7.6	7.6
Required return on wealth (%)	7.6	6.3	4.9
Required return on wealth net of capital growth p.c. (%)	5.8	4.8	3.6
Required return on wealth net of TFP growth (%)	7.3	5.7	4.5
Tobin's q	78.9	148.2	198.4

Note: This table reports moments for the U.S. nonfinancial corporate sector. The construction of each variable is detailed in Appendix C.2. Data are from the Bureau of Economic Activity (2023), FRED, and Feenstra, Inklaar, and Timmer (2015).



(a) Corporate bond yields (b) Bank lending rate

FIGURE C.3.—Required returns on corporate debt. *Notes.* Panel (a) plots the evolution of U.S. corporate bond yields by Moody's ratings. Panel (b) plots the evolution of the bank lending rate. Both are in real terms. Data are from Moody's (2023a), Moody's (2023b), and Board of Governors of the Federal Reserve System (US) (2023a).

required return from the interest rate paid by the corporate sector.[56] Figure C.3 plots the required returns on debt, using Moody's data on corporate bond yields for firms rated AAA and BAA (Moody's (2023a) and Moody's (2023b)) and the bank lending rate (Board of Governors of the Federal Reserve System (US) (2023a)). We deflate these required returns using a lagged 3-years average inflation (U.S. Bureau of Labor Statistics (2023)) in order to obtain real returns. We find that both rates have declined substantially over time.

Following Barkai (2020), we construct the required return on corporate debt by averaging the two series, with weights given by the relative quantity of each type of debt according to the *Integrated Macroeconomic Accounts*. Similar to the case of the required return on all corporate liabilities, what matters is the interest rate on debt relative to the growth rate of the economy. Figure C.4 plots the resulting interest rate using the same de-

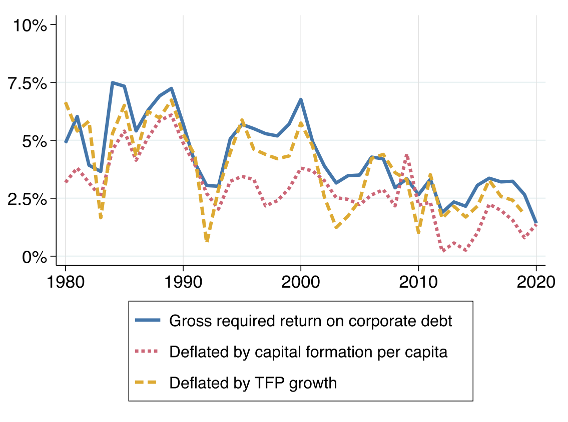


FIGURE C.4.—Required returns on corporate debt net of aggregate growth.

[56]More precisely, since we only need to estimate the decline in required returns, we only need to assume that the probability of default does not change over time.

TABLE C.IV

REQUIRED RETURNS ON CORPORATE DEBT.

Moment	1980–1985	1980–2020	2015–2020
Interest on corporate debt (%)	5.6	4.3	2.8
Interest on corporate debt net of capital growth p.c. (%)	3.8	2.9	1.5
Interest on corporate debt net of TFP growth (%)	5.2	3.8	2.4

flators as in Figure C.2, while Table C.IV reports the average of interest rates in different periods.

We obtain that the change in the real interest rate paid by firms −2.7 pp. Deflating by the growth rate of capital per capita gives a change of −2.3 pp., which is the same as the change in the required return on all corporate liabilities. This justifies our approach of considering a homogeneous decline in the rate of return across all securities.

C.3. *Estimating Pareto Inequality*

We now describe how we estimate Pareto inequality in the data. First, we use data from Smith, Zidar, and Zwick (2023), who use an improved version of the capitalization method developed in Saez and Zucman (2016) to construct wealth share estimates in the U.S. Relative to Saez and Zucman (2016), the methodology allows for more granular return heterogeneity.[57] Using this data, we construct three alternative estimates of Pareto inequality using the "top share estimator" defined in equation (22), with $p = 0.001\%$, $p = 0.01\%$, and $p = 0.1\%$.

As a robustness check, we also construct two alternative sets of estimates for Pareto inequality using Forbes data on the wealthiest 400 individuals.[58] We only use data on their rank in the list and on their stated wealth. First, we use the log-rank estimator proposed by Gabaix and Ibragimov (2011). The idea is to estimate a cross-sectional regression of log wealth on the log rank minus 1/2 and use the slope of this regression as an estimate of the Pareto exponent of the wealth distribution. To get an estimate of Pareto inequality, we simply take the inverse of this coefficient. Second, following Saez (2001), we use the mean-min estimator $\theta = 1 - \mathbb{E}[W|W > \underline{W}]/\underline{W}$, where $\mathbb{E}[W|W > \underline{W}]$ is the average wealth of households in the Forbes 400 list and $\underline{W}$ is the wealth of the last household in list.

Table C.V contains the beginning, average, and end value of the Pareto inequality estimates over our time period of interest (i.e., 1980 to 2020). Taking the log difference between the average value in the last 5 years of our sample minus the average value in the first 5 years of our sample, and averaging across the different measures of Pareto inequality, gives an estimate for the rise in Pareto inequality of 22 log points.

[57]The authors summarize the influence of return heterogeneity on estimated top wealth shares as follows: "In terms of top portfolios, we find that accounting for estimated return heterogeneity makes a difference. First, relative to an equal returns approach, we find a larger role for pass-through business wealth and a smaller role for fixed income wealth. Second, the fixed income portfolio share falls and the equity share rises with wealth at the top. Pass-through business and C-corporation equity wealth are the primary sources of wealth at the top. At the very top, C-corporation equity is the largest component, accounting for 53% of top 0.001% wealth, and pass-through business accounts for 22%. In contrast, pensions and housing account for almost all wealth of the bottom 90%. Third, we find that the fixed-income portfolio share at the very top remained relatively stable, whereas under equal returns, the fixed-income portfolio share increased substantially since 2000.".

[58]We use data from Gomez (2023).

TABLE C.V

ESTIMATES OF PARETO INEQUALITY.

Estimate	1980–1985	1980–2020	2015–2020
Ratio top shares (1%–0.1%)	0.59	0.64	0.67
Ratio top shares (0.1%–0.01%)	0.53	0.62	0.66
Ratio top shares (0.01%–0.001%)	0.47	0.59	0.65
Mean-min (Forbes 400)	0.58	0.67	0.71
Log-rank (Forbes 400)	0.57	0.69	0.73

Note: The table reports estimates of Pareto inequality using, successively, the log ratio between the top 0.1% and the top 1%, the log ratio between the top 0.1% and the top 0.01%, the log ratio between the top 0.001% and the top 0.1%, the log ratio between the average wealth in Forbes 400 and the wealth of the last person in Forbes 400, and the slope estimate in a regression of log rank minus 1/2 on log wealth. Data from Smith, Owen, and Eric (2022) and Forbes.

C.4. *Evidence Beyond the Top 100*

Private Businesses

We now use data from the 2016 wave of the *Survey of Consumer Finances* (SCF) from Federal Reserve Board (2016) to quantify the prevalence of entrepreneurs at the top of the wealth distributions (i.e., individuals who founded or acquired a business that they actively manage). Table C.VI presents summary statistics. First, notice that entrepreneurs are overrepresented at the top. As in Cagetti and De Nardi (2006), we find that wealthier individuals are much more likely to be entrepreneurs. In the full population, 11% of individuals are entrepreneurs while in the top 0.01%, the fraction increases to 66%. Second, the businesses founded by wealthy individuals tend to be pass-through entities, which is consistent with the evidence in Cooper et al. (2016). For instance, 93% of businesses owned by households in the top 0.01% are partnerships or S corporations. This is in a sharp contrast with the fact that roughly two-thirds of entrepreneurs in the top 100 own public firms (i.e., C corporations).

The effect of required returns on wealth inequality in our model depends on the extent to which these businesses rely on external financing (through either equity or debt financing). Due to data limitations, we are unable to produce estimates of the equity issuance and leverage of the firms owned by entrepreneurs in the top 1% in the U.S. However, we now present evidence from Kochen (2022) that firms in high-income countries frequently use external financing.

Kochen (2022) harmonizes data for 11 high-income countries (i.e., Austria, Belgium, Denmark, Finland, France, Germany, Italy, Norway, Spain, Sweden, and the United Kingdom) over the 1996–2018 period using the Orbis database. The data set contains firm-level

TABLE C.VI

ENTREPRENEURS IN THE TOP PERCENTILES (SCF, 2016).

Top percentile groups	Total	Top 1%	Top 0.1%	Top 0.01%
Entrepreneurs	0.11	0.43	0.59	0.66
Sole proprietorship	0.48	0.09	0.06	0.02
Partnership	0.35	0.60	0.64	0.63
S corporation	0.11	0.21	0.24	0.30
Other corporations	0.06	0.11	0.06	0.05

TABLE C.VII

DEBT AND EQUITY FINANCING (ORBIS).

Variable	Mean
Leverage	1.5
Frequency of equity issuance	0.08
Size of equity issuance	0.18

Note: "Leverage" represents the ratio of total asset (i.e., debt plus book equity) divided by the book value of equity, "Frequency of equity issuance" represents the share of firms that have issued equity in the current year; "Size of equity issuance" represents the ratio of equity issuance to capital, conditional on equity issuance being positive. All averages are weighted by capital and obtained in Appendix Table A.3. in Kochen (2022). While the paper reports the average debt-to-capital ratio, we transform this into an estimate of the average book leverage (capital-to-book-equity ratio) as $1/(1 -$ debt-to-capital ratio) Similarly, we report the size of equity issuances as a share of book equity while the paper reports it as a share of capital.

data on millions of companies, most of which are private. Table C.VII summarizes the importance of debt and equity financing. First, notice that firms use a substantial amount of leverage, which amounts (on average) to 1.5 using book values. This is somewhat higher than what we find among public firms in our sample (see Table II). Regarding equity issuances, on average 8% of firms issue equity in a given year and, conditional on doing an equity issuance, it amounts to roughly 18% of book equity. Putting together, this represents a roughly $8\% \times 18\% \approx 1.5\%$ annual net equity issuance yield (or a -1.5% buyback yield), which is roughly half as much as the firms in our sample (see Table II).

How would the sufficient statistic change after incorporating these entrepreneurs? On the one hand, the fact that they use less equity financing will tend to decrease the numerator in the sufficient statistic (19). On the other hand, the fact that their lifetime average growth rate is (most likely) smaller than the entrepreneurs in Forbes (i.e., most of them did not become billionaires) will tend to decrease the denominator in the sufficient statistic. As a result, the overall effect of incorporating these less successful entrepreneurs is ambiguous (see Appendix C.1.2 for the sensitivity of the sufficient statistic to the equity issuance and growth rate of entrepreneurs).

Venture Capital Backed Firms

Firms backed by venture capitalists (henceforth VCs) are an important part of the U.S. economy. According to Capshare, 10,400 companies received venture funding in 2018.[59] On average, the ownership share of the founders decreases by roughly 25% every funding round. Since funding rounds tend to happen every 18 months, this corresponds to an annual dilution rate of 16% (i.e., an equity payout yield of -16%, assuming that no dividends are paid out).

Another way to obtain a measure of the average dilution rate is to divide the total equity raised by firms funded by VC to their total market capitalization. Pitchbook estimates that VC-backed companies have a combined market capitalization of around $3 trillion in 2022 and that they collectively raised $130 billion that year.[60] Combining these two figures give a (market-capitalization weighted) dilution rate of 4.5% (i.e., an equity payout yield of -4.5%, assuming that no dividends are paid out). The fact that this estimate is lower than the previous figure reflects the fact that the largest dilution happens in early rounds, that is, in companies with smaller market capitalizations.

[59]Capshare is a "web application that helps businesses manage their stock and assets on one organized platform." Our statistics are taken from their "2018 Private Company Equity Statistics Report.".

[60]These two statistics are taken from their "2022 Quantitative Perspectives: US Market Insights.".

While the number of VC firms is small relative to the number of households, it is worth noting that many key employees of these firms receive a substantial proportion of their income in the form equity (i.e., equity grants, stock options, etc.). Equity compensation typically leads to concentrated portfolios due to a mix of vesting time, other restrictions on stock sales, and illiquidity (especially pre-IPO). Our notion of "entrepreneurs" in the model can be interpreted as including not only the founder of a firm, but also any individual who invests the majority of their wealth in the firm. In particular, it also includes employees that receive a substantial proportion of their income as equity. Eisfeldt, Falato, and Xiaolan (2019) reports that equity compensation represents almost 45% of total compensation to high-skilled labor in recent years and that employees working in VC-backed firms account for approximately 2% of the workforce. Despite the lack of data on the portfolio of such "human capitalists," we think that many wealthy, high-skilled employees have portfolios with concentrated holdings. We expect these concentrated holdings to be particularly important for firms that are net equity issuers.

APPENDIX D: APPENDIX FOR SECTION 5

D.1. *Characterization of Equilibrium*

Firm Policy Functions

The optimal labor and investment satisfies

$$w_t = (1-\alpha)(K_t/L_t)^\alpha,$$
$$q_{s,t} = 1 + \chi(g_{s,t} - \underline{g}_s),$$

where $q_s \equiv V_s(K)/K$. Notice that while firms $s = 1, 2$ choose different growth rates, they choose the same capital to output ratio. This is because their production technology is identical. The solution is

$$g_{s,t} = \underline{g}_s + \frac{1}{\chi}(q_{s,t} - 1),$$
$$L_{s,t} = (1-\alpha)^{-\frac{1}{\alpha}} w_t^{-\frac{1}{\alpha}} K_{s,t}.$$

Firm Valuations

Using the optimal policy functions and the definition $\text{MPK} \equiv F_K(K, L)$, we have

$$0 = \left(\text{MPK}_t - r_t + \tau(\psi q_{1,t} - 1) - (r_t + \tau - \underline{g}_0)(q_{0,t} - 1) + \frac{1}{2\chi}(q_{0,t} - 1)^2\right) \mathrm{d}t + \mathbb{E}_t[\mathrm{d}q_{0,t}], \tag{52}$$

$$0 = \left(\text{MPK}_t - r_t - (r_t - \underline{g}_1)(q_{1,t} - 1) + \frac{1}{2\chi}(q_{1,t} - 1)^2\right) \mathrm{d}t + \mathbb{E}_t[\mathrm{d}q_{1,t}], \tag{53}$$

Along a balanced growth path (i.e., $\text{MPK}_t = \text{MPK}$, $r_t = r$), we have

$$0 = \text{MPK} - r + \tau(\psi q_1 - 1) - (r + \tau - \underline{g}_0)(q_0 - 1) + \frac{1}{2\chi}(q_0 - 1)^2,$$

$$0 = \text{MPK} - r - (r - \underline{g}_1)(q_1 - 1) + \frac{1}{2\chi}(q_1 - 1)^2.$$

Implications of Labor Market Clearing

From the first-order condition for labor, we have that the capital to labor ratio is the same at both types of firms. Using the labor market clearing condition (i.e., $L_{0,t} + L_{1,t} = 1 - \pi$), the equilibrium wage and MPK must be

$$w_t = (1-\alpha)\left(\frac{K_t}{1-\pi}\right)^{1-\alpha}, \qquad \text{MPK}_t = \alpha\left(\frac{K_t}{1-\pi}\right)^{\alpha-1}.$$

Law of Motion for Capital

The law of motion for detrended capital by firm type is

$$\mathrm{d}K_{0,t} = (g_{0,t} - \tau - \eta)K_{0,t}\,\mathrm{d}t + \eta\pi\overline{K}\,\mathrm{d}t, \qquad \mathrm{d}K_{1,t} = (g_{1,t} - \eta)K_{1,t}\,\mathrm{d}t + \tau\psi K_{0,t}\,\mathrm{d}t.$$

In steady state, we have

$$K_0 = \frac{\eta}{\eta + \tau - g_0}\pi\overline{K}, \qquad K_1 = \frac{\tau\psi}{\eta - g_1}K_0, \qquad K = \frac{\eta - g_1 + \tau\psi}{\eta - g_1}\frac{\eta}{\eta + \tau - g_0}\pi\overline{K},$$

where K is aggregate capital and g_s is the steady-state growth rate of the firm of each type, that is, $g_s = \underline{g}_s + \frac{1}{\chi}(q_s - 1)$.

Duration of Aggregate Wealth

In steady state, the (aggregate) Tobin's Q is defined as the capital-weighted average of individuals qs:

$$Q = \frac{\eta - g_1}{\eta - g_1 + \tau\psi}q_0 + \left(1 - \frac{\eta - g_1}{\eta - g_1 + \tau\psi}\right)q_1.$$

In particular, the duration of aggregate wealth is given by

$$D \equiv -\frac{\partial_r Q}{Q} = -\frac{\partial_r(q_0K_0 + q_1K_1)}{QK} = \frac{q_0K_0}{QK}D_0 + \frac{q_1K_1}{QK}D_1,$$

where $D_s \equiv -\partial_r q_s/q_s$ denotes the duration of a firm in state s.

Entrepreneur Wealth

Let $\text{epy}_{s,t}$ be the equity payout yield of firm in state s at time t:

$$\text{epy}_{s,t} \equiv \frac{r_t - g_{s,t} + \lambda\big(\text{MPK}_t - \iota_s(g_{s,t}) - (r_t - g_{s,t})\big)}{q_{s,\lambda,t}},$$

where $q_{s,\lambda,t} \equiv 1 + \lambda(q_{s,t} - 1)$. Denoting by T the (random) time at which the firm matures. Assuming that $1 + \lambda(\psi q_{1,t} - 1) > 0$ (this ensures that the entrepreneur does not default when transitioning from a growth to a mature firm, and it will be satisfied in our calibration), the wealth of an entrepreneur evolves according to

$$\frac{\mathrm{d}W_t}{W_t} = \begin{cases} (\mathrm{epy}_{0,t} + g_{0,t} - \rho)\,\mathrm{d}t + \dfrac{\mathrm{d}q_{0,\lambda,t}}{q_{0,\lambda,t}} & \text{if } t < T, \\ \dfrac{1 + \lambda(\psi q_{1,t} - 1)}{1 + \lambda(q_{0,t} - 1)} - 1 & \text{if } t = T, \\ (\mathrm{epy}_{1,t} + g_{1,t} - \rho)\,\mathrm{d}t + \dfrac{\mathrm{d}q_{1,\lambda,t}}{q_{1,\lambda,t}} & \text{if } t > T. \end{cases} \tag{54}$$

Denote by $W_{E,s,t}$ the detrended total wealth of entrepreneurs owning firms in state $s \in \{0, 1\}$. Its law of motion is given by

$$\mathrm{d}W_{E,0,t} = \left((\mathrm{epy}_{0,t} + g_{0,t} - \rho - \tau - \eta)\,\mathrm{d}t + \frac{\mathrm{d}q_{0,\lambda,t}}{q_{0,\lambda,t}} \right) W_{E,0,t} + \eta\pi \frac{\overline{K} q_{0,\lambda,t}}{\lambda}\,\mathrm{d}t,$$

$$\mathrm{d}W_{E,1,t} = \left((\mathrm{epy}_{1,t} + g_{1,t} - \rho - \eta)\,\mathrm{d}t + \frac{\mathrm{d}q_{1,\lambda,t}}{q_{1,\lambda,t}} \right) W_{E,1,t} + \tau \frac{1 + \lambda(\psi q_{1,t} - 1)}{1 + \lambda(q_{0,t} - 1)} W_{E,0,t}\,\mathrm{d}t.$$

Mutual Fund Wealth

By Walras' law, labor and product market clearing implies financial market clearing. The mutual fund must therefore hold all wealth not held by entrepreneurs:

$$W_{M,t} = Q_t K_t - W_{E,0,t} - W_{E,1,t}.$$

Since the entrepreneurs own levered claims on firms (i.e., levered equity shares), it means that the mutual fund must hold debt. In a steady state, this is inconsequential, since all assets have the same return. But over a transition path, it means that the revaluation gains of the mutual fund will differ from those of entrepreneurs. We obtain the mutual fund's revaluation gains as a residual

$$\frac{\mathrm{d}q_{M,t}}{q_{M,t}} = \frac{Q_t K_t \dfrac{\mathrm{d}Q_t}{Q_t} - W_{E,0,t} \dfrac{\mathrm{d}q_{0,\lambda,t}}{q_{0,\lambda,t}} - W_{E,1,t} \dfrac{\mathrm{d}q_{1,\lambda,t}}{q_{1,\lambda,t}}}{Q_t K_t - W_{E,0,t} - W_{E,1,t}}.$$

Worker Wealth

Denote $W_{L,t}$ to be detrended worker wealth. Its law of motion is

$$\mathrm{d}W_{L,t} = \left((r_t - \rho_L - \eta)\,\mathrm{d}t + \frac{\mathrm{d}q_{M,t}}{q_{M,t}} - \mathbb{E}_t\left[\frac{\mathrm{d}q_{M,t}}{q_{M,t}} \right] \right) W_{L,t} + (1 - \pi) w_t\,\mathrm{d}t - \rho_L H_t\,\mathrm{d}t, \tag{55}$$

where $H_t \equiv \mathbb{E}_t[\int_0^\infty e^{-\int_0^s r_{t+h}\,\mathrm{d}h} w_{t+s}\,\mathrm{d}s]$ denotes the human wealth of a worker at time t.

Foreigner Wealth

The law of motion for detrended foreigner wealth is

$$\mathrm{d}W_{F,t} = S_{F,t}\,\mathrm{d}t + \left(\frac{\mathrm{d}q_{M,t}}{q_{M,t}} - \eta\,\mathrm{d}t \right) W_{F,t},$$

where $S_{F,t}$ is the flow of savings by foreigners.

Pareto Inequality

The formula for steady-state Pareto inequality is almost exactly as in the stylized model (see Section 2):

$$\theta = \max\left(\frac{\text{epy}_0 + g_0 - \rho}{\eta + \tau}, \frac{r - \rho}{\eta}\right). \tag{56}$$

The key difference is that, in the stylized model, $r - \rho$ corresponds to the return of rentiers (i.e., return on a diversified portfolio). Now, it corresponds to the return of holding a mature firm. Since the return of mature firms is deterministic, it must be r (both in expectation and ex post).

D.2. *Neoclassical Growth Model as a Limiting Case*

We now show that our model nests the neoclassical growth model as a special case where:
(1) Capital is fully elastic ($\chi = 0$) and there is no firm heterogeneity ($\psi = 0$);
(2) All agents are workers ($\pi = 1$) and there is no population renewal ($\eta = 0$).
For simplicity, we focus on a closed-economy steady-state equilibrium. Using the parameter restriction (1) and the firm valuation equations (52), we obtain

$$q_0 = 1, \qquad \text{MPK} - \tau = r.$$

In words, this says that there are no rents in equilibrium (i.e., the cost of capital r equals the net return on capital), which implies that Tobin's Q is one. Notice that the parameter τ now has the interpretation of a depreciation rate.

Using the parameter restriction (2), we have that existing agents own all of future wages and payments to capital, which means that their total wealth is $W = Y/r$, where $Y = K^\alpha L^{1-\alpha}$. Given the log utility assumption, their optimal consumption is $C = \rho_L Y/r$. Using the product market clearing condition (i.e., $C = Y$), we have that

$$r = \rho_L.$$

In words, this means that the required return is equal to the subjective discount factor.

Putting together, we obtain the steady-state allocation in the neoclassical growth model, where the net marginal product of capital is equal to the subjective discount factor (i.e., $\text{MPK} - \tau = \rho_L$). In the calibrated model, we relax (1) in order to generate a wedge between the return on capital and the cost of capital and relax (2) in order to have wealth inequality due to concentrated portfolios.

D.3. *Domestic Savings Glut*

In the baseIine model experiment, we generate an equilibrium decline in r by feeding in an exogenous rise in savings by foreigners. We now consider an asset-demand shock that originates domestically (i.e., a domestic savings glut). We do so by changing the subjective discount factor ρ of domestic agents. This captures, in a reduced-form way, a number of forces, such as rising longevity, that pushes up the desire to save. They key difference with the baseline model experiment is that a decline in ρ has a direct on top wealth inequality, a force that we now quantify.

TABLE D.I

MODEL EXPERIMENT WITH A DOMESTIC SAVINGS GLUT (LONG-RUN, PERCENTAGE POINTS).

Model	Δr	$\Delta \rho$	$\Delta \log \theta$
Baseline	−2.0	0.0	11
Domestic savings glut	−2.0	−1.5	16

To implement the model experiment, we consider a model extension where workers and entrepreneurs have time-varying subjective discount factors and where the flow of savings by foreigners is constant over time.

The model experiment consists of a steady-state comparative static where we shock the subjective discount factors of both workers and entrepreneurs by a common shifter ε. Other than that, all other model parameters are exactly as in the baseline calibration. The path of foreign savings is constant at some value S_F. We choose the value S_F as being equal to its value in the $r = 6\%$ (i.e., 1985–2015 average) steady state of the baseline model. That way, we match the fact that the NFA to domestic wealth ratio is 5% (i.e., a targeted moment in the baseline model calibration).

Table D.I reports the long-run change in the required return, the subjective discount factor, as well as the change in (log) Pareto inequality. Overall, we find that the rise in Pareto inequality is roughly 1.5 times larger than in the baseline model. To understand the forces at play, it is instructive to use an comparative statics formula for the change in Pareto inequality in response to an infinitesimal change in the subjective discount factor $\mathrm{d}\rho$ and the required return $\mathrm{d}r$: Totally differentiating the expression for Pareto inequality (56) in steady state, and assuming that we are in the entrepreneur regime, we have that

$$\mathrm{d}\log\theta = \partial_\rho \log\theta \,\mathrm{d}\rho + \partial_r \log\theta \,\mathrm{d}r$$

with

$$\partial_\rho \log\theta = -\frac{1}{\mathrm{epy}_0 + g_0 - \rho},$$

$$\partial_r \log\theta = \lambda \frac{q_0}{q_{0,\lambda}} \times \frac{\left(\mathrm{epy}_0 \times (-\partial_r \log q_0) - 1\right)}{\mathrm{epy}_0 + g_0 - \rho}.$$

Or, in words,

$$\partial_\rho \log\theta = \frac{-1}{\text{Growth rate of wealth}},$$

$$\partial_r \log\theta = \frac{1 + \text{Market leverage} \times (\text{Equity payout yield} \times \text{Duration} - 1)}{\text{Growth rate of wealth}}.$$

The formula expresses the change in Pareto inequality as a linear function of the change in the subjective discount factor ρ and in the required return r. In the model experiment, the required return r declines by 2 pp. while the subjective discount factors decline by roughly 1.5 pp.[61] However, the sensitivity of Pareto inequality to required returns is higher

[61]The reason why a 1.5 pp. decline in the subjective discount factors of both workers and entrepreneurs leads to a 2 pp. decline in the required return is that there is a reallocation of wealth toward entrepreneurs who have a lower subjective discount factor that workers (see Table IV).

than its sensitivity to the subjective discount factor. The reason is that a change in ρ moves the growth rate of wealth of wealth one-for-one for all entrepreneurs, while a change in r affects the growth rate of successful entrepreneurs more than one-for-one, due to the fact that these entrepreneurs use a lot of external financing and own high-duration firms (i.e., $1 +$ Market leverage $\times$ (Equity payout yield $\times$ Duration $- 1) > 1$, see Table IV).

D.4. *Quantifying the Intensive and Extensive Margins of Top Wealth Share Growth*

We now decompose the rise in top wealth shares in our model into an intensive and extensive margin. This allows us to measure the relative contribution of the growth rate of existing fortunes, as opposed to the inflow of new fortunes, in the rise in top wealth inequality.

Following Gomez (2023), we now decompose the growth rate of the share of aggregate wealth owned by a top percentile, at each time period, into an intensive and extensive term. The intensive term holds constant the composition of individuals in the top percentile over the period of time: it is defined as the wealth growth of individuals who are initially in the top percentile relative to the growth of the average wealth in the economy. In contrast, the extensive term, which is defined as a residual, accounts for all composition changes in the top percentile. More precisely, in our model, this extensive term is the sum of a positive force, that is, the flow of successful entrepreneurs in the top percentile (of type 0) who displace the less successful ones (of type 1), as well as a negative force, population growth.[62]

Figure D.1 plots the (annualized) growth of the top 0.1% wealth share in the baseline model experiment, as well as its decomposition into an intensive and an extensive margin, as discussed above. For the first 5 years, the rise in the intensive term explains most of the rise in the top wealth share. This is because the realized returns of individuals at the top (26) are high relative to the rest of the distribution. This comes from the fact that revaluation gains are particularly high for individuals at the top of the wealth distribution, who tend to own levered positions in high-duration firms. However, as realized returns start declining, the contribution of the intensive term declines. In fact, the intensive term

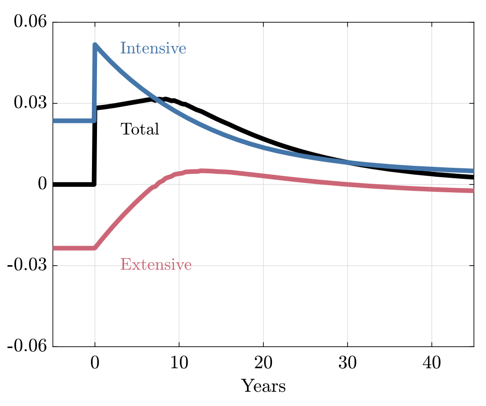


FIGURE D.1.—Decomposing the growth of the top 0.1% into an intensive and extensive term.

[62]Gomez (2023) further decomposes the extensive margin as the sum of a positive "between" term, which accounts for the dispersion in wealth growth within top individuals, and a negative "demography" term, which accounts for demographic changes such as a death and population growth.

is ultimately lower in the new steady state compared to the initial steady state, as the average return on wealth in this economy is now lower.

The rise in the top 0.1% wealth share is ultimately driven by a rise of the extensive term. This increase in the extensive term reflects the fact that, in a low-rate environment, the most successful entrepreneurs accumulate capital more quickly as they face a lower cost of capital. As shown in Figure D.1, this higher inflow of new fortunes in the top 0.1% more than compensates for the lower growth rate of existing fortunes. Overall, these results are consistent with evidence from Gomez (2023), Zheng (2019), and Atkeson and Irie (2022), who argue that an increase in the flow of new fortunes in top percentiles has played a substantial role in the recent rise in U.S. top wealth inequality.

Finally, this decomposition is useful to relate our theory to the central idea in Piketty and Zucman (2015), which is that Pareto inequality increases with "$r-g$" (i.e., the required return net of per capita growth). On the one hand, it is true that a decline in "$r-g$" leads to a decrease in the growth rate of existing fortunes relative to the economy, which tends to push down top wealth inequality. This is captured by the long-run decline in the intensive term in Figure D.1. However, what our decomposition shows is that the lower growth rate of existing fortunes in a low-rate environment is more than compensated by the larger inflow of new fortunes in the top percentiles (as the decline in the intensive term is more than compensated by the increase in the extensive term).

D.5. *Elastic Capital Calibrations*

Calibrations

Table D.II reports the model fit for the three elastic capital extensions (i.e., low-elasticity, medium-elasticity, and high-elasticity).

Evidence From Investment Regressions

In Section 5.5, we consider three alternative calibrations where we use the parameter χ—which governs the degree of investment adjustment costs–to match, respectively, a 0.5, 1, and 1.5 percentage point decline of the return on capital in the model experiment.

TABLE D.II

TARGETED MOMENTS (ELASTIC CAPITAL CALIBRATIONS).

Moment	Period	Data	Low	Medium	High
Conditional micro moments					
Equity payout yield	1985-2015	−0.022	−0.022	−0.022	−0.022
Growth rate of wealth	1985-2015	0.32	0.32	0.32	0.32
Market leverage	1985-2015	1.4	1.4	1.4	1.4
Duration	1985-2015	35	34	34	34
Macro moments					
Return on capital	1985	0.07	0.071	0.072	0.072
Depreciation rate	1985-2015	0.08	0.08	0.081	0.081
Pareto inequality	1985-2015	0.6	0.6	0.6	0.6
Aggregate duration	1985-2015	20	21	21	21
NFA to domestic wealth	1985-2015	−0.05	−0.05	−0.05	−0.05
Change in return on capital	1985-2015	0	−0.005	−0.01	−0.015

TABLE D.III

MODEL-IMPLIED REGRESSION COEFFICIENTS (PERCENTAGE POINTS).

Calibration	Δrok	$\Delta \log \theta$	$1/\chi$
Baseline	0.0	10.9	0
Low elasticity	−0.5	9.3	0.3
Medium elasticity	−1.0	7.9	0.6
High elasticity	−1.5	6.7	1
Very high elasticity	−4.0	3.5	3.6

Table D.III reports model objects for four calibrations of the model: the baseline calibration, the three elastic capital calibrations, as well as a "very high elasticity" calibration where we target a 4 percentage points decline of the return on capital.

The first column reports the targeted long-run decline of the return on capital (i.e., Δrok). As discussed earlier, we target values from 0 pp. (in the baseline model) to -4 pp. (in the very high elasticity calibration). The second column reports the long-run increase in (log) Pareto inequality (i.e., $\Delta \log \theta$). Notice that the rise in Pareto inequality is monotonically decreasing in the degree of capital elasticity. In the most aggressive calibration (i.e., the very high elasticity calibration), the rise is 4 log points, which is about a third of the rise in the baseline model.

The last column reports the inverse of the parameter $1/\chi$, which we will use to assess whether our calibrations are consistent with the existing empirical evidence on the sensitivity of firm-level investment to the cost of capital. Recall that, in the model, the following structural relationship holds:

$$g_{s,t} = \underline{g}_s + \frac{1}{\chi}(q_{s,t} - 1). \tag{57}$$

If the state s was observed, we could therefore consistently estimate $1/\chi$ by running a regression of the firm-level investment rate g on q with a state fixed-effect. Alternatively, if the state s is sufficiently persistent, the state fixed-effect could be proxied by a firm fixed-effect. For instance, Table 2 of Peters and Taylor (2017) reports comparable regression coefficients of investment rate on q (with year and firm fixed effects) using Compustat data on public firms from 1975–2011. They report the values for different types of investment: physical, intangibles, and R&D. In Panel B of Peters and Taylor (2017), the authors use the usual definition of q (i.e., enterprise value over physical capital) and report regression coefficient values ranging from 0.3 pp. (for R&D investment) to 0.6 pp. (for physical investment). Taking these numbers at face value, our baseline calibration (with an implied value of $1/\chi = 0$) is not too far off and the "medium elasticity" calibration compares favorably to the data. The authors also propose an improved measure of q, which accounts for the presence of intangible capital, and obtain larger regression coefficient values, ranging from 1.3 pp. (for R&D investment) to 2.9 pp. (for physical capital). Those values are almost an order of magnitude larger, and are closer to our high elasticity calibrations.

REFERENCES

ABEL, ANDREW B., N. GREGORY MANKIW, LAWRENCE H. SUMMERS, AND RICHARD J. ZECKHAUSER (1989): "Assessing Dynamic Efficiency: Theory and Evidence," *The Review of Economic Studies*, 56, 1–19. [12]

ATKESON, ANDREW G., AND MAGNUS IRIE (2022): "Rapid Dynamics of Top Wealth Shares and Self-Made Fortunes: What Is the Role of Family Firms?" *American Economic Review: Insights*, 4, 409–424. [25]

BARKAI, SIMCHA (2020): "Declining Labor and Capital Shares," *The Journal of Finance*, 75, 2421–2463. [2, 15]

BEARE, BRENDAN K., AND ALEXIS AKIRA TODA (2022): "Determination of Pareto Exponents in Economic Models Driven by Markov Multiplicative Processes," *Econometrica*, 90, 1811–1833. [3]

BEARE, BRENDAN K., WON-KI SEO, AND ALEXIS AKIRA TODA (2022): "Tail Behavior of Stopped Lévy Processes With Markov Modulation," *Econometric Theory*, 38, 986–1013. [4,5]

BOARD OF GOVERNORS OF THE FEDERAL RESERVE SYSTEM (US) (2023a): "Bank Prime Loan Rate DPRIME," Retrieved from FRED. Federal Reserve Bank of St. Louis, https://fred.stlouisfed.org/series/DPRIME, accessed on 2023-09-11. [15]

BOARD OF GOVERNORS OF THE FEDERAL RESERVE SYSTEM (US) (2023b): "Households; Net Worth, Level BOGZ1FL192090005A," Retrieved from FRED, Federal Reserve Bank of St. Louis, https://fred.stlouisfed.org/series/BOGZ1FL192090005A, accessed on 2023-09-11. [10]

BOARD OF GOVERNORS OF THE FEDERAL RESERVE SYSTEM (US) (2023c): "Population Growth for the United States SPPOPGROWUSA," Retrieved from FRED, Federal Reserve Bank of St. Louis, https://fred.stlouisfed.org/series/SPPOPGROWUSA, accessed on 2023-09-11. [13,14]

BOUDOUKH, JACOB, RONI MICHAELY, MATTHEW RICHARDSON, AND MICHAEL R. ROBERTS (2007): "On the Importance of Measuring Payout Yield: Implications for Empirical Asset Pricing," *The Journal of Finance*, 62, 877–915. [9]

BUREAU OF ECONOMIC ACTIVITY (2023): "Integrated Macroeconomic Accounts for the United States," Retrieved from https://www.bea.gov/data/special-topics/integrated-macroeconomic-accounts, accessed on 2023-09-11. [12,14]

CAGETTI, MARCO, AND MARIACRISTINA DE NARDI (2006): "Entrepreneurship, Frictions, and Wealth," *Journal of political Economy*, 114, 835–870. [8,17]

CAMPBELL, JOHN Y. (2017): *Financial Decisions and Markets: A Course in Asset Pricing*. Princeton University Press. [13]

CHETRITE, RAPHAËL, AND HUGO TOUCHETTE (2015): "Nonequilibrium Markov Processes Conditioned on Large Deviations," in *Annales Henri Poincaré*, Vol. 16. Springer, 2005–2057. [5]

COCHRANE, JOHN H. (1991): "Production-Based Asset Pricing and the Link Between Stock Returns and Economic Fluctuations," *The Journal of Finance*, 46, 209–237. [2]

COOPER, MICHAEL, JOHN MCCLELLAND, JAMES PEARCE, RICHARD PRISINZANO, JOSEPH SULLIVAN, DANNY YAGAN, OWEN ZIDAR, AND ERIC ZWICK (2016): "Business in the United States: Who Owns It, and How Much Tax Do They Pay?" *Tax Policy and the Economy*, 30, 91–128. [17]

EISFELDT, ANDREA L., ANTONIO FALATO, AND MINDY Z. XIAOLAN (2019): "Human Capitalists," Working paper. [19]

FAMA, EUGENE F., AND KENNETH R. FRENCH (2001): "Disappearing Dividends: Changing Firm Characteristics or Lower Propensity to Pay?" *Journal of Financial economics*, 60, 3–43. [9]

FEDERAL RESERVE BOARD (2016): "2016 Survey of Consumer Finances," Retrieved from Federal Reserve Board, https://www.federalreserve.gov/econres/scf_2016.htm, accessed on 2023-09-11. [17]

FEENSTRA, ROBERT C., ROBERT INKLAAR, AND MARCEL P. TIMMER (2015): "The Next Generation of the Penn World Table," *American economic review*, 105, 3150–3182. [13,14]

GABAIX, XAVIER, AND RUSTAM IBRAGIMOV (2011): "Rank- 1/2: A Simple Way to Improve the OLS Estimation of Tail Exponents," *Journal of Business & Economic Statistics*, 29, 24–39. [16]

GOMEZ, MATTHIEU (2023): "Decomposing the Growth of Top Wealth Shares," *Econometrica*, 91, 979–1024. [24,25]

——— (2023): "Replication Package for "Decomposing the Growth of Top Wealth Shares"." https://doi.org/10.5281/zenodo.10045152. [16]

GOUIN-BONENFANT, ÉMILIEN (2022): "Productivity Dispersion, Between-Firm Competition, and the Labor Share," *Econometrica*, 90, 2755–2793. [2]

KARABARBOUNIS, LOUKAS, AND BRENT NEIMAN (2019): "Accounting for Factorless Income," *NBER Macroeconomics Annual*, 33, 167–228. [2]

KOCHEN, FEDERICO (2022): "Finance Over the Life Cycle of Firms," Working paper. [17,18]

LECOMTE, VIVIEN (2007): "Thermodynamique des histoires et fluctuations hors d'équilibre," Ph.D. thesis. [5]

MOLL, BENJAMIN (2014): "Productivity Losses From Financial Frictions: Can Self-Financing Undo Capital Misallocation?" *American Economic Review*, 104, 3186–3221. [8]

MOODY'S (2023a): "Moody's Seasoned Aaa Corporate Bond Yield AAA," Retrieved from FRED, Federal Reserve Bank of St. Louis, https://fred.stlouisfed.org/series/AAA, accessed on 2023-09-11. [15]

——— (2023b): "Moody's Seasoned Baa Corporate Bond Yield BAA," Retrieved from FRED, Federal Reserve Bank of St. Louis, https://fred.stlouisfed.org/series/BAA, accessed on 2023-09-11. [15]

PETERS, RYAN H., AND LUCIAN A. TAYLOR (2017): "Intangible Capital and the Investment-q Relation," *Journal of Financial Economics*, 123, 251–272. [26]

PIKETTY, THOMAS, AND GABRIEL ZUCMAN (2015): "Wealth and Inheritance in the Long Run," in *Handbook of Income Distribution*, Vol. 2. Elsevier, 1303–1368. [25]

SAEZ, EMMANUEL (2001): "Using Elasticities to Derive Optimal Income Tax Rates," *The Review of Economic Studies*, 68, 205–229. [16]

SAEZ, EMMANUEL, AND GABRIEL ZUCMAN (2016): "Wealth Inequality in the United States Since 1913: Evidence From Capitalized Income Tax Data," *The Quarterly Journal of Economics*, 131, 519–578. [16]

SMITH, MATTHEW, ZIDAR OWEN, AND ZWICK ERIC (2022): "Replication Data for 'Top Wealth in America': New Estimates Under Heterogeneous Returns." https://doi.org/10.7910/DVN/JKFYMJ. [17]

SMITH, MATTHEW, OWEN ZIDAR, AND ERIC ZWICK (2023): "Top Wealth in America: New Estimates Under Heterogeneous Returns," *The Quarterly Journal of Economics*, 138, 515–573. [16]

SP GLOBAL MARKET INTELLIGENCE (2023): "Compustat Annual Data," Retrieved from WRDS, https://wrds-web.wharton.upenn.edu/, accessed on 2023-03-03. [8]

U.S. BUREAU OF LABOR STATISTICS (2023): "Consumer Price Index for All Urban Consumers: All Items in U.S. City Average CPIAUCSL," Retrieved from FRED, Federal Reserve Bank of St. Louis, https://fred.stlouisfed.org/series/CPIAUCSL, accessed on 2023-09-11. [12,14,15]

U.S. CENSUS BUREAU (2023): "Total Households TTLHH," Retrieved from FRED, Federal Reserve Bank of St. Louis, https://fred.stlouisfed.org/series/TTLHH, accessed on 2023-09-11. [10]

UNIVERSITY OF GRONINGEN AND UNIVERSITY OF CALIFORNIA, DAVIS (2023): "Total Factor Productivity at Constant National Prices for United States RTFPNAUSA632NRUG," Retrieved from FRED, Federal Reserve Bank of St. Louis, https://fred.stlouisfed.org/series/RTFPNAUSA632NRUG, accessed on 2023-09-11. [13,14]

ZHENG, GEOFFREY (2019): "Wealth Shares in the Long Run," Working paper. [25]

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