Econometrica Supplementary Material

SUPPLEMENT TO "A REMEDI FOR MICROSTRUCTURE NOISE"

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APPENDIX SA: ASSUMPTIONS AND NOTATIONS

IN THE PROOFS, K will be a constant that may change from line to line. When it depends on some parameter par, we write K_{par} instead. But it never depends on n or any parameters that depend on n. Let V be any Itô semimartingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ that has a Grigelionis representation as X in (2) of Li and Linton (2021) with coefficients b^V, σ^V, ϑ^V,[1] which satisfy the following:

ASSUMPTION K: *The processes b^V, σ^V, ϑ^V are bounded with $\vartheta^V(\omega, t, z) \leq J(z)$ for some bounded function J on E satisfying $\int J^2(z)\lambda(\mathrm{d}z) < \infty$.*

Then, for any V satisfying Assumption K and any $r \geq 2$, we have, for any finite $(\mathcal{F}_t)$-stopping times $S \leq T$,

$$\mathbb{E}\big(|V_T - V_S|^r | \mathcal{F}_S\big) \leq \mathbb{E}(T - S|\mathcal{F}_S), \tag{SA.1}$$

$$\big|\mathbb{E}(V_T - V_S|\mathcal{F}_S)\big| \leq \mathbb{E}(T - S|\mathcal{F}_S). \tag{SA.2}$$

ASSUMPTION SHON: *We have Assumptions* H, N, O *in* Li and Linton (2021) *and further assume that the processes X, α, $\overline{\alpha}$, and γ satisfy Assumption* K, *and the process $1/\alpha$ is bounded.*

According to a "localization procedure," we can always assume SHON below, which implies the existence of $\rho \in (1/2 + \eta, 1)$, such that

$$\delta(n, i) \leq K\delta_n^\rho, \qquad A_t \leq Kt, \qquad \mathbb{P}\big(\Omega_t^n\big) \to 1 \quad \text{if } \Omega_t^n := \big\{\delta_n N_t^n \leq 1 + Kt\big\}. \tag{SA.3}$$

In the sequel, we will assume $\mathbf{j} = (j_1, \dots, j_q)$, $\mathbf{j}' = (j_1', \dots, j_{q'}') \in \mathfrak{J}$, and $q \leq q'$. k_n and $\boldsymbol{k}_n$ are an integer and a vector of integers that will be specified later. Let

$$\begin{aligned} g(\mathbf{j}, \boldsymbol{k}_n)_i^n &:= \Delta_{\mathbf{j}}^{\boldsymbol{k}_n}(Y)_i^n - \big(\gamma_i^n\big)^q \Delta_{\mathbf{j}}^{\boldsymbol{k}_n}(\chi)_i^n, \qquad u(\mathbf{j}, \boldsymbol{k}_n)_i^n := \Delta_{\mathbf{j}}^{\boldsymbol{k}_n}(\chi)_i^n - \boldsymbol{r}(\mathbf{j}; \boldsymbol{k}_n), \\ q_n &:= 2^{q-1}k_n, \qquad d_i^n := \alpha_i^n \delta(n, i+1) - \delta_n, \\ \theta(\mathbf{j}, \boldsymbol{k}_n)_i^n &:= \sqrt{\delta_n}\big(\gamma_i^n\big)^q u(\mathbf{j}, \boldsymbol{k}_n)_i^n, \end{aligned} \tag{SA.4}$$

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[1]In this supplementary material, Arabic numbering of equations always refers to the equations in the main text Li and Linton (2021).

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where $\boldsymbol{r}(\mathbf{j};\boldsymbol{k}_n) := \mathbb{E}(\Delta_{\mathbf{j}}^{\boldsymbol{k}_n}(\chi)_0^n)$. When $\boldsymbol{k}_n$ satisfies the conditions specified in (17) of Li and Linton (2021), we write $g(\mathbf{j})_i^n$ and $u(\mathbf{j})_i^n$ instead of $g(\mathbf{j},\boldsymbol{k}_n)_i^n$ and $u(\mathbf{j},\boldsymbol{k}_n)_i^n$.

Following Jacod, Li, and Zheng (2017), we assume the processes X, α, $\overline{\alpha}$, γ and the observation times T_i^n are defined on a space $(\Omega^{(0)}, \mathcal{F}^{(0)}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}^{(0)})$; $\{\chi_i\}_{i\in\mathbb{Z}}$ is defined on another space $(\Omega^{(1)}, \mathcal{G}, (\mathcal{G}_i)_{i\in\mathbb{Z}}, \mathbb{P}^{(1)})$ with $\mathcal{G}_i := \sigma(\chi_k : k \leq i)$ and $\mathcal{G}^i := \sigma(\chi_k : k \geq i)$. Let $\Omega = \Omega^{(0)} \times \Omega^{(1)}$, $\mathcal{F} = \mathcal{F}^{(0)} \otimes \mathcal{G}$, $\mathbb{P} = \mathbb{P}^{(0)} \otimes \mathbb{P}^{(1)}$.

APPENDIX SB: SOME AUXILIARY RESULTS

LEMMA S1: *Let ξ, ξ' be two variables in the probability space $(\Omega^{(1)}, \mathcal{G}, \mathbb{P}^{(1)})$ so that ξ is $\mathcal{G}_i$-measurable and ξ' is $\mathcal{G}^{i+\ell}$-measurable, where $\ell \in \mathbb{N}_+$. Assume ξ, ξ' have bounded second moments. Under Assumption* N *in* Li and Linton (2021), *we have*

$$\left|\mathbb{E}(\xi\xi') - \mathbb{E}(\xi)\mathbb{E}(\xi')\right| \leq K\ell^{-v}. \tag{SB.1}$$

PROOF: By first conditioning on $\mathcal{G}_i$, plus an application of the Cauchy–Schwarz inequality, the LHS of (SB.1) is bounded by $\sqrt{\mathbb{E}((\xi-\mathbb{E}(\xi))^2)\mathbb{E}((\mathbb{E}(\xi'-\mathbb{E}(\xi')|\mathcal{G}_i))^2)}$. Now the result follows from the fact that the second moment is bounded and an application of Lemma VIII 3.102 of Jacod and Shiryaev (2003). *Q.E.D.*

Now assume $\boldsymbol{k}_n$ satisfies (15), and let $\underline{k}_n := \inf_{l\geq 2}(k_{l+1,n} - k_{l,n})$; $\overline{k}_n := k_{q,n} \vee (-k_{1,n}) = \sup\{|k_{l,n}| : 1 \leq l \leq q\}$.

LEMMA S2: *We have under Assumption* N *that*

$$\left|\boldsymbol{r}(\mathbf{j};\boldsymbol{k}_n) - \boldsymbol{r}(\mathbf{j})\right| \leq \frac{K}{\left(|k_{1,n}| \wedge \underline{k}_n\right)^v}, \tag{SB.2}$$

where $\boldsymbol{r}(\mathbf{j})$ is defined in (12).

PROOF: Let $\mathcal{Q}_q$ be the collection of all *proper* subsets of $\{1, 2, \ldots, q\}$:

$$\mathcal{Q}_q = \{Q : Q \subsetneq \{1, \ldots, q\}\}; \tag{SB.3}$$

thus, for any $Q \in \mathcal{Q}_q$, $Q^c \neq \emptyset$. Now we have, for $Q \in \mathcal{Q}_q$,

$$\left|\mathbb{E}\left(\prod_{l\in Q}\chi_{j_l}\prod_{l\in Q^c}\chi_{j_l-k_{l,n}}\right)\right| = \begin{cases} \left|\mathbb{E}\left(\chi_{j_1-k_{1,n}}\prod_{l\in Q}\chi_{j_l}\prod_{l\in Q^c, l\neq 1}\chi_{j_l-k_{l,n}}\right)\right| & \text{if } 1 \in Q^c, \\ \left|\mathbb{E}\left(\chi_{j_{\bar{l}}-k_{\bar{l},n}}\prod_{l\in Q}\chi_{j_l}\prod_{l\in Q^c, l\neq \bar{l}}\chi_{j_l-k_{l,n}}\right)\right| & \text{if } 1 \notin Q^c, \end{cases} \tag{SB.4}$$

where $\bar{l} = \max\{l : l \in Q^c\}$ if $1 \notin Q^c$. Apply Lemma S1 with $\xi = \prod_{l\in Q}\chi_{j_l}\prod_{l\in Q^c, l\neq 1}\chi_{j_l-k_{l,n}}$, $\xi' = \chi_{j_1-k_{1,n}}$, $i = j_1$, $\ell = |k_{1,n}|$ if $1 \in Q^c$, and $\xi = \chi_{j_{\bar{l}}-k_{\bar{l},n}}$, $\xi' = \prod_{l\in Q}\chi_{j_l}\prod_{l\in Q^c, l\neq\bar{l}}\chi_{j_l-k_{l,n}}$, $i = j_{\bar{l}} - k_{\bar{l},n}$, $\ell = \underline{k}_n$ if $1 \notin Q^c$; we get $|\mathbb{E}(\prod_{l\in Q}\chi_{j_l}\prod_{l\in Q^c}\chi_{j_l-k_{l,n}})| \leq C(|k_{1,n}| \wedge \underline{k}_n)^{-v}$. Now (SB.2) is proved since $\boldsymbol{r}(\mathbf{j};\boldsymbol{k}_n) - \boldsymbol{r}(\mathbf{j}) = \sum_{Q\in\mathcal{Q}_q}(-1)^{|Q^c|}\mathbb{E}(\prod_{l\in Q}\chi_{j_l}\prod_{l\in Q^c}\chi_{j_l-k_{l,n}})$. *Q.E.D.*

LEMMA S3: *Assume* $(\overline{k}_n \vee j_1)\delta_n^{\rho} \to 0$; *let*

$$\mathrm{ReMeDI}'(\chi;\mathbf{j},\boldsymbol{k}_n)_t^n := \sum_{i=2^{q-1}}^{N_t^n-k_n-j_1} (\gamma_i^n)^q \Delta_{\mathbf{j}}^{\boldsymbol{k}_n}(\chi)_i^n. \tag{SB.5}$$

Then, for any $r>1$, *there is some constant* $K_{r,q}>0$ *such that*

$$\mathbb{E}\big(\big|\mathrm{ReMeDI}(Y;\mathbf{j},\boldsymbol{k}_n)_t^n - \mathrm{ReMeDI}'(\chi;\mathbf{j},\boldsymbol{k}_n)_t^n\big|\mathbf{1}_{\{\Omega_t^n\}}\big) \le K_{r,q}(\overline{k}_n \vee j_1)^{\frac{1}{r}}\delta_n^{\frac{\rho}{r}-1}. \tag{SB.6}$$

PROOF: Let $\zeta_{i,l}^n := X_{i+j_l}^n - X_{i+j_l-k_{l,n}}^n + (\gamma_{i+j_l}^n - \gamma_i^n)\chi_{i+j_l} - (\gamma_{i+j_l-k_{l,n}}^n - \gamma_i^n)\chi_{i+j_l-k_{l,n}}$, and $\zeta'^n_{i,l} := \gamma_i^n(\chi_{i+j_l} - \chi_{i+j_l-k_{l,n}})$. Now it follows (recall $\mathcal{Q}_q$ defined in (SB.3)) that

$$\begin{aligned}
&\Delta_{\mathbf{j}}^{\boldsymbol{k}_n}(Y)_i^n = \prod_{l=1}^q(\zeta_{i,l}^n + \zeta'^n_{i,l}), \qquad (\gamma_i^n)^q\Delta_{\mathbf{j}}^{\boldsymbol{k}_n}(\chi)_i^n = \prod_{l=1}^q \zeta'^n_{i,l},\\
&g(\mathbf{j},\boldsymbol{k}_n)_i^n = \sum_{Q\in\mathcal{Q}_q}\prod_{l\in Q}\zeta'^n_{i,l}\prod_{l\in Q^c}\zeta_{i,l}^n.
\end{aligned} \tag{SB.7}$$

Apply (SA.1) for X and γ, and the fact that χ has bounded moments; we have, for any $k\ge 2$,

$$\mathbb{E}\big(|\zeta_{i,l}^n|^k\big) \le K(\overline{k}_n \vee j_1)\delta_n^{\rho}, \qquad \mathbb{E}\big(\big|\zeta'^n_{i,l}\big|^k\big) \le K, \quad \forall i,l. \tag{SB.8}$$

Let $\ell = |Q^c|$, whence $\ell \ge 1$ (recall (SB.3)). For $r\ge 2$, apply Hölder's inequality with exponents $(\underbrace{r\ell,\dots,r\ell}_{\ell}, \frac{r}{r-1})$; we have

$$\mathbb{E}\left(\left|\prod_{l\in Q^c}\zeta_{i,l}^n\prod_{l\in Q}\zeta'^n_{i,l}\right|\right) \le \prod_{l\in Q^c}\big(\mathbb{E}\big(\big|\zeta_{i,l}^n\big|^{r\ell}\big)\big)^{\frac{1}{r\ell}}\left(\mathbb{E}\left(\left|\prod_{l\in Q}\zeta'^n_{i,l}\right|^{\frac{r}{r-1}}\right)\right)^{\frac{r-1}{r}}, \tag{SB.9}$$

which is further bounded by $K_{r,q}((\overline{k}_n \vee j_1)\delta_n^{\rho})^{\frac{1}{r}}$ in view of (SB.8). Now (SB.6) follows immediately.

For $1<r\le 2$, we first note (SB.9) still holds if $\ell\ge 2$. For $\ell = 1$, we let $Q^c = \{l^*\}$. Let $\mathcal{H}_{i,l^*}^n := \mathcal{F}_{i+j_{l^*}-k_{l^*,n}}^n \otimes \mathcal{G}$ if $l^*>1$, and $\mathcal{H}_{i,l^*}^n := \mathcal{F}_i^n \otimes \mathcal{G}$ if $l^*=1$. Then, by the independence of $\mathcal{G}$ and $\mathcal{F}^{(0)}$, (SA.2) for γ, we have $|\mathbb{E}(\zeta_{i,l^*}^n|\mathcal{H}_{i,l^*}^n)| \le K(\overline{k}_n \vee j_1)\delta_n^{\rho}(1+|\chi_{i+j_{l^*}}|+|\chi_{i+j_{l^*}-k_{l^*,n}}|)$, which yields (recall that $\prod_{l\neq l^*}\zeta'^n_{i,l}$ is measurable with respect to $\mathcal{H}_{i,l^*}^n$)

$$\begin{aligned}
&\mathbb{E}\left(\left|\mathbb{E}\left(\zeta_{i,l^*}^n\prod_{l\neq l^*}\zeta'^n_{i,l}|\mathcal{H}_{i,l^*}^n\right)\right|\right)\\
&\quad\le K(\overline{k}_n \vee j_1)\delta_n^{\rho}\mathbb{E}\left(\big(1+|\chi_{i+j_{l^*}}|+|\chi_{i+j_{l^*}-k_{l^*,n}}|\big)\prod_{l\neq l^*}|\chi_{i+j_l}-\chi_{i+j_l-k_{l,n}}|\right)\\
&\quad\le K(\overline{k}_n \vee j_1)\delta_n^{\rho}.
\end{aligned} \tag{SB.10}$$

On the other hand, since $r > 1$, apply Hölder's inequality to get

$$\mathbb{E}\Bigl(\Bigl(\zeta^n_{i,l^*}\prod_{l\neq l^*}\zeta'^n_{i,l}\Bigr)^2\Bigr) \leq K\bigl((\overline{k}_n \vee j_1)\delta_n^\rho\bigr)^{1/r}. \tag{SB.11}$$

Note that $\zeta^n_{i,l^*}\prod_{l\neq l^*}\zeta'^n_{i,l}$ is measurable with respect to $\mathcal{F}^n_{i+j_1-k_{1,n}}\otimes\mathcal{G}$; combined with (SB.10) and (SB.11), we can apply Lemma A.6 of Jacod, Li, and Zheng (2017) and obtain

$$\mathbb{E}\Biggl(\Biggl|\sum_{i=2^{q-1}}^{N_t^n-k_n-j_1}\zeta^n_{i,l^*}\prod_{l\neq l^*}\zeta'^n_{i,l}\Biggr|\Biggr) \leq K\bigl((\overline{k}_n \vee j_1)\delta_n^{\rho-1} + (\overline{k}_n\vee j_1)^{\frac{r+1}{2r}}\delta_n^{\frac{\rho-r}{2r}}\bigr),$$

and it is further bounded by $K(\overline{k}_n\vee j_1)^{\frac{1}{r}}\delta_n^{\frac{\rho}{r}-1}$ since $\delta_n^\rho(\overline{k}_n\vee j_1)\to 0$. *Q.E.D.*

In this sequel, we assume $\boldsymbol{k}_n$ is specified as follows for a given integer k_n: $k_{l,n} = -k_n$ if $l = 1$, $k_{l,n} = 2^{l-1}k_n$ if $l \geq 2$. In line with the notations in Li and Linton (2021), we will write $\Delta_{\mathbf{j}}(\cdot)^n_i$ instead of $\Delta_{\mathbf{j}}^{\boldsymbol{k}_n}(\cdot)^n_i$ when $\boldsymbol{k}_n$ is specified as above. Moreover, we will replace $\boldsymbol{r}(\mathbf{j},\boldsymbol{k}_n)$ by $\boldsymbol{r}(\mathbf{j},k_n)$. We further denote $h_{\mathbf{j}}(i,l)_n := i + j_l - k_{l,n}$. For $Q_q \subset \{1,2,\dots,q\}$, let $\widetilde{\chi}(Q_q,\mathbf{j})^n_i := \prod_{j_l\in\mathbf{j}}\chi_{i+j_l} - \boldsymbol{r}(\mathbf{j})$ if $Q_q^c = \emptyset$ and $(-1)^{|Q_q^c|}\prod_{l\in Q_q}\chi_{i+j_l}\prod_{l\in Q_q^c}\chi_{h_{\mathbf{j}}(i,l)_n}$ otherwise. $\widetilde{\chi}(Q'_{q'},\mathbf{j}')^n_i$ is defined in a similar manner for $\mathbf{j}', Q'_{q'}\subset\{1,2,\dots,q'\}$. We have for any i, k that (recall $u(\mathbf{j})^n_i$ defined in (SA.4)): $u(\mathbf{j})^n_i = \sum_{Q_q\subset\{1,2,\dots,q\}}\widetilde{\chi}(Q_q,\mathbf{j})^n_i$; $u(\mathbf{j}')^n_{i+k} = \sum_{Q'_{q'}\subset\{1,2,\dots,q'\}}\widetilde{\chi}(Q'_{q'},\mathbf{j}')^n_{i+k}$. Now we introduce four mutually exclusive categories of pairs of $(Q_q, Q'_{q'})$, or their complements $(Q_q^c, (Q'_{q'})^c)$:

$$\begin{cases} Q_q^c = (Q'_{q'})^c, & \text{(SB.12)}\\ Q_q^c = \{l\}, \quad (Q'_{q'})^c = \{l'\}, \quad l \neq l', & \text{(SB.13)}\\ (Q'_{q'})^c = \emptyset, \quad Q_q^c = \{l\}, & \text{(SB.14)}\\ Q_q^c = \emptyset, \quad (Q'_{q'})^c = \{l'\}. & \text{(SB.15)}\end{cases}$$

First, we show the following.

LEMMA S4: *For any pair $(Q_q, Q'_{q'})$ that does* not *satisfy (SB.12) to (SB.15), we define the following sets of indices for any integers i, k: $\mathbb{I}(Q_q^c)_i = \{h_{\mathbf{j}}(i,l)_n : l\in Q_q^c\}$, $\mathbb{I}((Q'_{q'})^c)_{i+k} = \{h_{\mathbf{j}'}(i+k,l')_n : l'\in(Q'_{q'})^c\}$, $\mathbb{I}(Q_q)_i = \{i+j_l : l\in Q_q\}$, $\mathbb{I}(Q'_{q'})_{i+k} = \{i+k+j'_{l'} : l'\in Q'_{q'}\}$. Then there exists at least one index in $\mathbb{I}(Q_q^c)_i\cup\mathbb{I}((Q'_{q'})^c)_{i+k}$ that is at least $k_n/3$ apart from the remaining indices in $\mathbb{I}(Q_q^c)_i\cup\mathbb{I}((Q'_{q'})^c)_{i+k}\cup\mathbb{I}(Q_q)_i\cup\mathbb{I}(Q'_{q'})_{i+k}$.*

PROOF: We first consider pairs of $(Q_q, Q'_{q'})$ that do *not* satisfy (SB.12) to (SB.15) but *satisfy* $|Q_q^c| = |(Q'_{q'})^c|$. If this were true, then violating (SB.12) and (SB.13) implies $|Q_q^c| = |(Q'_{q'})^c| \geq 2$. Now suppose Lemma S4 is not true. Denote $(l_\tau)_{1\leq\tau\leq|Q_q^c|}$ so that $h_{\mathbf{j}}(i,l_\tau)_n\in\mathbb{I}(Q_q^c)_i$, and they are in an ascending order, that is, $h_{\mathbf{j}}(i,l_1)_n < h_{\mathbf{j}}(i,l_2)_n < \cdots < h_{\mathbf{j}}(i,l_{|Q_q^c|})_n$, or equivalently, $l_1 > l_2 > \cdots > l_{|Q_q^c|}$.[2] $(l'_{\tau'})_{1\leq\tau'\leq|(Q'_{q'})^c|}$ are defined similarly for the indices

[2]In this proof, many inequalities hold up to adding a constant. For example, we conclude $z_n > z'_n$ if $z_n + c_1 > z'_n + c_2$, where c_1, c_2 are some constant and z_n, z'_n are large when n is large.

$h_{\mathbf{j}'}(i+k, l'_{\tau'})_n \in \mathbb{I}((Q'_{q'})^c)_{i+k}$. Since the minimal distance between any index in $\mathbb{I}(Q^c_q)_i$ (or $\mathbb{I}((Q'_{q'})^c)_{i+k}$) and the remaining indices in $\mathbb{I}(Q^c_q)_i \cup \mathbb{I}(Q_q)_i$ (or $\mathbb{I}((Q'_{q'})^c)_{i+k} \cup \mathbb{I}((Q'_{q'}))_{i+k}$) is k_n, we conclude that each pair of indices

$$\left|h_{\mathbf{j}}(i, l_\tau)_n - h_{\mathbf{j}'}\left(i+k, l'_\tau\right)_n\right| \le k_n/3, \quad \tau = 1, \dots, \left|Q^c_q\right|, \tag{SB.16}$$

were Lemma S4 not true.

Assume $l_1 > l'_1$. Then by (SB.16) we have (since $|Q^c_q| \ge 2$):

$$\left|\left(h_{\mathbf{j}}(i, l_1)_n - h_{\mathbf{j}}(i, l_2)_n\right) - \left(h_{\mathbf{j}'}\left(i+k, l'_1\right)_n - h_{\mathbf{j}'}\left(i+k, l'_2\right)_n\right)\right| \le 2k_n/3, \tag{SB.17}$$

which implies

$$\begin{cases} 2^{l_1-1} = 2^{l'_1-1}, & \text{if } l_2 = 1, l'_2 = 1; \\ 2^{l_1-1} = 2^{l'_1-1} - 2^{l'_2-1} - 1, & \text{if } l_2 = 1, l'_2 > 1; \\ 2^{l_1-1} = 2^{l_2-1} + 2^{l'_1-1} + 1, & \text{if } l_2 > 1, l'_2 = 1; \\ 2^{l_1-1} = 2^{l_2-1} + 2^{l'_1-1} - 2^{l'_2-1}, & \text{if } l_2 > 1, l'_2 > 1. \end{cases}$$

But it contradicts the fact that $l_1 > \max(l_2, l'_1, l'_2)$. Therefore, we have $l_1 \le l'_1$; similarly, we get $l_1 \ge l'_1$. Thus, we conclude $l_1 = l'_1$. We also have $l_2 = l'_2$ since (SB.16) (with $\tau = 1$) implies $|k| \le k_n/3$. We can proceed to prove $l_\tau = l'_\tau$ for all $l_\tau \in Q^c_q$, that is, $Q^c_q = (Q'_{q'})^c$, which is a contradiction. Therefore, we conclude that for any pair of $(Q_q, Q'_{q'})$ that does *not* satisfy (SB.12) to (SB.15), we have $|Q^c_q| \ne |(Q'_{q'})^c|$.

Now we consider pairs of $(Q_q, Q'_{q'})$ that do *not* satisfy (SB.12) to (SB.15) but satisfy $|Q^c_q| > |(Q'_{q'})^c|$. Equation (SB.14) implies $|Q^c_q| \ge 2$. Consider the following scenarios:

1. If $|Q^c_q| > |(Q'_{q'})^c| + 1$, apply the *Pigeonhole Principle*: consider $|Q^c_q|$ "containers" centered at $\{h_{\mathbf{j}}(i, l)_n : h_{\mathbf{j}}(i, l)_n \in \mathbb{I}(Q^c_q)_i\}$ with "radius" $k_n/3$. Were Lemma S4 not true, we need to place the $|(Q'_{q'})^c| + 1$ "items" $\{h_{\mathbf{j}'}(i+k, l')_n \in \mathbb{I}((Q'_{q'})^c)_{i+k}, \mathbb{I}(Q'_{q'})_{i+k}\}$[3] into the "containers." The Pigeonhole Principle implies at least one of the "containers" is empty, thus Lemma S4 must be true.
2. If $|Q^c_q| = |(Q'_{q'})^c| + 1 \ge 2$ and Lemma S4 is false, there is one-to-one correspondence between the $|(Q'_{q'})^c| + 1$ "items" $\{h_{\mathbf{j}'}(i+k, l')_n \in \mathbb{I}((Q'_{q'})^c)_{i+k}, \mathbb{I}(Q'_{q'})_{i+k}\}$ and $|Q^c_q|$ "items" $\{h_{\mathbf{j}}(i, l)_n : l \in Q^c_q\}$ so that each pair has a distance less than $k_n/3$ (recall a representation of such correspondence by (SB.16)). Now we need to consider the following two cases:
   (a) $\mathbb{I}((Q'_{q'})^c)_{i+k} = \{h_{\mathbf{j}}(i+k, 1)_n\}$, that is, $(Q'_{q'})^c = \{1\}$. Let us fix the index of $\mathbb{I}(Q'_{q'})_{i+k}$ at $i+k$.[4] Let $Q^c_q = \{l_1, l_2\}$. Apply the similar arguments to obtain (SB.17); we have the estimate that $|i+k+j'_1+k_n-(i+k)-(h_{\mathbf{j}}(i, l_1)_n - h_{\mathbf{j}}(i, l_2)_n)| \le 2k_n/3$. This contradicts to $|i+k+j'_1+k_n-(i+k)-(h_{\mathbf{j}}(i, l_1) - h_{\mathbf{j}}(i, l_2))| \ge |(h_{\mathbf{j}}(i, l_1)_n - h_{\mathbf{j}}(i, l_2)_n)| - k_n$, which is no smaller than k_n.
   (b) Now assume $\exists l' \in (Q'_{q'})^c, l' > 1$; we can apply the arguments used above to show that, for each $l'_\tau > 1$, $l'_\tau \in (Q'_{q'})^c$, there is some $l_\tau \in Q^c_q$ such that $l'_\tau = l_\tau$. We also conclude $|k| \le k_n/3$. Let l^*, l^*_+ be the two indices satisfying (recall $|Q^c_q| \ge 2$):

[3] Asymptotically, we treat $\mathbb{I}(Q'_{q'})_{i+k}$ as one "item" since the distances between the indices in $\mathbb{I}(Q'_{q'})_{i+k}$ are independent of n, thus "fixed."

[4] It can be any of $\{i+k+j'_{l'} : l' \in Q'_{q'}\}$, but asymptotically they are equivalent.

$l^* = \text{argmax}_{\{l>1: l\in(Q'_{q'})^c\}} h_{\mathbf{j}}(i,l)_n$; $l^*_+ = \text{argmax}_{\{l\neq l^*: h_{\mathbf{j}}(i,l)_n \in \mathbb{I}(Q^c_q)_i\}} h_{\mathbf{j}}(i,l)_n$. (Note that l^*_+ could be 1.) Now we have $|h_{\mathbf{j}'}(i+k,l^*)_n - (i+k) - (h_{\mathbf{j}}(i,l^*)_n - h_{\mathbf{j}}(i,l^*_+)_n)| \le 2k_n/3$. But this contradicts to

$$\begin{aligned}
&\left|h_{\mathbf{j}'}(i+k,l^*)_n - (i+k) - \left(h_{\mathbf{j}}(i,l^*)_n - h_{\mathbf{j}}(i,l^*_+)_n\right)\right| \\
&\quad \ge \begin{cases} \left|\left(h_{\mathbf{j}}(i,l^*)_n - h_{\mathbf{j}}(i,l^*_+)_n\right)\right| - \left|h_{\mathbf{j}'}(i+k,l^*)_n - (i+k)\right| & \text{if } l^*_+ = 1, \\ \left|h_{\mathbf{j}'}(i+k,l^*)_n - (i+k)\right| - \left|\left(h_{\mathbf{j}}(i,l^*)_n - h_{\mathbf{j}}(i,l^*_+)_n\right)\right| & \text{if } l^*_+ > 1 \end{cases} \\
&\quad \ge k_n.
\end{aligned}$$

This finishes the proof of Lemma S4 for the case $|Q^c_q| > |(Q'_{q'})^c|$. The conclusion for $|Q^c_q| < |(Q'_{q'})^c|$ can be proved analogously, and the proof now is complete. *Q.E.D.*

LEMMA S5: *For any pair $(Q_q, Q'_{q'})$ that does* not *satisfy* (SB.12) *to* (SB.15), *we have*

$$\left|\mathbb{E}\left(\widetilde{\chi}(Q_q,\mathbf{j})^n_i \widetilde{\chi}\left(Q'_{q'},\mathbf{j}'\right)^n_{i+k}\right)\right| \le Ck_n^{-v}, \quad \forall k \in \mathbb{Z}. \tag{SB.18}$$

PROOF: Let one of the indices satisfying Lemma S4 be h^*. Write

$$\widetilde{\chi}(Q_q,\mathbf{j})^n_i \widetilde{\chi}\left(Q'_{q'},\mathbf{j}'\right)^n_{i+k} = \hat{\chi}^n_{h^*-\frac{k_n}{3}} \chi_{h^*} \hat{\chi}^n_{h^*+\frac{k_n}{3}},$$

where $\hat{\chi}^n_{h^*-\frac{k_n}{3}}$ and $\hat{\chi}^n_{h^*+\frac{k_n}{3}}$ are the products of the remaining factors in $\widetilde{\chi}(Q_q,\mathbf{j})^n_i \widetilde{\chi}(Q'_{q'},\mathbf{j}')^n_{i+k}$ (other than χ_{h^*}) that are measurable with respect to $\mathcal{G}_{h^*-[k_n/3]}$ and $\mathcal{G}^{h^*+[k_n/3]}$, respectively. Since $\hat{\chi}^n_{h^*-\frac{k_n}{3}}$, χ_{h^*} and $\hat{\chi}^n_{h^*+\frac{k_n}{3}}$ are integrable, we can apply Lemma S1 to get

$$\begin{aligned}
\left|\mathbb{E}\left(\hat{\chi}^n_{h^*-\frac{k_n}{3}} \chi_{h^*} \hat{\chi}^n_{h^*+\frac{k_n}{3}}\right) - \mathbb{E}\left(\hat{\chi}^n_{h^*-\frac{k_n}{3}} \chi_{h^*}\right)\mathbb{E}\left(\hat{\chi}^n_{h^*+\frac{k_n}{3}}\right)\right| &\le Kk_n^{-v}; \\
\left|\mathbb{E}\left(\hat{\chi}^n_{h^*-\frac{k_n}{3}} \chi_{h^*}\right)\right| &\le Kk_n^{-v}.
\end{aligned} \tag{SB.19}$$

This finishes the proof of (SB.18). *Q.E.D.*

LEMMA S6: *For all pairs of $(Q_q, Q'_{q'})$ that* satisfy (SB.12), *we have, for any $k \in \mathbb{Z}$,*

$$\left|\sum_{(Q_q,Q'_{q'})}^{\text{(SB.12)}} \mathbb{E}\left(\widetilde{\chi}(Q_q,\mathbf{j})^n_i \widetilde{\chi}\left(Q'_{q'},\mathbf{j}'\right)^n_{i+k}\right) - \boldsymbol{s}_0\left(\mathbf{j},\mathbf{j}';k\right) - \boldsymbol{s}_1\left(\mathbf{j},\mathbf{j}';k\right)\right| \le Kk_n^{-v}. \tag{SB.20}$$

PROOF: If $Q^c_q = (Q'_{q'})^c = \emptyset$, we have $\mathbb{E}(\widetilde{\chi}(Q_q,\mathbf{j})^n_i \widetilde{\chi}(Q'_{q'},\mathbf{j}')^n_{i+k}) = \boldsymbol{s}_0(\mathbf{j},\mathbf{j}';k)$. Now consider $Q^c_q = (Q'_{q'})^c \neq \emptyset$ so that $Q'_{q'} = Q_{q'}$ (recall $Q_{q'}$ defined in (25)), and

$$\mathbb{E}\left(\widetilde{\chi}(Q_q,\mathbf{j})^n_i \widetilde{\chi}\left(Q'_{q'},\mathbf{j}'\right)^n_{i+k}\right) = \mathbb{E}\left(\prod_{l\in Q_q} \chi_{i+j_l} \prod_{l'\in Q_{q'}} \chi_{i+k+j'_{l'}} \prod_{l\in Q^c_q} \chi_{h_{\mathbf{j}}(i,l)_n} \chi_{h_{\mathbf{j}'}(i+k,l)_n}\right).$$

Let $|k| \le \frac{k_n}{2}$; by successive conditioning as we did to obtain (SB.19), we obtain

$$\left|\mathbb{E}\left(\prod_{l\in Q^c_q} \chi_{h_{\mathbf{j}}(i,l)_n} \chi_{h_{\mathbf{j}'}(i+k,l)_n}\right) - \prod_{l\in Q^c_q} \boldsymbol{r}\left(j_l, j'_l + k\right)\right| \le Kk_n^{-v}.$$

This yields, together with the fact that $|\boldsymbol{r}(\mathbf{j}_{Q_q} \oplus \mathbf{j}'_{Q'_{q'}}(+k))|$ is bounded, that

$$\begin{aligned}&\left|\mathbb{E}\left(\prod_{l\in Q_q}\chi_{i+j_l}\prod_{l'\in Q'_{q'}}\chi_{i+k+j'_{l'}}\prod_{l\in Q_q^c}\chi_{h_{\mathbf{j}}(i,l)_n}\chi_{h_{\mathbf{j}'}(i+k,l)_n}\right)-\boldsymbol{r}\big(\mathbf{j}_{Q_q}\oplus\mathbf{j}'_{Q'_{q'}}(+k)\big)\prod_{l\in Q_q^c}\boldsymbol{r}\big(j_l,j'_l+k\big)\right|\\&\quad\le\left|\mathbb{E}\left(\left(\prod_{l\in Q_q}\chi_{i+j_l}\prod_{l'\in Q'_{q'}}\chi_{i+k+j'_{l'}}-\boldsymbol{r}\big(\mathbf{j}_{Q_q}\oplus\mathbf{j}'_{Q'_{q'}}(+k)\big)\right)\prod_{l\in Q_q^c}\chi_{h_{\mathbf{j}}(i,l)_n}\chi_{h_{\mathbf{j}'}(i+k,l)_n}\right)\right|+Kk_n^{-v}.\end{aligned}$$

Apply Lemma S1; by successive conditioning, we get that the expectation after the inequality is also bounded by Kk_n^{-v}, since the indices $\{i+j_l, i+k+j'_{l'} : l \in Q_q, l' \in Q'_{q'}\}$ are at least $k_n/2$ apart from the indices $\{h_{\mathbf{j}}(i,l)_n, h_{\mathbf{j}'}(i+k,l)_n : l \in Q_q^c\}$. This proves

$$\left|\sum_{Q_q^c=(Q'_{q'})^c\neq\emptyset}\mathbb{E}\big(\widetilde{\chi}(Q_q,\mathbf{j})_i^n\widetilde{\chi}\big(Q'_{q'},\mathbf{j}'\big)_{i+k}^n\big)-\boldsymbol{s}_1\big(\mathbf{j},\mathbf{j}';k\big)\right|\le Kk_n^{-v},\tag{SB.21}$$

for $|k| \le k_n/2$. For $|k| \ge k_n/2$, we also have $|\mathbb{E}(\widetilde{\chi}(Q_q,\mathbf{j})_i^n\widetilde{\chi}(Q'_{q'},\mathbf{j}')_{i+k}^n)| \le Kk_n^{-v}$ and $|\boldsymbol{s}_1(\mathbf{j},\mathbf{j}';k)| \le Kk_n^{-v}$, thus (SB.21) holds for $|k| \ge k_n/2$ as well. This completes the proof. *Q.E.D.*

LEMMA S7: *For all pairs $(Q_q, Q'_{q'})$ that satisfy (SB.13) to (SB.15), we have*

$$\left|\sum_{(Q_q,Q'_{q'})}^{\text{(SB.13)}\sim\text{(SB.15)}}\mathbb{E}\big(\widetilde{\chi}(Q_q,\mathbf{j})_i^n\widetilde{\chi}\big(Q'_{q'},\mathbf{j}'\big)_{i+k}^n\big)-\boldsymbol{s}_{2,k_n}\big(\mathbf{j},\mathbf{j}';k\big)\right|\le Kk_n^{-v},\tag{SB.22}$$

where

$$\begin{aligned}\boldsymbol{s}_{2,k_n}\big(\mathbf{j},\mathbf{j}';k\big):=&\sum_{\substack{j_l\in\mathbf{j},j'_{l'}\in\mathbf{j}'\\ l\neq l'}}\boldsymbol{r}_{k_n}\big(j_l,j'_{l'}+k\big)\boldsymbol{r}(\mathbf{j}_{-l})\boldsymbol{r}\big(\mathbf{j}'_{-l'}\big)-\sum_{j_l\in\mathbf{j}}\boldsymbol{r}_{k_n}\big(\{j_l\}\oplus\mathbf{j}'(+k)\big)\boldsymbol{r}(\mathbf{j}_{-l})\\&-\sum_{j'_{l'}\in\mathbf{j}'}\boldsymbol{r}_{k_n}\big(\big\{j'_{l'}+k\big\}\oplus\mathbf{j}\big)\boldsymbol{r}\big(\mathbf{j}'_{-l'}\big),\end{aligned}$$

with

$$\boldsymbol{r}_{k_n}\big(j_l,j'_{l'}+k\big):=\begin{cases}\boldsymbol{r}\big(j_l,j'_{l'}+k-\big(2^{l'-1}+1\big)k_n\big) & \textit{if } l=1, l'>1,\\ \boldsymbol{r}\big(j_l,j'_{l'}+k+\big(2^{l-1}+1\big)k_n\big) & \textit{if } l'=1, l>1,\\ \boldsymbol{r}\big(j_l,j'_{l'}+k-\big(2^{l'-1}-2^{l-1}\big)k_n\big) & \textit{if } l>1, l'>1, l\neq l',\end{cases}$$

$$\boldsymbol{r}_{k_n}\big(\{j_l\}\oplus\mathbf{j}'(+k)\big):=\begin{cases}\boldsymbol{r}\big(\{j_l\}\oplus\mathbf{j}'\big(+(k-k_n)\big)\big) & \textit{if } l=1,\\ \boldsymbol{r}\big(\{j_l\}\oplus\mathbf{j}'\big(+\big(k+2^{l-1}k_n\big)\big)\big) & \textit{if } l>1,\end{cases}$$

$$\boldsymbol{r}_{k_n}\big(\big\{j'_{l'}+k\big\}\oplus\mathbf{j}\big):=\begin{cases}\boldsymbol{r}\big(\big\{j'_{l'}+k\big\}\oplus\mathbf{j}\big(+(-k_n)\big)\big) & \textit{if } l'=1,\\ \boldsymbol{r}\big(\big\{j'_{l'}+k\big\}\oplus\mathbf{j}\big(+\big(2^{l-1}k_n\big)\big)\big) & \textit{if } l'>1.\end{cases}$$

PROOF: First, we prove

$$\left|\sum_{(Q_q,Q'_{q'})}^{\text{(SB.13)}} \mathbb{E}\big(\widetilde{\chi}(Q_q,\mathbf{j})_i^n\widetilde{\chi}(Q'_{q'},\mathbf{j}')_{i+k}^n\big) - \sum_{j_l\in\mathbf{j},j'_{l'}\in\mathbf{j}',l\neq l'} \boldsymbol{r}_{k_n}\big(j_l, j'_{l'}+k\big)\boldsymbol{r}(\mathbf{j}_{-l})\boldsymbol{r}\big(\mathbf{j}'_{-l'}\big)\right| \leq Kk_n^{-v}. \quad \text{(SB.23)}$$

Let us assume $l' > l = 1$. Then, for $(\frac{1}{2}+2^{l'-1})k_n \leq k \leq (\frac{3}{2}+2^{l'-1})k_n$, $\chi_{h_{\mathbf{j}}(i,l)_n}\chi_{h_{\mathbf{j}'}(i+k,l')}$, $\prod_{\bar{l}\neq p}\chi_{i+j_{\bar{l}}}$, and $\prod_{\bar{l}'\neq l'}\chi_{i+k+j'_{\bar{l}'}}$ are asymptotically at least $k_n/2$ away from each other. Apply Lemma S1; we can separate the terms with an error bounded by Kk_n^{-v}:

$$\big|\mathbb{E}\big(\widetilde{\chi}(Q_q,\mathbf{j})_i^n\widetilde{\chi}(Q'_{q'},\mathbf{j}')_{i+k}^n\big) - \boldsymbol{r}\big(j_l, j'_{l'}+k-\big(2^{l'-1}+1\big)k_n\big)\boldsymbol{r}(\mathbf{j}_{-l})\boldsymbol{r}\big(\mathbf{j}'_{-l'}\big)\big| \leq Kk_n^{-v}. \quad \text{(SB.24)}$$

For $k < (\frac{1}{2}+2^{l'-1})k_n$ or $k > (\frac{3}{2}+2^{l'-1})k_n$, at least one of $h_{\mathbf{j}}(i,l)_n$, $h_{\mathbf{j}'}(i+k,l')$ is at least $k_n/2$ from the remaining factors in $\widetilde{\chi}(Q_q,\mathbf{j})_i^n\widetilde{\chi}(Q'_{q'},\mathbf{j}')_{i+k}^n$, thus we can show

$$\big|\mathbb{E}\big(\widetilde{\chi}(Q_q,\mathbf{j})_i^n\widetilde{\chi}(Q'_{q'},\mathbf{j}')_{i+k}^n\big)\big| \vee \big|\boldsymbol{r}\big(j_l, j'_{l'}+k-\big(2^{l'-1}+1\big)k_n\big)\big| \leq Kk_n^{-v},$$

thus (SB.24) still holds. Similarly, we have for $l > l' = 1$ that

$$\big|\mathbb{E}\big(\widetilde{\chi}(Q_q,\mathbf{j})_i^n\widetilde{\chi}(Q'_{q'},\mathbf{j}')_{i+k}^n\big) - \boldsymbol{r}\big(j_l, j'_{l'}+k+\big(2^{l-1}+1\big)k_n\big)\boldsymbol{r}(\mathbf{j}_{-l})\boldsymbol{r}\big(\mathbf{j}'_{-l'}\big)\big| \leq Kk_n^{-v}, \quad \text{(SB.25)}$$

for $-(\frac{3}{2}+2^{l-1})k_n \leq k \leq -(\frac{1}{2}+2^{l-1})k_n$, and

$$\big|\mathbb{E}\big(\widetilde{\chi}(Q_q,\mathbf{j})_i^n\widetilde{\chi}(Q'_{q'},\mathbf{j}')_{i+k}^n\big)\big| \vee \big|\boldsymbol{r}\big(j_l, j'_{l'}+k+\big(2^{l-1}+1\big)k_n\big)\big| \leq Kk_n^{-v},$$

for $k < -(\frac{3}{2}+2^{l-1})k_n$ or $k > -(\frac{1}{2}+2^{l-1})k_n$. Now assume $l' \neq l$, $l > 1$, $l' > 1$. For $(2^{l'-1} - 2^{l-1} - \frac{1}{2})k_n \leq k \leq (2^{l'-1} - 2^{l-1} + \frac{1}{2})k_n$, we have

$$\big|\mathbb{E}\big(\widetilde{\chi}(Q_q,\mathbf{j})_i^n\widetilde{\chi}(Q'_{q'},\mathbf{j}')_{i+k}^n\big) - \boldsymbol{r}\big(j_l, j'_{l'}+k-\big(2^{l'-1}-2^{l-1}\big)k_n\big)\boldsymbol{r}(\mathbf{j}_{-l})\boldsymbol{r}\big(\mathbf{j}'_{-l'}\big)\big| \leq Kk_n^{-v}.$$

For $k > (2^{l'-1} - 2^{l-1} + \frac{1}{2})k_n$ or $k < (2^{l'-1} - 2^{l-1} - \frac{1}{2})k_n$, we have

$$\big|\mathbb{E}\big(\widetilde{\chi}(Q_q,\mathbf{j})_i^n\widetilde{\chi}(Q'_{q'},\mathbf{j}')_{i+k}^n\big)\big| \vee \big|\boldsymbol{r}\big(j_l, j'_{l'}+k-\big(2^{l'-1}-2^{l-1}\big)k_n\big)\big| \leq Kk_n^{-v}.$$

This completes the proof of (SB.23).

The proofs of

$$\left|\sum_{(Q_q,Q'_{q'})}^{\text{(SB.14)}} \mathbb{E}\big(\widetilde{\chi}(Q_q,\mathbf{j})_i^n\widetilde{\chi}(Q'_{q'},\mathbf{j}')_{i+k}^n\big) + \sum_{j_l\in\mathbf{j}} \boldsymbol{r}_{k_n}\big(\{j_l\}\oplus\mathbf{j}'(+k)\big)\boldsymbol{r}(\mathbf{j}_{-l})\right| \leq Kk_n^{-v},$$

$$\left|\sum_{(Q_q,Q'_{q'})}^{\text{(SB.15)}} \mathbb{E}\big(\widetilde{\chi}(Q_q,\mathbf{j})_i^n\widetilde{\chi}(Q'_{q'},\mathbf{j}')_{i+k}^n\big) + \sum_{j'_{l'}\in\mathbf{j}'} \boldsymbol{r}_{k_n}\big(\big\{j'_{l'}+k\big\}\oplus\mathbf{j}\big)\boldsymbol{r}\big(\mathbf{j}'_{-l'}\big)\right| \leq Kk_n^{-v},$$

are similar (in fact, simpler), and this completes the proof. *Q.E.D.*

LEMMA S8: *For any integers i, k, we have*

$$\left|\mathbb{E}\big(u(\mathbf{j})_i^n u(\mathbf{j}')_{i+k}^n\big) - \boldsymbol{s}_{k_n}(\mathbf{j},\mathbf{j}';k)\right| \le K k_n^{-v}, \tag{SB.26}$$

where $\boldsymbol{s}_{k_n}(\mathbf{j},\mathbf{j}';k) := \boldsymbol{s}_0(\mathbf{j},\mathbf{j}';k) + \boldsymbol{s}_1(\mathbf{j},\mathbf{j}';k) + \boldsymbol{s}_{2,k_n}(\mathbf{j},\mathbf{j}';k)$, *and* $\boldsymbol{s}_0(\mathbf{j},\mathbf{j}';k)$, $\boldsymbol{s}_1(\mathbf{j},\mathbf{j}';k)$ *are introduced in Appendix* A *in Li and Linton* (2021).

PROOF: Equation (SB.2) implies we can replace $\boldsymbol{r}(\mathbf{j};k_n)$, $\boldsymbol{r}(\mathbf{j}';k_n)$ by $\boldsymbol{r}(\mathbf{j})$, $\boldsymbol{r}(\mathbf{j}')$ with errors no larger than Kk_n^{-v}. Now (SB.26) follows from (SB.20) and (SB.22). *Q.E.D.*

Next, we will present and prove a key result on stable convergence.

THEOREM S1: *Let*

$$G_t^n := \sum_{i=q_n}^{N_t^n-k_n-j_1} \theta(\mathbf{j},\boldsymbol{k}_n), \qquad G_t'^n := \sum_{i=q_n'}^{N_t^n-k_n-j_1'} \theta(\mathbf{j}',\boldsymbol{k}_n);$$

$$H_t^n := \frac{1}{\sqrt{\delta_n}} \sum_{i=q_n}^{N_t^n-k_n-j_1} (\gamma_i^n)^q d_i^n, \qquad H_t'^n := \frac{1}{\sqrt{\delta_n}} \sum_{i=q_n'}^{N_t^n-k_n-j_1'} (\gamma_i^n)^{q'} d_i^n;$$

$$\mathbf{G}_t^n := (G_t^n, G_t'^n), \qquad \mathbf{H}_t^n := (H_t^n, H_t'^n).$$

Assume (17); *we have* $(\mathbf{G}_t^n, \mathbf{H}_t^n)$ *converges* $\mathcal{F}_\infty$*-stably in law to* $(\mathbf{G}_t, \mathbf{H}_t)$ *with components* $\mathbf{G}_t = (G_t, G_t')$, $\mathbf{H}_t = (H_t, H_t')$ *that is defined on an extension* $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ *of* $(\Omega, \mathcal{F}, \mathbb{P})$, *which, conditionally on* $\mathcal{F}$, *is a centered Gaussian martingale with conditional covariances*

$$\widetilde{\mathbb{E}}(G_t G_t'|\mathcal{F}) = \boldsymbol{s}(\mathbf{j},\mathbf{j}')\int_0^t \gamma_s^{q+q'}\,\mathrm{d}A_s, \qquad \widetilde{\mathbb{E}}(H_t H_t'|\mathcal{F}) = \int_0^t \gamma_s^{q+q'}\overline{\alpha}_s\,\mathrm{d}A_s; \tag{SB.27}$$

$$\widetilde{\mathbb{E}}(G_t H_t|\mathcal{F}) = \widetilde{\mathbb{E}}(G_t H_t'|\mathcal{F}) = \widetilde{\mathbb{E}}(G_t' H_t|\mathcal{F}) = \widetilde{\mathbb{E}}(G_t' H_t'|\mathcal{F}) = 0. \tag{SB.28}$$

Since $\{u(\mathbf{j})_i^n\}_i$ are serially dependent, we will employ the "block splitting" technique that is often used in the literature (see, e.g., Jacod, Li, and Zheng (2019)): we will divide the observations into "big blocks" of size pk_n separated by "small blocks" of size $K_{\mathbf{j},\mathbf{j}'}k_n$, where p will eventually grow to infinity and $K_{\mathbf{j},\mathbf{j}'}$ is a constant that depends on $\mathbf{j}$, $\mathbf{j}'$.

Now we consider small blocks of size $(2+2^{q-1})k_n$, and we need to introduce a sequence of notations associated with the block splitting techniques. By polarization, we will consider $\mathbf{j}=\mathbf{j}'$; moreover, $\boldsymbol{k}_n$ satisfies (17), thus is also fixed. We therefore write θ_i^n instead of $\theta(\mathbf{j},\boldsymbol{k}_n)_i^n$ in the sequel:

$$m(p,q) := p+2+2^{q-1}, \qquad J_n(p,t) := 1 + \left[\frac{N_t^n}{m(p,q)k_n}\right],$$

$$I_n(p,t) := q_n + J_n(p,t)m(p,q)k_n - 1;$$

$$\mathcal{H}_i^n := \mathcal{F}_i^n \otimes \mathcal{G}_{i-q_n-k_n+j_1}, \qquad \mathcal{H}(p)_j^n := \mathcal{H}_{jm(p,q)k_n+q_n}^n, \qquad \mathcal{H}'(p)_j^n := \mathcal{H}_{(jm(p,q)+p)k_n+q_n}^n;$$

$$\zeta(p)_i^n := \sum_{j=i}^{i+pk_n-1} \theta_j^n, \qquad R(p)_t^n := \sum_{i=N_t^n-k_n-j_1+1}^{I_n(p,t)} \theta_i^n;$$

$$\eta(p)^n_j := \zeta(p)^n_{(j-1)m(p,q)k_n+q_n}, \qquad \eta'(p)^n_j := \zeta\big(2+2^{q-1}\big)^n_{((j-1)m(p,q)+p)k_n+q_n};$$
$$\overline{\eta}(p)^n_j := \mathbb{E}\big(\eta(p)^n_j|\mathcal{H}(p)^n_{j-1}\big), \qquad \overline{\eta}'(p)^n_j := \mathbb{E}\big(\eta'(p)^n_j|\mathcal{H}'(p)^n_{j-1}\big);$$
$$F(p)^n_t := \sum_{j=1}^{J_n(p,t)} \overline{\eta}(p)^n_j, \qquad M(p)^n_t := \sum_{j=1}^{J_n(p,t)} \big(\eta(p)^n_j - \overline{\eta}(p)^n_j\big);$$
$$F'(p)^n_t := \sum_{j=1}^{J_n(p,t)} \overline{\eta}'(p)^n_j, \qquad M'(p)^n_t := \sum_{j=1}^{J_n(p,t)} \big(\eta'(p)^n_j - \overline{\eta}'(p)^n_j\big).$$

Since $p \geq 2+2^{q-1}$, we conclude that $\eta(p)^n_j$ is $\mathcal{H}(p)^n_j$-measurable and $\eta'(p)^n_j$ is $\mathcal{H}'(p)^n_j$-measurable. Now it follows that

$$G^n_t = F(p)^n_t + F'(p)^n_t + M(p)^n_t + M'(p)^n_t - R(p)^n_t. \tag{SB.29}$$

LEMMA S9: *For fixed $p \geq 2+2^{q-1}$, we have*

$$\big|\mathbb{E}\big(\zeta(p)^n_i|\mathcal{H}^n_i\big)\big| \leq K_p \delta_n^{\frac{1}{2}} k_n^{1-v}; \qquad \big|\mathbb{E}\big(\big(\zeta(p)^n_i\big)^4|\mathcal{H}^n_i\big)\big| \leq K_p \delta_n^2 k_n^4.$$

PROOF: By the independence of $\mathcal{G}, \mathcal{F}^{(0)}$, the boundedness of γ, and Lemma S1, we have for $j \geq i$ that $|\mathbb{E}(\theta^n_j|\mathcal{H}^n_i)| \leq K\sqrt{\delta_n}|\mathbb{E}(u(\mathbf{j})^n_j|\mathcal{G}_{i-q_n-k_n+j_1})| \leq K\sqrt{\delta_n}(k_n+j-i)^{-v}$. Thus, we have $|\mathbb{E}(\zeta(p)^n_i|\mathcal{H}^n_i)| \leq K\sqrt{\delta_n}\sum_{j=i}^{i+pk_n-1}(k_n+j-i)^{-v} \leq K_p\sqrt{\delta_n}k_n^{1-v}$. The second estimate follows immediately from

$$\big|\mathbb{E}\big(\big(\zeta(p)^n_i\big)^4|\mathcal{H}^n_i\big)\big| \leq K \sum_{\ell_1\leq\ell_2\leq\ell_3\leq\ell_4} \big|\mathbb{E}\big(\theta^n_{\ell_1}\theta^n_{\ell_2}\theta^n_{\ell_3}\theta^n_{\ell_4}|\mathcal{H}^n_i\big)\big| \leq K_p\delta_n^2 k_n^4.$$

This completes the proof. *Q.E.D.*

In the following lemma, we omit $\mathbf{j}$ and simply write $\boldsymbol{s}_\ell(k)$, $\ell = 0, 1$, $\boldsymbol{s}_{k_n}(k)$, and $\boldsymbol{s}_{2,k_n}(k)$ instead of $\boldsymbol{s}_\ell(\mathbf{j},\mathbf{j};k)$, $\ell=0,1$, $\boldsymbol{s}_{k_n}(\mathbf{j},\mathbf{j};k)$, and $\boldsymbol{s}_{2,k_n}(\mathbf{j},\mathbf{j};k)$.

LEMMA S10: *Let $v > 2$; for any $p \geq 2+2^{q-1}$, we have*

$$\left|\sum_{k=1}^{pk_n-1} \frac{k}{pk_n}\big(\boldsymbol{s}_{k_n}(k)+\boldsymbol{s}_{k_n}(-k)\big) - \frac{\mathfrak{C}_\mathbf{j}}{p}\right| \leq \frac{K_p}{k_n}, \tag{SB.30}$$

where $\mathfrak{C}_\mathbf{j} := \sum_{(l,l'):j_l,j_{l'}\in\mathbf{j},l\neq l'} \boldsymbol{r}(\mathbf{j}_{-l})\boldsymbol{r}(\mathbf{j}_{-l'})\mathfrak{K}_{l,l'} - 2\sum_{l:j_l\in\mathbf{j}}\boldsymbol{r}(\mathbf{j}_{-l})\mathfrak{K}_l$; *and*

$$\mathfrak{K}_{l,l'} := \sum_{k=-\infty}^{\infty} \boldsymbol{r}(0,k) \times \begin{cases} 2^{l'-1}+1 & \textit{if } l=1, l'>1; \\ 2^{l-1}+1 & \textit{if } l'=1, l>1; \\ 2^{l\vee l'-1} - 2^{l\wedge l'-1} & \textit{if } l>1, l'>1, l\neq l'; \end{cases}$$

$$\mathfrak{K}_l := 2^{l-1}\sum_{k=-\infty}^{\infty} \boldsymbol{r}\big(\{0\}\oplus\mathbf{j}(+k)\big).$$

PROOF: Since $v > 2$, we have $|\sum_{k=1}^{pk_n-1} \frac{k}{pk_n}(\boldsymbol{s}_\ell(k)+\boldsymbol{s}_\ell(-k))| \leq K|\sum_{k=1}^{pk_n-1} \frac{k}{pk_n}\frac{1}{k^v}| \leq \frac{K_p}{k_n}$ for $\ell = 0, 1$. Thus, it suffices to show

$$\left|\sum_{k=1}^{pk_n-1} \frac{k}{pk_n}\big(\boldsymbol{s}_{2,k_n}(k)+\boldsymbol{s}_{2,k_n}(-k)\big) - \frac{\mathfrak{C}_{\mathbf{j}}}{p}\right| \leq \frac{K_p}{k_n}. \tag{SB.31}$$

To see this, we will first show for $j_l, j_{l'} \in \mathbf{j}$, $l \neq l'$ (recall $\boldsymbol{r}_{k_n}$ defined in Lemma S7)

$$\left|\sum_{k=1}^{pk_n-1} \frac{k}{pk_n}\big(\boldsymbol{r}_{k_n}(j_l, j_{l'}+k)+\boldsymbol{r}_{k_n}(j_l, j_{l'}-k)\big) - \frac{\mathfrak{K}_{l,l'}}{p}\right| \leq \frac{K_p}{k_n}. \tag{SB.32}$$

Let

$$k'_n := \begin{cases} \big(2^{l'-1}+1\big)k_n + j_l - j_{l'} & \text{if } l=1, l'>1; \\ \big(2^{l-1}+1\big)k_n + j_{l'} - j_l & \text{if } l'=1, l>1; \\ \big(2^{l\vee l'-1} - 2^{l\wedge l'-1}\big)k_n + j_{l\wedge l'} - j_{l\vee l'} & \text{if } l>1, l'>1, l\neq l'. \end{cases}$$

Then (SB.32) follows from

$$\sum_{k=1}^{pk_n-1} \big(\boldsymbol{r}_{k_n}(j_l, j_{l'}+k)+\boldsymbol{r}_{k_n}(j_l, j_{l'}-k)\big)k = \sum_{k=1-k'_n}^{pk_n-1-k'_n} \big(\boldsymbol{r}(0,k)+\boldsymbol{r}\big(0,k+2k'_n\big)\big)\big(k+k'_n\big),$$

and the easy estimates that $|\sum_{k=1-k'_n}^{pk_n-1-k'_n} \boldsymbol{r}(0,k) - \sum_{k=-\infty}^{\infty}\boldsymbol{r}(0,k)| \leq \frac{K}{k_n^{v-1}}$, $|\sum_{k=1-k'_n}^{pk_n-1-k'_n} \boldsymbol{r}(0,k)k| \leq K$, and $|\sum_{k=1-k'_n}^{pk_n-1-k'_n} \boldsymbol{r}(0,k+2k'_n)(k+k'_n)| \leq \frac{K}{k_n^{v-2}}$. We can prove in a similar manner that, for $j_l \in \mathbf{j}$,

$$\left|\sum_{k=1}^{pk_n-1} \frac{k}{pk_n}\big(\boldsymbol{r}_{k_n}\big(\{j_l+k\}\oplus\mathbf{j}\big)+\boldsymbol{r}_{k_n}\big(\{j_l-k\}\oplus\mathbf{j}\big)\big) - \frac{K_l}{p}\right| \leq \frac{K_p}{k_n},$$

$$\left|\sum_{k=1}^{pk_n-1} \frac{k}{pk_n}\big(\boldsymbol{r}_{k_n}\big(\{j_l\}\oplus\mathbf{j}(+k)\big)+\boldsymbol{r}_{k_n}\big(\{j_l\}\oplus\mathbf{j}(-k)\big)\big) - \frac{K_l}{p}\right| \leq \frac{K_p}{k_n}.$$

This finishes the proof of (SB.31) and the proof is now complete. *Q.E.D.*

LEMMA S11: *Let $v > 2$; for any $p \geq 2 + 2^{q-1}$, we have*

$$\left|\mathbb{E}\big(\big(\zeta(p)_i^n\big)^2|\mathcal{H}_i^n\big) - pk_n\delta_n\big(\gamma_i^n\big)^{2q}\left(\boldsymbol{s}(\mathbf{j},\mathbf{j}) - \frac{\mathfrak{C}_{\mathbf{j}}}{p}\right)\right| \leq K_p k_n\delta_n\big(k_n\delta_n^{\frac{1}{2}+\kappa} \vee k_n\delta_n^{\rho} \vee k_n^{-1}\big). \tag{SB.33}$$

PROOF: We have

$$\big(\zeta(p)_i^n\big)^2 = \sum_{j=i}^{\mu(p,0)_i^n} \big(\theta_j^n\big)^2 + 2\sum_{k=1}^{pk_n-1}\sum_{j=i}^{\mu(p,k)_i^n} \theta_j^n\theta_{j+k}^n, \quad \text{with } \mu(p,k)_i^n := i-k+pk_n-1.$$

Thus, $\mathbb{E}((\zeta(p)_i^n)^2|\mathcal{H}_i^n) = \sum_{\ell=0}^{7} \mathfrak{E}(\ell)_{i,0}^{n,p} + 2\sum_{k=1}^{pk_n-1}\sum_{\ell=0}^{7} \mathfrak{E}(\ell)_{i,k}^{n,p}$, where, for any nonnegative integer k,

$$\mathfrak{E}(0)_{i,k}^{n,p} := p\delta_n k_n \big(\gamma_i^n\big)^{2q} \boldsymbol{s}_{k_n}(k)\bigg(1 - \frac{k}{pk_n}\bigg);$$

$$\mathfrak{E}(1)_{i,k}^{n,p} := pk_n \boldsymbol{s}_{k_n}(k)\big(\gamma_i^n\big)^{2q}\mathbb{E}\big(d_i^n|\mathcal{H}_i^n\big); \qquad \mathfrak{E}(2)_{i,k}^{n,p} := -k\boldsymbol{s}_{k_n}(k)\big(\gamma_i^n\big)^{2q}\mathbb{E}\big(d_i^n|\mathcal{H}_i^n\big);$$

$$\mathfrak{E}(3)_{i,k}^{n,p} := \boldsymbol{s}_{k_n}(k)\sum_{j=i}^{\mu(p,k)_i^n} \mathbb{E}\big(\big(\gamma_j^n\big)^{2q}\alpha_j^n\delta(n, j+1) - \big(\gamma_i^n\big)^{2q}\alpha_i^n\delta(n, i+1)|\mathcal{H}_i^n\big);$$

$$\mathfrak{E}(4)_{i,k}^{n,p} := \boldsymbol{s}_{k_n}(k)\sum_{j=i}^{\mu(p,k)_i^n} \mathbb{E}\big(\big(\gamma_j^n\big)^{q}\big(\big(\gamma_{j+k}^n\big)^q - \big(\gamma_j^n\big)^q\big)\alpha_j^n\delta(n, j+1)|\mathcal{H}_i^n\big);$$

$$\mathfrak{E}(5)_{i,k}^{n,p} := -\boldsymbol{s}_{k_n}(k)\sum_{j=i}^{\mu(p,k)_i^n} \mathbb{E}\big(\big(\gamma_j^n\big)^{q}\big(\gamma_{j+k}^n\big)^q d_j^n|\mathcal{H}_i^n\big);$$

$$\mathfrak{E}(6)_{i,k}^{n,p} := \sum_{j=i}^{\mu(p,k)_i^n} \delta_n\big(\mathbb{E}\big(\theta_j^n\theta_{j+k}^n|\mathcal{F}^{(0)}\big) - \boldsymbol{s}_{k_n}(k)\mathbb{E}\big(\big(\gamma_j^n\big)^q\big(\gamma_{j+k}^n\big)^q|\mathcal{H}_i^n\big)\big);$$

$$\mathfrak{E}(7)_{i,k}^{n,p} := \sum_{j=i}^{\mu(p,k)_i^n} \delta_n\big(\mathbb{E}\big(\theta_j^n\theta_{j+k}^n|\mathcal{H}_i^n\big) - \mathbb{E}\big(\theta_j^n\theta_{j+k}^n|\mathcal{F}^{(0)}\big)\big).$$

First, we note by (4) that $\sum_{k=0}^{pk_n-1}|\mathfrak{E}(1)_{i,k}^{n,p}| \le K_p\delta_n^{\frac{3}{2}+\kappa}k_n$, and an application of Lemma S10 yields a similar estimate $\sum_{k=0}^{pk_n-1}|\mathfrak{E}(2)_{i,k}^{n,p}| \le K_p\delta_n^{\frac{3}{2}+\kappa}k_n$. Next, we show

$$\sum_{k=0}^{pk_n-1}\big|\mathfrak{E}(3)_{i,k}^{n,p}\big| \le K_p k_n\delta_n\big(\delta_n^{\frac{1}{2}+\kappa} \vee k_n\delta_n^{\rho}\big). \tag{SB.34}$$

Let $z(1)_{i,j}^n := ((\gamma_j^n)^{2q}\alpha_j^n - (\gamma_i^n)^{2q}\alpha_i^n)\delta(n, j+1)$, $z(2)_{i,j}^n := (\gamma_i^n)^{2q}\alpha_i^n(\delta(n, j+1) - \delta(n, i+1))$; then we have $\mathfrak{E}(3)_{i,k}^{n,p} = \boldsymbol{s}_{k_n}(k)\sum_{j=i}^{\mu(p,k)_i^n}\mathbb{E}(z(1)_{i,j}^n + z(2)_{i,j}^n|\mathcal{H}_i^n)$. By first conditioning on $\mathcal{H}_i^n \vee \sigma(\delta(n, j+1))$, (SA.2), and (4), we have $|\mathbb{E}(z(1)_{i,j}^n|\mathcal{H}_i^n)| \le K(j-i)\delta_n^{1+\rho}$; similarly, we get $|\mathbb{E}((\alpha_i^n - \alpha_j^n)\delta(n, j+1)|\mathcal{H}_i^n)| \le K(j-i)\delta_n^{1+\rho}$; together with the simple estimate (using again (4)) $|\mathbb{E}(\alpha_j^n\delta(n, j+1) - \alpha_i^n\delta(n, i+1)|\mathcal{H}_i^n)| \le K\delta_n^{\frac{3}{2}+\kappa}$, we have $|\mathbb{E}(z(2)_{i,j}^n|\mathcal{H}_i^n)| \le K((j-i)\delta_n^{1+\rho} \vee \delta_n^{3/2+\kappa})$. This proves (SB.34). Next, since for any $k > 0$, $\delta(n, j+1)$ is independent of γ_{j+k}^n conditional on $\mathcal{H}_j^n$, by first conditioning on $\mathcal{H}_j^n$, (SA.1) yields that $|\mathbb{E}((\gamma_j^n)^q((\gamma_{j+k}^n)^q - (\gamma_j^n)^q)\alpha_j^n\delta(n, j+1)|\mathcal{H}_i^n)| \le K\delta_n^{1+\rho}k$, which implies $\sum_{k=0}^{pk_n-1}|\mathfrak{E}(4)_{i,k}^{n,p}| \le K_p k_n^2\delta_n^{1+\rho}$. Similarly, we have $\sum_{k=0}^{pk_n-1}|\mathfrak{E}(5)_{i,k}^{n,p}| \le K_p k_n^2\delta_n^{\frac{3}{2}+\kappa}$. Next, we can apply Lemma S1 and Lemma S8, which yields the following: $\sum_{k=0}^{pk_n-1}(|\mathfrak{E}(6)_{i,k}^{n,p}| + |\mathfrak{E}(7)_{i,k}^{n,p}|) \le K_p\delta_n k_n^{2-v}$. Now let $\boldsymbol{s}(p)_{k_n} := \sum_{k=-(pk_n-1)}^{pk_n-1}\boldsymbol{s}_{k_n}(k)$. We have (1) $|\boldsymbol{s}(p)_{k_n} - \boldsymbol{s}(\mathbf{j},\mathbf{j})| \le K_p k_n^{1-v}$, (2) $\sum_{k=-(pk_n-1)}^{pk_n-1}(1 - \frac{|k|}{pk_n})\boldsymbol{s}_{k_n}(k) = \boldsymbol{s}(p)_{k_n} - \sum_{k=1}^{pk_n-1}\frac{k}{pk_n}(\boldsymbol{s}_{k_n}(k) + \boldsymbol{s}_{k_n}(-k))$, and (3) $\boldsymbol{s}_{2,k_n}(\mathbf{j},\mathbf{j};k) = \boldsymbol{s}_{2,k_n}(\mathbf{j},\mathbf{j};-k)$. The first estimate is obtained from Lemma S10.

Therefore, we have by the above three estimates that $|\mathfrak{E}(0)^{n,p}_{i,0} + 2\sum_{k=1}^{pk_n-1}\mathfrak{E}(0)^{n,p}_{i,k} - p\delta_n k_n(\gamma^n_i)^{2q}(\boldsymbol{s}(\mathbf{j},\mathbf{j}) - \frac{\mathfrak{C}_\mathbf{j}}{p})| \leq K_p\delta_n$. This finishes the proof of the lemma. *Q.E.D.*

LEMMA S12: *If V is a càdlàg process, $p \geq 2 + 2^{q-1}$, and $\delta^\rho_n k_n \to 0$, we have, for all $t > 0$, $k_n\delta_n \sum_{j=1}^{J_n(p,t)} V_{(j-1)m(p,q)k_n+q_n} \xrightarrow{\mathbb{P}} \frac{\int_0^t V_s\,\mathrm{d}A_s}{m(p,q)}$.*

PROOF: We only need to prove $k_n\delta_n \sum_{j=1}^{J_n(p,t)} V_{(j-1)m(p,q)k_n} \xrightarrow{\mathbb{P}} \int_0^t V_s\,\mathrm{d}A_s/m(p,q)$, since $k_n\delta_n \sum_{j=1}^{J_n(p,t)} \mathbb{E}(|V_{(j-1)m(p,q)k_n+q_n} - V_{(j-1)m(p,q)k_n}|) \leq K\sqrt{k_n\delta^\rho_n} \to 0$. Let $u^n_t := k_n\delta_n J_n(p,t)$. Then we have $k_n\delta_n \sum_{j=1}^{J_n(p,t)} V^n_{(j-1)m(p,q)k_n} = \int_0^{t+h_n} V_s\,\mathrm{d}u^n_s$, where $h_n := T^n_{J_n(p,t)m(p,q)k_n} - T^n_{N^n_t}$. It suffices to prove $h_n \xrightarrow{\mathbb{P}} 0$ since $u^n_t \xrightarrow{\mathbb{P}} A_t/m(p,q)$. Since $J_n(p,t)m(p,q)k_n - N^n_t \leq m(p,q)k_n$, we have, for any $\epsilon > 0$, $\limsup_n \mathbb{P}(|h_n| > \epsilon) \leq \limsup_n \mathbb{P}(A_{t+\epsilon} - A_t \leq m(p,q)k_n\delta_n) \to 0$, as $A_{t+\epsilon} > A_t$, $k_n\delta_n \to 0$. *Q.E.D.*

LEMMA S13: *Let $\delta_n k_n^2 \to 0$, $v > 2$, $\delta_n k_n^{2v} \to \infty$ as $n \to \infty$; for all $p \geq 2 + 2^{q-1}$, we have*

$$\sum_{j=1}^{J_n(p,t)} \mathbb{E}\big((\eta(p)^n_j)^2|\mathcal{H}(p)^n_{j-1}\big) \xrightarrow{\mathbb{P}} \frac{p}{m(p,q)}\left(\boldsymbol{s}(\mathbf{j},\mathbf{j}) - \frac{\mathfrak{C}_\mathbf{j}}{p}\right)\int_0^t \gamma_s^{2q}\,\mathrm{d}A_s; \qquad \text{(SB.35)}$$

$$F(p)^n_t \xrightarrow{\mathbb{P}} 0, \qquad F'(p)^n_t \xrightarrow{\mathbb{P}} 0; \qquad \text{(SB.36)}$$

$$\mathbb{E}\big((M'(p)^n_t)^2\big) \leq \frac{K_t}{p}, \qquad R(p)^n_t \xrightarrow{\mathbb{P}} 0. \qquad \text{(SB.37)}$$

PROOF: Equation (SB.35) follows directly from Lemma S11 and Lemma S12. Since $J_n(p,t) \leq \frac{K_p t}{k_n\delta_n}$, we have by Lemma S9 that $\mathbb{E}(|F(p)^n_t|) \leq \frac{K_p t}{\delta_n^{1/2}k_n^v} \to 0$; the same result applies to $F'(p)^n_t$. This proves (SB.36). By the martingale property, we have $\mathbb{E}((M'(p)^n_t)^2) \leq \sum_{j=1}^{J_n(p,t)} \mathbb{E}((\zeta(2+2^{q-1})^n_{((j-1)m(p,q)+p)k_n+q_n})^2) \leq \frac{K_t}{p}$. The last inequality follows from Lemma S11 and $J_n(p,t) \leq \frac{Kt}{p\delta_n k_n}$. Note that $I_n(p,t) - (N^n_t - k_n - j_1) \leq (p+2(2^{q-1}+2))k_n$, therefore, we have $\mathbb{E}((R(p)^n_t)^2) \leq K_p k_n \sum_{i=N^n_t-k_n-j_1+1}^{N^n_t+(p+2(2^{q-1}+2))k_n} \mathbb{E}((\theta^n_i)^2) \leq K_p\delta_n k_n^2 \to 0$. This proves (SB.37). *Q.E.D.*

PROPOSITION S1: *Let $v > 2$, $\delta_n k_n^3 \to 0$. For any fixed $p \geq 2 + 2^{q-1}$, the sequence of processes $M(p)^n$ converges $\mathcal{F}_\infty$-stably in law to the process $G(p)_t$ defined on an extension $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ of the original space, conditionally on $\mathcal{F}$, is centered Gaussian with (conditional) variance $Z(p,\mathbf{j})_t := \frac{p}{m(p,q)}(\boldsymbol{s}(\mathbf{j},\mathbf{j}) - \frac{\mathfrak{C}_\mathbf{j}}{p})\int_0^t \gamma_s^{2q}\,\mathrm{d}A_s$.*

PROOF: Let $\widehat{\eta}(p)^n_j := \eta(p)^n_j - \overline{\eta}(p)^n_j$. Let $\Delta(V,p)^n_j := V^n_{jm(p,q)k_n+q_n} - V^n_{(j-1)m(p,q)k_n+q_n}$ for any process V. We also set $\mathcal{M} = \mathcal{M}_1 \cup W$, where W is the Brownian motion driving X and $\mathcal{M}_1$ denotes the class of all bounded $(\mathcal{F}_t)$-martingales orthogonal to W. By a standard stable convergence theorem for triangular arrays (see, e.g., Theorem IX 7.28 in Jacod and Shiryaev (2003)), it suffices to prove the following three convergences:

$$\sum_{j=1}^{J_n(p,t)} \mathbb{E}\big((\widehat{\eta}(p)^n_j)^2|\mathcal{H}(p)^n_{j-1}\big) \xrightarrow{\mathbb{P}} \frac{p}{m(p,q)}\left(\boldsymbol{s}(\mathbf{j},\mathbf{j}) - \frac{\mathfrak{C}_\mathbf{j}}{p}\right)\int_0^t \gamma_s^{2q}\,\mathrm{d}A_s; \qquad \text{(SB.38)}$$

$$\sum_{j=1}^{J_n(p,t)} \mathbb{E}\big((\widehat{\eta}(p)^n_j)^4 | \mathcal{H}(p)^n_{j-1}\big) \xrightarrow{\mathbb{P}} 0; \tag{SB.39}$$

$$\forall V \in \mathcal{M}, \qquad \sum_{j=1}^{J_n(p,t)} \mathbb{E}\big(\widehat{\eta}(p)^n_j \Delta(V,p)^n_j | \mathcal{H}(p)^n_{j-1}\big) \xrightarrow{\mathbb{P}} 0. \tag{SB.40}$$

(1) Note that $\mathbb{E}((\widehat{\eta}(p)^n_j)^2|\mathcal{H}(p)^n_{j-1}) = \mathbb{E}((\eta(p)^n_j)^2|\mathcal{H}(p)^n_{j-1}) - \mathbb{E}((\overline{\eta}(p)^n_j)^2|\mathcal{H}(p)^n_{j-1})$, and from Lemma S9, we have $(\overline{\eta}(p)^n_j)^2 \le K_p \delta_n k_n^{2-2v}$. Since $J_n(p,t) \le \frac{K_p t}{\delta_n k_n}$, we conclude that $\sum_{j=1}^{J_n(p,t)} \mathbb{E}((\overline{\eta}(p)^n_j)^2|\mathcal{H}(p)^n_{j-1}) \le K_p k_n^{1-2v} \to 0$. Now (SB.38) follows from the first part of Lemma S13.

(2) By Lemma S9, we have $\sum_{j=1}^{J_n(p,t)} \mathbb{E}((\eta(p)^n_j)^4|\mathcal{H}(p)^n_{j-1}) \le K_p \delta_n k_n^3$, and we further have $\sum_{j=1}^{J_n(p,t)} \mathbb{E}((\overline{\eta}(p)^n_j)^4|\mathcal{H}(p)^n_{j-1}) \le K_p \delta_n k_n^{3-4v}$. Now (SB.39) is proved.

(3) It suffices to show

$$\sum_{j=1}^{J_n(p,t)} \mathbb{E}\big(\eta(p)^n_j \Delta(V,p)^n_j | \mathcal{H}(p)^n_{j-1}\big) \xrightarrow{\mathbb{P}} 0, \tag{SB.41}$$

since $\mathbb{E}(\Delta(V,p)^n_j|\mathcal{H}(p)^n_{j-1}) = 0$ for any $V \in \mathcal{M}$. Consider any i in the range $[(j-1)m(p,q)k_n + q_n, jm(p,q) - 2k_n - 1]$. We have $\mathbb{E}(\theta^n_i \Delta(V,p)^n_j|\mathcal{H}(p)^n_{j-1}) = \mathfrak{X}(1)^n_{i,j}\mathfrak{X}(2)^n_{i,j}$, where $\mathfrak{X}(1)^n_{i,j} := \mathbb{E}(\sqrt{\delta_n}(\gamma^n_i)^q \Delta(V,p)^n_j|\mathcal{F}^n_{(j-1)m(p,q)k_n+q_n})$, and $\mathfrak{X}(2)^n_{i,j}$ is given by $\mathfrak{X}(2)^n_{i,j} := \mathbb{E}(u(\mathbf{j})^n_i|\mathcal{G}_{((j-1)m(p,q)-1)k_n+j_1})$. By Lemma S1, we have $\sum_i |\mathfrak{X}(2)^n_{i,j}| \le K_p k_n^{1-v}$; now we have $|\mathfrak{X}(1)^n_{i,j}| \le K\sqrt{\delta_n \mathbb{E}((\Delta(V,p)^n_j)^2|\mathcal{F}^n_{(j-1)m(p,q)k_n+q_n})}$ since γ is bounded. Thus, we have

$$\big|\mathbb{E}\big(\eta(p)^n_j \Delta(V,p)^n_j|\mathcal{H}(p)^n_{j-1}\big)\big| \le K_p k_n^{1-v} \sqrt{\delta_n \mathbb{E}\big(\big(\Delta(V,p)^n_j\big)^2 | \mathcal{F}^n_{(j-1)m(p,q)k_n+q_n}\big)},$$

and an application of the Cauchy–Schwarz inequality and the martingale property yield $\mathbb{E}((\sum_{j=1}^{J_n(p,t)} \mathbb{E}(\eta(p)^n_j \Delta(V,p)^n_j|\mathcal{H}(p)^n_{j-1}))^2) \le K_{p,t} k_n^{1-2v} \mathbb{E}(\sum_{j=1}^{J_n(p,t)} (\Delta(V,p)^n_j)^2)$, and it is further bounded by $K_{p,t} k_n^{1-2v} \mathbb{E}((V_{T^n_{J_n(p,t)m(p,q)k_n+q_n}} - V_0)^2)$. Now we have that if $V \in \mathcal{M}_1$, $\mathbb{E}((V_{T^n_{J_n(p,t)m(p,q)k_n+q_n}} - V_0)^2)$ is further bounded by $\mathbb{E}((V_\infty - V_0)^2) < K$, and this proves (SB.41) with $V \in \mathcal{M}_1$. When $V = W$, $T^n_{J_n(p,t)m(p,q)k_n+q_n} \le t+1$ for n large enough on the set Ω^n_t (recall (SA.3)). Thus, (SB.41) is proved with $V = W$ on the set Ω^n_t. Since $\mathbb{P}(\Omega^n_t) \to 1$, the proof is complete for $V = W$. *Q.E.D.*

THEOREM S2: *$(M(p)^n_t, H^n_t)$ converges $\mathcal{F}_\infty$-stably in law to $(G(p)_t, H_t)$ that is defined on an extension $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ of $(\Omega, \mathcal{F}, \mathbb{P})$, which, conditionally on $\mathcal{F}$, is a centered Gaussian martingale with conditional covariances*

$$\widetilde{\mathbb{E}}\big(G(p)_t G(p)_t|\mathcal{F}\big) = Z(p,\mathbf{j})_t, \qquad \widetilde{\mathbb{E}}\big(H_t H'_t|\mathcal{F}\big) = \int_0^t \gamma_s^{q+q'} \overline{\alpha}_s \, \mathrm{d}A_s,$$

$$\widetilde{\mathbb{E}}\big(G(p)_t H_t|\mathcal{F}\big) = 0.$$

PROOF: Lemma S11 yields an estimate that $\mathbb{E}((\widehat{\eta}(p)^n_j)^2) \le K_p \delta_n k_n$. Now we have $\mathbb{E}(\sum_{j=1}^{J_n(p,t)} \mathbb{E}((\widehat{\eta}(p)^n_j)^2 \mathbf{1}_{\{\widehat{\eta}(p)^n_j > \epsilon\}}|\mathcal{H}(p)^n_{j-1})) \to 0$ by Lebesgue's dominated convergence theorem and the fact that $\delta_n k_n \to 0$. This in turn leads to the following convergence: for any $\epsilon > 0$,

$$\sum_{j=1}^{J_n(p,t)} \mathbb{E}\Big(\big(\widehat{\eta}(p)_j^n\big)^2 \mathbf{1}_{\{\widehat{\eta}(p)_j^n > \epsilon\}} \big| \mathcal{H}(p)_{j-1}^n\Big) \xrightarrow{\mathbb{P}} 0. \tag{SB.42}$$

On the other hand, we have $e^{iuM(p)_t^n} = g(u,p)_t^n \mathfrak{M}(u,p)_t^n$, $e^{iuG(p)_t} = g(u,p)_t \mathfrak{M}(u,p)_t$, where $g(u,p)_t^n$ and $g(u,p)_t$ are predictable with finite variation, and $\mathfrak{M}(u,p)_t^n$ and $\mathfrak{M}(u,p)_t$ are martingales (see, e.g., Theorem II.2.47 in Jacod and Shiryaev (2003)). According to the proof of Theorem VIII.2.4 in Jacod and Shiryaev (2003) (see also the proof of Theorem A.4 in Jacod, Li, and Zheng (2017)), (SB.38) and (SB.42) imply $g(u,p)_t^n \xrightarrow{\mathbb{P}} g(u,p)_t$. Now the joint convergence follows from Proposition S1 and Theorem A.4 of Jacod, Li, and Zheng (2017). *Q.E.D.*

PROOF OF THEOREM S1: By polarization, it suffices to consider $\mathbf{j} = \mathbf{j}'$. The process $V(p)^n := G_t^n - M(p)^n$ satisfies $\lim_{p\to\infty} \limsup_{n\to\infty} \mathbb{P}(|V(p)_t^n| > \epsilon) = 0$ for all $\epsilon > 0$, $t > 0$. This follows from (SB.29) and Lemma S13. On the other hand, $Z(p,\mathbf{j})_t(\omega) \le K$ and $Z(p,\mathbf{j})_t(\omega) \to Z(\mathbf{j},\mathbf{j})_t(\omega)$ for all $t > 0$ and ω; we thus have $G(p)_t \xrightarrow{\mathbb{P}} G_t$. Now Theorem S1 follows from Theorem S2. *Q.E.D.*

The next lemma will be used to prove the consistency of the proposed estimators for the asymptotic variances and covariances.

We first introduce notations. Let $\mathbf{j}_\ell \in \mathfrak{J}$, $\mathbf{j}_\ell = (j_{\ell,1}, \dots, j_{\ell,q_\ell})$, $q_\ell = |\mathbf{j}_\ell|$, $\ell = 1, 2, \dots, d$. $\{w_\ell^n\}_{\ell=1}^d$ is a sequence of integers satisfying $w_{n,1} = 0$, $w_{\ell+1}^n - w_\ell^n \ge 2^{q_{\ell+1}-1}k_n + j_{\ell,1} + 2k_n$ for $\ell \ge 1$. Let $\overline{w}_d^n := w_d^n \vee k_n \vee \overline{j}$, where $\overline{j} := \max\{j_{\ell,p} : 1 \le \ell \le d, 1 \le p \le q_\ell\}$. Let

$$\mathfrak{U}_{d,t}^n := \sum_{i=2^{q_1-1}k_n}^{N_t^n - (j_{d,1} + w_d^n \vee k_n)} \prod_{\ell=1}^d \Delta_{\mathbf{j}_\ell}(Y)_{i+w_\ell^n}^n, \qquad \mathfrak{U}'^n_{d,t} := \sum_{i=2^{q_1-1}k_n}^{N_t^n - (j_{d,1} + w_d^n \vee k_n)} \big(\gamma_i^n\big)^{\overline{q}} \prod_{\ell=1}^d \Delta_{\mathbf{j}_\ell}(\chi)_{i+w_\ell^n}^n,$$

$$\mathfrak{U}''^n_{d,t} := \sum_{i=2^{q_1-1}k_n}^{N_t^n - (j_{d,1} + w_d^n \vee k_n)} \big(\gamma_i^n\big)^{\overline{q}} \left(\prod_{\ell=1}^d \Delta_{\mathbf{j}_\ell}(\chi)_{i+w_\ell^n}^n - \prod_{\ell=1}^d \boldsymbol{r}(\mathbf{j}_\ell; k_n)\right).$$

LEMMA S14: *Assume $\delta_n^\rho \overline{w}_d^n \to 0$ and $v > 1$. Then we have*

$$\mathbb{E}\big(\big|\mathfrak{U}_{d,t}^n - \mathfrak{U}'^n_{d,t}\big| \mathbf{1}_{\{\Omega_t^n\}}\big) \le K_r \big(\overline{w}_d^n\big)^{1/r} \delta_n^{\rho/r-1}, \tag{SB.43}$$

$$\mathbb{E}\big(\big(\mathfrak{U}''^n_{d,t}\big)^2 \mathbf{1}_{\{\Omega_t^n\}}\big) \le K\big(\overline{w}_d^n \delta_n^{-1} + \big(\delta_n k_n^v\big)^{-2}\big). \tag{SB.44}$$

PROOF: Let $s_0 = 0$, $s_\ell := s_{\ell-1} + q_\ell$, $\ell \ge 1$. Let $\{\iota_l\}_{l=1}^{\overline{q}}$ be an enumeration of $\{j_{\ell,p} : 1 \le \ell \le d, 1 \le p \le q_\ell\}$ such that $\iota_l = j_{\ell, l-s_{\ell-1}}$ if $s_{\ell-1} < l \le s_\ell$. That is, for each $1 \le l \le \overline{q}$, there is a unique pair $(\ell(l), p(l))$ such that $\iota_l = j_{\ell(l),p(l)}$.

Let $\zeta_{i,m,m'}^n := X_{i+m}^n - X_{i+m'}^n + (\gamma_{i+m}^n - \gamma_i^n)\chi_{i+m} - (\gamma_{i+m'}^n - \gamma_i^n)\chi_{i+m'}$, $\zeta'^n_{i,m,m'} := \gamma_i^n(\chi_{i+m} - \chi_{i+m'})$. For any integer $p \ge 1$, we let $k_{p,n} = -k_n$ if $p = 1$ and $k_{p,n} = 2^{p-1}k_n$ if $p > 1$. Now let $m_l := w_{\ell(l)}^n + j_{\ell(l),p(l)}$; $m'_l := w_{\ell(l)}^n + j_{\ell(l),p(l)} - k_{p(l),n}$. Using the notations, we obtain (recall

(SB.3) for $\mathcal{Q}_{\overline{q}}$)

$$\prod_{\ell=1}^{d} \Delta_{\mathbf{j}_\ell}(Y)^n_{i+w^n_\ell} = \prod_{l=1}^{\overline{q}} \big(\zeta^n_{i,m_l,m'_l} + \zeta'^n_{i,m_l,m'_l}\big),$$

$$(\gamma^n_i)^{\overline{q}} \prod_{\ell=1}^{d} \Delta_{\mathbf{j}_\ell}(\chi)^n_{i+w^n_\ell} = \prod_{l=1}^{\overline{q}} \zeta'^n_{i,m_l,m'_l}; \tag{SB.45}$$

$$\prod_{\ell=1}^{d} \Delta_{\mathbf{j}_\ell}(Y)^n_{i+w^n_\ell} - (\gamma^n_i)^{\overline{q}} \prod_{\ell=1}^{d} \Delta_{\mathbf{j}_\ell}(\chi)^n_{i+w^n_\ell} = \sum_{Q\in\mathcal{Q}_{\overline{q}}} \prod_{l\in Q} \zeta'^n_{i,m_l,m'_l} \prod_{l\in Q^c} \zeta^n_{i,m_l,m'_l}.$$

Apply (SA.1) for X and γ, and the fact that χ has bounded moments; we get for any $k \geq 2$ that $\mathbb{E}(|\zeta^n_{i,m_l,m'_l}|^k) \leq K\delta^\rho_n \overline{w}^n_d$; $\mathbb{E}(|\zeta'^n_{i,m,m'}|^k) \leq K$.

For a fixed $Q \in \mathcal{Q}_{\overline{q}}$, let $\mu = |Q^c|$ whence $\mu \geq 1$. For $r \geq 2$, apply Hölder's inequality with exponents $(\underbrace{r\mu, \dots, r\mu}_{\mu}, \frac{r}{r-1})$; we get

$$\mathbb{E}\bigg(\bigg|\prod_{l\in Q} \zeta'^n_{i,m_l,m'_l} \prod_{l\in Q^c} \zeta^n_{i,m_l,m'_l}\bigg|\bigg) \leq \prod_{l\in Q^c} \big(\mathbb{E}\big(|\zeta^n_{i,m_l,m'_l}|^{r\mu}\big)\big)^{\frac{1}{r\mu}} \bigg(\mathbb{E}\bigg(\bigg|\prod_{l\in Q} \zeta'^n_{i,m_l,m'_l}\bigg|^{\frac{r}{r-1}}\bigg)\bigg)^{\frac{r-1}{r}}$$
$$\leq K\big(\delta^\rho_n \overline{w}^n_d\big)^{1/r}. \tag{SB.46}$$

For $1 < r \leq 2$, we note (SB.46) still holds. Now let us consider $Q^c = \{l^*\}$. Let (ℓ^*, p^*) be the associated pair such that $\iota_{l^*} = j_{\ell^*,p^*}$. Let $\mathcal{H}^n_{i,l^*} := \mathcal{F}^n_{i+m'_{l^*}} \otimes \mathcal{G}$ if $p^* = 1$, and $\mathcal{H}^n_{i,l^*} := \mathcal{F}^n_{i+w_{n,\ell^*}} \otimes \mathcal{G}$ if $p^* > 1$. Thus, we have $|\mathbb{E}(\zeta^n_{i,m_{l^*},m'_{l^*}}|\mathcal{H}^n_{i,l^*})| \leq K\delta^\rho_n \overline{w}^n_d(1 + |\chi_{i+m_{l^*}}| + |\chi_{i+m'_{l^*}}|)$, which yields $\mathbb{E}(|\mathbb{E}(\zeta^n_{i,m_{l^*},m'_{l^*}} \prod_{l\neq l^*} \zeta'^n_{i,m_l,m'_l}|\mathcal{H}^n_{i,l^*})|) \leq K\delta^\rho_n \overline{w}^n_d$, since it is bounded by $K\overline{w}^n_d \delta^\rho_n \mathbb{E}((1 + |\chi_{i+m_{l^*}}| + |\chi_{i+m'_{l^*}}|) \prod_{l\neq l^*} |\chi_{i+m_l} - \chi_{i+m'_l}|)$. On the other hand, we have by Hölder's inequality (since $r > 1$) that $\mathbb{E}((\zeta^n_{i,m_{l^*},m'_{l^*}} \prod_{l\neq l^*} \zeta'^n_{i,m_l,m'_l})^2) \leq K(\delta^\rho_n(\overline{w}^n_d))^{1/r}$. Also note that $\zeta^n_{i,m_{l^*},m'_{l^*}} \prod_{l\neq l^*} \zeta'^n_{i,m_l,m'_l}$ is measurable with respect to $\mathcal{F}^n_{i+w_{n,\ell^*}+j_{\ell^*,1}-k_{1,n}} \otimes \mathcal{G}$; we thus have by Lemma A.6 in Jacod, Li, and Zheng (2017) that

$$\mathbb{E}\Bigg(\Bigg|\sum_{i=2^{q_1-1}k_n}^{N^n_t-(j_{d,1}+\overline{w}^n_d)k_n} \zeta^n_{i,m_{l^*},m'_{l^*}} \prod_{l\neq l^*} \zeta'^n_{i,m_l,m'_l}\Bigg| \mathbf{1}_{\{\Omega^n_t\}}\Bigg) \leq K_r\big(\overline{w}^n_d \delta^{\rho-1}_n + \big(\overline{w}^n_d\big)^{\frac{r+1}{2r}} \delta_n^{\frac{\rho-r}{2r}}\big),$$

which is further bounded by $K_r(\overline{w}^n_d)^{1/r}\delta_n^{\rho/r-1}$ since $\delta^\rho_n \overline{w}^n_d \to 0$. This proves (SB.43).

Now we prove (SB.44). Let $\varpi^n_i := \prod_{\ell=1}^d \Delta_{\mathbf{j}_\ell}(\chi)^n_{i+w^n_\ell} - \prod_{\ell=1}^d \boldsymbol{r}(\mathbf{j}_\ell; k_n)$ (When the index set is empty, we let the product be 1, e.g., for $\ell = 1$, $\prod_{\ell'=1}^{\ell-1} \Delta_{\mathbf{j}_{\ell'}}(\chi)^n_{i+w_{n,\ell'}} = 1$.) Then $\varpi^n_i = \sum_{\ell=1}^d \varpi^n_{i,\ell}$, where $\varpi^n_{i,\ell} := u(\mathbf{j}_\ell)^n_{i+w^n_\ell} \prod_{\ell'=1}^{\ell-1} \Delta_{\mathbf{j}_{\ell'}}(\chi)^n_{i+w_{n,\ell'}} \prod_{\ell''=\ell+1}^d \boldsymbol{r}(\mathbf{j}_{\ell''}; k_n)$. By Lemma S1, we have $|\mathbb{E}(\varpi^n_{i,\ell})| \leq Kk_n^{-v}$. Next, using again Lemma S1, we have for any $1 \leq \ell, \ell' \leq d$ that $|\mathbb{E}(\varpi^n_{i,\ell}\varpi^n_{i+l,\ell'})| \leq Kk_n^{-2v} + K((l - h_n) \vee 1)^{-v}$, where $h_n := \overline{w}^n_d + \overline{j} + (2^{q_1-1} + 1)k_n$. It yields (SB.44) since γ is bounded and $v > 1$. *Q.E.D.*

LEMMA S15: *Assume all conditions of Theorem 3 hold. We have*

$$\frac{\widehat{\sigma}_1(\mathbf{j},\mathbf{j}')^n_t}{N^n_t} \xrightarrow{\mathbb{P}} \frac{\boldsymbol{s}(\mathbf{j},\mathbf{j}')}{A_t}\int_0^t \gamma_s^{q''}\,\mathrm{d}A_s. \tag{SB.47}$$

PROOF: For $\ell=0,1,2$, let $S_\ell(\mathbf{j},\mathbf{j}';i_n)_1 := s_\ell(\mathbf{j},\mathbf{j}';0)+\sum_{k=1}^{i_n}(s_\ell(\mathbf{j},\mathbf{j}';k)+s_\ell(\mathbf{j}',\mathbf{j};k))$, and $S_\ell(\mathbf{j},\mathbf{j}';i_n)_2 := \sum_{k>i_n}(s_\ell(\mathbf{j},\mathbf{j}';k)+s_\ell(\mathbf{j}',\mathbf{j};k))$. Let $\boldsymbol{s}_\ell(\mathbf{j},\mathbf{j}') := \sum_{k=1}^{2} S_\ell(\mathbf{j},\mathbf{j}';i_n)_k$. We first prove

$$\delta_n\Bigg(U\big(7,0;\mathbf{j},\mathbf{j}'\big)^n_t+\sum_{k=1}^{i_n}\big(U\big(7,k;\mathbf{j},\mathbf{j}'\big)^n_t+U\big(7,k;\mathbf{j}',\mathbf{j}\big)^n_t\big)+(2i_n+1)U\big(4,\mathbf{j},\mathbf{j}'\big)^n_t\Bigg)$$
$$\xrightarrow{\mathbb{P}} \boldsymbol{s}_0\big(\mathbf{j},\mathbf{j}'\big)\int_0^t \gamma_s^{q''}\,\mathrm{d}A_s. \tag{SB.48}$$

Since $v>1$, γ is bounded, we have $S_\ell(\mathbf{j},\mathbf{j}';i_n)_2\int_0^t\gamma_s^{q''}\,\mathrm{d}A_s \le K i_n^{1-v}\to 0$. It is therefore sufficient to replace $\boldsymbol{s}_0(\mathbf{j},\mathbf{j}')\int_0^t\gamma_s^{q''}\,\mathrm{d}A_s$ by $S_\ell(\mathbf{j},\mathbf{j}';i_n)_1\int_0^t\gamma_s^{q''}\,\mathrm{d}A_s$ on the RHS of (SB.48). Using the decomposition (SC.1) (for $\mathbf{j}\oplus\mathbf{j}'(+k)$), for $k\le i_n$, we have $\mathbb{E}(\delta_n(G^n_t)^2)\le K\delta_n(k_n\vee i_n)$. Lemma S3 gives $\mathbb{E}(|\sqrt{\delta_n}\mathfrak{R}(\mathbf{j}\oplus\mathbf{j}'(+k),2)|)\le K_r(\delta_n^\rho(k_n\vee i_n))^{1/r}$; since $\rho>1/2$, we can find some $r>1$ such that $(\delta_n^\rho(k_n\vee i_n))^{1/r}\le\sqrt{\delta_n(k_n\vee i_n)}$ whence $\mathbb{E}(|\sqrt{\delta_n}\mathfrak{R}(\mathbf{j}\oplus\mathbf{j}'(+k),2)|)\le K\sqrt{\delta_n(k_n\vee i_n)}$; we also have $\mathbb{E}(|\sqrt{\delta_n}\mathfrak{R}(\mathbf{j}\oplus\mathbf{j}'(+k),3)|)\le Kk_n^{-v}$ by Lemma S1. Since $i_n^2\delta_n\to 0$, we have by Lemma A.2 and Lemma A.7 in Jacod, Li, and Zheng (2017) that $(|H^n_t|+|\mathfrak{R}(\mathbf{j}\oplus\mathbf{j}'(+k),1)|)\sqrt{\delta_n}i_n\xrightarrow{\mathbb{P}}0$ since $(k_n\vee i_n)\delta_n^{\rho-1/2}\to 0$. Therefore, we have

$$\sum_{k=1}^{i_n}\bigg(\delta_n U\big(7,k;\mathbf{j},\mathbf{j}'\big)^n_t-\boldsymbol{r}\big(\mathbf{j}\oplus\mathbf{j}'(+k)\big)\int_0^t\gamma_s^{q''}\,\mathrm{d}A_s\bigg)\xrightarrow{\mathbb{P}}0. \tag{SB.49}$$

Let $U'(4;\mathbf{j},\mathbf{j}')^n_t := -\sum_{i=2^{q-1}k_n}^{N^n_t-w(4)_n}(\gamma^n_i)^{q''}\Delta_{\mathbf{j}}(\chi)^n_i\Delta_{\mathbf{j}'}(\chi)^n_{i+w(4)^n_2}$, $C_4(\mathbf{j},\mathbf{j}')^n_t := \delta_n U(4,\mathbf{j},\mathbf{j}')^n_t+\boldsymbol{r}(\mathbf{j})\boldsymbol{r}(\mathbf{j}')\int_0^t\gamma_s^{q''}\,\mathrm{d}A_s$. $C_4(\mathbf{j},\mathbf{j}')^n_t$ can be decomposed into

$$C_4\big(\mathbf{j},\mathbf{j}'\big)^n_t=\sum_{\ell=1}^{5}\mathfrak{D}_4(\ell)^n_t, \tag{SB.50}$$

where

$$\mathfrak{D}_4(1)^n_t := \delta_n\big(U\big(4,\mathbf{j},\mathbf{j}'\big)^n_t-U'\big(4,\mathbf{j},\mathbf{j}'\big)^n_t\big);$$

$$\mathfrak{D}_4(2)^n_t := \big(\boldsymbol{r}(\mathbf{j})\boldsymbol{r}\big(\mathbf{j}'\big)-\boldsymbol{r}(\mathbf{j};k_n)\boldsymbol{r}\big(\mathbf{j}';k_n\big)\big)\int_0^t\gamma_s^{q''}\,\mathrm{d}A_s;$$

$$\mathfrak{D}_4(3)^n_t := \boldsymbol{r}(\mathbf{j};k_n)\boldsymbol{r}\big(\mathbf{j}';k_n\big)\Bigg(\int_0^t\gamma_s^{q''}\,\mathrm{d}A_s-\sum_{i=2^{q-1}k_n}^{N^n_t-w(4)_n}\big(\gamma^n_i\big)^{q''}\alpha^n_i\delta(n,i+1)\Bigg);$$

$$\mathfrak{D}_4(4)^n_t := \boldsymbol{r}(\mathbf{j};k_n)\boldsymbol{r}\big(\mathbf{j}';k_n\big)\sum_{i=2^{q-1}k_n}^{N^n_t-w(4)_n}\big(\gamma^n_i\big)^{q''}\big(\alpha^n_i\delta(n,i+1)-\delta_n\big);$$

$$\mathfrak{D}_4(5)_t^n := \delta_n \sum_{i=2^{q-1}k_n}^{N_t^n - w(4)_n} \left(\gamma_i^n\right)^{q''} \left(\Delta_{\mathbf{j}}(\chi)_i^n \Delta_{\mathbf{j}'}(\chi)_{i+w(4)_2^n}^n - \boldsymbol{r}(\mathbf{j};k_n)\boldsymbol{r}(\mathbf{j}';k_n)\right).$$

Now we will prove the convergence $i_n(\delta_n U(4,\mathbf{j},\mathbf{j}')_t^n + \boldsymbol{r}(\mathbf{j})\boldsymbol{r}(\mathbf{j}')\int_0^t \gamma_s^{q''}\,\mathrm{d}A_s) \xrightarrow{\mathbb{P}} 0$ by almost repeating the analysis to obtain (SB.49): $\mathbb{E}(|\mathfrak{D}_4(1)_t^n|\mathbf{1}_{\{\Omega_t^n\}}) \le K_r(\delta_n^\rho k_n)^{1/r}$ by Lemma S14; since χ has bounded moments of all orders and γ is bounded, Lemma S2 leads to $|\mathfrak{D}_4(2)_t^n| \le K k_n^{-v}$. Next, Lemma A.2 and Lemma A.7 in Jacod, Li, and Zheng (2017) imply $\sqrt{\delta_n}(\mathfrak{D}_4(3)_t^n + \mathfrak{D}_4(4)_t^n)i_n \xrightarrow{\mathbb{P}} 0$; a direct application of the second part of Lemma S14 gives $\mathbb{E}((\mathfrak{D}_4(5)_t^n)^2) \le K(\delta_n k_n + \delta_n k_n^{-2v})$. By the above convergence together with (SB.49), we have (SB.48). We can prove in a similar manner

$$\delta_n\left(U(5,0;\mathbf{j},\mathbf{j}')_t^n + \sum_{k=1}^{i_n}\left(U(5,k;\mathbf{j},\mathbf{j}')_t^n + U(5,k;\mathbf{j}',\mathbf{j})_t^n\right)\right) \xrightarrow{\mathbb{P}} \boldsymbol{s}_1(\mathbf{j},\mathbf{j}')\int_0^t \gamma_s^{q''}\,\mathrm{d}A_s,$$

$$\delta_n\left(U(6,0;\mathbf{j},\mathbf{j}')_t^n + \sum_{k=1}^{i_n}\left(U(6,k;\mathbf{j},\mathbf{j}')_t^n + U(6,k;\mathbf{j}',\mathbf{j})_t^n\right)\right) \xrightarrow{\mathbb{P}} \boldsymbol{s}_2(\mathbf{j},\mathbf{j}')\int_0^t \gamma_s^{q''}\,\mathrm{d}A_s,$$

which together with (5) and (SB.48) imply (SB.47). *Q.E.D.*

LEMMA S16: *Assume all conditions of Theorem 3 hold. We have*

$$\frac{\widehat{\sigma}_2(\mathbf{j},\mathbf{j}')_t^n}{N_t^n} \xrightarrow{\mathbb{P}} \frac{\boldsymbol{r}(\mathbf{j})\boldsymbol{r}(\mathbf{j}')}{A_t}\int_0^t \gamma_s^{q''}\overline{\alpha}_s\,\mathrm{d}A_s; \tag{SB.51}$$

$$\frac{\widehat{\sigma}_3(\mathbf{j},\mathbf{j}')_t^n}{N_t^n} \xrightarrow{\mathbb{P}} \frac{\mathbf{R}(\mathbf{j})\mathbf{R}(\mathbf{j}')}{A_t}\int_0^t \overline{\alpha}_s\,\mathrm{d}A_s - \frac{\mathbf{R}(\mathbf{j})\boldsymbol{r}(\mathbf{j}')}{A_t}\int_0^t \gamma_s^{q'}\overline{\alpha}_s\,\mathrm{d}A_s - \frac{\boldsymbol{r}(\mathbf{j})\mathbf{R}(\mathbf{j}')}{A_t}\int_0^t \gamma_s^{q}\overline{\alpha}_s\,\mathrm{d}A_s. \tag{SB.52}$$

PROOF: We prove (SB.51), and

$$\frac{U(\mathbf{j})_t^n}{N_t^n} \xrightarrow{\mathbb{P}} \frac{\boldsymbol{r}(\mathbf{j})}{A_t}\int_0^t \gamma_s^{q}\overline{\alpha}_s\,\mathrm{d}A_s \tag{SB.53}$$

can be proved analogously. In view of (3.17) in Jacod, Li, and Zheng (2017), (SB.52) follows immediately from (SB.53).

Let $B_t^n := \delta_n U(3,\mathbf{j},\mathbf{j}')_t^n - \sum_{i=q_n}^{N_t^n - w(3)_n} \delta_n \boldsymbol{r}(\mathbf{j};k_n)\boldsymbol{r}(\mathbf{j}';k_n)\overline{\alpha}_i^n(\gamma_i^n)^{q''}$, $\widetilde{\mathcal{H}}_i^n := \mathcal{F}_i^n \otimes \mathcal{G}_{i+2+2k_n}$, and

$$\mathfrak{B}(1)_i^n := \widehat{\delta}_i^n\left(\Delta_{\mathbf{j}}(Y)_{i+w(3)_2^n}^n \Delta_{\mathbf{j}'}(Y)_{i+w(3)_3^n}^n - \left(\gamma_i^n\right)^{q''}\Delta_{\mathbf{j}}(\chi)_{i+w(3)_2^n}^n \Delta_{\mathbf{j}'}(\chi)_{i+w(3)_3^n}^n\right);$$

$$\mathfrak{B}(2)_i^n := \widehat{\delta}_i^n\left(\gamma_i^n\right)^{q''}\left(\Delta_{\mathbf{j}}(\chi)_{i+w(3)_2^n}^n \Delta_{\mathbf{j}'}(\chi)_{i+w(3)_3^n}^n - \boldsymbol{r}(\mathbf{j};k_n)\boldsymbol{r}(\mathbf{j}';k_n)\right);$$

$$\mathfrak{B}(3)_i^n := \boldsymbol{r}(\mathbf{j};k_n)\boldsymbol{r}(\mathbf{j}';k_n)\left(\gamma_i^n\right)^{q''}\left(\widehat{\delta}_i^n - \overline{\alpha}_i^n\right).$$

Then we have an easy estimate by the independence of $\mathcal{F}^{(0)}$ and $\mathcal{G}$ and Lemma S1 that $\mathbb{E}(|\mathbb{E}(\mathfrak{B}(2)_i^n|\widetilde{\mathcal{H}}_i^n)|) \le K k_n^{-v}$. Moreover, we have $\mathbb{E}((\mathfrak{B}(2)_i^n)^2) \le K$ as $\mathbb{E}((\widehat{\delta}_i^n)^2|\mathcal{F}_i^n) \le$

K (see the proof of Lemma A.10 in Jacod, Li, and Zheng (2017)), since $\mathfrak{B}(2)_i^n$ is $\widetilde{\mathcal{H}}^n_{i+q_n+q'_n+3k_n+j_1}$-measurable. Now Lemma A.6 in Jacod, Li, and Zheng (2017) yields $\mathbb{E}(|\sum_{i=0}^{N_t^n-w(3)_n}\mathfrak{B}(2)_i^n|\mathbf{1}_{\{\Omega_t^n\}}) \leq Kk_n^{1/2}\delta_n^{-1/2}$. Using the decomposition (SB.45), and applying Hölder's inequality, we obtain

$$\mathbb{E}\Big(\big(\Delta_{\mathbf{j}}(Y)^n_{i+w(3)_2^n}\Delta_{\mathbf{j}'}(Y)^n_{i+w(3)_3^n} - \big(\gamma_i^n\big)^{q''}\Delta_{\mathbf{j}}(\chi)^n_{i+w(3)_2^n}\Delta_{\mathbf{j}'}(\chi)^n_{i+w(3)_3^n}\big)^2\Big) \leq K_r\big(\delta_n^\rho k_n\big)^{1/r}.$$

Apply the Cauchy–Schwarz inequality; we have $\mathbb{E}(|\sum_{i=q_n}^{N_t^n-w(3)_n}\mathfrak{B}(1)_i^n|\mathbf{1}_{\{\Omega_t^n\}}) \leq K_{r,q}k_n^{\frac{1}{2r}}\delta_n^{\frac{\rho}{2r}-1}$. Next, we have by Lemma A.10 in Jacod, Li, and Zheng (2017) that $\mathbb{E}(|\sum_{i=q_n}^{N_t^n-w(3)_n}\mathfrak{B}(3)_i^n| \times \mathbf{1}_{\{\Omega_t^n\}}) \leq \frac{K}{\delta_n k_n}$. Since $|B_t^n| \leq \delta_n|\sum_{\ell=1}^3\sum_{i=q_n}^{N_t^n-w(3)_n}\mathfrak{B}(\ell)_i^n|$ and $\mathbb{P}(\Omega_t^n) \to 1$, we have $B_t^n \xrightarrow{\mathbb{P}} 0$. Now the proof of (SB.51) is complete. *Q.E.D.*

APPENDIX SC: PROOF OF THE MAIN THEOREMS

Let

$$Z(\mathbf{j})_t^n =: G_t^n - \boldsymbol{r}(\mathbf{j}; k_n)H_t^n + \sum_{\ell=1}^3 \mathfrak{R}(\mathbf{j}, \ell)_t^n, \tag{SC.1}$$

where $\mathfrak{R}(\mathbf{j}, 1)_t^n := -\frac{\boldsymbol{r}(\mathbf{j};k_n)}{\sqrt{\delta_n}}(\int_0^t \gamma_s^q \,\mathrm{d}A_s - \sum_{i=q_n}^{N_t^n-k_n-j_1}(\gamma_i^n)^q\alpha_i^n\delta(n, i+1))$; $\mathfrak{R}(\mathbf{j}, 2)_t^n := \sqrt{\delta_n}(\mathrm{ReMeDI}(Y; \mathbf{j}, \boldsymbol{k}_n)_t^n - \mathrm{ReMeDI}'(\chi; \mathbf{j}, \boldsymbol{k}_n)_t^n)$; $\mathfrak{R}(\mathbf{j}, 3)_t^n := \frac{\boldsymbol{r}(\mathbf{j};k_n)-\boldsymbol{r}(\mathbf{j})}{\sqrt{\delta_n}}\int_0^t \gamma_s^q \,\mathrm{d}A_s$.

PROOF OF THEOREM 1: It suffices to show $\sqrt{\delta_n}Z(\mathbf{j})_t^n \xrightarrow{\mathbb{P}} 0$ in view of (5). Since γ is bounded, a direct application of Lemma S1 yields $\mathbb{E}(\delta_n(G_t^n)^2) \leq K\delta_n\overline{k}_n \to 0$; Lemma A.2 and Lemma A.7 of Jacod, Li, and Zheng (2017) imply $\sqrt{\delta_n}(|\boldsymbol{r}(\mathbf{j}; k_n)H_t^n| + |\mathfrak{R}(\mathbf{j}, 1)_t^n|) \xrightarrow{\mathbb{P}} 0$; Lemma S3 yields $\sqrt{\delta_n}\mathbb{E}(|\mathfrak{R}(\mathbf{j}, 2)_t^n|\mathbf{1}_{\{\Omega_t^n\}}) \leq K(\delta_n^\rho\overline{k}_n)^{1/r} \to 0$ whence $\sqrt{\delta_n}\mathfrak{R}(\mathbf{j}, 2)_t^n \xrightarrow{\mathbb{P}} 0$ since $\mathbb{P}(\Omega_t^n) \to 1$; Lemma S2 gives $\sqrt{\delta_n}|\mathfrak{R}(\mathbf{j}, 3)_t^n| \to 0$. This completes the proof. *Q.E.D.*

PROOF OF THEOREM 2: It is immediate that $\mathfrak{R}(\mathbf{j}, 1)_t^n \xrightarrow{\mathbb{P}} 0$, $\mathbb{E}(|\mathfrak{R}(\mathbf{j}, 2)_t^n||\mathbf{1}_{\{\Omega_t^n\}}) \leq Kk_n^{1/r}\delta_n^{\rho/r-1/2}$, $|\mathfrak{R}(\mathbf{j}, 3)_t^n| \leq Kk_n^{-v}\delta_n^{-1/2}$, which follow from Lemma A.7 in Jacod, Li, and Zheng (2017), Lemma S3, and Lemma S2, respectively. Thus, we have $\mathfrak{R}(\mathbf{j}, 2)_t^n \xrightarrow{\mathbb{P}} 0$ for r close to 1 and $\mathbb{P}(\Omega_t^n) \to 1$, and $\mathfrak{R}(\mathbf{j}, 3)_t^n \xrightarrow{\mathbb{P}} 0$ in view of (17). Now the first part of Theorem 2 is a simple consequence of Theorem S1 and part (b) follows directly the proof of Theorem 3.4 in Jacod, Li, and Zheng (2017). *Q.E.D.*

PROOF OF THEOREM 3: The convergence is an immediate result of Theorem 2, Lemma S15, and Lemma S16. *Q.E.D.*

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