SUPPLEMENT TO "EXPERIMENTATION AND APPROVAL MECHANISMS"

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APPENDIX D: EXISTENCE OF SOLUTIONS

WE NOW TURN to establishing the existence of a solution to $H_N(V)$ and H_h. Throughout this section, we take $b_k = \inf\{x \leq X^k : x \in \mathcal{D}_k\}$ and $B_k = \sup\{x \geq X^k : x \in \mathcal{D}_k\}$ as in Appendix A.

Let $\mathcal{X}_k^d = \{x \geq X^{k+1} : d = \arg\max_{d'} g(x, k, d')\}$, $\mathcal{D}_k^d = \mathcal{D}_k \cap \mathcal{X}_k^d$ be the set of $x \in \mathcal{D}_k$ at which action d is optimal and $\mathcal{D}'_{k,d}$ be the set of $x \in \mathcal{D}_k^d$ for which there exists a τ such that $\mathbb{P}(\tau > 0|X_0 = x) > 0$ and $F_k(x) = \mathbb{E}^x[e^{-r(\tau\wedge\tau(X^{k+1}))}G_k(X_{\tau\wedge\tau(X^{k+1})})]$; that is, for $x \in \mathcal{D}'_{k,d}$ it is optimal both to stop immediately and to continue according to some stopping rule which (with positive probability) continues for some positive amount of time. Let $\mathcal{D}^{\mathrm{o}}_{k,d} = \mathcal{D}_k^d \backslash \mathcal{D}'_{k,d}$ and $\mathcal{D}^{\mathrm{o}}_k = \mathcal{D}^{\mathrm{o}}_{k,0} \cup \mathcal{D}^{\mathrm{o}}_{k,1}$. It is strictly optimal to immediately stop at any history h_t with $X_t \in \mathcal{D}^{\mathrm{o}}_{\kappa(M_t)}$. Our next result provides sufficient conditions under which the solution to the Lagrangian in our general stopping problem (rewritten below) is unique:

$$\sup_{(\tau,d_\tau)} \mathbb{E}^x\left[e^{-r\tau}g\big(X_\tau, \kappa(M_\tau), d_\tau\big) + \sum_{k=1}^{P} e^{-r\tau(X^k)}\xi^k \mathbb{1}\big(\tau \geq \tau\big(X^k\big)\big)\right]. \tag{11}$$

PROPOSITION 7: *Suppose $g(x, k, 1) - g(x, k, 0)$ and $g(x, k, 1)$ are strictly increasing in x. Then $\mathcal{D}'_{k,1} = \emptyset$ for all k. If $\mathcal{D}'_{k,0} \neq \emptyset$, then it is a singleton. If $\mathcal{D}'_{k,0} = \emptyset$ for all k, then (τ^*, d^*_τ) as defined in Proposition 4 is the unique solution to (11).*

PROOF: We first argue that $x \in \mathcal{D}_k$ implies $G_k(x) = g(x, k, d_k^x) \geq 0$. Suppose $g(x, k, d_k^x) < 0$. Take $\epsilon > 0$ such that $x - \epsilon > X^{k+1}$ and $\max\{g(x-\epsilon, k, d_k^x), g(x+\epsilon, k, d_k^x)\} < 0$. Define $\tau^\epsilon = \tau_+(x+\epsilon) \wedge \tau(x-\epsilon)$. Because $g(X_t, k, d_k^x)$ is a martingale, $\mathbb{E}^x[g(X_{\tau^\epsilon}, k, d_k^x)] = g(x, k, d_k^x)$ by Doob's optional stopping theorem and

$$\begin{aligned} F_k(x) &\geq \mathbb{E}^x\big[e^{-r\tau^\epsilon}G_k(X_{\tau^\epsilon})\big] \geq \mathbb{E}^x\big[e^{-r\tau^\epsilon}g\big(X_{\tau^\epsilon}, k, d_k^x\big)\big] \\ &> \mathbb{E}^x\big[g\big(X_{\tau^\epsilon}, k, d_k^x\big)\big] = g\big(x, k, d_k^x\big) = G_k(x), \end{aligned}$$

a contradiction of $x \in \mathcal{D}_k$.

Because $g(x, k, 1) - g(x, k, 0)$ is increasing in x, $\mathcal{X}_k^d$ is either empty or a connected set. Thus, for any $x_1, x_2 \in \mathcal{D}_k^d$ and $x_3 \in (x_1, x_2)$, $x_3 \in \mathcal{D}_k$ implies $x_3 \in \mathcal{D}_k^d$.

For each d and k, we argue $\mathcal{D}_k^d$ must be a connected set or empty. Suppose not; then there exists a d and $x \notin \mathcal{D}_k^d$ and $x_1, x_2 \in \mathcal{D}_k^d$ such that $x \in (x_1, x_2)$. Since $x_1 \in \mathcal{D}_k^d$ implies $x_1 > X^{k+1}$ and X is continuous, X must enter $\mathcal{D}_k^d$ before $\tau(X^{k+1})$ when $X_0 = x$. Stopping at $\inf\{t : X_t \in \mathcal{D}_{\kappa(M_t)}\}$ is an optimal stopping rule, so when $(X_0, M_0) = (x, m)$ with m such that $\kappa(m) = k$, $\tau' = \inf\{t : X_t \in \mathcal{D}_k\}$ is an optimal stopping rule. Because x is bounded

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above and below by elements of $\mathcal{D}_k^d$, $\mathbb{P}(X_{\tau'} \in \mathcal{D}_k^d | X_0 = x) = 1$ and, using $g(X_{\tau'}, k, d) \geq 0$, we have

$$F_k(x) = \mathbb{E}^x\left[e^{-r\tau'} g(X_{\tau'}, k, d)\right] \leq \mathbb{E}^x\left[g(X_{\tau'}, k, d)\right] = g(x, k, d) = G_k(x), \tag{12}$$

which contradicts $F_k(x) > G_k(x)$ by $x \notin \mathcal{D}_k$.

Because $g(x, k, 1) - g(x, k, 0)$ is increasing in x, if $x \in \mathcal{D}_k^1$, then for all $x' > x$, $x' \in \mathcal{D}_k$ implies $x' \in \mathcal{D}_k^1$. Suppose $\mathcal{D}_k^1 \neq \emptyset$ and $\sup\{x \in \mathcal{D}_k^1\} < \infty$. Let $B' = \sup\{x \in \mathcal{D}_k^1\}$, so $\inf\{t : X_t \in \mathcal{D}_k\} = \tau(B')$ when $X_0 = x > B'$. For such x, $F_k(x) = \mathbb{E}^x[e^{-r\tau(B')} G_k(B')]$. Because $\lim_{x\to\infty} \mathbb{E}^x[e^{-r\tau(B')}] = 0$, we have

$$\lim_{x\to\infty} F_k(x) = \lim_{x\to\infty} \mathbb{E}^x\left[e^{-r\tau(B')} G_k\left(B'\right)\right] = 0 < \lim_{x\to\infty} G_k(x), \tag{13}$$

a contradiction of $F_k(x) \geq G_k(x)$. Therefore, either $\mathcal{D}_k^1 = \emptyset$ or $\sup\{x \in \mathcal{D}_k^1\} = \infty$.

We next argue that if $\mathcal{D}'_{k,d} \neq \emptyset$, then $\mathcal{D}_k^d = \mathcal{D}'_{k,d}$ and must be a singleton. For any d and $x \in \mathcal{D}'_{k,d}$, $\inf\{|x - y| : y \in \mathcal{D}^{\mathrm{o}}_{k,d}\} > 0$; otherwise, with probability one, X immediately enters $\mathcal{D}^{\mathrm{o}}_{k,d}$ when $X_0 = x$, where stopping immediately is strictly optimal. This contradicts that there was an optimal stopping rule, which did not stop for a positive length of time with positive probability when $(X_0, M_0) = (x, m)$ for some m with $\kappa(m) = k$.

If $\mathcal{D}'_{k,d} \neq \emptyset$ and $\mathcal{D}^{\mathrm{o}}_{k,d} \neq \emptyset$, then $\inf\{|x - y| : x \in \mathcal{D}'_{k,d}, y \in \mathcal{D}^{\mathrm{o}}_{k,d}\} = 0$; otherwise, $\mathcal{D}_k^d$ would not be a connected set. Because $\inf\{|x - y| : y \in \mathcal{D}^{\mathrm{o}}_{k,d}\} > 0$ for all $x \in \mathcal{D}'_{k,d}$, $\mathcal{D}'_{k,d}$ is a nonempty interval with at least one open end. Then there exists a nonempty interval $(x_1, x_2) \subseteq \mathcal{D}'_{k,d}$. Because it is not strictly optimal to stop immediately at $x \in \mathcal{D}'_{k,d}$, there exists an optimal strategy that never stops at $x \in \mathcal{D}'_{k,d}$.[1] Letting $\tau' = \inf\{t : X_t \notin (x_1, x_2)\}$, because continuing is both weakly optimal at $x \in \mathcal{D}'_{k,d}$ and $F_k(x') = g(x', k, d)$ for all $x' \in \mathcal{D}'_{k,d}$, we have

$$\begin{aligned} F_k(x) &= \mathbb{E}^x\left[e^{-r\tau'} F_k(X_{\tau'})\right] = \mathbb{E}^x\left[e^{-r\tau'} g(X_{\tau'}, k, d)\right] < \mathbb{E}^x\left[g(X_{\tau'}, k, d)\right] = g(x, k, d) \\ &= G_k(x), \end{aligned}$$

a contradiction. Thus, either $\mathcal{D}'_{k,d} = \emptyset$ or $\mathcal{D}^{\mathrm{o}}_{k,d} = \emptyset$. If $\mathcal{D}'_{k,d} \neq \emptyset$, then $\mathcal{D}'_{k,d} = \mathcal{D}_k^d$.

Next, we argue that $\mathcal{D}'_{k,d} = \mathcal{D}_k^d$ implies $\mathcal{D}'_{k,d}$ is a singleton. Suppose not; then there exists a nonempty interval $(x_1, x_2) \subset \mathcal{D}'_{k,d}$, which we have just argued cannot be. Given our previous characterization of $\mathcal{D}_k^1$ as being either empty or an interval, we conclude $\mathcal{D}'_{k,1} = \emptyset$ and $\mathcal{D}^{\mathrm{o}}_{k,1} = \mathcal{D}_k^1$. If $\mathcal{D}'_{k,0} = \emptyset$ for all k, then $\mathcal{D}^{\mathrm{o}}_{k,0} = \mathcal{D}_k^0$ as well. In this case $\mathcal{D}_k = \mathcal{D}_k^{\mathrm{o}}$, so it is strictly optimal to stop the first time $X_t \in \mathcal{D}_{\kappa(M_t)}$; thus, τ^* as defined in Proposition 4 is the unique solution to (11). *Q.E.D.*

Proof of Proposition 5

We first prove a useful auxiliary lemma.

LEMMA 20: *Let $b' < x < B$. If $\widetilde{V}(B, b', x) \geq 0$, then $\widetilde{V}(B, b'', x) > 0$ $\forall b'' \in (b', x)$.*

[1] We can always replace stopping at a history h_t with $X_t \in \mathcal{D}'_{k,d}$ with a continuation mechanism at h_t that continues with positive probability and achieves the same payoff as stopping immediately.

PROOF: Suppose $\widetilde{V}(B, b', x) \geq 0$. By single-peakedness, $\widetilde{V}(B, b, x)$ is decreasing in b on $[b^*(B), x]$. Since $\widetilde{V}(B, x, x) = 0$, we have $\widetilde{V}(B, b'', x) > 0$ for all $b'' \in [b^*(B), x)$. The only remaining case is $b' < b'' < b^*(B)$. By single-peakedness, $\widetilde{V}(B, b, x)$ is increasing in b on $(-\infty, b^*(B)]$. Thus, $0 \leq \widetilde{V}(B, b', x) < \widetilde{V}(B, b'', x)$. *Q.E.D.*

Let $\widehat{\Lambda} \in \arg\min_{\Lambda \in \mathbb{R}_-^{N+1}} \mathcal{L}^*(\Lambda)$. With some abuse of notation, we let $\mathcal{B}_N = \{X^1, \dots, X^P\}$ be the set of X_n such that $\widehat{\lambda}_n < 0$, keeping the dependence of $\mathcal{B}_N$ on $\widehat{\Lambda}$ implicit. After dropping constant terms, we can write $\sup_{(\tau, d_\tau)} \mathcal{L}(\tau, d_\tau, \widehat{\Lambda})$ in the form of (11) by taking $g(x, k, d) = u(x, d) - (\widehat{\gamma} + \sum_{j=1}^k \widehat{\lambda}^j) v(x, d)$ and $\xi^k = \widehat{\lambda}^k \frac{c_A}{r}$. Both $g(x, k, 1) - g(x, k, 0)$ and $g(x, k, 1)$ are then strictly increasing in x. Because $\tilde{u}(X_t), \tilde{v}(X_t)$ are martingales in X_t, $g(X_t, k, d)$ is also a martingale. Note that $d_k^x = 1$ if and only if $\tilde{u}(x) - (\widehat{\gamma} + \sum_{j=1}^k \widehat{\lambda}^j)\tilde{v}(x) \geq 0$. Because $\tilde{u}(x) \leq \tilde{v}(x)$, $d_k^x = 1$ implies $\tilde{v}(x) \geq 0$. Thus,

$$g\big(X^{k+1}, k+1, d_k^{X^{k+1}}\big) - g\big(X^{k+1}, k, d_k^{X^{k+1}}\big) = -\widehat{\lambda}^{k+1}\left(\tilde{v}\big(X^{k+1}\big)\, d_k^{X^{k+1}} + \frac{c_A}{r}\right) \geq -\widehat{\lambda}^{k+1}\frac{c_A}{r},$$

meeting all the assumptions on g in the general stopping problem. By Proposition 4, a solution to $\sup_{(\tau, d_\tau)} \mathcal{L}(\tau, d_\tau, \widehat{\Lambda})$ exists.

For each k, $\lim_{x\to\infty} g(x, k, 1) = 1 + \frac{c_R}{r} - (\widehat{\gamma} + \sum_{j=1}^k \widehat{\lambda}^j)(1 + \frac{c_A}{r}) > 0$. By a similar argument as in (13), if $\mathcal{D}_k^1 = \emptyset$, then $\lim_{x\to\infty} F_k(x) = 0$, contradicting $F_k(x) \geq g(x, k, 1)$. Therefore, $\mathcal{D}_k^1 \neq \emptyset$.

Let $\mathcal{M}^*(\Lambda) = \arg\max_{(\tau, d_\tau)} \mathcal{L}(\tau, d_\tau, \Lambda)$. If stopping at $t = 0$ is strictly optimal, then the optimal mechanism is unique. Suppose stopping at $t = 0$ is not strictly optimal. For arbitrary $\widehat{\Lambda} \in \arg\min_{\Lambda \in \mathbb{R}_-^{N+1}} \mathcal{L}^*(\Lambda)$, let $X^L = \min\{X^k \in \mathcal{B}_N : \exists(\tau, d_\tau) \in \mathcal{M}^*(\widehat{\Lambda}) \text{ s.t. } \mathbb{P}(\tau > \tau(X^k)) > 0\}$ if $\mathcal{B}_N \neq \emptyset$; otherwise, take $X^L = 0$ (we keep the dependence on $\widehat{\Lambda}$ implicit). For each $X^k \in \mathcal{B}_N$, $X^k < X^L$ implies that $\tau \leq \tau(X^k)$ for all $(\tau, d_\tau) \in \mathcal{M}^*(\widehat{\Lambda})$. Our next proof shows that for every optimal mechanism, its continuation mechanism at $\tau(X^L)$ is the same. In this case, we say the optimal continuation mechanism at $\tau(X^L)$ is unique.

LEMMA 21: *Suppose stopping at $t = 0$ is not strictly optimal. For each $\widehat{\Lambda}$ and corresponding X^L, $\mathcal{D}'_{L,0} = \emptyset$ and the unique optimal continuation mechanism at $\tau(X^L)$ is (τ^L, d_τ^L) where $\tau^L = \inf\{t : X_t \notin (b_L, B_L)\}$ and $d_\tau^L = \mathbb{1}(X_t \geq B_L)$.*

PROOF: It suffices to show $\mathcal{D}'_{L,0} = \emptyset$; if this is so, then the same arguments as in Proposition 7 imply the optimal continuation mechanism (τ^L, d_τ^L) is unique and $\tau^L = \inf\{t : X_t \notin (b_L, B_L)\}$ and $d_\tau^L = \mathbb{1}(X_t \geq B_L)$. That $\tau \geq \tau(X^k)$ for all $X^k < X^L$ means either $X^L = X^P$ or there is a lower stopping threshold in $b_L \in (X^{L+1}, X^L]$ at which stopping is strictly optimal (namely, $b_L \in \mathcal{D}_L^{\circ}$). In the latter case, if $b_L \in \mathcal{D}_L^1$, then because $\mathcal{D}_L^1$ is an interval unbounded above, $X^L \in \mathcal{D}_L^1$ and so it is strictly optimal to stop immediately at $\tau(X^L)$, a contradiction of the definition of X^L. Therefore, $b_L \in \mathcal{D}_{L,0}^{\circ}$, which as shown in the proof of Proposition 7, implies $\mathcal{D}'_{L,0} = \emptyset$.

Suppose $X^L = X^P$. The payoff to rejecting at $t > \tau(X^L)$ is $\widehat{c} = \frac{c_R}{r} - (\widehat{\gamma} + \sum_{k=1}^L \widehat{\lambda}^k)\frac{c_A}{r}$. If $\widehat{c} = 0$, then it is never optimal to stop and reject as there is always always a positive option value of continued experimentation, so $\mathcal{D}'_{k,0} = \emptyset$. Suppose $\widehat{c} > 0$. If $\mathcal{D}'_{L,0} \neq \emptyset$, then $\mathcal{D}'_{L,0} = \mathcal{D}_L^0 = \{b_L\}$. For $X_0 < b_L$, $\inf\{t : X_t \in \mathcal{D}_L\} = \inf\{t : X_t \geq b_L\}$ and so

$$\lim_{x\to-\infty} F_L(x) = \lim_{x\to-\infty} \mathbb{E}^x\big[e^{-r\tau_+(b_L)} G_L(b_L)\big] = 0 < \widehat{c} \leq \lim_{x\to-\infty} G_L(x),$$

a contradiction. This argument implies that stopping is strictly optimal at sufficiently low x and so $b_L > -\infty$. Thus, $\mathcal{D}'_{L,0} = \emptyset$. *Q.E.D.*

Our next lemma looks at complementary slackness conditions. We note that $RDP(X_n)$ can be rewritten as $\mathbb{E}^{X_n}[e^{-r\tau[h_{\tau(X_n)}]}v(X_{\tau[h_{\tau(X_n)}]}, d_\tau[h_{\tau(X_n)}])] \geq 0$ and so only depends on the continuation mechanism at $\tau(X_n)$. When the optimal continuation mechanism at $\tau(X_n)$ is unique, we will will simply say that $RDP(X_n)$ binds (or is violated), keeping the dependence of $RDP(X_n)$ on the optimal continuation mechanism at $\tau(X_n)$ implicit.

LEMMA 22: *There exists $\widehat{\Lambda} \in \arg\min_{\Lambda \in \mathbb{R}_-^{N+1}} \mathcal{L}^*(\Lambda)$ and $(\tau, d_\tau) \in \mathcal{M}^*(\widehat{\Lambda})$ such that (τ, d_τ) and $\widehat{\Lambda}$ satisfy complementary slackness for all $RDP(X_n)$ with $X_n \leq X^L$.*

PROOF: Take some $\widehat{\Lambda} \in \arg\min_{\Lambda \in \mathbb{R}_-^{N+1}} \mathcal{L}^*(\Lambda)$. $\mathcal{L}^*(\widehat{\Lambda})$ can be written as

$$\begin{aligned}
&\max_{(\tau, d_\tau)} \mathbb{E}\Bigg[e^{-r\tau}\Bigg(u(X_\tau, d_\tau) - \Bigg(\widehat{\gamma} + \sum_{k=1}^{L-1} \widehat{\lambda}^k\Bigg) v(X_\tau, d_\tau)\Bigg)\mathbb{1}\big(\tau < \tau(X^L)\big) \\
&\quad + \sum_{k=1}^{L-1} e^{-r(\tau \wedge \tau(X^k))} \widehat{\lambda}^k v\big(X_{\tau \wedge \tau(X_k)}, d_\tau(X^k)\big) \\
&\quad + e^{-r\tau(X^L)}\Bigg(F_L\big(X^L; \widehat{\Lambda}\big) + \widehat{\lambda}^L \frac{c_A}{r}\Bigg)\mathbb{1}\big(\tau \geq \tau(X^L)\big)\Bigg] + \widehat{\gamma}\Bigg(V + \frac{c_A}{r}\Bigg),
\end{aligned}$$

where F_L is defined as in our general stopping problem only now making the dependence on $\widehat{\Lambda}$ explicit. Any change to $\widehat{\Lambda}$ that decreases $F_L(X^L; \widehat{\Lambda}) + \widehat{\lambda}^L \frac{c_A}{r}$ will weakly decrease $\mathcal{L}^*(\widehat{\Lambda})$, strictly so if $b_k = -\infty$ for all $k < L$.[2]

$RDP(X_n)$ binds for $X_n < b_L$ since $\mathbb{P}(\tau > \tau(X_n)) = 0$, so complementary slackness conditions hold. Suppose $RDP(X^L)$ is violated. We can apply the same arguments as in Lemma 5 to show[3] that A's continuation value under any $(\tau, d_\tau) \in \mathcal{M}^*(\widehat{\Lambda})$ at $\tau(X^L)$, namely $\widetilde{V}(B_L, b_L, X^L)$, must be strictly negative. $F_L(X^L; \widehat{\Lambda})$ is then equal to

$$\begin{aligned}
&\mathbb{E}^{X^L}\Bigg[e^{-r\tau_+(B_L; b_L)}\Bigg(u(B_L, 1) - \Bigg(\widehat{\gamma} + \sum_{k=1}^{L} \widehat{\lambda}^k\Bigg) v(B_L, 1)\Bigg) \\
&\qquad + e^{-r\tau(b_L; B_L)}\Bigg(\frac{c_R}{r} - \Bigg(\widehat{\gamma} + \sum_{k=1}^{L} \widehat{\lambda}^k\Bigg)\frac{c_A}{r}\Bigg)\Bigg] \\
&\quad = \widetilde{J}\big(B_L, b_L, X^L\big) + \frac{c_R}{r} - \Bigg(\widehat{\gamma} + \sum_{k=1}^{L} \widehat{\lambda}^k\Bigg)\Bigg(\widetilde{V}\big(B_L, b_L, X^L\big) + \frac{c_A}{r}\Bigg). \qquad (14)
\end{aligned}$$

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[2]If $b_k = -\infty$ for all $k < L$, then continuing at $\tau(b_k)$ is strictly optimal and a small change in $F_L(X^L; \widehat{\Lambda}) + \widehat{\lambda}^L \frac{c_A}{r}$ will still preserve $b_k = -\infty$. If $b_k > \infty$, then $b_k \in \mathcal{D}'_k$, so stopping and continuing are both optimal at $\tau(b_k)$. In this case, reducing $F_L(X^L; \widehat{\Lambda}) + \widehat{\lambda}^L \frac{c_A}{r}$ lowers the value of continuing at $\tau(b_k)$ and so would make stopping at $\tau(b_k)$ strictly optimal. Since stopping at $\tau(b_k)$ was optimal before, the value of the Lagrangian is the same.

[3]The proof of Lemma 5 for $X_n = X^L$ only depends on the continuation mechanism of $(\tau_N^*, d_{N,\tau}^*)$ at $\tau(X^L)$ being unique and so applies here.

Suppose $b_k = -\infty$ for all $k < L$. By the theorem of the maximum, the optimal thresholds and decisions at each threshold are continuous in Λ at $\widehat{\Lambda}$. Applying the envelope theorem, we have

$$\frac{d}{d\widehat{\lambda}^1}\left[F_L(X^L;\widehat{\Lambda}) + \widehat{\lambda}^L \frac{c_A}{r}\right] = -\left[\widetilde{V}(B_L, b_L, X^L) + \frac{c_A}{r}\right] + \frac{c_A}{r} > 0.$$

Thus, decreasing $\widehat{\lambda}^L$ will lower $\mathcal{L}^*(\widehat{\Lambda})$, a contradiction of $\widehat{\Lambda} \in \arg\min_{\Lambda\in\mathbb{R}_-^{N+1}} \mathcal{L}^*(\Lambda)$. Therefore, $RDP(X^L)$ cannot be violated at $\widehat{\Lambda}$. A similar argument holds if $RDP(X^L)$ is slack, only now we derive a contradiction by increasing $\widehat{\lambda}^L$ instead of decreasing $\widehat{\lambda}^L$. Because the optimal continuation mechanism is also unique at $\tau(X_n)$ for $X_n \in (b_L, X^L)$, an analogous argument implies $RDP(X_n)$ cannot be violated.

Suppose there exists a j such that $b_j > -\infty$ for $j < L$ and let k be the largest such j. Decreasing $F_L(X^L;\widehat{\Lambda}) + \widehat{\lambda}^L \frac{c_A}{r}$ reduces the value continuing at $\tau(b_k)$, which then makes stopping at $\tau(b_k)$ strictly optimal. The continuation mechanism at $\tau(X^k)$ is now unique. Thus, if $RDP(X^L)$ is not binding or $RDP(X_n)$ is violated for some $X_n \leq X^L$, then by changing $\widehat{\Lambda}$ as in the previous paragraph to some $\widehat{\Lambda}'$ which lowers $F_L(X^L;\Lambda) + \lambda^L \frac{c_A}{r}$, we have not decreased the Lagrangian so $\mathcal{L}^*(\widehat{\Lambda}) = \mathcal{L}^*(\widehat{\Lambda}')$ and $\widehat{\Lambda}' \in \arg\min_{\Lambda\in\mathbb{R}_-^{N+1}} \mathcal{L}^*(\Lambda)$. We can apply the same arguments as above, taking X^k to replace X^L, to conclude that if $b_j = -\infty$ for all $j < k$, then $\widehat{\Lambda}'$ and any $(\tau, d_\tau) \in \mathcal{M}^*(\widehat{\Lambda}')$ must satisfy complementary slackness conditions for $RDP(X_n)$ for $X_n \leq X^k$. If there exists a $j > k$ such that $b_j > -\infty$, then we can apply the same arguments as above until we have reached an k' such that $b_j = -\infty$ for all $j < k'$. In this case, complementary slackness conditions must hold for all $RDP(X_n)$ with $X_n \leq X^{k'}$ and $X^{k'}$ takes the role of X^L for our corresponding choice of Λ derived from $\widehat{\Lambda}$ using the above procedure. *Q.E.D.*

Take $\widehat{\Lambda} \in \arg\min_{\Lambda\in\mathbb{R}_-^{N+1}} \mathcal{L}^*(\Lambda)$ such that for some $(\tau, d_\tau) \in \mathcal{M}^*(\widehat{\Lambda})$, complementary slackness conditions hold for all $RDP(X_n)$ with $X_n \leq X^L$. By the same arguments as in Lemma 5, that $RDP(X^L)$ binds implies $\widetilde{V}(B_L, b_L, X^L) = 0$. Using (14), because $G_L(x) \geq \frac{c_R}{r} - (\widehat{\gamma} + \sum_{k=1}^{L} \widehat{\lambda}^k)\frac{c_A}{r}$ and $\widetilde{V}(B_L, b_L, X^L) = 0$, $F_L(X^L) > G_L(X^L)$[4] implies $\widetilde{J}(B_L, b_L, X^L) > 0$.

We now argue that rejection at $X_t = x > X^L$ is strictly suboptimal. Because X has independent increments conditional on θ, we have

$$\begin{aligned}
\widetilde{V}(B, b, x; z_0) &= \frac{e^{z_x}}{1+e^{z_x}}\widetilde{V}(B, b, x; \infty) + \frac{1}{1+e^{z_x}}\widetilde{V}(B, b, x; -\infty) \\
&= \frac{e^{z_x}}{1+e^{z_x}}\widetilde{V}(B-x, b-x, 0; \infty) + \frac{1}{1+e^{z_x}}\widetilde{V}(B-x, b-x, 0; -\infty) \\
&= \widetilde{V}(B-x, b-x, 0; z_x).
\end{aligned}$$

Thus, $\widetilde{V}(B_N(X^L), b_L, X^L; z_0) = 0$ implies $\widetilde{V}(B_N(X^L) - X^L, b_L - X^L, 0, z_{X^L}) = 0$. By Lemma 3, $\widetilde{V}(B_N(X^L) - X^L, b_L - X^L, 0, z_x) > 0$ for all $x > X^L$. Then, by Lemma 20, $\widetilde{V}(B_N(X^L) - X^L, -\epsilon, 0, z_x) > 0$ for any $\epsilon \in (0, X^L - b_L)$ and all $x > X^L$. A similar argument holds for R so that $\widetilde{J}(B_N(X^L) - X^L, -\epsilon, 0, z_x) > 0$.

[4]By the definition of X^L, we must have $X^L \notin \mathcal{D}_L$; otherwise, $\mathbb{P}(\tau > \tau(X^L)) = 0$ for all $(\tau, d_\tau) \in \mathcal{M}^*(\widehat{\Lambda})$.

Take some (x,m) and $\epsilon > 0$ with $m > b_L$, $x > X^L$ and $x - \epsilon > X^{\kappa(m)+1}$. Let $B' = B_N(X^L) - X^L + x$ and define $\tau' = \inf\{t : X_t \notin (x-\epsilon, B')\}$ and $d'_\tau = \mathbb{1}(X_{\tau'} \geq B')$. The continuation value in our Lagrangian from using (τ', d'_τ) at a history h_t such that $(X_t, M_t) = (x,m)$ is

$$\begin{aligned}
&\mathbb{E}_{x,m}\Bigg[e^{-r\tau_+(B';x-\epsilon)}\Bigg(u(B',1) - \Bigg(\widehat{\gamma} + \sum_{k=1}^{\kappa(m)} \widehat{\lambda}^k\Bigg) v(B',1)\Bigg) \\
&\quad + e^{-r\tau(x-\epsilon;B')}\Bigg(\frac{c_R}{r} - \Bigg(\widehat{\gamma} + \sum_{k=1}^{\kappa(m)} \widehat{\lambda}^k\Bigg)\frac{c_A}{r}\Bigg)\Bigg] \\
&= \widetilde{J}\big(B_N(X^L) - X^L, -\epsilon, 0; z_x\big) + \frac{c_R}{r} \\
&\quad - \Bigg(\widehat{\gamma} + \sum_{k=1}^{\kappa(m)} \widehat{\lambda}^k\Bigg)\Bigg(\widetilde{V}\big(B_N(X^L) - X^L, -\epsilon, 0; z_x\big) + \frac{c_A}{r}\Bigg),
\end{aligned}$$

which is strictly greater than $\frac{c_R}{r} - (\widehat{\gamma} + \sum_{k=1}^{\kappa(m)} \widehat{\lambda}^k)\frac{c_A}{r}$, the payoff at x of rejecting. Thus, rejection when $(X_t, M_t) = (x,m)$ cannot be optimal and so $\mathcal{D}_k^0 = \mathcal{D}'_{k,0} = \emptyset$ for all $k < L$. $\mathcal{D}'_{L,1} = \emptyset$ by Lemma 21, so Proposition 7 implies the solution to the Lagrangian, call it $(\tau_N^*, d_{N,\tau}^*)$, is unique.

By analogous arguments those in Lemma 22, $(\tau_N^*, d_{N,\tau}^*)$ and $\widehat{\Lambda}$ must satisfy complementary slackness conditions for all $RDP(X_n)$ and $PK(V)$. We conclude that $(\tau_N^*, d_{N,\tau}^*)$ solves $H_N(V)$. Finally, if $B_N(0) > 0$, but $B_N(m) = m$ for some $m < 0$, then the stopping rule approves with probability one. It is easy to see that immediate approval at $t = 0$ strictly dominates this mechanism.

Although complementary slackness conditions imply $RDP(X^k)$ binds under $(\tau_N^*, d_{N,\tau}^*)$ for all $X^k \in \mathcal{B}_N$, they do not imply that $RDP(X_n)$ is slack for all $X_n \notin \mathcal{B}_N$. However, we can add into $\mathcal{B}_N$ any X_n such that $RDP(X_n)$ binds but $\widehat{\lambda}_n = 0$ without changing the statement of Proposition 5.

Proof of Proposition 6

PROOF: Let $\widehat{\Lambda} \in \arg\min_{\Lambda \in \mathbb{R}_-^{N'}} \mathcal{L}^*(\Lambda)$ with $\{X^1, \dots, X^P\} = \{X_n : \widehat{\lambda}_n < 0\}$. For $0 \leq k \leq P-1$, take $g(x,k,1) = \tilde{u}(x) + \sum_{j=k+1}^{P} \widehat{\lambda}^j \hat{v}_\ell(x,1)$, $g(x,P,1) = \tilde{u}(x)$ and $\xi^k = \widehat{\lambda}^k \hat{v}_\ell(X^k, 0)$. We rule out the choice of $d = 0$ by setting $g(x,k,d)$ to be a sufficiently low constant.[5] It is straightforward to verify that $g(X_t, k, 1)$ is a martingale. Then

$$\begin{aligned}
g\big(X^{k+1}, k+1, 1\big) - g\big(X^{k+1}, k, 1\big) &= -\widehat{\lambda}^{k+1} \hat{v}_\ell\big(X^{k+1}, 1\big) \geq -\widehat{\lambda}^{k+1} \widehat{v}_\ell\big(X^{k+1}, 0\big) \\
&= -\xi^{k+1}.
\end{aligned}$$

A solution τ_N^* to $\sup_\tau \mathcal{L}(\tau, \widehat{\Lambda})$ exists by Proposition 4.

[5]We can safely ignore all conditions on g for $d = 0$ since $d = 0$ will never be optimal.

We next to show $g(x,k,1)$ is increasing in x at $\widehat{\Lambda}$. Note that

$$\frac{\partial g(x,k,1)}{\partial x} = \frac{\frac{2\mu}{\sigma^2} e^{z_h(x)}}{\left(1+e^{z_h(x)}\right)^2}\left[1-f+\sum_{j=k+1}^{P}\widehat{\lambda}^j \frac{1+e^{z_h}}{1+e^{z_\ell}}\left(e^{-\Delta_z}\left(1+\frac{c_A}{r}\right)-a-\frac{c_A}{r}\right)\right].$$

If $e^{-\Delta_z}(1+\frac{c_A}{r})-a-\frac{c_A}{r}\le 0$, then $\frac{\partial g(x,k,d)}{\partial x}>0$.

Suppose $e^{-\Delta_z}(1+\frac{c_A}{r})-a-\frac{c_A}{r}>0$. The sign of $\frac{\partial g(x,k,1)}{\partial x}$ is the same for all x, but may be negative for arbitrary $\widehat{\Lambda}$. In this case, $\frac{\partial g(x,k,1)}{\partial x}$ is increasing in k, so it suffices to show $\frac{\partial g(x,1,1)}{\partial x}>0$. Suppose $\frac{\partial g(x,1,1)}{\partial x}\le 0$. The limit $\lim_{x\to-\infty} g(x,1,1)<0$, so $\frac{\partial g(x,1,1)}{\partial x}\le 0$ implies $g(x,1,1)<0$ for all x, in which case it is never optimal to stop at $t<\tau(X^1)$ and

$$\mathcal{L}^*(\widehat{\Lambda}) = \mathbb{E}\left[e^{-r\tau(X^1)}\left(F_1\big(X^1;\widehat{\Lambda}\big)+\widehat{\lambda}^1\frac{c_A}{r}\right))\right]-\sum_{k=1}^{P}\widehat{\lambda}^k\left(V_\ell+\frac{c_A}{r}\right).$$

$\widehat{\lambda}^1$ does not appear in $F_1(X^1;\widehat{\Lambda})$, so changing $\widehat{\lambda}^1$ has no impact on the continuation value $F_1(X^1;\widehat{\Lambda})$ and $\frac{\partial\mathcal{L}^*(\widehat{\Lambda})}{\partial\widehat{\lambda}^1}=\mathbb{E}[e^{-r\tau(X^1)}\frac{c_A}{r}]-V_\ell-\frac{c_A}{r}<0$, a contradiction of $\widehat{\lambda}^1<0$ and $\widehat{\Lambda}\in\arg\min_{\Lambda\in\mathbb{R}_-^{N'}}\mathcal{L}^*(\Lambda)$. Therefore, we must have $\frac{\partial g(x,1,1)}{\partial x}>0$. We conclude that $g(x,k,1)$ is strictly increasing in x for all k.

We next argue that $\lim_{x\to\infty} g(x,k,1)>0$. If $\lim_{x\to\infty} g(x,1,1)\le 0$, then $g(x,1,1)<0$ for all x. A similar contradiction can be derived from the fact that stopping at $t<\tau(X^1)$ would never be optimal and so $\lim_{x\to\infty} g(x,1,1)>0$. Because $g(x,k,1)$ is increasing in k, we conclude $\lim_{x\to\infty} g(x,k,1)>0$ for all k. As argued in the proof of Proposition 7, this implies $\mathcal{D}_k^1\neq\emptyset$ for all k.

By ruling out $d_\tau=0$, we know $\mathcal{D}_k^0=\mathcal{D}'_{k,0}=\emptyset$ and we can apply Proposition 7 to conclude that τ_N^* is the unique solution to $\mathcal{L}^*(\widehat{\Lambda})$. Let $B_N(m)=B_{\kappa(m)}$. To show $\tau_N^*=\inf\{t: X_t\ge B_N(M_t)\}$, it suffices to show that if $B_k>X^k$, then $b_k=-\infty$. Suppose not, so that $b_k>X^{k+1}$ for some k. By the same arguments as in Proposition 7, $b_k\in\mathcal{D}_k=\mathcal{D}_k^1$ implies $x\in\mathcal{D}_k^1$ for all $x>b_k$, contradicting $B_k>X_k$. Therefore, $b_k=-\infty$ for all k with $B_k>X^k$.

We can apply an analogous argument as in Lemma 22 to show that $(\tau_N^*,1)$ and $\widehat{\Lambda}$ must satisfy complementary slackness conditions for all $RDIC(X_n)$ constraints. Theorem 1 of Balzer and Janßen (2002) then shows that $(\tau_N^*,1)$ solves H_h. As we did in Proposition 5, we let $\mathcal{B}_N=\{X^1,\dots,X^P\}$ and then add into $\mathcal{B}_N$ any X_n such that $RDP(X_n)$ binds but $\widehat{\lambda}_n=0$ without changing the result. *Q.E.D.*

Satisfying Conditions of Theorem 1 of Balzer and Janßen (2002)

Balzer and Janßen (2002) make two restrictions on the choice of (τ,d_τ), requiring $\mathbb{P}(\tau>0)=1$ and $\mathbb{P}(\tau<\infty)=1$. The restriction $\mathbb{P}(\tau>0)=1$ can be dropped as we allow τ to depend on the randomization device Y_0. The restriction $\mathbb{P}(\tau<\infty)=1$ can be dropped as well, because for each $d\in\{0,1\}$, both $e^{-rt}u(X_t,d)$ and $e^{-rt}v(X_t,d)$ go to 0 as $t\to\infty$.

Their theorem also requires a Slater condition that there exists a mechanism for which all constraints are slack. To construct such a mechanism for $H_N(V)$, let B_A^{FB} be the static approval threshold in A's first best mechanism. Take a mechanism, which approves with probability $1-\epsilon$ at $\tau(X_n)$ for each $X_n\ge B_A^{FB}$ and uses A's first-best mechanism as its continuation mechanism at $\tau(\bar{x})$ where $\bar{x} = \sup\{X_n : X_n < B_A^{FB}\}$. For small enough ϵ, all constraints will be slack. For the problem in H_h, the Slater condition is satisfied by τ' with $\mathbb{P}(\tau' = \infty) = 1$.

APPENDIX E: STATIC THRESHOLD MECHANISM PROOFS

Let $\Phi(B, b, x) = \mathbb{E}^x[e^{-r\tau_+(B;b)}]$ be the discounted probability of reaching B before b when $(X_0, Z_0) = (x, z_x)$ and $\phi(B, b, x) = \mathbb{E}^x[e^{-r\tau(b;B)}]$ be the discounted probability of reaching b before B when $(X_0, Z_0) = (x, z_x)$.[6] In both functions, we restrict attention to $B > b$. It is easy to see that Φ is decreasing in B and b, ϕ is increasing b and B. $\Phi(B, b, x) + \phi(B, b, x)$ is strictly less than 1 and is decreasing in B for $x \in (b, B)$.[7] Φ, ϕ are differentiable in all arguments. Let $\Phi_B(B, b, x) := \frac{\partial \Phi(B,b,x)}{\partial B}$ and let $\Phi_b(B, b, x) := \frac{\partial \Phi(B,b,x)}{\partial x}$, with a similar definition for $\phi_B(B, b, x)$, $\phi_b(B, b, x)$.

Proof of Lemma 1

PROOF: We first present the proof of single-peakedness in B for $\widetilde{V}(B, b, x)$. The proof for $\widetilde{J}$ is analogous. Fix $b < x$. If $\widetilde{V}$ is not single-peaked in B, then there exist $B^3 > B^2 > B^1 \geq x$ such that $\widetilde{V}(B^1, b, x) = \widetilde{V}(B^3, b, x) \geq \widetilde{V}(B^2, b, x)$. For threshold B^1, we have

$$\widetilde{V}\big(B^1, b, x\big) = \Phi\big(B^1, b, x\big)v\big(B^1, 1\big) + \phi\big(B^1, b, x\big)v(b, 0) - \frac{c_A}{r}.$$

For B^2, by standard dynamic programming arguments, we have

$$\begin{aligned}
\widetilde{V}\big(B^2, b, x\big) &= \mathbb{E}\bigg[e^{-r\tau_+(B^1;b)}\mathbb{E}^{B^1}\bigg[e^{-r\tau(B^2;b)}v\big(B^2, 1\big) + e^{-r\tau(b;B^2)}\frac{c_A}{r}\bigg] + e^{-r\tau(b;B^1)}v(b, 0)\bigg] - \frac{c_A}{r} \\
&= \Phi\big(B^1, b, x\big)\bigg(\widetilde{V}\big(B^2, b, B^1\big) + \frac{c_A}{r}\bigg) + \phi\big(B^1, b, x\big)v(b, 0) - \frac{c_A}{r}, \qquad (15)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\widetilde{V}\big(B^3, b, x\big) &= \Phi\big(B^1, b, x\big)\bigg(\widetilde{V}\big(B^3, b, B^1\big) + \frac{c_A}{r}\bigg) + \phi\big(B^1, b, x\big)v(b, 0) - \frac{c_A}{r}, \\
\widetilde{V}\big(B^2, b, x\big) &= \Phi\big(B^2, b, x\big)v\big(B^2, 1\big) + \phi\big(B^2, b, x\big)v(b, 0) - \frac{c_A}{r}, \\
\widetilde{V}\big(B^3, b, x\big) &= \Phi\big(B^2, b, x\big)\bigg(\widetilde{V}\big(B^3, b, B^2\big) + \frac{c_A}{r}\bigg) + \phi\big(B^2, b, x\big)v(b, 0) - \frac{c_A}{r}.
\end{aligned}$$

Using the above expressions and $\widetilde{V}(B^1, b, x) = \widetilde{V}(B^3, b, x) \geq \widetilde{V}(B^2, b, x)$, we get $\widetilde{V}(B^3, b, B^1) + \frac{c_A}{r} = v(B^1, 1) \geq \widetilde{V}(B^2, b, B^1) + \frac{c_A}{r}$ and $\widetilde{V}(B^3, b, B^2) + \frac{c_A}{r} \geq v(B^2, 1)$.

[6]Stokey (2009) gives closed-form formula for these discounted probabilities conditional on θ, which can then be used to calculate Φ, ϕ explicitly based on the belief about θ implies by x.

[7]That $\Phi + \phi$ is decreasing in B follows from the observation that $\Phi + \phi = \mathbb{E}^x[e^{-r(\tau_+(B) \wedge \tau(b))}]$ and for $B < B'$, $\tau(B) \wedge \tau(b) \leq \tau(B') \wedge \tau(b)$.

Suppose $v(B^1,1)\geq 0$. Using $\widetilde{V}(B^3,b,B^1)+\frac{c_A}{r}=v(B^1,1)$, by similar dynamic programming arguments as in (15), we have

$$\begin{aligned}\widetilde{V}(B^3,b,B^2)&=\Phi(B^3,B^1,B^2)v(B^3,1)+\phi(B^3,B^1,B^2)\left(\widetilde{V}(B^3,b,B^1)+\frac{c_A}{r}\right)-\frac{c_A}{r}\\&=\Phi(B^3,B^1,B^2)v(B^3,1)+\phi(B^3,B^1,B^2)v(B^1,1)-\frac{c_A}{r}\\&=\mathbb{E}^{B^2}\big[e^{-r(\tau_+(B^3)\wedge\tau(B^1))}v(X_{\tau_+(B^3)\wedge\tau(B^1)},1)\big]-\frac{c_A}{r}\\&<\mathbb{E}^{B^2}\big[v(X_{\tau_+(B^3)\wedge\tau(B^1)},1)\big]-\frac{c_A}{r}=v(B^2,1)-\frac{c_A}{r},\end{aligned}$$

contradicting $\widetilde{V}(B^3,b,B^2)+\frac{c_A}{r}\geq v(B^2,1)$. The first inequality above follows from $v(B^1,1)\geq 0$ and $v(B^3,1)>0$ while the last equality follows by an application of Doob's optional stopping theorem and that $v(X_t,1)$ is a martingale.

Now suppose $v(B^1,1)<0$. Because $\Phi(B^2,b,B^1)+\phi(B^2,b,B^1)<1$, multiplying both sides by $v(B^1,1)<0$, we have

$$\begin{aligned}v(B^1,1)&<\Phi(B^2,b,B^1)v(B^1,1)+\phi(B^2,b,B^1)v(B^1,1)\\&<\Phi(B^2,b,B^1)v(B^2,1)+\phi(B^2,b,B^1)\frac{c_A}{r}=\widetilde{V}(B^2,b,B^1)+\frac{c_A}{r},\end{aligned}$$

a contradiction of $v(B^1,1)\geq\widetilde{V}(B^2,b,B^1)+\frac{c_A}{r}$. It must be that $\widetilde{V}$ is strictly single-peaked in B. By interchanging the roles of B with b and $v(B,1)$ with $v(b,0)$, an analogous argument shows single-peakedness with respect to b. *Q.E.D.*

Proof of Lemma 2

PROOF: Fix $b<x$ and let $B'=\arg\max_B\widetilde{V}(B,b,x)$. Given the single-peakedness of $\widetilde{J}$, if $B'>\arg\max_B\widetilde{J}(B,b,x)$, then $\frac{\partial\widetilde{J}(B,b,x)}{\partial B}|_{B=B'}<0=\frac{\partial\widetilde{V}(B,b,x)}{\partial B}|_{B=B'}$. To generate a contradiction, it suffices to show $\frac{\partial\widetilde{J}(B,b,x)}{\partial B}\geq\frac{\partial\widetilde{V}(B,b,x)}{\partial B}$. This follows from

$$\begin{aligned}\frac{\partial\widetilde{J}(B,b,x)}{\partial B}&=\Phi_B(B,b,x)u(B,1)+\Phi(B,b,x)\frac{\partial u(B,1)}{\partial B}+\phi_B(B,b,x)\frac{c_R}{r}\\&=\Phi_B(B,b,x)\tilde{u}(B)+\Phi(B,b,x)\tilde{u}'(B)+\big(\Phi_B(B,b,x)+\phi_B(B,b,x)\big)\frac{c_R}{r}\\&\geq\Phi_B(B,b,x)\tilde{v}(B)+\Phi(B,b,x)\tilde{v}'(B)+\big(\Phi_B(B,b,x)+\phi_B(B,b,x)\big)\frac{c_A}{r}\\&=\frac{\partial\widetilde{V}(B,b,x)}{\partial B}.\end{aligned}$$

The inequality follows from $\Phi_B\leq 0$, $\tilde{u}\leq\tilde{v}$, $\tilde{v}'\leq\tilde{u}'$, $\Phi_B+\phi_B\leq 0$ and $c_A\geq c_R$. *Q.E.D.*

Proof of Lemma 3

PROOF: Suppose $\widetilde{V}(B,b,x;z)\geq 0$. $\widetilde{V}(B,b,x;z)$ is a convex combination of $\widetilde{V}(B,b,x;\infty)$ (with weight $\frac{e^{z+\frac{2\mu}{\sigma^2}x}}{1+e^{z+\frac{2\mu}{\sigma^2}x}}$) and $\widetilde{V}(B,b,x;-\infty)$ (with weight $\frac{1}{1+e^{z+\frac{2\mu}{\sigma^2}x}}$), so the proof is immediate if $\widetilde{V}(B,b,x;\infty) > \widetilde{V}(B,b,x;-\infty)$. Let $\Psi = \mathbb{E}^x[e^{-r\tau_+(B;b)}|H]$ and $\psi = \mathbb{E}^x[e^{-r\tau(b;B)}|H]$. Then $\widetilde{V}(B,b,x;\infty) = \Psi(1+\frac{c_A}{r}) + \psi\frac{c_A}{r} - \frac{c_A}{r}$. Stokey (2009) shows $\mathbb{E}^x[e^{-r\tau_+(B;b)}|L] = \Psi e^{\frac{2\mu}{\sigma^2}(x-B)}$ and $\mathbb{E}^x[e^{-r\tau(b;B)}|L] = \psi e^{\frac{2\mu}{\sigma^2}(x-b)}$, so $\widetilde{V}(B,b,x;-\infty) = \Psi e^{\frac{2\mu}{\sigma^2}(x-B)}(a+\frac{c_A}{r}) + \psi e^{\frac{2\mu}{\sigma^2}(x-b)}\frac{c_A}{r} - \frac{c_A}{r}$.

In order for $\widetilde{V}(B,b,x;z) \geq 0$, either $\widetilde{V}(B,b,x;\infty) \geq 0$ or $\widetilde{V}(B,b,x;-\infty) \geq 0$. Therefore, we only need to show $\widetilde{V}(B,b,x;-\infty) < \max\{0,\widetilde{V}(B,b,x;\infty)\}$. Suppose $\widetilde{V}(B,b,x;-\infty) \geq \max\{0,\widetilde{V}(B,b,x;\infty)\}$. Then

$$\Psi e^{\frac{2\mu}{\sigma^2}(x-B)}\left(a+\frac{c_A}{r}\right) + \psi e^{\frac{2\mu}{\sigma^2}(x-b)}\frac{c_A}{r} - \frac{c_A}{r} \geq \max\left\{0, \Psi\left(1+\frac{c_A}{r}\right) + \psi\frac{c_A}{r} - \frac{c_A}{r}\right\}.$$

The LHS is increasing in a so it suffices to show a contradiction when $a=1$. For $a=1$, we can rearrange this inequality to get

$$\frac{c_A}{r+c_A}\frac{1-\psi e^{\frac{2\mu}{\sigma^2}(x-b)}}{e^{\frac{2\mu}{\sigma^2}(x-B)}} \leq \Psi \leq \frac{c_A}{r+c_A}\frac{\psi\left(e^{\frac{2\mu}{\sigma^2}(x-b))}-1\right)}{1-e^{\frac{2\mu}{\sigma^2}(x-B)}}.$$

Simplifying the LHS and RHS of these inequalities, we get $\psi \geq \frac{e^{-\frac{2\mu}{\sigma^2}(x-B)}-1}{e^{-\frac{2\mu}{\sigma^2}(b-B)}-1}$. Stokey (2009) shows $\psi = \frac{e^{R_1(x-B)}-e^{R_2(x-B)}}{e^{R_1(b-B)}-e^{R_2(b-B)}}$ where $R_1 = \frac{-\mu-\sqrt{\mu^2+2r\sigma^2}}{\sigma^2}$, $R_2 = \frac{-\mu+\sqrt{\mu^2+2r\sigma^2}}{\sigma^2}$. At $r=0$, we have $R_1 = -\frac{2\mu}{\sigma^2}$ and $R_2 = 0$, which implies $\psi = \frac{e^{-\frac{2\mu}{\sigma^2}(x-B)}-1}{e^{-\frac{2\mu}{\sigma^2}(b-B)}-1}$. As is easily seen from its definition, ψ is strictly decreasing in r. Thus, for any $r>0$, we have $\psi < \frac{e^{-\frac{2\mu}{\sigma^2}(x-B)}-1}{e^{-\frac{2\mu}{\sigma^2}(b-B)}-1}$, a contradiction. *Q.E.D.*

Proof of Lemma 4

PROOF: The same arguments as in Lemma 1 imply $\check{J}$ is single-peaked in B and b. Given this, it suffices to show that $\frac{\partial \check{J}(B,b,x,U')}{\partial B} \geq \frac{\partial \check{J}(B,b,x,U)}{\partial B}$ for $U' > U \geq 0$. Using $\phi_B \geq 0$, we have

$$\begin{aligned}\frac{\partial \check{J}(B,b,x,U')}{\partial B} &= \Phi_B(B,b,x)u(B,1) + \Phi(B,b,x)\tilde{u}'(B) + \phi_B(B,b,x)\left(U' + \frac{c_R}{r}\right)\\ &\geq \Phi_B(B,b,x)u(B,1) + \Phi(B,b,x)\tilde{u}'(B) + \phi_B(B,b,x)\left(U + \frac{c_R}{r}\right)\\ &= \frac{\partial \check{J}(B,b,x,U)}{\partial B}.\end{aligned}$$

Q.E.D.

For our next two proofs, it is useful to define the function $\bar{V}(B,x) := \max_b \widetilde{V}(B,b,x)$, which gives A's continuation value at $X_t = x$ when A is allowed to choose optimally when to quit but R fixes the approval threshold at B.

Proof of Lemma 7

PROOF: Take $m^1 < m^2$. As shown in the proof of Proposition 5, $\widetilde{V}(\underline{B}_N(m^i), m^i - \delta_N, m^i; z_0) = \widetilde{V}(\underline{B}_N(m^i) - m^i, -\delta_N, 0; z_{m^i})$. By Lemma 3,

$$\widetilde{V}\big(\underline{B}_N(m^1) - m^1, -\delta_N, 0; z_{m^2}\big) > \widetilde{V}\big(\underline{B}_N(m^1) - m^1, -\delta_N, 0; z_{m^1}\big) = 0.$$

Because $\lim_{B\to\infty} \widetilde{V}(B, b, 0; z) < 0$ for any $b < 0$ and $\widetilde{V}$ is single-peaked in B, we can find a unique $B' > \underline{B}_N(m^1) - m^1$ such that $\widetilde{V}(B', -\delta_N, 0; z_{m^2}) = 0$. It must then be that $\underline{B}_N(m^2) = B' + m^2 > \underline{B}_N(m^1) + m^2 - m^1 > \underline{B}_N(m^1)$, so $\underline{B}_N(m)$ is increasing.

Suppose there is a discontinuity in $\underline{B}_N$ at m'. For sufficiently small ϵ, continuity of $\widetilde{V}$ implies

$$0 = \widetilde{V}\big(\underline{B}_N(m' + \epsilon), m' + \epsilon - \delta_N, m' + \epsilon\big) \approx \widetilde{V}\big(\underline{B}_N(m' + \epsilon), m' - \epsilon - \delta_N, m' - \epsilon\big).$$

$\widetilde{V}(B, m' - \epsilon - \delta_N, m' - \epsilon)$ is strictly decreasing in B for $B \geq \underline{B}_N(m' - \epsilon)$. Because $\lim_{\epsilon\to 0}(\underline{B}_N(m'+\epsilon) - \underline{B}_N(m'-\epsilon)) > 0$, we have $\lim_{\epsilon\to 0} \widetilde{V}(\underline{B}_N(m'+\epsilon), m'-\epsilon-\delta_N, m'-\epsilon) < 0$, a contradiction. Therefore, $\underline{B}_N$ must be continuous.

Because $\widetilde{V}(\underline{B}_N(m), m, m) = 0$ and $\widetilde{V}$ is single-peaked with respect to b, in order for $\widetilde{V}(\underline{B}_N(m), m - \delta_N, m) = 0$, it must be that $b^*(\underline{B}_N(m)) \in (m - \delta_N, m)$; taking the limit, we get $b^*(\underline{B}_\infty(m)) = m$ where $\underline{B}_\infty(m) = \lim_{N\to\infty} \underline{B}_N(m)$.

Take any $m' > b_A^{FB} + \delta_N$. Choosing $B_A^{FB} = \arg\max_B \bar{V}(B, x)$ maximizes $\bar{V}(B, x)$ for all x,[8] and so increases $\inf\{x : \bar{V}(B, x) > 0\} = b^*(B)$. Thus, $\widetilde{V}(B_A^{FB}, b_A^{FB}, m') > 0$, which implies $\widetilde{V}(B_A^{FB}, m' - \delta_N, m') > 0$ by Lemma 20. Since $\lim_{B\to\infty} \widetilde{V}(B, b, x) < 0$ for all $b < x$, we can find a $B' > B_A^{FB}$ such that $\widetilde{V}(B', m' - \delta_N, m') = 0$. Thus, $\underline{B}_N(m') > B_A^{FB}$ and so $\underline{B}_\infty(m') \geq B_A^{FB}$.

We now show $b^*(B)$ is increasing and continuous in B for $B > B_A^{FB}$. Uniqueness of A's optimal stopping thresholds (and so $b^*(B)$) follows from the same arguments in Lemma 21. Continuity of $b^*(B)$ follows from the theorem of the maximum. For $x' \in (x, B]$, $\bar{V}(B, x) = \mathbb{E}^x[e^{-r\tau_+(x';b^*(B))}\bar{V}(B, x') + e^{-r\tau(b^*(B);x')}\frac{c_A}{r}] - \frac{c_A}{r}$. Because A prefers immediate approval whenever above B_A^{FB}, we know $\bar{V}(B, x') < \bar{V}(x', x')$ for each $B > x' \geq B_A^{FB}$. Thus, increasing $B \geq B_A^{FB}$ reduces A's continuation value at all $x < B$ and so must increase $b^*(B)$. Because $b^*(B)$ is increasing in $B \geq B_A^{FB}$, there is a unique $B \geq B_A^{FB}$ such that $b^*(B) = m'$. Since $\underline{B}_\infty(m') > B_A^{FB}$, $\underline{B}_\infty(m')$ is this unique B. We conclude that $\underline{B}_\infty(m) = \underline{B}(m)$. Continuity of $\underline{B}(m)$ follows from continuity of $b^*(B)$. *Q.E.D.*

Continuity in Limit of Optimal Mechanisms

Here, we verify $\lim_{N\to\infty} J(\tau_N^*, d_{N,\tau}^*, z_0) = J(\tau^*, d_\tau^*, z_0)$. Take $\epsilon \in (0, \min_m B(m) - m)$, $K < \max\{u(-\infty, 1), 0\}$, $\underline{\tau}_N = \tau^* \wedge \tau_N^*$, $\overline{\tau}_N = \tau^* \vee \tau_N^*$, $\underline{d}_N = d_\tau^* \mathbb{1}(\underline{\tau}_N = \tau^*) + d_{N,\tau}^* \mathbb{1}(\underline{\tau}_N = \tau_N^*)$. Define $\overline{d}_N$ analogously but replacing $\underline{\tau}_N$ with $\overline{\tau}_N$. Let $\overline{B}_{\underline{\tau}_N} = B(M_{\underline{\tau}_N}) \vee B_N(M_{\underline{\tau}_N})$ and $\overline{b}_N = \underline{b} \wedge \underline{b}_N$. Then $|J(\tau^*, d_\tau^*, z_0) - J(\tau_N^*, d_{N,\tau}^*, z_0)|$ is equal to

$$\begin{aligned}
&\mathbb{E}\big[e^{-r\underline{\tau}_N}\big|u(X_{\underline{\tau}_N}, \underline{d}_N) - \mathbb{E}_{X_{\underline{\tau}_N}, M_{\underline{\tau}_N}}\big[e^{-r(\overline{\tau}_N - \underline{\tau}_N)}u(X_{\overline{\tau}_N}, \overline{d}_N)\big]\big|\big] \\
&\quad \leq \mathbb{E}[e^{-r\underline{\tau}_N}\underline{d}_N|u(X_{\underline{\tau}_N}, 1) - \mathbb{E}^{X_{\underline{\tau}_N}}\big[e^{-r\tau_+(\overline{B}_{\underline{\tau}_N}; X_{\underline{\tau}_N} - \epsilon)}u(X_{\underline{\tau}_N} - \epsilon, 1) + e^{-r\tau(X_{\underline{\tau}_N} - \epsilon; \overline{B}_{\tau_N})}K|\big]
\end{aligned}$$

[8]Standard dynamic programming arguments imply that A's optimal threshold can be chosen independent of x.

$$+\mathbb{E}[e^{-r\underline{\tau}_N}(1-\underline{d}_N)|u(X_{\underline{\tau}_N},0)-\mathbb{E}^{X_{\underline{\tau}_N}}[e^{-r\tau(\underline{b}_N;X_{\underline{\tau}_N}+\epsilon)}u(\underline{b}_N,0)+e^{-r\tau_+(X_{\underline{\tau}_N}+\epsilon;\underline{b}_N)}K|].$$

Because $X_{\underline{\tau}_N}=B(M_{\underline{\tau}_N})\wedge B_N(M_{\underline{\tau}_N})$ when $\underline{d}_N=1$ and $\lim_{N\to\infty}|B(m)-B_N(m)|=0$, it is easily verified that for each history $h_{\underline{\tau}_N}$, $\lim_{N\to\infty}\mathbb{E}^{B(M_{\underline{\tau}_N})\wedge B_N(M_{\underline{\tau}_N})}[e^{-r\tau_+(\overline{B}_{\underline{\tau}_N};X_{\underline{\tau}_N}-\epsilon)}]=1$ and $\lim_{N\to\infty}\mathbb{E}^{B(M_{\underline{\tau}_N})\wedge B_N(M_{\underline{\tau}_N})}[e^{-r\tau(X_{\underline{\tau}_N}-\epsilon;\overline{B}_{\underline{\tau}_N})}]=0$, so the first absolute value after the inequality above converges to $\underline{d}_N(u(X_{\underline{\tau}_N},1)-u(X_{\underline{\tau}_N}-\epsilon,1))$ as $N\to\infty$. Since ϵ is arbitrary, the first expectation can be made to converge to 0. A similar argument holds for the second expectation after the inequality. We conclude that $\lim_{N\to\infty}|J(\tau^*,d^*_\tau,z_0)-J(\tau^*_N,d^*_{N,\tau},z_0)|=0$. Analogous arguments show the difference in A's continuation value after history h_t from τ^*_N and τ^* goes to 0 as $N\to\infty$.

APPENDIX F: ADDITIONAL RESULTS FROM SECTION 4

We now show that DP is a relaxation of the dynamic participation constraint.

LEMMA 23: *If (τ,d_τ) satisfies the dynamic participation constraint, it satisfies DP.*

PROOF: Suppose (τ,d_τ) satisfies the dynamic participation constraint. For any τ', $V(\tau,d_\tau,z_0)-V(\tau\wedge\tau',d_\tau\mathbb{1}(\tau<\tau'),z_0)$ is equal to

$$\mathbb{E}\left[e^{-r\tau'}\mathbb{1}(\tau\geq\tau')\left\{\mathbb{E}^{X_{\tau'}}\left[e^{-r\tau[h_{\tau'}]}v\left(X_{\tau[h_{\tau'}]},d_\tau[h_{\tau'}]\right)\right]-\frac{c_A}{r}\right\}\right]$$

DP holds if $\mathbb{E}[e^{-r\tau'}\mathbb{1}(\tau\geq\tau')\{\mathbb{E}^{X_{\tau'}}[e^{-r\tau[h_{\tau'}]}v(X_{\tau[h_{\tau'}]},d_\tau[h_{\tau'}])]-\frac{c_A}{r}\}]\geq 0$ for all τ', which follows because $\mathbb{E}^{X_{\tau'}}[e^{-r\tau[h_{\tau'}]}v(X_{\tau[h_{\tau'}]},d_\tau[h_{\tau'}])]-\frac{c_A}{r}$ is A's continuation value under (τ,d_τ) at $h_{\tau'}$ and is positive by the dynamic participation constraint. *Q.E.D.*

We next prove the result mentioned in the Introduction of Section 4 in which we consider R's problem with only a time-zero participation constraint.

PROPOSITION 8: *For any $W\in[0,\sup_{(\tau,d_\tau)}V(\tau,d_\tau,z_0))$, the solution to $\sup_{(\tau,d_\tau)}J(\tau,d_\tau,z_0)$ subject to $V(\tau,d_\tau,z_0)\geq W$ is a static threshold mechanism.*

PROOF: Using Theorem 1 of Balzer and Janßen (2002), there exists a $\widehat{\lambda}\leq 0$ such that the value of R's problem is equal to

$$\sup_{(\tau,d_\tau)}\mathbb{E}\left[e^{-r\tau}\left(u(X_\tau,d_\tau)-\widehat{\lambda}v(X_\tau,d_\tau)\right)\right]-\frac{c_R}{r}+\widehat{\lambda}\left(W+\frac{c_A}{r}\right).$$

Given that $u(x,1)-\widehat{\lambda}v(x,1)$ is increasing in x, by standard optimal stopping arguments, the optimal stopping rule takes the form $\tau^*=\inf\{t:X_t\notin(b^*,B^*)\}$ for some $b^*\leq 0\leq B^*$ and $d^*_\tau=\mathbb{1}(X_\tau\geq B^*)$ (we allow for $b^*=-\infty$ if it is never optimal to reject). The same arguments as in the proof of Proposition 5 show that (τ^*,d^*_τ) will solve R's problem for an appropriate choice of $\widehat{\lambda}$. *Q.E.D.*

Proof of Proposition 1

PROOF: Compare the optimal mechanisms (in Z-space) for $Z_0\in\{z^1,z^2\}$ with $z^1>z^2$. Let $(\tau^{Z,i},d^{Z,i}_\tau)$ be the optimal mechanism when $Z_0=z^i$ and let $B^Z_i(m)$ be the approval threshold from $(\tau^{Z,i}, d_\tau^{Z,i})$ in Z-space when $M_t^Z = m$. Define $b_Z^*(\cdot)$ and $\underline{B}_Z(\cdot)$ analogously to the $b^*(\cdot)$, $\underline{B}(\cdot)$. Let $\tau_+^Z(B) = \inf\{t : Z_t \geq B\}$ and $\tau^Z(b) = \inf\{t : Z_t \leq b\}$.

We start by arguing that the rejection threshold in all optimal mechanisms is equal to the highest z, call it $\underline{z}$, such that $\sup_{(\tau,d_\tau)} J(\tau, d_\tau, z)$ subject to $DP(z)$ is equal to 0. It is never optimal to reject at $\tau^Z(z)$ for $z > \underline{z}$ as, for each $(\tau^{Z,i}, d_\tau^{Z,i})$, there exists a continuation mechanism at $\tau^Z(z)$ that makes both R and A better off.[9] If R does not reject at $\tau^Z(\underline{z})$ under $(\tau^{Z,i}, d_\tau^{Z,i})$, then A's continuation value at $\tau^Z(\underline{z})$ is strictly positive; otherwise R could reject at $\tau^Z(\underline{z})$ and be better off without making A worse off.

Suppose A's continuation value was strictly positive at $\tau^Z(\underline{z})$ under $(\tau^{Z,i}, d_\tau^{Z,i})$. The approval threshold must be constant prior to $\tau^Z(\underline{z})$ and $b_Z^*(B_i^Z(\underline{z})) < \underline{z}$. R would be better off increasing the rejection threshold to $\underline{z}$. By the same arguments as in Lemma 20, doing so will not violate DP,[10] contradicting the optimality of $(\tau^{Z,i}, d_\tau^{Z,i})$. We conclude that all optimal mechanisms will use the rejection threshold $\underline{z}$.

We now show $B_1^Z(m) = B_2^Z(m)$ for all $m \leq z^2$. Once the approval threshold begins to decrease, it is pinned down as $\underline{B}_Z$. Therefore, it suffices to show that $\bar{B}_1^Z := B_1^Z(z^2) = B_2^Z(z^2) =: \bar{B}_2^Z$. Suppose $\bar{B}_1^Z \neq \bar{B}_2^Z$. Let $J_i(z)$ be R's continuation value under $(\tau^{Z,i}, d^{Z,i})$ at $\tau^Z(z)$. Because the continuation mechanism for $(\tau^{Z,1}, d_\tau^{Z,1})$ at $\tau^Z(z^2)$ satisfies DP, we must have $J_1(z^2) \leq J_2(z^2)$ by the optimality when $Z_0 = z^2$ of using $(\tau^{Z,2}, d_\tau^{Z,2})$ rather than the continuation mechanism for $(\tau^{Z,1}, d_\tau^{Z,1})$ at $\tau^Z(z^2)$.

Suppose $J_1(z^2) < J_2(z^2)$. If A's continuation value is 0 at $\tau^Z(z^2)$ under $(\tau^{Z,1}, d_\tau^{Z,1})$, then R is strictly better off changing the continuation mechanism of $(\tau^{Z,1}, d_\tau^{Z,1})$ at $\tau^Z(z^2)$ to $(\tau^{Z,2}, d_\tau^{Z,2})$ because it (weakly) increases both players' continuation values, strictly so for R.

Suppose A's continuation value under $(\tau^{Z,1}, d_\tau^{Z,1})$ at $\tau^Z(z^2)$ is strictly positive. Then $z^2 > b_Z^*(\bar{B}_1^Z)$. Construct a mechanism (τ', d'_τ) that only stops prior to $\tau^Z(z^2)$ if $Z_t \geq \bar{B}_1^Z$ and then uses $(\tau^{Z,2}, d_\tau^{Z,2})$ as its continuation mechanism at $\tau^Z(z^2)$. When $Z_0 = z^1$, (τ', d'_τ) leads to the same outcomes as $(\tau^{Z,1}, d_\tau^{Z,1})$ if $\tau^{Z,1} < \tau^Z(z^2)$ and increases R's continuation value at $\tau^Z(z^2)$. Because $(\tau^{Z,2}, d_\tau^{Z,2})$ satisfies DP, to show that DP is satisfied under (τ', d'_τ) we need only verify that A has no incentive to quit before $\tau^Z(z^2)$. Because A's continuation value under $(\tau^{Z,1}, d_\tau^{Z,1})$ is weakly positive at $\tau^Z(z^2)$, A's continuation value under (τ', d'_τ) at h_t with $t < \tau^Z(z^2)$ is bounded below his value of a static threshold mechanism (with thresholds in Z-space) with approval threshold $\bar{B}_1^Z$ and rejection threshold z^2. Because $z^2 > b_Z^*(\bar{B}_1^Z)$, A's value of this static threshold mechanism is positive by the arguments in Lemma 20. Thus, (τ', d'_τ) satisfies DP and is a strict improvement for R over $(\tau^{Z,1}, d_\tau^{Z,1})$ when $Z_0 = z^1$, contradicting the optimality of $(\tau^{Z,1}, d_\tau^{Z,1})$. Therefore, $J_1(z^2) = J_2(z^2)$. Using the continuation mechanism from $(\tau^{Z,1}, d_\tau^{Z,1})$ at $\tau^Z(z^2)$ when $Z_0 = z^2$ is therefore optimal, meaning the $\bar{B}_1^Z = \bar{B}_2^Z$. *Q.E.D.*

This result implies that, in X space, the approval threshold function in the optimal SI-mechanism when $(X_0, Z_0) = (0, z_0)$ and in the optimal SI-mechanism when $(X_0, Z_0) = (x, z_x)$ are the same when $x < 0$.

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[9]To see this, note that by fixing the optimal mechanism at some Z_0 as a function of (X, M) and increasing Z_0, we will slacken DP and raise R's expected utility.

[10]The same properties in Lemma 20 hold when we write $\widetilde{V}$ in terms of Z_t rather than X_t.

General Utility Functions

As mentioned at the end of Section 4, we can extend Theorem 1 to allow for more general utility functions than presented in the main body of the text. We place the following assumptions on $\tilde{u}$ and $\tilde{v}$.

ASSUMPTION 1: *$\tilde{u}, \tilde{v}$ are bounded, differentiable, and such that $\tilde{v}(x) \geq \tilde{u}(x)$, $\tilde{u}'(x) \geq \tilde{v}'(x) \geq 0$ and $\tilde{u}(X_t), \tilde{v}(X_t)$ are supermartingales.*

In our main specification of the model, $\tilde{v}(x) \geq \tilde{u}(x)$, $\tilde{u}'(x) \geq \tilde{v}'(x) \geq 0$ are captured by $a \in [f, 1]$. Translating from X_t into π_t, because π_t is a martingale, $\tilde{u}$ and $\tilde{v}$ are supermartingales if they are weakly concave in π_t. This condition holds in our main model, in which $\tilde{u}$ and $\tilde{v}$ are linear in π_t.

The proof when $\tilde{u}$ and $\tilde{v}$ are supermartingales changes only slightly; in particular, we only need to change the equalities that result when we apply Doob's optional stopping theorem and the fact $\tilde{u}$ and $\tilde{v}$ are martingales to inequalities going in the needed direction when they are supermartingales.

No Commitment

We first specify the details of the model without commitment. We assume A can temporarily stop experimenting at any time. No flow cost is paid while experimentation is stopped and R can approve at any time.[11]

A strategy for A is a process $\alpha = \{\alpha_t : 0 \leq t < \infty\}$ that is measurable with respect to the filtration generated by X. A continuation strategy of α^* at history h_t is $\alpha^*[h_t]$ defined by, for each ω with history h_t, $\alpha^*[h_t](\chi_t \omega) = \alpha^*(\omega)$. Both agents observe X, which solves to stochastic differential equation $dX_t = \alpha_t(\mu_\theta dt + \sigma\, dW_t)$. R's strategy is given as before by a stopping time and decision rule (τ, d_τ).[12]

DEFINITION 8: A pair $(\alpha^*, (\tau^*, d_\tau^*))$ is an equilibrium if for every history h_t, the continuation actions $\alpha^*[h_t]$ and $(\tau^*[h_t], d_\tau^*[h_t])$ satisfy

- $\alpha^*[h_t] \in \arg\max_\alpha \mathbb{E}^{X_t}[e^{-r\tau^*[h_t]}\tilde{v}(X_{\tau^*[h_t]})\, d_\tau^*[h_t] - \int_0^{\tau^*[h_t]} e^{-rs}\alpha_s c_A\, ds | \alpha]$.
- $(\tau^*[h_t], d_\tau^*[h_t]) \in \arg\max_{\tau, d_\tau} \mathbb{E}^{X_t}[e^{-r\tau}\tilde{u}(X_\tau)\, d_\tau - \int_0^\tau e^{-rs}\alpha_s^*[h_t] c_R\, ds | \alpha^*[h_t]]$.

PROPOSITION 9: *The optimal mechanism can be implemented as an equilibrium.*

PROOF: Suppose R uses (τ^*, d_τ^*) from Theorem 1 and A uses the following strategy: experiment until τ^*, at which immediately stop and never restart experimenting, and if experimentation has stopped before τ^*, immediately restart experimenting and keep experimenting until τ^*.

We claim this is an equilibrium. First, consider the incentives of R to deviate. Suppose the equilibrium calls for R to approve at time τ^*. If she does not approve at τ^*, A quits experimenting at τ^* forever. Because no new learning occurs, R prefers to approve immediately at τ^* because $\tilde{u}(X_{\tau^*}) \geq 0$. Suppose R had a profitable deviation to stop at some

[11]The case when A can irrevocably quit experimenting has been studied in Kolb (2019) and Henry and Ottaviani (2019). Using the Markov perfect equilibrium as the solution concept, they find an equilibrium in which R's approval decision takes a static threshold form.

[12](τ, d_τ) is taken to be measurable with respect to the sigma algebra generated by $\{\alpha_s, X_s : 0 \leq s \leq t\}$.

τ' such that $\tau' \leq \tau^*$. R will never find it profitable to reject earlier than τ^*. If R's continuation value was negative at some history h_t with $X_t \geq M_t > \underline{b}$, then R's continuation value would be negative at $\tau(M_t)$ and R would be better off under rejecting at $\tau(M_t)$; by similar arguments as those made in the proof of Proposition 1, rejection at $\tau(M_t)$ would still satisfy DP, a contradiction of the optimality of (τ^*, d^*_τ). Therefore, R must approve at τ' when $\tau' < \tau^*$. If R is better off approving at a history $h_{\tau'}$ with $X_{\tau'} \in [M_{\tau'}, B(M_{\tau'}))$, then R would better off lowering the approval threshold $B(M_{\tau'})$, which would increase A's utility as well by the arguments in Lemma 8, contradicting the optimality of τ^*. Therefore, no such deviation can exist.

Next, we consider the incentives of A to deviate from the proposed equilibrium. Under the proposed approval rule, the dynamic participation constraint implies A has no incentive to quit early. If he were to quit early, R would believe A will restart experimenting immediately and, therefore, not find it optimal to approve. Moreover, A has an incentive to stop experimenting at τ^* because he believes R will approve immediately. In the off-path event that R does not approve, A believes R will approve in the next instant and has no incentive to restart experimentation because it is costly and will not increase the probability of approval. Because neither A nor R have an incentive to deviate, (τ^*, d^*_τ) is an equilibrium. *Q.E.D.*

APPENDIX G: OMITTED PROOFS FROM SECTION 5

Proof of Lemma 11

PROOF: Because $a \geq 0$, $v_i(x,1) > v_i(x,0) > 0$ for all x and $i \in \{\ell, h\}$. Because $v_i(X_t, 1)$ is a strictly positive martingale, for any $b < x < B$ we have

$$\begin{aligned}\widetilde{V}_i(B,b,x) &\leq \mathbb{E}^{x,z_i(x)}\big[e^{-r(\tau_+(B)\wedge\tau(b))}v_i(X_{\tau_+(B)\wedge\tau(b)},1)\big] - \frac{c_A}{r} \\ &< \mathbb{E}^{x,z_i(x)}\big[v_i(X_{\tau_+(B)\wedge\tau(b)},1)\big] - \frac{c_A}{r} = \tilde{v}_i(x) = \widetilde{V}_i(x,b,x).\end{aligned}$$

Take any $B' \in (x, B)$. Using $\widetilde{V}_i(B,b,B') < \tilde{v}_i(B')$, we have

$$\begin{aligned}\widetilde{V}_i(B,b,x) &= \mathbb{E}^{x,z_i(x)}\big[e^{-r\tau_+(B';b)}\big]\left(\widetilde{V}_i(B,b,B') + \frac{c_A}{r}\right) + \mathbb{E}^{x,z_i(x)}\big[e^{-r\tau(b;B')}\big]\frac{c_A}{r} - \frac{c_A}{r} \\ &< \mathbb{E}^{x,z_i(x)}\big[e^{-r\tau_+(B';b)}\big]\left(\tilde{v}_i(B') + \frac{c_A}{r}\right) + \mathbb{E}^{x,z_i(x)}\big[e^{-r\tau(b;B')}\big]\frac{c_A}{r} - \frac{c_A}{r} = \widetilde{V}_i(B',b,x).\end{aligned}$$

Thus, $\widetilde{V}_i$ is decreasing in B. *Q.E.D.*

Proof of Lemma 12

PROOF: Given $\underline{B}_i(m) = b_i^{*-1}(m)$, it suffices to show $b^*(B;z)$ is decreasing in z. For the sake of contradiction, suppose $b^*(B;\infty) > b^*(B;-\infty)$. Without loss, assume $0 \in (b^*(B;\infty), B)$. As in Lemma 3, let $\Psi(b) = \mathbb{E}[e^{-r\tau_+(B;b)}|H]$ and $\psi(b) = \mathbb{E}[e^{-r\tau(b;B)}|H]$; we will drop dependence on b when $b = b^*(B;\infty)$. By single-peakedness of $\widetilde{V}$ with respect to b, $\frac{\partial \widetilde{V}(B,b,0;\infty)}{\partial b}|_{b=b^*(B;\infty)} = 0 > \frac{\partial \widetilde{V}(B,b,0;-\infty)}{\partial b}|_{b=b^*(B;\infty)}$. By the definitions of $\widetilde{V}(B,b,0;\infty)$ and $\widetilde{V}(B,b,0;-\infty)$ provided in Lemma 3, $\frac{\partial \widetilde{V}(B,b,0;\infty)}{\partial b}|_{b=b^*(B;\infty)} = \frac{d\Psi}{db}(1+\frac{c_A}{r}) + \frac{d\psi}{db}\frac{c_A}{r} = 0$, which implies $\frac{d\Psi}{db} = -\frac{d\psi}{db}\frac{c_A}{r+c_A}$ and $0 > \frac{\partial \widetilde{V}(B,b,0;-\infty)}{\partial b}|_{b=b^*(B;\infty)}$ implies

$$0 > \frac{d\Psi}{db} e^{-\frac{2\mu}{\sigma^2}B}\left(a + \frac{c_A}{r}\right) + \frac{d\psi}{db} e^{-\frac{2\mu}{\sigma^2}b^*(B;\infty)}\frac{c_A}{r} - \frac{2\mu}{\sigma^2}\psi e^{-\frac{2\mu}{\sigma^2}b^*(B;\infty)}\frac{c_A}{r}.$$

Let $\Delta = B - b^*(B;\infty)$. Using $\frac{d\Psi}{db} = -\frac{d\psi}{db}\frac{c_A}{r+c_A}$, the above inequality is equivalent to

$$\frac{d\psi}{db}\left(1 - \frac{ar+c_A}{r+c_A}e^{-\frac{2\mu}{\sigma^2}\Delta}\right) - \frac{2\mu}{\sigma^2}\psi < 0.$$

Using the formula for ψ provided in Lemma 3, $\frac{d\psi}{db} = \psi\frac{R_2 e^{-R_2\Delta} - R_1 e^{-R_1\Delta}}{e^{-R_1\Delta} - e^{-R_2\Delta}}$. Plugging this into the above inequality and simplifying, we have

$$\frac{R_2 e^{-R_2\Delta} - R_1 e^{-R_1\Delta}}{e^{-R_1\Delta} - e^{-R_2\Delta}} < \frac{2\mu}{\sigma^2\left(1 - \frac{ar+c_A}{r+c_A}e^{-\frac{2\mu}{\sigma^2}\Delta}\right)} \le \frac{2\mu}{\sigma^2\left(1 - e^{-\frac{2\mu}{\sigma^2}\Delta}\right)}.$$

Recall from Lemma 3 that $R_1 + R_2 = -\frac{2\mu}{\sigma^2}$ and $R_2 \ge 0$. If $R_2 = 0$, then $\frac{R_2 e^{-R_2\Delta} - R_1 e^{-R_1\Delta}}{e^{-R_1\Delta} - e^{-R_2\Delta}} = \frac{2\mu}{\sigma^2(1-e^{-\frac{2\mu}{\sigma^2}\Delta})}$. The derivative of $\frac{R_2 e^{-R_2\Delta} - R_1 e^{-R_1\Delta}}{e^{-R_1\Delta} - e^{-R_2\Delta}}$ with respect to R_2 when $R_1 = -\frac{2\mu}{\sigma^2} - R_2$ is $\frac{\sinh(\Delta(\frac{2\mu}{\sigma^2}+2R_2)) - \Delta(\frac{2\mu}{\sigma^2}+2R_2)}{\cosh(\Delta(\frac{2\mu}{\sigma^2}+2R_2)) - 1} \ge 0$. Thus, $\frac{R_2 e^{-R_2\Delta} - R_1 e^{-R_1\Delta}}{e^{-R_1\Delta} - e^{-R_2\Delta}} \ge \frac{2\mu}{\sigma^2(1-e^{-\frac{2\mu}{\sigma^2}\Delta})}$, a contradiction. We conclude that $b^*(B;\infty) < b^*(B;-\infty)$.

$b^*(B;z)$ is characterized by the first-order condition

$$\frac{e^{zx}}{1+e^{zx}}\frac{\partial \widetilde{V}(B,b,x;\infty)}{\partial b}\bigg|_{b=b^*(B;z)} + \frac{1}{1+e^{zx}}\frac{\partial \widetilde{V}(B,b,x;-\infty)}{\partial b}\bigg|_{b=b^*(B;z)} = 0. \tag{16}$$

Given $b^*(B;\infty) < b^*(B;-\infty)$ and the single-peakedness of $\widetilde{V}$ with respect to b, if $\frac{\partial \widetilde{V}(B,b,x;\infty)}{\partial b} > 0$, then $\frac{\partial \widetilde{V}(B,b,x;-\infty)}{\partial b} > 0$. To satisfy (16) at $b = b^*(B;z)$, we must have $\frac{\partial \widetilde{V}(B,b,x;\infty)}{\partial b} < 0 < \frac{\partial \widetilde{V}(B,b,x;-\infty)}{\partial b}$. Taking the derivative of our first-order condition with respect to z and doing a bit of algebra, we get that the sign of $\frac{\partial b^*(B;z)}{\partial z}$ is equal to $\frac{\partial V(B,b,x;\infty)}{\partial b}|_{b=b^*(B;z)} - \frac{\partial V(B,b,x;-\infty)}{\partial b}|_{b=b^*(B;z)} < 0$. *Q.E.D.*

Proof of Lemma 13

PROOF: Let $m' = \max\{m \le 0 : V^*(\tau', d'_\tau, m, z_\ell(m)) = 0\}$, because (τ', d'_τ) rejects at $\tau(\underline{b})$, $m' \ge \underline{b}$. Thus, $V^*(\tau', d'_\tau, m, z_\ell(m)) > 0$ for all $m \in (m', 0]$ if $m' < 0$.

Suppose $m' < 0$. Take some small $\epsilon > 0$. Because ℓ always prefers a lower approval threshold and his continuation value is 0 at $\tau(\underline{m})$, $V^*(\tau', d'_\tau, m' + \epsilon, z_\ell(m' + \epsilon))$ is bounded above by his expected utility from the static threshold mechanism with approval threshold $B(m')$ and rejection threshold m', namely $\widetilde{V}_\ell(B(m'), m', m' + \epsilon) \ge V^*(\tau', d'_\tau, m' + \epsilon, z_\ell(m' + \epsilon)) \ge 0$. As is shown in in the proof of Lemma 7, $b_i^*(B)$ is increasing in B for $B > B_A^{FB}$. Because $B_A^{FB} = -\infty$ when $a \ge 0$ and $B(m') > \underline{B}_\ell(m')$, $b_\ell^*(B(m')) > m'$, which implies that ℓ's continuation value in the static threshold mechanism at $\tau(m'')$ for $m'' \in (m', b_\ell^*(B(m'))$ is strictly negative. For $\epsilon \in (0, b_\ell^*(B(m')) - m')$, we have $\widetilde{V}_\ell(B(m'), m', m' + \epsilon) < 0$, a contradiction. Therefore, $m' = 0$, which implies $V^*(\tau', d'_\tau, z_\ell) = 0$. *Q.E.D.*

Proof of Lemma 14

For this proof, we will use the characterization of the optimal mechanism when $DIC(h)$ is dropped. None of the proofs when deriving the optimal mechanism when $DIC(h)$ was dropped relied on this lemma. In the next two proofs, we will use X_c^i to denote the value of X_c when $z_0 = z_i$.

PROOF: Take z_h sufficiently large and let (τ^i, d_τ^i) be type i's mechanism when $DIC(h)$ is dropped. As $z_h \to \infty$, $X_c^h \to -\infty$. By the arguments in Lemma 15, R will never reject h while $X_t > X_c^h$. Thus, the probability that R rejects h goes to 0 as $z_h \to \infty$.

Suppose h weakly prefers ℓ's mechanism. It is straightforward to verify that h would never quit prior to $\tau(\underline{b}_\ell)$ under (τ^ℓ, d_τ^ℓ). Consider a modification of ℓ's mechanism, call it $(\tilde{\tau}^\ell, \tilde{d}_\tau^\ell)$, that uses the same approval threshold as (τ^ℓ, d_τ^ℓ) prior to $\tau(\underline{b}_\ell)$ but uses a continuation mechanism (τ', d'_τ) at $\tau(\underline{b}_\ell)$ with $\tau' = \inf\{t : X_t \notin (X_c^h, B'(M_t))\}$ and $d'_\tau = \mathbb{1}(X_{\tau'} \geq B'(M_{\tau'}))$ for some function B' with $B'(m) \in (\underline{B}_\ell(m), \underline{B}_h(m))$. By Lemma 13, ℓ will find it optimal to quit at $\tau(\underline{b}_\ell)$ under $(\tilde{\tau}^\ell, \tilde{d}_\tau^\ell)$, so $V^*(\tilde{\tau}^\ell, \tilde{d}_\tau^\ell, z_\ell) = V(\tau^\ell, d_\tau^\ell, z_\ell)$. Thus, replacing (τ^h, d_τ^h) with $(\tilde{\tau}^\ell, \tilde{d}_\tau^\ell)$ will satisfy $DIC(\ell)$ and increase the discounted probability of approval. It is easy to see that $\underline{b}_\ell$ is finite in the limit as $z_h \to \infty$, so this increase in the discounted probability of approval is bounded away from 0 as $z_h \to \infty$.

h's continuation value under $(\tilde{\tau}^\ell, \tilde{d}_\tau^\ell)$ is strictly positive at $\tau(\underline{b}_\ell)$, so h will now strictly prefer $(\tilde{\tau}^\ell, \tilde{d}_\tau^\ell)$ to (τ^h, d_τ^h). Because the discounted probability of rejection is approximately 0 under both (τ^h, d_τ^h) and $(\tilde{\tau}^\ell, \tilde{d}_\tau^\ell)$, for h to strictly prefer $(\tilde{\tau}^\ell, \tilde{d}_\tau^\ell)$ to (τ^h, d_τ^h), it must be that $\mathbb{E}^{0,z_h}[e^{-r\tau^h} d_\tau^h(1 + \frac{c_A}{r})] < \mathbb{E}^{0,z_h}[e^{-r\tilde{\tau}^\ell} \tilde{d}_\tau^\ell(1 + \frac{c_A}{r})]$, which implies $\mathbb{E}^{0,z_h}[e^{-r\tau^h} d_\tau^h] < \mathbb{E}^{0,z_h}[e^{-r\tilde{\tau}^\ell} \tilde{d}_\tau^\ell]$.

For z_h sufficiently large, R's expected utility from (τ, d_τ) is approximately $\mathbb{E}^{0,z_h}[e^{-r\tau} d_\tau]$. Because offering h $(\tilde{\tau}^\ell, \tilde{d}_\tau^\ell)$ would satisfy $DIC(\ell)$, $(\tilde{\tau}^\ell, \tilde{d}_\tau^\ell)$ represents an improvement for R over (τ^h, d_τ^h), a contradiction. Therefore, h must strictly prefer (τ^h, d_τ^h) to (τ^ℓ, d_τ^ℓ). *Q.E.D.*

Proof of Lemma 17

PROOF: Suppose, for the sake of contradiction, $X^k > \underline{X}_\ell^N$ and $X^{k+1} < X^k - \delta_N$, so that $X^{k+1} + \delta_N \notin \mathcal{B}_N$. By Lemma 16, $\rho(X^{k+1} + \delta_N) > 0 > \rho(X^k)$. Because $B_N(X^{k+1} + \delta_N) = B_N(X^k)$ and $\rho_\ell(X_n) = \widetilde{V}_\ell(B_N(X_n), X_n - \delta_N, X_n)$, we have

$$\widetilde{V}_\ell\big(B_N\big(X^k\big), X^{k+1}, X^{k+1} + \delta_N\big) > 0 > \widetilde{V}_\ell\big(B_N\big(X^k\big), X^k - \delta_N, X^k\big). \tag{17}$$

Because $\widetilde{V}_\ell$ is strictly decreasing in B, Lemma 17 implies $\underline{B}_{N,\ell}(X^{k+1} + \delta_N) > B_N(X^k) > \underline{B}_{N,\ell}(X^k)$, which contradicts that $\underline{B}_{N,\ell}$ is increasing (Lemma 7). *Q.E.D.*

PROPOSITION 10: *If $z_\ell > \log(-f)$, then $B_h^1 \leq B_\ell^1$ and $B_h^1 < B_\ell^1$ implies $b_\ell^1 < b_h^1$.*

PROOF: Because R would like to approve h immediately, $DIC(\ell)$ must bind. $z_\ell > \log(-f)$ implies $X_c^\ell < 0$. By the same arguments made in the example in Section 4, R will never reject at any history h_t with $B^\ell(M_t;\eta_\ell) > X_c^\ell$. Because $B_\ell^1 \geq 0 > X_c^\ell$ and $B^\ell(m;\eta_\ell)$ only decreases at $m < b_\ell^*(B_\ell^1)$, we must have $\underline{b}_\ell < b_\ell^*(B_\ell^1)$. ℓ's expected utility from (τ^ℓ, d_τ^ℓ) is $\widetilde{V}_\ell(B_\ell^1, b_\ell^*(B_\ell^1), 0)$.

Suppose $B_h^1 > B_\ell^1$. Because ℓ's continuation value at $\tau(b_h^1)$ under (τ^h, d_τ^h) when optimally choosing when to quit is zero, $V^*(\tau^h, d_\tau^h, z_\ell) = \widetilde{V}_\ell(B_h^1, b_h^1, 0)$ and so

$$V^*\big(\tau^h, d_\tau^h, z_\ell\big) = \widetilde{V}_\ell\big(B_h^1, b_h^1, 0\big) < \widetilde{V}_\ell\big(B_\ell^1, b_h^1, 0\big) \leq \widetilde{V}_\ell\big(B_\ell^1, b_\ell^*\big(B_\ell^1\big), 0\big),$$

contradicting that $DIC(\ell)$ binds. We conclude that $B_h^1 \leq B_\ell^1$.

Suppose $B_h^1 < B_\ell^1$ and $b_h^1 < b_\ell^1$. If ℓ chooses to misreport his type and quit at $\tau(b_\ell^*(B_\ell^1))$, his expected utility is $\widetilde{V}_\ell(B_h^1, b_\ell^*(B_\ell^1), 0)$ since $B^h(m;\eta_h)$ is constant for $m \geq b_h^1 > b_\ell^*(B_\ell^1)$. We then have

$$V^*\big(\tau^h, d_\tau^h, z_\ell\big) \geq \widetilde{V}_\ell\big(B_h^1, b_\ell^*\big(B_\ell^1\big), 0\big) > \widetilde{V}_\ell\big(B_h^1, b_\ell^*\big(B_\ell^1\big), 0\big),$$

a contradiction of $DIC(\ell)$. Thus, $B_h^1 < B_\ell^1$ implies $b_h^1 < b_\ell^1$. *Q.E.D.*

Comparative Statics

We begin with a proposition that will be useful later. It shows that, when $c_A = 0$, the optimal mechanism must pool h and ℓ. Let $\pi_i = \frac{e^{z_i}}{1+e^{z_i}}$.

PROPOSITION 11: *R's value of the optimal mechanism when $c_A = 0$ and $a = 1$ is equal to the optimal mechanism in the R's single decision-maker problem with prior $\mathbb{P}(z_h)\pi_h + (1 - \mathbb{P}(z_h))\pi_\ell$.*

PROOF: Let $\alpha_i = \mathbb{E}[e^{-r\tau^i} d_\tau^i | \theta = H]$and $\beta_i := \mathbb{E}[e^{-r\tau^i} d_\tau^i | \theta = L]$. Incentive compatibility for h implies

$$\begin{aligned} &\pi_h\alpha_h + (1-\pi_h)\beta_h a \geq \pi_h\alpha_\ell + (1-\pi_h)\beta_\ell a, \\ \Rightarrow \quad &\pi_h\frac{\alpha_h}{a} + (1-\pi_h)\beta_h \geq \pi_h\frac{\alpha_\ell}{a} + (1-\pi_h)\beta_\ell. \end{aligned} \tag{18}$$

Because R does not offer ℓ's mechanism to h, we also must have

$$\begin{aligned} &\pi_h\alpha_h + f(1-\pi_h)\beta_h \geq \pi_h\alpha_\ell + f(1-\pi_h)\beta_\ell \\ \Rightarrow \quad &\pi_h\frac{\alpha_h}{|f|} - (1-\pi_h)\beta_h \geq \pi_h\frac{\alpha_\ell}{|f|} - (1-\pi_h)\beta_\ell. \end{aligned} \tag{19}$$

Adding (19) with (18) and simplifying, we get $\alpha_h \geq \alpha_\ell$. A similar argument using incentive compatibility for ℓ implies $\alpha_h \leq \alpha_\ell$. Therefore, we conclude $\alpha_h = \alpha_\ell$ and, to preserve incentive compatibility, $\beta_h = \beta_\ell$. It is without loss to offer both types the same mechanism, which corresponds to R's optimal solution with prior $\mathbb{P}(z_h)\pi_h + (1-\mathbb{P}(z_h))\pi_\ell$. *Q.E.D.*

Proof of Proposition 2

PROOF: Suppose $z_h = \infty$, $z_\ell = -\infty$. We first examine a limiting case where the signal to noise ratio $\frac{2\mu}{\sigma^2} \to 0$ and $c_A = 0$. By Proposition 11, we know the value of the optimal mechanism converges R's single decision-maker problem with prior $\mathbb{P}(z_h)\pi_h + (1-\mathbb{P}(z_h))\pi_\ell$.

As $\frac{2\mu}{\sigma^2} \to 0$, learning becomes slow, and for any $\epsilon > 0$, the expected time for beliefs to move by more than ϵ goes to infinity. If $\mathbb{P}(z_\ell) > \frac{\mathbb{P}(z_h)}{-f}$, then R's expected utility will converge to zero.

Next, we want to show that for c_A large enough, we can find an approval rule such that ℓ will drop out immediately and h will be approved with strictly positive probability. Suppose R offers h a mechanism $(\tau, 1)$ with $\tau = \inf\{t : X_t \geq \underline{B}_h(M_t)\}$ and rejects ℓ immediately. This satisfies *DIC* and approves h with probability one. Moreover, as $c_A \to \infty$, the function $\underline{B}_\ell(m) \to m$, and so the expected length of experimentation time goes to 0, giving R a strictly positive utility. *Q.E.D.*

Proof of Proposition 3

PROOF: Suppose A learns θ and R offers the SI-mechanism for $\pi = \mathbb{P}(z_h)$ to both h, ℓ. Call this SI-mechanism (τ^S, d^S_τ). Because h is more optimistic about the state than he would be under symmetric information, h will never have an incentive to quit early. By an analogous argument, ℓ will choose to quit earlier than A would under symmetric information. Let us define $(\tau^h, d^h_\tau) = (\tau^S, d^S_\tau)$, and (τ^ℓ, d^ℓ_τ) to be the same as (τ^S, d^S_τ) except that it rejects immediately whenever ℓ would find it optimal to quit.

This menu of mechanisms is clearly incentive compatible. We argue that it yields a strictly higher utility than the optimal mechanism in the symmetric-information model. R's utility is the same when $\theta = H$ in both the symmetric mechanism and under (τ^h, d^h_τ), since the distribution of approval and rejection times is the same. R's utility is strictly higher when $\theta = L$ from using (τ^ℓ, d^ℓ_τ) when compared to (τ^S, d^S_τ). With positive probability, R approves when $\theta = L$ under (τ^S, d^S_τ) and rejects under (τ^ℓ, d^ℓ_τ) before she would have approved under (τ^S, d^S_τ). Moreover, every ω that leads to approval under (τ^ℓ, d^ℓ_τ) will also lead to approval in (τ^S, d^S_τ) and $\tau^S(\omega) = \tau^\ell(\omega)$. Thus, R's value of this mechanism when A is informed about θ is higher than under symmetric information. *Q.E.D.*

APPENDIX H: GENERAL VALUES OF z_h

We consider R's asymmetric information problem for arbitrary z_h. In this case, both $DIC(h)$ and $DIC(\ell)$ may bind and so we must solve AM_h with the $PK_h(V_h')$ constraint for some value of V_h'. Consider the problem of characterizing the Pareto frontier of R and h's expected utility across all mechanisms that satisfy $DIC(\ell, V_\ell)$ and h's dynamic participation constraint. Solving AM_h with the $PK_h(V_h')$ is equivalent to finding the mechanism that generates the point with V_h' utility for h on the Pareto frontier.

Each point on the Pareto frontier is generated by the mechanism that solves, for some weight γ_h, the problem of a social planner placing weight γ_h on R's utility and $1 - \gamma_h$ on h's utility, namely maximizing $\mathbb{E}[e^{-r\tau}(\gamma_h u(X_\tau, d_\tau) + (1 - \gamma_h)v_h(X_\tau, d_\tau))]$ subject to $DIC(\ell, V_\ell)$ and h's dynamic participation constraint. This is equivalent to solving AM_h when $DIC(h, V_h')$ is dropped but R's utility u is replace with $\gamma_h u + (1 - \gamma_h)v_h$. All arguments continue to apply as in the proof of Theorem 2 and so we get the same structure to the optimal mechanism for h in the solution to this social planner's problem.[13]

[13]The only important difference with this new utility is that costs of experimentation in our objective function are no longer 0. However, the only point at which we used $c_R = 0$ in the proof of Theorem 2 is in Lemma 15 to ensure R's continuation value from (τ', d'_τ) was strictly positive. But if we replace R's utility function with a weighted sum of R's and h's utility function, the same argument applies since h's continuation value under (τ', d'_τ) was equal to 0.

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