Econometrica Supplementary Material

SUPPLEMENT TO "PAIRWISE STABLE MATCHING IN LARGE ECONOMIES"

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MICHAEL GREINECKER
Department of Economics, University of Graz

CHRISTOPHER KAH
Mercedes-Benz AG

APPENDIX A: MATHEMATICAL APPENDIX

HERE, WE COLLECT SOME MATHEMATICAL BACKGROUND INFORMATION used throughout the paper without much ado. The reader is assumed to be familiar with basic notions of general topology and a bit of measure and integration theory. The material on weak convergence of measures can be found in Parthasarathy (1967) and Billingsley (1999), with the caveat that these books only deal with probability measures. Nonprobability measures are dealt with in Bogachev (2007, Chapter II.8), but that book is considerably less accessible. However, there is a mechanical way to identify a family of uniformly bounded measures with a family of probability measures that allows one to transfer results on probability measures to the more general case. Let $\mathcal{F}$ be a family of measures on a measurable space $(X, \mathcal{X})$ such that for some $b > 0$, $\mu(X) < b$ for all $\mu \in \mathcal{F}$. Define a new measurable space $(X^*, \mathcal{X}^*)$ such that for some $* \notin X$, $X^* = X \cup \{*\}$ and $\mathcal{X}^* = \mathcal{X} \cup \{A \cup \{*\} \mid A \in \mathcal{X}\}$. For each $\mu \in \mathcal{F}$, let μ^* be the probability measure on $(X^*, \mathcal{X}^*)$ such that $\mu^*(A) = \mu(A)/b$ for $A \in \mathcal{X}$ and $\mu^*(\{*\}) = 1 - \mu(X)/b$. The function $\mu \mapsto \mu^*$ identifies measures in $\mathcal{F}$ with probability measures. If X has a Polish topology, to be defined below, there is a unique Polish topology on X^* such that X is a subspace and $*$ an isolated point. A continuous real-valued function on X can then be identified with a continuous real-valued function on X^* that vanishes on $*$. With these tools at hand, the reader should be able to obtain the general results from the special case of probability measures.

A topological space is *metrizable* if there exists a metric that induces the topology; such a metric is then *compatible*. A topological space is *completely metrizable* if there exists a complete metric that induces the topology. A subset of a topological space is *dense* if it intersects every nonempty open set or, equivalently, its closure is the whole space. A topological space is *separable* if there is some countable dense subset. A metrizable topological space is separable if and only if it has a countable basis, that is, if there is a countable family of open sets such that every open set is a union of open sets from this family. A topological space is *Polish* if it is separable and completely metrizable. The distinction between Polish spaces and separable complete metric spaces is not just nitpicking. A metric subspace S of a separable complete metric space is a separable complete metric space if and only if S is closed. But a topological subspace S of a Polish space is Polish if and only if S is the countable intersection of open sets (which includes closed sets). The countable topological product of Polish spaces is again Polish. We usually view products of topological spaces as being endowed with the product topology without further comment. A topological space is *locally compact* if every point is in the interior of a compact set. Euclidean spaces are locally compact. Examples of Polish spaces that fail to be locally compact are infinite-dimensional separable Banach spaces.

Michael Greinecker: michael.greinecker@uni-graz.at
Christopher Kah: christopher.s.kah@gmail.com

We endow each Polish space X with the *Borel σ-algebra*, the smallest σ-algebra that includes all open sets. Measurable sets in this σ-algebra are *Borel sets*. We only consider measures with real values (∞ is not allowed as the value of a measure). A measure defined on the Borel σ-algebra is a *Borel measure*. It is a *Borel probability measure* if X has measure 1. A Borel measure μ on a Polish space is always *regular*, that is, for each Borel set $B \subseteq X$,

$$\mu(B) = \sup\{\mu(K) \mid K \text{ is compact and } K \subseteq B\} = \inf\{\mu(O) \mid O \text{ is open and } O \supseteq B\}.$$

If X is a Polish space, we let $\mathcal{M}(X)$ be the corresponding set of Borel measures and $\mathcal{P}(X)$ be the corresponding space of Borel probability measures. We endow $\mathcal{M}(X)$ with the *topology of weak convergence*. This is the weakest topology such that for every bounded continuous function $g : X \to \mathbb{R}$, the function $\mu \mapsto \int g \,\mathrm{d}\mu$ is continuous. Endowed with the topology of weak convergence, $\mathcal{M}(X)$ is again a Polish space and $\mathcal{P}(X)$ a closed subspace. Convergence of sequences of measures will always be understood to be with respect to this topology. Write ∂B for the *boundary* of B, that is, the set of closure points of B that are not interior points. If μ is a Borel measure, the set B is a *μ-continuity set* if $\mu(\partial B) = 0$. Note that X itself has an empty boundary and is therefore always a μ-continuity set. The so-called *Portmanteau theorem* states that the following are equivalent for a sequence $\langle \mu_n \rangle$ in $\mathcal{M}(X)$ and a measure $\mu \in \mathcal{M}(X)$:

(i) the sequence $\langle \mu_n \rangle$ converges to μ,
(ii) $\limsup_n \mu_n(F) \leq \mu(F)$ for every closed set $F \subseteq X$ and $\lim_n \mu_n(X) = \mu(X)$,
(iii) $\liminf_n \mu(O) \geq \mu(O)$ for every open set $O \subseteq X$ and $\lim_n \mu_n(X) = \mu(X)$,
(iv) $\lim_n \mu_n(B) = \mu(B)$ for every μ-continuity set $B \subseteq X$.

We say that a family $\mathcal{F} \subseteq \mathcal{M}(X)$ of Borel measures is *tight* if for each $\epsilon > 0$ there is a compact set $K_\epsilon \subseteq X$ such that $\mu(X \setminus K_\epsilon) < \epsilon$ for all $\mu \in \mathcal{F}$. Similarly, we say that a sequence $\langle \mu_n \rangle$ of elements of $\mathcal{M}(X)$ is *tight* if the family $\{\mu_n \mid n \in \mathbb{N}\}$ is. *Prohorov's theorem* states that $\mathcal{F} \subseteq \mathcal{M}(X)$ is relatively compact (has compact closure) if and only if $\mathcal{F}$ is tight and $\sup_{\mu \in \mathcal{F}} \mu(X) < \infty$. The *support* $\operatorname{supp} \mu$ of a Borel measure $\mu \in \mathcal{M}(X)$ is the largest closed set whose complement has μ-measure zero. In particular, μ has *full support* if $\operatorname{supp} \mu = X$; this is equivalent to every open set of μ-measure zero being empty. The family of all Borel measures with finite support is dense in $\mathcal{M}(X)$.

If $(X_i, \mathcal{X}_i)_{i \in I}$ is a family of measurable spaces, the *product σ-algebra* $\otimes_i \mathcal{X}_i$ on $\prod_i X_i$ is the smallest σ-algebra that makes the coordinate projections measurable. Alternatively, it is the smallest σ-algebra on $\prod_i X_i$ that includes every *measurable rectangle*, where a measurable rectangle is a set of the form $\prod_i A_i$ with $A_i \in \mathcal{X}_i$ for all i and $A_i = X_i$ for all but finitely many i. For the countable topological product of Polish spaces, the Borel σ-algebra of the topological product coincides with the product σ-algebra of the individual Borel σ-algebras. If we look at only two measurable spaces $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$, we write $\mathcal{X} \otimes \mathcal{Y}$ for the product σ-algebra. If $(X, \mathcal{X}, \nu)$ and $(Y, \mathcal{Y}, \mu)$ are measure spaces, there is a unique measure $\nu \otimes \mu$ defined on the product σ-algebra, the *product measure*, such that $\nu \otimes \mu(A \times B) = \nu(A)\mu(B)$ for each measurable rectangle $A \times B$. We heavily rely on the fact that for two Polish spaces X and Y and sequences $\langle \mu_n \rangle$ in $\mathcal{M}(X)$ and $\langle \nu_n \rangle$ in $\mathcal{M}(Y)$, $\langle \mu_n \rangle$ converges to $\mu \in \mathcal{M}(X)$ and $\langle \nu_n \rangle$ converges to $\nu \in \mathcal{M}(Y)$ if and only if the sequence $\langle \mu_n \otimes \nu_n \rangle$ converges to $\mu \otimes \nu \in \mathcal{M}(X \times Y)$ (see Billingsley (1999, Theorem 2.8)).

For probability measures, product measures can be defined even with infinitely many factors. If $(X_i, \mathcal{X}_i, \mu_i)_{i \in I}$ is a family of probability spaces, there is a unique probability measure $\otimes_i \mu_i$, the *independent product* or, again, *product measure*, defined on $\otimes_i \mathcal{X}_i$ such that $\mu(\prod_i A_i) = \prod_{i: A_i \neq X_i} \mu_i(A_i)$ for every measurable rectangle $\prod_i A_i$. If X is a Polish space and $\omega = \langle \omega_n \rangle$ a sequence in X and n a natural number, we let $\mu_n^\omega \in \mathcal{P}(X)$ be the *n-th sample distribution* given by

$$\mu_n^\omega(B) = n^{-1} \# \{m \leq n \mid \omega_m \in B\}$$

for each Borel set $B \subseteq X$ ($\# A$ is the cardinality of A). The Varadarajan (1958) version of the Glivenko–Cantelli theorem says that for each $\mu \in \mathcal{P}(X)$, the random sequence $\langle \mu_n^\omega \rangle$ converges to μ for $\otimes_n \mu$-almost all $\omega \in \prod_n X$.

APPENDIX B: NONEXISTENCE OF AN INDIVIDUALISTIC MATCHING UNDER TRANSFERABLE UTILITY

EXAMPLE 3: Let again the set of agents on one side of the market be $A_W = [0, 1]$ and the set of agents on the other side of the market be $A_M = [0, 1/2] \times \{1, 2\}$ as in Example 1. But now we consider a transferable utility model in which the surplus generated by $a \in A_W$ and $(x, m) \in A_M$ when matched is ax. Matched agents are free to divide the surplus between them any way they see fit. The utility of an unmatched agent is simply 0, so there is no reason for anyone to stay unmatched. A matching can now be represented by a bijection $f : A_W \to A_M$ and functions $V_W : A_W \to \mathbb{R}_+$ and $V_M : A_M \to \mathbb{R}_+$ such that $V_M(x, m) + V_W(f^{-1}(x, m)) = xf^{-1}(x, m)$ for all $(x, m) \in A_M$. The matching f is stable if there is no blocking pair, that is, there are no $a \in A_W$ and $(x, m) \in A_M$ such that $V_W(a) + V_M(x, m) < ax$.

Let f, V_W, and V_M be any matching; we show it is not stable. Pick any $x \in [0, 1/2]$. Without loss of generality, assume that $f^{-1}(x, 1) < f^{-1}(x, 2)$. Let $a \in A_W$ satisfy $f^{-1}(x, 1) < a < f^{-1}(x, 2)$ and let $(y, m) = f(a)$. We must either have $y > x$ or $y < x$. We look at the case $y > x$. If f were stable, we would have both $V_W(a) + V_M(x, 2) \geq ax$ and $V_W(f^{-1}(x, 2)) + V_M(y, m) \geq f^{-1}(x, 2)y$. But then, using the supermodularity of the surplus function,

$$\begin{aligned} &V_W(a) + V_M(x, 2) + V_W\big(f^{-1}(x, 2)\big) + V_M(y, m) \\ &\quad \geq ax + f^{-1}(x, 2)y > ay + f^{-1}(x, 2)x \\ &\quad = V_W(a) + V_M(y, m) + V_W\big(f^{-1}(x, 2)\big) + V_M(x, 2), \end{aligned}$$

which is impossible. A similar argument shows $y < x$ is not compatible with f being stable. Hence, f is not stable.

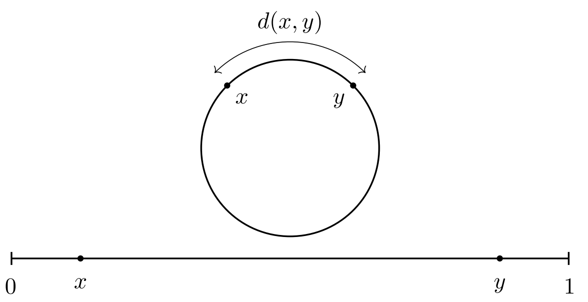
Other examples for the nonexistence of stable matchings can be obtained from the optimal transport literature using Kantorovich duality. One constructs examples in which every solution of the primal optimal transport problem is nondeterministic. By duality, no dual problem, and hence no stable matching is deterministic. Examples in mathematics are Santambrogio (2015, p. 21) or Villani (2009, Example 4.9); examples in economics are Gretsky, Ostroy, and Zame (1992, Example 5) and Chiappori, McCann, and Nesheim (2010, Example 8). Our Example 3 is somewhat simpler; it relies only on the order structure of the unit interval and makes no use of measure theory. Also, the same argument works verbatim for $A_W = [0, 1] \cap \mathbb{Q}$ and $A_M = [0, 1/2] \times \{1, 2\} \cap \mathbb{Q} \times \{1, 2\}$. The nonexistence of stable matchings does not depend on there being a continuum of agents.

APPENDIX C: FAILURE OF LOWER HEMICONTINUITY OF THE SOLUTION CORRESPONDENCE

EXAMPLE 4: We are in a setting without transfers, so we suppress the space C. The types of women and men are simply points on a circle. Formally, we let $W = M = [0, 1)$, endowed with the metric d given by

$$d(x, y) = \min\{|x - y|, 1 - |x - y|\}.$$

Basically, we wrap the half-open unit interval $[0, 1)$ around a circle of circumference 1, or equivalently, take the closed interval $[0, 1]$ and glue the endpoints together.



Now we let $u_W : W \times M_\emptyset \to \mathbb{R}$ be given by $u_W(w, m) = d(w, m)$ and $u_W(w, \emptyset) = -1$ for all $w \in W$ and $m \in M$, and we let $u_M : W_\emptyset \times M \to \mathbb{R}$ be given by $u_M(w, m) = -d(w, m)$ and $u_M(\emptyset, m) = -1$ for all $w \in W$ and $m \in M$. So every woman wants a man whose type is as far away as possible from her type on the circle, every man wants a woman whose type is as close as possible to his type on the circle, and everyone is desperate to avoid loneliness.

For any real number r, we let $[r]$ be the largest integer not larger than r. We also let $(r) = r - [r]$. Note that $(r) \in [0, 1)$ for ever real number r. Fix some irrational number θ. We define population measures ν_W^n and ν_M^n for each natural number n by

$$\nu_W^n = \nu_M^n = \frac{1}{n} \sum_{l=1}^{n} \delta_{(\theta l)},$$

where δ_x denotes the probability measure with support $\{x\}$. This means, we are allocating n points on a circle of circumference 1, with clockwise distance (θ) between consecutive points. The irrationality of θ ensures that all these points will be different. Let μ_n be the matching for ν_W^n and ν_M^n that pairs a man of type (θl) with a woman of type (θl). Since every man gets his top choice, μ_n is stable. We let $\nu_W = \nu_M$ be the uniform distribution on $[0, 1)$. It can be shown that $\langle \nu_W^n \rangle$ converges to ν_W and $\langle \nu_M^n \rangle$ converges to ν_M; see Kuipers and Niederreiter (1974, Theorem 1.1 and Example 2.1). Let μ be the uniform distribution on the diagonal $D = \{(x, y) \mid x = y, x, y \in [0, 1)\}$. Then $\langle \mu_n \rangle$ converges to μ, so

$$0 = \lim_{n \to \infty} \int_{W \times M} d \, \mathrm{d}\mu_n = \int_{W \times M} d \, \mathrm{d}\mu.$$

We now define a new sequence $\langle \nu_M^{n\prime} \rangle$ of population measures for men, adding a single man of type 0 to each ν_M^n. That is,

$$\nu_M^{n\prime} = \frac{1}{n}\delta_0 + \frac{1}{n}\sum_{l=1}^{n}\delta_{(\theta l)}.$$

Clearly, $\langle \nu_M^{n\prime} \rangle$ still converges to ν_M; the presence of one more man is not observable in the limit. But there will be no sequence $\langle \mu_n' \rangle$ such that μ_n' is a stable matching for the population measures μ_W^n and $\mu_M^{n\prime}$ for each natural number n, and such that $\langle \mu_n' \rangle$ converges to μ.

To see this, take any sequence $\langle \mu_n' \rangle$ such that μ_n' is a stable matching for population distributions μ_W^n and $\mu_M^{n\prime}$ for each natural number n. Since ν_W is uniformly distributed on $[0,1)$, we get $\int d(\cdot, m)\,\mathrm{d}\nu_W = 1/4$, the average distance to any point on the circle under the given metric, for each $m \in [0,1)$. Now in each matching μ_n' exactly one man will stay single, and for large n, the average d-distance between the types of the women in the population and the type of the unmatched man will be close to $1/4$. Since μ_n' is stable, no woman prefers the unmatched man to her current partner, and almost every woman must be matched with a partner whose type is at least as good. So the average distance between the type of a woman and her partner is close to at least $1/4$. Therefore,

$$1/4 \le \liminf_n \int_{W\times M} d\,\mathrm{d}\mu_n' \neq \int_{W\times M} d\,\mathrm{d}\mu = 0.$$

APPENDIX D: THE NEED FOR COMPACTNESS IN THEOREM 3

EXAMPLE 5: First, let $W = [0,1)$, ν_W be the uniform distribution, $M = \mathbb{R}_+$, and let ν_M be any continuous full support distribution with infinite expectation, such as a truncated Cauchy distribution. We consider a transferable utility setting with a surplus function given by $S(w,m) = w \cdot m$. The value of staying single is zero for everyone; there is no need for anyone to stay alone. A stable matching continues to be stable when restricted to some subpopulation by Lemma 4 and by restricting the matching to types in given compact sets we can apply Galichon (2016, Theorem 4.7), a result for compact type spaces,[1] together with Theorem 4 to conclude the matching is positive assortative and supported on the graph of the function $T : W \to M$, where T is the quantile function of ν_M. Similarly, it follows using Galichon (2016, Theorem 4.8) and outside options being zero that almost every woman w obtains a payoff of $\int_0^w T\,\mathrm{d}t$. This gives us a nice continuous increasing value function $V_W : W \to \mathbb{R}$. Since ν_W is uniform and T the quantile function of ν_M, we have $\nu_M = \nu_W \circ T^{-1}$. Since the first moment does not exist for ν_M, the continuous increasing function V_W cannot be bounded above.

We now change the model in a seemingly innocuous way: We let $W = [0,1]$, and ν_W be the uniform distribution. We let everything else be as before and extend S in the natural way by letting $S(1,m) = m$. The only thing we changed is that we added a type of woman that has zero measure in the population distribution. We still get exactly the same stable matching μ. If there were a continuous function $V_W : [0,1] \to \mathbb{R}$ such that $V_W(w) = u_W(w,m,c)$ for μ-almost all (w,m,c), it would have to agree with the function constructed before on $[0,1)$. But there is no way to extend an increasing, continuous, unbounded real-valued function on $[0,1)$ to a continuous real-valued function on all of

[1]What matters is not so much the compactness of the type spaces but the integrability of the surplus function.

[0, 1]. Intuitively, we would have to choose $V_W(1) = \infty$, and that is not an available option. The function V_W is necessarily continuous almost everywhere, but there is no continuous function that V_W equals almost everywhere.

In Example 5, we cannot find a continuous version of V_W because M is not compact. W being compact is not essential to obtaining a continuous version of V_W. We show in Proposition E1 below that we can find a continuous version of V_W when W is locally compact (this covers Euclidean spaces) and M compact.

APPENDIX E: A SUFFICIENT CONDITION FOR ONE-SIDED EQUAL TREATMENT

In order to obtain strong equal treatment for women in a stable matching, we do not need the assumption that W is compact in its full strength; it suffices that W is locally compact. Recall that a topological space is locally compact if every point has a compact neighborhood. The following proposition generalizes Lemma 9.

PROPOSITION E1: *Let W be locally compact and M be compact and let μ be a stable matching. Then there exists a unique continuous function $V_W : \operatorname{supp} \nu_W \to \mathbb{R}$ such that*

$$V_W(w) = u_W(w, m, c)$$

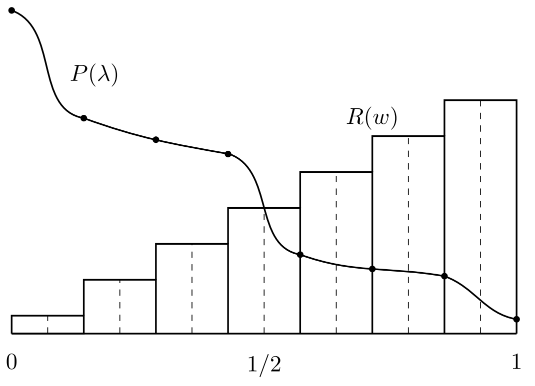
for μ-almost all $(w, m, c) \in W \times M_\emptyset \times [0, 1]$.

The assumption that W is locally compact is fairly weak and will be satisfied if W is a closed or open subspace of a Euclidean space. Still, there are known examples where natural spaces of characteristics are not locally compact. In the model of Chiappori and Reny (2016), each agent is endowed with a random variable representing a stochastic income, and natural spaces of random variables are not locally compact.

To see the kind of restriction local compactness imposes, consider a quasilinear environment so that u_W can be interpreted in terms of some numeraire commodity. Let M be compact. The function $(w, c) \mapsto u_W(w, \cdot, c)$ from W to the space $C(M_\emptyset)$ of continuous functions on the compact space $M_\emptyset$ endowed with the uniform topology is continuous. It follows that the range of this function is a countable union of compact sets since the local compactness of W implies that W is the countable union of compact sets. However, $C(M_\emptyset)$ is not the countable union of compact sets unless M is finite. It follows that $\{\succ_w \mid w \in W\}$ can only include a proper subset of all conceivable preferences.

APPENDIX F: NONEXISTENCE OF STABLE MATCHINGS WITH FINITELY MANY AGENTS

EXAMPLE 6: Let $W = [0, 1]$. We now interpret the members of this side of the market as high school graduates who have to decide whether to go to college or join the labor force directly. There is little use in making the other side explicit. The net value for a high school graduate of going to college is αw for some $\alpha > 0$, so $R(w) = \alpha w$ is their reservation wage. There is a uniform wage paid on the labor market that depends on the fraction λ of high school graduates joining the labor force; the decreasing inverse labor demand function is given by P. We consider a finite population of n students placed on the points $1/(2n) + (2k)/(2n)$ for $k = 0, \dots, n-1$. Assume that $R(1/2) = \alpha/2 = P(1/2)$, so that the market would clear exactly when the mass of students on the labor market is exactly 1/2. But for n odd, this is impossible. There will be a high school graduate placed exactly at 1/2, but since they have mass $1/n$ in the population, the mass on the labor market would be strictly below or above the market clearing mass, depending on whether they join the labor market or go to college. The figure below illustrates. The black dots on the inverse labor demand function correspond to wages that are feasible depending on who joins the labor market. Clearly, none of them exactly clears the market.



There is no stable matching if agents are indivisible.

But the population distribution converges to the uniform distribution on $[0, 1]$ as n increases to infinity,[2] and market clearing is possible in the limit.

APPENDIX G: THE SET OF STABLE MATCHINGS FOR THE MARRIAGE MODEL

In this section, we study the solution correspondence for the classical marriage problem. Under the assumption that W and M are compact, that preferences are negatively transitive, and that type spaces and preferences are compatible with small indifference curves in a measure-theoretic sense, we obtain for a topologically large set of matching problems the existence of extremal matchings in Theorem G1 and a version of the lone wolf theorem in Theorem G2.[3]

Let $\phi : \mathcal{M}(W) \times \mathcal{M}(M) \to 2^{\mathcal{M}(W_\emptyset \times M_\emptyset)}$ be the correspondence in the marriage model that maps a matching problem to the corresponding set of stable matchings and assume that W and M are both compact. The correspondence ϕ is upper hemicontinuous with nonempty and compact values. That the values are nonempty is simply Theorem 1. Moreover, the proof of Theorem 1 shows that ϕ is compact-valued and has a closed graph. It then follows from the closed-graph theorem that ϕ is upper hemicontinuous, Aliprantis and Border (2006, Theorem 17.11).[4] We know from Example 4 that ϕ need not be lower hemicontinuous at every point. However, the failure of lower hemicontinuity in Example 4 is a knife edge case and not robust. By a result in Fort (1949), an upper hemicontinuous correspondence with nonempty compact values in a metrizable space is continuous

[2]This follows from Kuipers and Niederreiter (1974, Theorem 1.1).

[3]Recall that the lone wolf theorem says that for the classical marriage problem with finitely many agents, the set of unmatched agents is the same in every stable matching.

[4]Aliprantis and Border (2006, Theorem 17.11) require the range of the correspondence to be compact, which will in general not hold for ϕ. However, upper hemicontinuity is a local property, so it suffices to show that we can find for each matching problem (ν_W, ν_M) a neighborhood $U \subseteq \mathcal{M}(W) \times \mathcal{M}(M)$ and a compact set $K \subseteq \mathcal{M}(W_\emptyset \times M_\emptyset)$ such that $\phi(\nu'_W, \nu'_M) \subseteq K$ for all $(\nu'_W, \nu'_M) \in U$. We can take U to be the set of matching problems (ν'_W, ν'_M) such that $\nu'_W(W) < \alpha$ and $\nu'_M(M) < \alpha$ for some α sufficiently large. As the corresponding K, we can take the set of all $\mu \in \mathcal{M}(W_\emptyset \times M_\emptyset)$ such that $\mu(W_\emptyset \times M_\emptyset) \leq 3\alpha$.

on a *residual set*, that is, a countable intersection of open dense sets. By the Baire category theorem, each residual set must be dense in a completely metrizable space. Residual sets are sometimes used as a topological notion of genericity, but this interpretation is not without problems. A residual subset of a Euclidean space can have Lebesgue measure zero; Hunt, Sauer, and Yorke (1992) provide a number of non-artificial examples. However, we can at least say that a property cannot be robust and hold on a nonempty open set if it fails on a residual, and hence dense set.

Before we can discuss the existence of extremal stable matchings, we have to define them and this requires us to define an appropriate ordering. We assume that all preferences are negatively transitive, so preferences can be represented by continuous functions $u_W : W \times M_\emptyset \to \mathbb{R}$ and $u_M : W_\emptyset \times M \to \mathbb{R}$ by Mas-Colell (1977). Let μ and μ' be matchings. We write $\mu \succeq_W \mu'$ if

$$\mu\big(B \times M_\emptyset \cap u_W^{-1}\big([r, \infty)\big)\big) \geq \mu'\big(B \times M_\emptyset \cap u_W^{-1}\big([r, \infty)\big)\big)$$

for every Borel set $B \subseteq W$ and $r \in \mathbb{R}$. We define $\succeq_M$ analogously. It is not hard to show that $\succeq_W$ and $\succeq_M$ do not depend on the specific utility representations u_W and u_M, respectively, they only depend on the underlying preferences.

LEMMA G1: *The graph of $\succeq_W$ is closed.*

The next lemma does most of the mathematical heavy lifting for proving Theorem G1 and Theorem G2. It shows that the set of matching problems that have full support and under which all indifference curves have measure zero is residual, provided at least one such matching problem exists. A sufficient condition for the existence of such a matching problem is that both W and M are convex compact subsets of Euclidean spaces with nonempty interior and that preferences are strictly increasing in every coordinate. In that case, one can take ν_W and ν_M both to be Lebesgue measures restricted to W and M, respectively.

LEMMA G2: *If W and M are compact and if there exists a full support measure ν_M^* on M such that*

$$\nu_M^*\big(\big\{m' \in M \mid u_W\big(w, m'\big) = u_W(w, m)\big\}\big) = 0$$

for all $w \in W$ and $m \in M$, then the set of $\nu_M \in \mathcal{M}(M)$ such that

$$\nu_M\big(\big\{m' \in M \mid u_W\big(w, m'\big) = u_W(w, m)\big\}\big) = 0$$

for all $w \in W$ and $m \in M$ is residual.

We are now ready to show that extremal matchings exist for a residual set of matching problems.

THEOREM G1: *Suppose that W and M are compact and there exists a matching problem (ν_W^*, ν_M^*) such that*

$$\nu_M^*\big(\big\{m' \in M \mid u_W\big(w, m'\big) = u_W(w, m)\big\}\big) = 0$$

and

$$\nu_W^*\big(\big\{w' \in W \mid u_M\big(w', m\big) = u_W(w, m)\big\}\big) = 0$$

for all $w \in W$ and $m \in M$. Then for a residual set of matching problems, there exist stable matchings μ^W and μ^M such that for all stable matchings μ:

(i) $\mu^W \succeq_W \mu \succeq_W \mu^M$,
(ii) $\mu^M \succeq_M \mu \succeq_M \mu^W$.

PROOF: By Fort (1949), the upper hemicontinuous and compact-valued correspondence ϕ is continuous on a residual sets of matching problems and by Lemma G2, we have

$$\nu_M\bigl(\bigl\{m' \in M \mid u_W\bigl(w, m'\bigr) = u_W(w, m)\bigr\}\bigr) = 0$$

and

$$\nu_W\bigl(\bigl\{w' \in W \mid u_M\bigl(w', m\bigr) = u_W(w, m)\bigr\}\bigr) = 0$$

for all $w \in W$ and $m \in M$ for a residual set of matchings problem (ν_W, ν_M). The intersection of the two residual sets is again residual and will serve as the desired residual set.

So take any (ν_W, ν_M) in the intersection. Since all indifference curves have ν_M-measure zero or ν_W-measure zero, respectively, so does every singleton. This implies that ν_W and ν_M are atomless. Using the Varadarajan version of the Glivenko–Cantelli theorem, we can find, via sampling and rescaling, a sequence of finite matching problems $\langle \nu_W^n, \nu_M^n\rangle$ converging to (ν_W^n, ν_M^n) such that no type in the support of ν_W^n is indifferent between any two distinct types in the support of ν_M^n and *vice versa* for any n. By Baïou and Balinski (2002, Theorem 5 and Theorem 6), we can find for each n stable matchings μ_n^W and μ_n^M for the matching problem (ν_W^n, ν_M^n) such that for any stable matching μ of the same matching problem, both $\mu_n^W \succeq_W \mu \succeq_W \mu_n^M$ and $\mu^M \succeq_M \mu \succeq_M \mu_n^W$ hold. As in the proof of Theorem 1, we can by passing to a subsequence assume that the sequences $\langle \mu_n^W\rangle$ and $\langle \mu_n^M\rangle$ converge to stable matchings μ^W and μ^M, respectively, for the matching problem (ν_W, ν_M).

We show that μ^W and μ^M have the desired properties. Let μ be any stable matching for the matching problem (ν_W, ν_M). Since ϕ is lower hemicontinuous at (ν_W, ν_M), there exists a sequence $\langle \mu_n\rangle$ of measures on $W_\emptyset \times M_\emptyset$ such that $\mu_n \in \phi(\nu_W^n, \nu_M^n)$ for all but finitely many n. This implies that for all but finitely many n, both $\mu_n^W \succeq_W \mu_n \succeq_W \mu_n^M$ and $\mu^M \succeq_M \mu_n \succeq_M \mu_n^W$ hold. The result now follows from Lemma G1. *Q.E.D.*

Next, we show that the conclusion of the lone wolf theorem holds for a residual set of matching problems.

THEOREM G2: *Suppose that W and M are compact and there exists a matching problem (ν_W^*, ν_M^*) such that*

$$\nu_M^*\bigl(\bigl\{m' \in M \mid u_W\bigl(w, m'\bigr) = u_W(w, m)\bigr\}\bigr) = 0$$

and

$$\nu_W^*\bigl(\bigl\{w' \in W \mid u_M\bigl(w', m\bigr) = u_W(w, m)\bigr\}\bigr) = 0$$

for all $w \in W$ and $m \in M$. Then for a residual set of matching problems, all stable matchings have the same traces on $W \times \{\emptyset\}$ and $\{\emptyset\} \times M$.

PROOF: We do one case, the other one works the same way. We can proceed exactly as in the proof of Theorem G1 up to, and including, the choice of the approximating sequence of matching problems (ν_W^n, ν_M^n). Now let μ and μ' be stable matchings for the matching problem (ν_W^n, ν_M^n). We have to show that $\mu(A \times \{\emptyset\}) = \mu'(A \times \{\emptyset\})$ for all Borel sets $A \subseteq W$. As in the proof of Lemma 2, it suffices to show this for Borel sets A such that $\mu(\partial A \times \{\emptyset\}) = \mu'(\partial A \times \{\emptyset\}) = 0$, where we use the fact that $\partial(A \times \{\emptyset\}) = \partial A \times \{\emptyset\} \cup A \times \partial\{\emptyset\} = \partial A \times \{\emptyset\}$, the last equality following from $\emptyset$ being an isolated point.

So let $A \subseteq W$ be a Borel set such that $\mu(\partial A \times \{\emptyset\}) = \mu'(\partial A \times \{\emptyset\}) = 0$. Since ϕ is lower hemicontinuous at (ν_W, ν_M), there exists sequences $\langle \mu_n \rangle$ and $\langle \mu'_n \rangle$ of measures on $W_\emptyset \times M_\emptyset$ such that $\mu_n \in \phi(\nu_W^n, \nu_M^n)$ and $\mu'_n \in \phi(\nu_W^n, \nu_M^n)$ for all but finitely many n, $\langle \mu_n \rangle$ converges to μ, and $\langle \mu'_n \rangle$ converges to μ'. By Baïou and Balinski (2002, Lemma 8), a version of the rural hospital theorem, we have $\mu_n(A \times \{\emptyset\}) = \mu'_n(A \times \{\emptyset\})$ for all but finitely many n and, therefore, by the Portmanteau theorem, $\mu(A \times \{\emptyset\}) = \mu'(A \times \{\emptyset\})$. *Q.E.D.*

APPENDIX H: PROOFS

PROOF OF PROPOSITION E1: We know from Lemma 9 that the result holds for W compact. But the only place where compactness of W was used was in invoking Lemma 8. It suffices, therefore, to show that Lemma 8 holds even if we replace the compact space K by a locally compact Polish space L.

Without loss of generality, we can take μ to have support L. Indeed, every closed subspace of a locally compact Hausdorff space is easily shown to be locally compact. Now by Aliprantis and Border (2006, 2.76 and 2.77), there exists an increasing sequence $\langle K_n \rangle$ of compact sets such that $\bigcup_n K_n = L$ and such that K_n is a subset of the interior of K_{n+1} for each natural number n. Let the Borel measure μ_n be defined by $\mu_n(B) = \mu(B \cap K_n)$ for each natural number n and each Borel set $B \subseteq L$. By Lemma 8, there exists a continuous function $g_n : K_n \to \mathbb{R}$ such that μ_n is supported on the graph g_n and any two continuous functions with this property must agree on the support of μ_n. Now for every point $x \in L$, there is some natural number n such that $x \in K_n$ and we let $n(x)$ be the smallest natural number with this property. We define $g : L \to \mathbb{R}$ by $g(x) = g_{n(x)+1}(x)$.

Next, we show that $g(x) = g_l(x)$ for each $l \geq n(x) + 1$. Indeed, x is in the interior of K_l for $l \geq n(x) + 1$. By the full support assumption, the interior of K_l is a subset of the support of μ_l. The support of $\mu_{n(x)+1}$ is a subset of the support of μ_l for each $l \geq n(x) + 1$. Now g_l restricted to the support of $\mu_{n(x)+1}$ is a continuous function such that $\mu_{n(x)+1}$ is supported on its graph. But then this restriction must coincide with $g_{n(x)+1}$ on the support of $\mu_{n(x)+1}$. It follows that $g(x) = g_l(x)$ for each $l \geq n(x) + 1$.

To see that g is continuous, take any $x \in L$. By assumption, $x \in K_{n(x)}$ and $K_{n(x)}$ is a subset of the interior of $K_{n(x)+1}$. So there is an open neighborhood U of x that is wholly included in the interior of $K_{n(x)+1}$. This implies $n(y) \leq n(x) + 1$ and, therefore, $g(y) = g_{n(x)+2}(y)$ for all $y \in U$. So g is continuous at x because $g_{n(x)+2}$ is.

We are almost done; two details are left. First, note that $\mu(B) = \lim_n \mu_n(B)$ for every Borel set $B \subseteq L$, so the measure μ is supported on the graph of g. Second, note that g is the only continuous functions whose graph supports μ since the uniqueness argument in the proof of Lemma 8 does not rely on L being compact. *Q.E.D.*

PROOF OF LEMMA G1: Let $\langle \mu_n \rangle$ be a sequence of matchings converging to μ and $\langle \mu'_n \rangle$ be a sequence of matchings converging to μ' such that $\mu_n \geq_G \mu'_n$ for all n. We have to show that $\mu \geq_G \mu'$.

Observe that the functions

$$B \mapsto \mu\big(B \times M_\emptyset \cap u_W^{-1}([r, \infty))\big) \quad \text{and} \quad B \mapsto \mu'\big(B \times M_\emptyset \cap u_W^{-1}([r, \infty))\big)$$

are finite Borel measures, which we shall denote by λ and λ', respectively. Since finite Borel measures on Polish spaces are regular, it suffices to show that the relevant inequalities are satisfied for compact A. So fix a compact set $B \subseteq W$ and some $r \in \mathbb{R}$.

As in the proof of Lemma 2, we can construct a decreasing sequence $\langle B^m \rangle$ of Borel sets that decreases to B such that $\lambda(\partial B^m) = \lambda'(\partial B^m) = 0$. Also, $\partial u_W^{-1}([r', \infty)) \subseteq u_W^{-1}(\{r'\})$ for each real number r'. Since sets of the form $u_W^{-1}(\{r'\})$ are disjoint, only countably many can have positive μ-measure or μ'-measure. It follows that there is a sequence $\langle r_m \rangle$ of real numbers that increases to r such that

$$\mu\big(\partial u_W^{-1}\big([r_m, \infty)\big)\big) = \mu'\big(\partial u_W^{-1}\big([r_m, \infty)\big)\big) = 0.$$

The intersection of two sets whose boundary has measure zero has a boundary of measure zero, so

$$\mu\big(\partial\big(B^m \times M_\emptyset \cap u_W^{-1}\big([r_m, \infty)\big)\big)\big) = \mu'\big(\partial\big(B^m \times M_\emptyset \cap u_W^{-1}\big([r_m, \infty)\big)\big)\big) = 0.$$

By assumption, we have

$$\mu_n\big(B^m \times M_\emptyset \cap u_W^{-1}\big([r_m, \infty)\big)\big) \geq \mu_n'\big(B^m \times M_\emptyset \cap u_W^{-1}\big([r_m, \infty)\big)\big)$$

and, therefore, by the Portmanteau theorem

$$\mu\big(B^m \times M_\emptyset \cap u_W^{-1}\big([r_m, \infty)\big)\big) \geq \mu'\big(B^m \times M_\emptyset \cap u_W^{-1}\big([r_m, \infty)\big)\big).$$

Now the sequence $\langle B^m \times M_\emptyset \cap u_W^{-1}([r_m, \infty))\rangle$ decreases to $B \times M_\emptyset \cap u_W^{-1}([r, \infty))$. So by the continuity of measures,

$$\mu\big(B \times M_\emptyset \cap u_W^{-1}\big([r, \infty)\big)\big) \geq \mu'\big(B \times M_\emptyset \cap u_W^{-1}\big([r, \infty)\big)\big). \qquad \textit{Q.E.D.}$$

PROOF OF LEMMA G2: Let $\mathcal{C}_M$ be the family of nonempty compact subsets of M endowed with the topology of closed convergence; see Aliprantis and Border (2006, p. 121). We let $\mathcal{I} \subseteq \mathcal{C}_M$ be the family of indifference curves given by

$$\mathcal{I} = \big\{\big\{m' \in M \mid u_W(w, m) = u_W(w, m)\big\} \mid w \in W, m \in M\big\}.$$

We first show that $\mathcal{I}$ is compact. We view the set $\{(w, m, m') : u_W(w, m) = u_W(w', m)\}$ as the graph of a correspondence from $W \times M$ to 2^M. Since the graph is clearly compact, this correspondence is upper hemicontinuous with nonempty compact values. So, by Aliprantis and Border (2006, Theorem 17.15), we can identify the correspondence with a continuous function from the compact set $W \times M$ to $\mathcal{C}_I$ whose range $\mathcal{I}$ is therefore compact.

By Dubins and Freedman (1964, Result 3.8), the function $\eta : \mathcal{M}(M) \times \mathcal{C}_M \to [0, 1]$ given by $\eta(\nu, F) = \nu(F)$ is upper semicontinuous. Write $\mathcal{M}^r(M)$ for the set of $\mu \in \mathcal{M}(M)$ such that $\mu(M) \leq r$ and $\mathcal{M}^{<r}(M)$ for the set of $\mu \in \mathcal{M}(M)$ such that $\mu(M) < r$. Note that $\mathcal{M}^r(M)$ is compact for each natural number r and its interior is $\mathcal{M}^{<r}(M)$. Let

$$O_n = \big\{\nu \in \mathcal{M}(M) \mid \nu(I) < 1/n \text{ for all } I \in \mathcal{I}\big\}.$$

We show that each O_n is open and dense, so that $\bigcap_n O_n \subseteq \mathcal{M}(M)$ provides us with the desired residual set.

To show that each O_n is open, it suffices to show for each $r > 0$ that the set

$$\big\{\nu \in \mathcal{M}^{<r}(M) \mid \nu(I) < 1/n \text{ for all } I \in \mathcal{I}\big\}$$

is open. Since η is upper semicontinuous and $\mathcal{M}^r(M) \times \mathcal{I}$ compact, the closed set

$$\{(\nu, I) \in \mathcal{M}^r(M) \times \mathcal{I} \mid \nu(I) \geq 1/n\}$$

is compact, too, and so is its projection to $\mathcal{M}^r(M)$, which is simply

$$\{\nu \in \mathcal{M}^r(M) \mid \nu(I) \geq 1/n \text{ for some } I \in \mathcal{I}\}.$$

The relative complement of this projection in $\mathcal{M}^r(M)$ is therefore relatively open and its intersection with $\mathcal{M}^{<r}(M)$ is plainly open. But this is exactly the set

$$\{\nu \in \mathcal{M}^{<r}(M) \mid \nu(I) < 1/n \text{ for all } I \in \mathcal{I}\}.$$

Finally, we show that each O_n is dense in $\mathcal{M}(M)$. Note first that ν_M^* is in O_n, and so is every measure in $\mathcal{M}(M)$ absolutely continuous with respect to ν_M^*. We also know that the set of measures with finite support is dense in $\mathcal{M}(M)$. It therefore suffices to show that such measures with finite support can be well approximated by measures that are absolutely continuous with respect to ν_M^*. Let μ be such a measure with finite support. By the definition of the topology of weak convergence, it suffices to show that there exists for every finite family $\mathcal{G}$ of continuous functions on M and every $\epsilon > 0$ some $\nu \in \mathcal{M}(M)$ absolutely continuous with respect to ν_M^* such that

$$\left| \int g \, \mathrm{d}\mu - \int g \, \mathrm{d}\nu \right| < \epsilon$$

for all $g \in \mathcal{G}$. Let S be the finite support of μ. For each $m \in S$, let V_m be an open neighborhood on which each $g \in \mathcal{F}$ varies by less than $\epsilon/\#S$, where $\#S$ is the number of elements of S. We can and do choose the V_m to be disjoint. Since ν_M^* has full support, we have $\nu_M^*(V_m) > 0$ for all $m \in S$. Define a measurable function $h : M \to \mathbb{R}$ by letting $h(x) = \mu(m)/\nu_M^*(V_m)$ for $x \in V_m$ and $h(x) = 0$ if x is in no V_m. Let ν be the measure that has Radon–Nikodym derivative h with respect to ν_M^*. Take any $g \in \mathcal{G}$. Then

$$\begin{aligned}
\left| \int g \, \mathrm{d}\mu - \int g \, \mathrm{d}\nu \right| &= \left| \int g \, \mathrm{d}\mu - \int hg \, \mathrm{d}\nu_M^* \right| \\
&= \left| \sum_{m \in S} \left(g(m)\mu(m) - \int_{V_m} hg \, \mathrm{d}\nu_M^* \right) \right| \\
&\leq \sum_{m \in S} \left| \left(g(m)\mu(m) - \int_{V_m} hg \, \mathrm{d}\nu_M^* \right) \right| \\
&< \sum_{m \in S} \mu(m)\epsilon/\#S = \epsilon.
\end{aligned}$$

Q.E.D.

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