Econometrica Supplementary Material

SUPPLEMENT TO “LUMPY DURABLE CONSUMPTION DEMAND AND THE LIMITED AMMUNITION OF MONETARY POLICY”

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APPENDIX A: COMPUTATIONAL APPENDIX

WE SOLVE THE MODEL BUILDING on the methods described in Achdou, Han, Lasry, Lions, and Moll (2021).

A.1. *Steady State*

Define $k = a + \lambda(1-f)pd$ as the distance from the borrowing limit. Construct tensor grids over the state variables (k, d, z). Then the steady state policy function is constructed as follows:

1. Start with an initial guess of the value function $v(k, d, z)$ and the value conditional on making an adjustment $v^*(k, d, z)$.
2. Solve for the optimal consumption and saving decisions when not adjusting. Compute v_k both as a forward difference v_k^f and as a backward difference v_k^b. At the boundaries of v_k^f and v_k^b, impose that the drift of k is zero. Invert $v_k(k, d, z) = U_c(c, d)$ to solve for $c^f(k, d, z)$ and $c^b(k, d, z)$, and the corresponding drift of k, $s^f(k, d, z)$ and $s^b(k, d, z)$. Finally, let $c^0(k, d, z)$ be the consumption consistent with zero drift. Pick among the candidates based on the following rule:
   (a) If $s^f < 0$ and $s^b < 0$, pick c^b, s^b.
   (b) If $s^f > 0$ and $s^b > 0$, pick c^f, s^f.
   (c) If $s^f < 0$ and $s^b > 0$, pick c^0, s^0.
   (d) If $s^f > 0$ and $s^b < 0$, pick the candidate that yields a larger value for the Hamiltonian.

   Using the solution, compute the felicity function $u(c, d)$.
3. Construct the transition matrix A based on the endogenous drifts of k and the exogenous drifts and shocks to d, z. See Achdou et al. (2021) for details.
4. The HJB equation can now be written as $\min\{\rho v - u - Av, v - v^*\} = 0$, and solved using an LCP solver for v. We use Yuval Tassa’s solver http://www.mathworks.com/matlabcentral/fileexchange/20952.
5. Compute optimal choice of d' conditional on adjusting and the corresponding $v^* = \max_{d'} v(k', d', z)$, where $k' = k + (1-f)(1-\lambda)pd - (1-\lambda(1-f))pd'$.
6. Repeat steps 1–5 until convergence.
7. To obtain the steady-state distribution, convert the policy functions for k' and d' conditional on an adjustment to index form. Fractions of an index determine the weights we assign to each index.

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8. Create a matrix $C^{\text{noadj}} = A - \text{diag}(\theta)$. Then set all the columns in C^{noadj} that correspond to adjustment points to zero. Define $A^{\text{adj}} = A - C^{\text{noadj}}$. This matrix contains the mass at adjustment points that needs to be reallocated to the nodes of the optimal k', d'. We assign this mass to the nodes surrounding k', d' based on the index fractions in the previous step. This yields a matrix of adjustments C^{adj}. The transition matrix is then $C = C^{\text{noadj}} + C^{\text{adj}}$.
9. Solve $0 = C\Phi$ for the steady-state distribution Φ.

A.2. *Jacobians*

The kth column of the Jacobians hold the impulse response function with respect to a shock $k-1$ periods in the future. The dimension of the Jacobian described here will be $T \times T$. The procedure largely follows Auclert, Bardóczy, Rognlie, and Straub (2019) but we need to compute the Jacobian numerically since our model features nondifferentiable policy rules.

1. Start with a shock T periods in the future and solve the policy function backwards, by repeating steps 1–5 given the terminal condition $v_{T+1} = v$. Each iteration reduces t by dt. Continue until $t = 0$ is reached.
2. Take the whole sequence of policy functions from v_0 to v_T. Repeat steps 7–8 for each period of the IRF and record outcomes for each period. This yields the IRF for the last column (T) of the Jacobian. Note that the initial distribution Φ_0 requires a modification if $p_0 \neq 1$. The distribution of k needs to shift since $k = a - \lambda pd$ and a, d are fixed in that instant.
3. For each t, if the adjustment thresholds change, then all the mass in Φ_t that is in the new adjustment region must be immediately shifted to its new location using the procedure in step 8. Call the new distribution $\hat{\Phi}_t$. Then compute $\Phi_{t+dt} = \hat{\Phi}_t + C_t\hat{\Phi}_t\,\mathrm{d}t$. Repeat this step until $t = T$.
4. Repeat the previous two steps using the sequence of policy functions for v_k to v_T followed by $k-1$ periods of the steady state policy function v_{T+1}. This yields the IRF for the $T-k$ column.
5. Conduct this procedure for a shock to G, p, Y, r, r^b.
6. For the productivity shock only the initial distribution gets rescaled, so there is no need to compute a policy function backward.

A.3. *General Equilibrium*

Following Auclert et al. (2019), we compute the partial equilibrium Jacobians for all outcome variables given news at time 0 to one-time deviations to r_s, r_s^b, G_s, Y_s, p_s, with rows corresponding to the quarter in which the outcome is measured and columns corresponding to periods in which the deviation occurs. Since we express the model variables relative to productivity Z_t, the productivity shock causes a rescaling of the initial aggregate distribution, which we capture by a Jacobian with a single column. Using matrix algebra, we can then solve for the impulse response functions in general equilibrium by incorporating the persistence of exogenous variables and the necessary endogenous price and income movements that satisfy (2) and (3).

APPENDIX B: ESTIMATION

The data selection and estimation strategy largely follows Berger and Vavra (2015).

B.1. *Data*

Observables. We use PSID data from 1999 through 2009. Our set of observables from the PSID, $\mathbf{Z}_{it}^{\text{data}}$, are net liquid assets a_{it}, the value of the durable stock d_{it}, and annualized consumption expenditures over the following wave $\bar{c}_{i,t,t+2}$.

Real nondurable consumption is nominal nondurable consumption in the PSID deflated by the BEA price index for nondurables (NIPA Table 1.1.4). Nominal nondurable consumption is the sum of food expenditures, utility expenditures, home insurance, transportation expenditures, property taxes, health expenditures, child care expenditures, and education expenditures. We exclude any loan or lease payments from transportation expenditures to align the definition of nondurables with our model.

Real durable holdings are the sum of real house values and real vehicle values. Real house values are reported nominal house values deflated by the OFHEO national house price index. For renters, we convert rent to a house value using the national house-to-rent ratio from Davis, Lehnert, and Martin (2008) available at http://www.aei.org/housing/land-price-indicators/. The PSID records the net wealth of up to three vehicles per household. We sum these values, add total vehicle debt (detailed below), and deflate the sum with the BEA price index for motor vehicles (NIPA Table 1.2.4).

Real liquid asset holdings are the sum of cash and deposit holdings, stock holdings, and bond holdings, deflated by the nondurables price index.

We construct net real liquid assets by subtracting real debt from housing and vehicles. Mortgage debt is directly reported and we deflate it using the nondurables price index. We construct existing vehicle debt from the initial loan amount on all three cars and subtract the number of payments made times the average payment amount. In less than 1% of cases, this results in a negative debt value, in which case we set vehicle debt to zero.

Housing adjustments come from either moving or a significant addition or repair. The PSID records the month and year of the most recent move since either the last interview (pre-2003) or since January 2 years ago. If a move is recorded and the move falls after the previous interview, then we code it as a housing adjustment for the current wave; otherwise, it is an adjustment in the previous wave. When the move falls in the month of the interview, we break the tie based on whether the interview was in the first- or second-half of the month. For significant additions and repairs, we record them as housing adjustments in the wave that they are reported.

Car adjustments are set to one if any one of the three reported cars has been acquired since the previous wave. This is the case if the most recent car's acquisition date is after the previous wave's interview date, or (if there is insufficient information using the date) a new car has been acquired less than 3 years ago and it was not reported in the previous wave. We weight a housing adjustment by 0.9 and car adjustments by 0.1.

Sample Selection. We only keep head of households since the data is reported at the household level. We drop heads of households 21 and younger, as well as households present for fewer than 3 waves. This selection helps with the estimation of household fixed effects. We drop households with zero durable holdings, or those with missing information on any variable. We winsorize all variables at the 5th and 95th percentile. The sample weight is the household weight in the PSID.

Household Fixed Effects. We demean durable holdings by the households average durable holdings over the sample. This accounts for permanent differences in tastes for durables across households, which are not part of the model. We also divide nondurable consumption, liquid asset holdings, and real debt holdings by a household's average nondurable consumption over the sample. This helps account for permanent differences in income, which are again not part of the model.

Consistency With National Aggregates. We divide all variables by average nondurable consumption in the sample. We then multiply each scaled variable (durables, liquid assets, debt) by a factor so that the sample average aligns with national aggregates from the fixed asset tables (durable-to-nondurable-consumption ratio) and the flow of funds (liquid-asset-to-nondurable-consumption and debt-to-nondurable-consumption). The rescaling is necessary because the PSID collects data for 72% of nondurable expenditures on average (Li, Schoeni, Danziger, and Charles (2010)). Further, households appear to overestimate the value of their vehicles (Czajka, Jacobson, and Cody (2003)).

B.2. *Estimation Algorithm*

1. Pick a given intensity of match quality shocks θ. Calibrate the discount rate ρ, the fixed cost f, and the preference for durables ψ to match the targets for net assets, the probability of adjustment, and the durable-stock-to-nondurable-consumption ratio.
2. Forecast the probability of adjustment $P(a, d, y)$ over the next 2 years. Also forecast the average nondurable consumption expenditure $\bar{c}$ for each initial state (a, d, y) over the next 2 years. From the latter, we obtain a steady-state distribution $G(a, d, \bar{c})$.
3. Regress the optimal durable stock d^* in the model on a, a^2, d, $\bar{c}$, $d/\bar{c}$ weighted using the steady-state distribution. The vector of estimated coefficient is β.
4. Add measurement error to the model variables a, d, $\bar{c}$ using three independent Gaussian quadratures. This yields a new distribution $\hat{G}(a, d, \bar{c})$, which includes measurement error.
5. Compute gaps $\omega = d^* - d$ for each point in the distribution $\hat{G}$. Integrating over ω using $\hat{G}$ yields the pdf $f(\omega)$ in the model. Similarly, integrating the probability of adjustment $P(a, d, y)$ over ω using $\hat{G}$ yields the hazard rate $h(\omega)$ in the model.
6. In the data, combine reported a, d, $\bar{c}$, and our estimates β to predict d^* and the durable gap $\omega = d^* - d$. Use the sample weights to compute $f(\omega)$ and the adjustment hazard $h(\omega)$.
7. Compute loss function $L = \sum_\omega w(\omega)[|f^{\text{model}}(\omega) - f^{\text{data}}(\omega)| + |h^{\text{model}}(\omega) - h^{\text{data}}(\omega)|]$ where the weight is $w(\omega) = \frac{1}{4}(f^{\text{model}}(\omega) + f^{\text{data}}(\omega))(h^{\text{model}}(\omega) + h^{\text{data}}(\omega))$. This weighting function attaches more weight to bins the greater the fraction of adjustments accounted for by that bin. Conversely, we attach little weight to regions in which both model and data predict few adjustments.
8. Repeat steps 4, 5, and 7 using a range of values for the standard deviation of the measurement error. Then pick the value that results in the smallest loss in 7.
9. Repeat steps 1–8 using a range of values for θ. Pick the θ with the smallest loss in 8.
10. To construct standard errors, sample 1000 new data sets with replacement from the original data set. Repeat steps 6 and 7 for each data set, record the loss-minimizing value for θ and the associated density and hazard function from both data and model.

Figure A.1 displays the density of gaps at our estimated parameters.

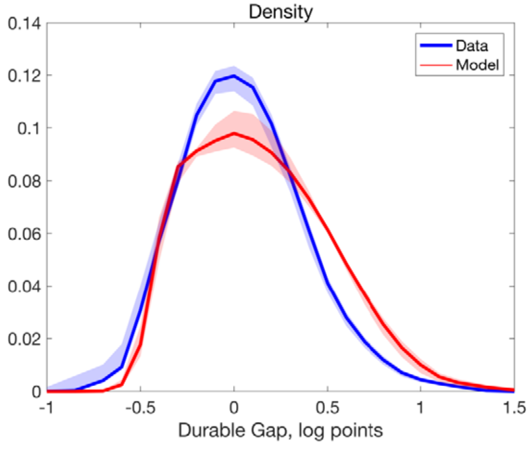


FIGURE A.1.—Density of the durable gap $\omega = d^* - d$, where d^* is the optimal durable choice conditional on adjusting and d the initial durable stock. Shaded areas are 95% confidence bands.

APPENDIX C: DATA APPENDIX

C.1. *Variables for Impulse Response Functions*

In this section, we detail how we construct the variables for the empirical impulse response functions to monetary policy shocks in Figures 2, 3, and 4 of Section 3. We obtained the data from the the St. Louis Fed FRED database. The variable identifiers are listed in Table A.I.

To construct real durable and nondurable expenditure, we proceed as follows. The problem is one where we have two components of nominal expenditure $Y_t = X_{1t} + X_{2t}$

TABLE A.I

VARIABLE NAMES AND FRED SERIES CODE.

Variable Name	FRED Series Code
Population	B230RC0Q173SBEA
Income (GDP)	GDPC1
Federal Funds Rate	FEDFUNDS
Consumer Durable Expenditure	PCDG
Residential Investment	PRFI
Consumer Nondurable Expenditure	PCEND
Consumer Service Expenditure	PCES
Consumer Housing Services Expenditure	DHSGRC0
Durable Price Index	DDURRD3Q086SBEA
Residential Investment Price Index	B011RG3Q086SBEA
Nondurable Price Index	DNDGRG3M086SBEA
Services Price Index	DSERRG3M086SBEA
Services Price Index: Housing	DHUTRG3Q086SBEA
Consumer Expenditure: Motor Vehicles	DMOTRC1Q027SBEA
Motor Vehicles Price Index	DMOTRG3Q086SBEA
House Price Index	USSTHPI
Residential Investment: Permanent Site	A943RC1Q027SBEA
Residential Investment: Other	A863RC1Q027SBEA
Residential Investment Price Index: Other	A863RG3Q086SBEA

(e.g., durable expenditure equals consumer durables plus residential investment), and their respective price indices P_{1t} and P_{2t}. We want to construct the price index P_t for Y_t.

We first construct the growth rate of nominal spending, $\Delta y_t = \Delta \ln(Y_t) = \ln(Y_t) - \ln(Y_{t-1})$, and of the price indeces, Δp_{1t} and Δp_{1t}. Define the share of good 1 in nominal expenditure, $s_{1t} = \frac{X_{1t}}{Y_t}$. Then the growth rate of the aggregate price index is $\Delta p_t = s_{1,t-1}\Delta p_{1t} + (1 - s_{1,t-1})\Delta p_{2t}$, from which we can construct the aggregate price index P_t. The growth rate of real expenditure is $\Delta y_t - \Delta p_t$, from which we can construct aggregate real expenditure. We convert all real expenditure to per capita by dividing by population.

For the price series of residential investment and consumer services, we make specific modifications. We separate residential investment into investment into new structures and other residential investment. For investment into new structures, we use the FHFA national house price index to capture changes in the price of land as well as the price of the new structure. For other residential investment, we use the associated price index from the BEA. The weights are based on nominal expenditures in new residential structures and other residential investment and calculated as above.

For consumption of services, we remove housing services because housing services in the model are obtained from durables and not counted in C_t. To do so, we follow the same procedure as above for the housing and nonhousing component of services. But rather than adding, we subtract the housing component, $Y_t = X_{2t} - X_{1t}$. The share of rent in nondurable expenditure is $s_{1t} = -\frac{X_{1t}}{Y_t}$. With these two modifications, we can calculate real expenditure and the price index as above.

The relative price series for durables is the price of durables divided by the price of nondurables and services. The real interest rate is defined in terms of nondurables. It is the federal funds rate net of realized nondurable inflation over the next 4 quarters.

C.2. *PSID: Housing Adjustment Probability*

We use the Panel Survey of Income Dynamics (PSID) to construct a time series for the probability of housing adjustments. We use data from 1969–1997 when the survey frequency is annual. We keep only people who are heads of household and those who are in the Survey Research Center (SRC) sample.

We use the moved since spring series to create a record of adjustments. If moved since spring is true, we record an adjustment for that year. If moved since spring is false, we record no adjustment for that year.

Following Bachmann and Cooper (2014), we set values to missing if the observation does not have a tenure status or is lag does not have a tenure status. For example, if their observation is in the year 1992, we will set the adjustment series to missing if we do not know whether the head of household was owning or renting in either 1991 or 1992. We create a time series of the probability of adjustment by aggregating the panel using the family weight.

C.3. *CEX: Car Acquisition Probability*

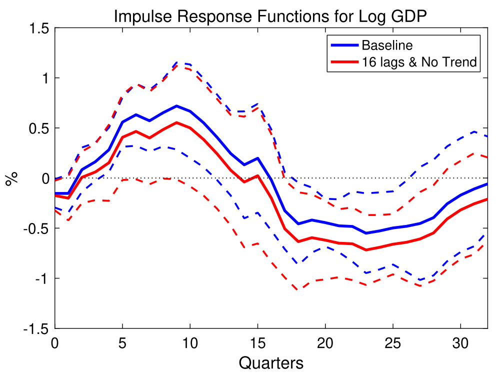
We use the consumer expenditure (CEX) survey from 1980–2017 to construct a quarterly time series of the probability of a household acquiring a car or truck (used or new). We download precompiled files from the BLS for 1996 onwards and earlier raw files from ICPSR. We clean the microdata files following Coibion, Gorodnichenko, Kueng, and Silvia (2017).

In the expenditure files, we sum the UCC codes 450110 (new cars), 450210 (new trucks), 460110 (used cars), 460901 (used trucks). All expenditure series are net of trade-in value. This definition aligns with the BEA definition of motor vehicle expenditure. Using the household weights, total motor vehicle expenditure implies by the CEX tracks BEA personal consumption motor vehicle expenditure very well.

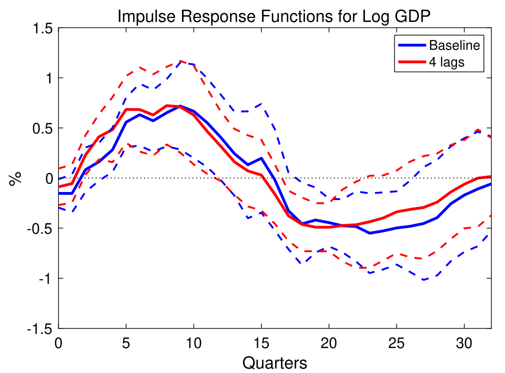
We construct the probability of adjustment by setting an indicator equal to 1 whenever a household's motor vehicle expenditures are positive, and aggregating the indicator using household weights.

APPENDIX D: ROBUSTNESS OF IMPULSE RESPONSE FUNCTIONS

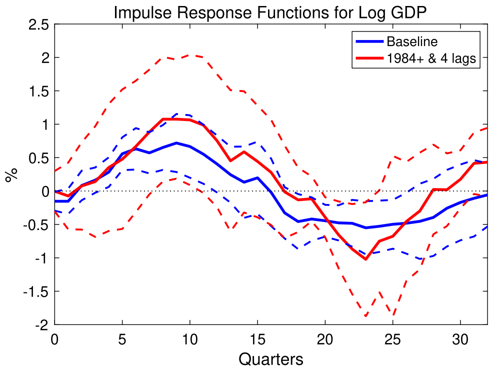
In each plot of Figure A.2, we compare our baseline impulse response function for GDP (blue line) against an alternative specification (red line). In Figure A.2a, we drop the deterministic trend. This helps allay concerns that we are biasing the model toward



(a) No trend



(b) Four lags



(c) Sample 1984-2016

FIGURE A.2.—Robustness of the impulse response function for GDP estimated in Section 3. The blue line depicts the baseline specification and the red line the alternative specification. Dashed lines are 95% confidence intervals. In panel (a), the alternative specification drops the deterministic time trend. In panel (b), the alternative specification includes four lags of the dependent variable and the monetary policy shock (as opposed to 16 in the baseline). In panel (c), the alternative specification is estimated over 1984–2006 and with four lags.

stationarity (Sims (1996)). In Figure A.2b, we include only four lags of the dependent variable and the monetary shock (vs 16 in the baseline) to address concerns that we may overfit the data. In Figure A.2c, we restrict the sample to the post-Volcker period, 1984–2016. Due to the shorter sample, we reduce the lag length to four in that last case. For each of these three alternative specifications, the estimated response is close to our baseline estimates both in economic and statistical terms. In particular, all alternative specifications display an initial increase in GDP and subsequent reversal.

APPENDIX E: IMPULSE RESPONSE FUNCTIONS FOR r_t AND p_t

We estimate the impulse response of the real interest rate in terms of nondurable goods, r_t and the relative durable price, p_t to a Romer–Romer monetary policy shock. For these impulse responses, we make use the equivalence result from (Plagborg-Møller and Wolf (2021)) who show that VARs and local projections yield the same impulse response up to the horizon of included lags (16 quarters in our case). The benefit of using a VAR is that it generates smoother impulse responses beyond 16 quarters, which is useful when feeding these paths into the household decision problem in Section 3. Other than generating smoother response beyond 16 quarters, the impulse response functions estimated by the VAR are very similar to impulse response functions estimated by local projections.

We estimate two bivariate VARs, in which the monetary shock is ordered first. The second variable is respectively the real interest rate and the change in the relative price of durables. The VAR also includes a time-trend and the standard errors are block-bootstrapped and bias-corrected following Kilian (1998). Figure A.3 plots these impulse response functions.

APPENDIX F: DETAILS OF THE GENERAL EQUILIBRIUM MODEL

F.1. *The Labor Market*

The labor demand curve of each labor type j is

$$l_{jt} = L_t \left(\frac{W_{jt}}{W_t} \right)^{-\varphi},$$

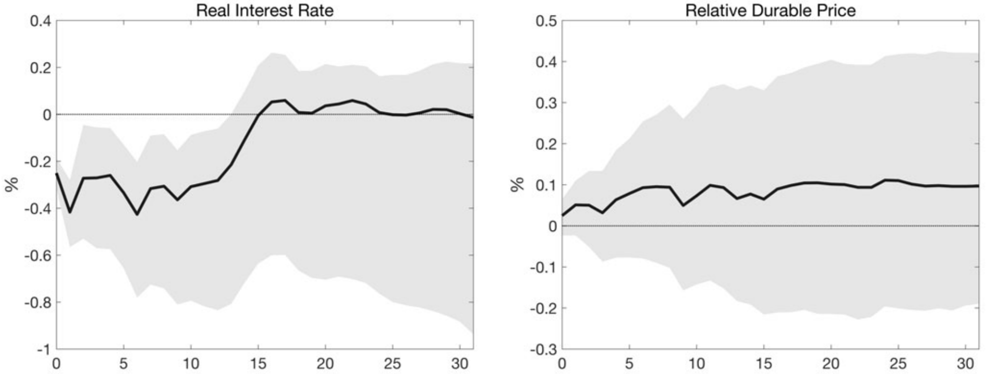


FIGURE A.3.—Impulse response function of the real interest rate in terms of nondurables (left panel) and the relative durable price (right panel) to a Romer–Romer monetary policy shock.

where the aggregate wage is equal to

$$W_t = \left(\int_0^1 W_{jt}^{1-\varphi}\, dj\right)^{\frac{1}{1-\varphi}}.$$

The union's problem can be stated in terms of piece rates $\tilde{W}_{jt} = W_{jt}/Z_t$,

$$\max_{\{\mu_{jt}\}} \int_{t=0}^{\infty} e^{-\rho t} \int_0^1 \left[u_c(c_{it}, s_{it}) \frac{\tilde{W}_{jt} Z_t}{P_t} l_{jt} z_{it} - \Omega_t v(l_{jt}) - \frac{\Psi}{2} \Omega_t L_t \mu_{jt}^2 \right] \mathrm{d}i\, \mathrm{d}t$$

subject to

$$\begin{aligned} \mathrm{d} \ln \tilde{W}_{jt} &= \mu_{jt}\, \mathrm{d}t, \\ l_{jt} &= L_t \left(\frac{\tilde{W}_{jt}}{\tilde{W}_t} \right)^{-\varphi}. \end{aligned}$$

Using the definition of Ω_t, we can rewrite the objective as

$$\max_{\{\mu_{jt}\}} \int_{t=0}^{\infty} e^{-\rho t} \Omega_t \left[\frac{\tilde{W}_{jt}}{P_t} l_{jt} - v(l_{jt}) - \frac{\Psi}{2} L_t \mu_{jt}^2 \right] \mathrm{d}t.$$

To analyze the union's problem, treat $q_{jt} \equiv \ln \tilde{W}_{jt}$ as the state and μ_{jt} as the control. The Hamiltonian is

$$H = \Omega_t \left[\frac{e^{q_{jt}}}{P_t} L_t \left(\frac{e^{q_{jt}}}{\tilde{W}_t} \right)^{-\varphi} - v\left(L_t \left(\frac{e^{q_{jt}}}{\tilde{W}_t} \right)^{-\varphi} \right) - \frac{\Psi}{2} L_t \mu_{jt}^2 \right] + \lambda_{jt} \mu_{jt},$$

where λ_{jt} is the costate. The necessary conditions for optimality are

$$\begin{aligned} \lambda_{jt} &= \Psi \Omega_t L_t \mu_{jt}, \\ \mathrm{d}\lambda_{jt} - \rho \lambda_{jt}\, \mathrm{d}t &= -(1-\varphi) \Omega_t \frac{\tilde{W}_{jt}}{P_t} l_{jt}\, \mathrm{d}t - \varphi \Omega_t v'(l_{jt}) l_{jt}\, \mathrm{d}t. \end{aligned}$$

Imposing symmetry and the relationships $P_t = \tilde{W}_t$ and $Y_t = Z_t L_t$ yields the nonlinear Phillips curve

$$\mathrm{d}\pi_t = \left[\rho\, \mathrm{d}t - \frac{\mathrm{d}Y_t}{Y_t} - \frac{\mathrm{d}\Omega_t}{\Omega_t} \right] \pi_t - \frac{(\varphi - 1)}{\Psi} \left[\frac{\varphi}{\varphi - 1} v'(L_t) - 1 \right] \mathrm{d}t.$$

Linearizing around a zero inflation steady state in which $\bar{L} = v'^{-1}(\frac{\varphi-1}{\varphi})$ and $\bar{Y}_t = Z_t \bar{L}$ yields

$$\mathrm{d}\pi_t = \rho \pi_t\, \mathrm{d}t - \frac{\varphi}{\Psi} v'(\bar{L}) \eta \left(\frac{Y_t - \bar{Y}_t}{\bar{Y}_t} \right) \mathrm{d}t,$$

where $1/\eta$ is the Frisch elasticity. Letting $\kappa = \frac{(\varphi-1)\eta}{\Psi}$ gives (1).

F.2. *Market Clearing*

Nondurables market clearing:

$$Y_t = \int_0^1 c_{it}\,\mathrm{d}i + M_t + G_t + \left(r_t^b - r_t\right)\int_0^1 a_{it} I_{(a_{it}<0)}\,\mathrm{d}i.$$

Durable goods market clearing:

$$X_t = \int_0^1 \left(\frac{\mathrm{d}d_{it}}{\mathrm{d}t} - \delta d_{it}\right)\mathrm{d}i + f\int_0^1 I_{d'_{it}\neq d_{it}} d_{it} + \nu \int_0^1 d_{it}\,\mathrm{d}i.$$

Bond market clearing:

$$\int_0^1 a_{it}\,\mathrm{d}i = A_t.$$

APPENDIX G: DATA FILTERING USING THE *MA* REPRESENTATION

In this Appendix, we implement a restricted version of the Kalman filter to recover aggregate shocks. We impose three restrictions on the standard Kalman filtering framework. First, we do not allow for measurement error in the observation equation. Second, we assume that either (a) the system is initially in steady state at the start of the sample or (b) the researcher knows the initial state with certainty and knows the transition path of the model back to steady state. If the system is stable, this restriction is not costly in situations where the researcher has a sufficient burn-in period at the start of the sample so that the effect of the initial state dissipates before the sample of interest begins. Third, we assume that there are at least as many states as there are observables. Under these restrictions, the Kalman smoother coincides with the Kalman filter.

The Filtering Algorithm. Consider a dynamic system with a state space representation

$$X_t = AX_{t-1} + B\epsilon_t, \tag{1}$$

$$Y_t = CX_t, \tag{2}$$

where X is the state, ϵ is a vector of i.i.d. mean-zero innovations and Y is the observed data. ϵ and Y are dimension $N \times 1$ and X is dimension $M \times 1$. A, B, and C are conformable matrices. We will require that CB is invertible, which requires that there are at least as many states as there are observables.

We assume that this internal description of the model is unknown to the researcher. Instead, the researcher has access to an external description of the system, that is, impulse response functions. Let $R(\tau, i)$ be the response of Y_τ to a unit change in the ith element of ϵ_0. The impulse responses are given by

$$R(\tau, i) = CA^\tau B\mathbf{1}_i,$$

where $\mathbf{1}_i$ is the standard basis vector in the ith dimension. $R(\tau, i)$ is a $N \times 1$ vector. Let $R(\tau)$ be a $N \times N$ matrix where the ith column is $R(\tau, i)$. Notice that $R(\tau) = CA^\tau B$. The researcher may also have access to an estimate of the effects of the initial state of the system $S(\tau) = CA^{\tau+1}X_{-1}$ for $\tau \geq 0$. In practice, one may wish to assume that the system is initially in steady state so $S(\tau) = 0$ for all τ. For a stationary system, where $A^t \to 0$ as $t \to \infty$, the role of the initial state will diminish over time so if one has a sufficient burn-in period of data assuming the system starts in steady state will have limited effect on the results.

The researcher has data $\{Y_t\}_{t=0}^T$ and wishes to recover an estimate of $\{\epsilon_t\}_{t=0}^T$. The filtering then proceeds recursively as follows: Let $Q_0 = S(0)$. At date 0, solve (1) and (2) for $\epsilon_0 = (CB)^{-1}(Y_0 - CAX_{-1})$ and notice that we can rewrite this as $\epsilon_0 = R(0)^{-1}(Y_0 - Q_0)$. Now suppose that we have solved for $\{\epsilon_\tau\}_{\tau=0}^{t-1}$ and we wish to solve for ϵ_t. Let $Q_t = CAX_{t-1}$ and by repeated substitution of (1) we have $Q_t = \sum_{\tau=0}^{t-1} R(t-\tau)\epsilon_\tau + S(t)$. From (1) and (2), we then have

$$\epsilon_t = R(0)^{-1}(Y_t - Q_t). \tag{3}$$

Relationship to the Kalman Filter. Let $\hat{X}_{t|t-1}$ be the point estimate of X_t given information through $t-1$. The Kalman filter updates this estimate as (Hamilton (1994, equation (13.2.15)))

$$\hat{X}_{t|t} = \hat{X}_{t|t-1} + P_{t|t-1}C'\big(CP_{t|t-1}C'\big)^{-1}(Y_t - C\hat{X}_{t|t-1}),$$

where $P_{t|t-1}$ is the covariance matrix of $\hat{X}_{t|t-1}$. Because we assume that the initial state (or rather its effects) is known and there is no measurement error, once Y_{t-1} is observed, ϵ_{t-1} is known and, therefore, the only reason $\hat{X}_{t|t-1}$ is uncertain is because of ϵ_t. Therefore, $P_{t|t-1} = B\Sigma B'$ where Σ is the covariance matrix of ϵ. Plugging this in above, we have

$$\begin{aligned}\hat{X}_{t|t} &= \hat{X}_{t|t-1} + B\Sigma B'C'\big(CB\Sigma B'C'\big)^{-1}(Y_t - C\hat{X}_{t|t-1}) \\ &= \hat{X}_{t|t-1} + B\underbrace{(CB)^{-1}(Y_t - C\hat{X}_{t|t-1})}_{=\epsilon_t}.\end{aligned}$$

Now notice that the update to $\hat{X}_{t|t-1}$ is just $B\epsilon_t$ so we have

$$\begin{aligned}\epsilon_t &= (CB)^{-1}(Y_t - C\hat{X}_{t|t-1}) \\ &= (CB)^{-1}(Y_t - CA\hat{X}_{t-1|t-1}),\end{aligned}$$

where the second line follows from Hamilton equation (13.2.17). Using the logic above, X_{t-1} is known after Y_{t-1} is observed so $\hat{X}_{t-1|t-1} = X_{t-1}$ so the above equation becomes

$$\epsilon_t = R(0)^{-1}(Y_t - Q_t)$$

in the notation of our filtering algorithm, which is the same as (3).

Incorporating the ELB. Let $\mathcal{R}$ be a $T \times T$ matrix that maps a path of output gaps into a path of real interest rates that satisfy the monetary rule. The Phillips curve and monetary policy response to inflation is embedded inside $\mathcal{R}$. With the ELB, we have

$$\overrightarrow{r}_t = \max\big\{\mathcal{R}(\mathcal{M}\overrightarrow{r} + \mathcal{Q}\epsilon_t + \overrightarrow{\hat{Y}}_{t|t-1}) + S(\eta_{t-1}, \epsilon_t), \underline{r}\big\}, \tag{4}$$

where $\overrightarrow{r}_t \equiv (r_t, \mathbb{E}_t r_{t+1}, \dots, \mathbb{E}_t r_{t+T-1})'$, $\mathcal{Q}$ maps the current shock to a path of output gaps under constant real rates, $\overrightarrow{\hat{Y}}_{t|t-1}$ is the forecast of output gaps given past shocks and monetary news, and S is a function that captures the effect of the exogenous term in the interest rate rule. Given the $N \times 1$ data vector Y_t, we solve a system of $N + T$ equations such that $Y_t = CB\epsilon_t + Q_t$ as above and such that (4) holds where the unknowns are the elements of ϵ_t and $\overrightarrow{r}_t$. We solve this system iteratively using partial updates.

APPENDIX H: DERIVATION AND DECOMPOSITION OF r^*

Derivation of Equation (4). Consider an abstract representation of our model expressed in discrete time steps corresponding to the time intervals on which we compute the model:

$$\begin{aligned} \hat{Y}_t &= \mathcal{Y}(h_t, \Phi_t), \\ h_t &= \mathcal{H}(\vec{r}_t, \eta_t), \\ \Phi_{t+1} &= \mathcal{T}(\Phi_t, h_t), \\ \eta_{t+1} &= \mathcal{F}(\eta_t, \epsilon^\eta_{t+1}). \end{aligned}$$

The first equation states that the output gap, $\hat{Y}_t$, is a function, $\mathcal{Y}$, of the household policy rules, h_t, and the distribution of households over individual states, Φ_t. In our model, a household chooses whether or not to adjust its durable stock and if so the level of durables, how much to consume in nondurables, and how much to save in liquid assets. All of these decision rules are contained in the collection h_t. The second equation states that the policy rules depend the vector of current and expected future real interest rates, $\vec{r}_t \equiv (r_t, \mathbb{E}_t r_{t+1}, \dots)'$, and the exogenous aggregate states, $\eta_t \equiv (g_t, G_t - \bar{G}, r^b_t - \bar{r}^b_t)'$. We extend the analysis to allow for prices other than real interest rates to affect the decision rules below, but we begin with a simpler formulation here for ease of exposition. The third equation shows how the distribution of individual states evolves as a function of the household decisions. In heterogeneous agent models, the evolution of the distribution depends on individual decisions as well as the stochastic process of idiosyncratic shocks. In our formulation, the effect of idiosyncratic shocks is embedded within the function $\mathcal{T}$. Finally, the fourth equation gives the law of motion for the exogenous aggregate states where ϵ^η_{t+1} is the vector of innovations to the aggregate stochastic processes, which are uncorrelated.[1]

Current and future real rates affect the policy rules at t. Previous real interest rates do not affect the policy rules because the policy rules are conditional on individual states. However, past interest rates affect the output gap at t through their effect on the distribution of individual states Φ_t. For example, if low interest rates in the past caused households to stock up on durables, then this is reflected in the distribution of households over levels of durables.

We linearize the system around steady state:

$$\begin{aligned} \hat{Y}_t &= \mathcal{Y}_h h_t + \mathcal{Y}_\Phi(\Phi_t - \bar{\Phi}), \\ h_t &= \mathcal{H}_r \vec{r}_t + \mathcal{H}_\eta \eta_t, \\ \Phi_{t+1} - \bar{\Phi} &= \mathcal{T}_\Phi(\Phi_t - \bar{\Phi}) + \mathcal{T}_h h_t, \\ \eta_{t+1} &= \mathcal{F}_\eta \eta_t + \epsilon^\eta_{t+1}. \end{aligned}$$

[1]For example, the first element is $\sigma_Z(\mathcal{W}^Z_{t+1} - \mathcal{W}^Z_t)$.

As with Φ_t, h_t can be interpreted as a vector that gives a discrete representation of the decision rules as in the Reiter (2009) method. Using the linearized system, the forecast at date t of the output gap at date $t+s$ for $s \geq 0$ is given by

$$\mathbb{E}_t \hat{Y}_{t+s} = \mathcal{Y}_h\big(\mathcal{H}_r \mathbb{E}_t \vec{r}_{t+s} + \mathcal{H}_\eta \mathcal{F}_\eta^s \eta_t\big) + \sum_{k=0}^{s-1} \mathcal{Y}_\Phi \mathcal{T}_\Phi^{s-k-1} \mathcal{T}_h\big(\mathcal{H}_r \mathbb{E}_t \vec{r}_{t+k} + \mathcal{H}_\eta \mathcal{F}_\eta^k \eta_t\big) \\ + \mathcal{Y}_\Phi \mathcal{T}_\Phi^s (\Phi_t - \bar{\Phi}),$$

where $\mathcal{Y}_h$ is the partial Jacobian of $\hat{Y}_t$ with respect to h_t and so on. As $\vec{r}_t \equiv (r_t, \mathbb{E}_t r_{t+1}, \mathbb{E}_t r_{t+2}, \ldots)$ we can write $\mathbb{E}_t \vec{r}_{t+s} = S_s \vec{r}_t$ where S_s is a shift operator that chops off the first s elements of $\vec{r}_t$. Using this shift operator and rearranging yields

$$\mathbb{E}_t \hat{Y}_{t+s} = \left[\mathcal{Y}_h \mathcal{H}_r S_s + \sum_{k=0}^{s-1} \mathcal{Y}_\Phi \mathcal{T}_\Phi^{s-k-1} \mathcal{T}_h \mathcal{H}_r S_k\right] \vec{r}_t \\ + \left[\mathcal{Y}_h \mathcal{H}_\eta \mathcal{F}_\eta^s + \sum_{k=0}^{s-1} \mathcal{Y}_\Phi \mathcal{T}_\Phi^{s-k-1} \mathcal{T}_h \mathcal{H}_\eta \mathcal{F}_\eta^k\right] \eta_t + \big[\mathcal{Y}_\Phi \mathcal{T}_\Phi^s\big](\Phi_t - \bar{\Phi}). \tag{5}$$

This equation shows that the forecast of the output gap at $t+s$ is (to a first-order approximation) a linear function of the expected real interest rate path, the exogenous states η_t, and the distribution Φ_t. Stacking equation (5) for $s \geq 0$ then yields equation (4) with the terms in square brackets forming the rows of $\mathcal{M}$, $\mathcal{Q}$, and $\mathcal{D}$, respectively.

Derivation of Equation (5). As shown in the text, the solution for r^* for a given set of states η_r, Φ_t is

$$\vec{r}_t^* = -\mathcal{M}^{-1}\big(\mathcal{Q}\eta_t + \mathcal{D}(\Phi_t - \bar{\Phi})\big).$$

To determine r^* as a function of past real rates and the exogenous states η, we solve out for the endogenous state Φ_t. Solving the state backwards yields

$$\begin{aligned} \Phi_t - \bar{\Phi} &= \mathcal{T}_h h_{t-1} + \mathcal{T}_\Phi(\Phi_{t-1} - \bar{\Phi}) \\ &= \mathcal{T}_h \mathcal{H}_r \vec{r}_{t-1} + \mathcal{T}_h \mathcal{H}_\eta \eta_{t-1} + \mathcal{T}_\Phi\big(\mathcal{T}_h h_{t-2} + \mathcal{T}_\Phi(\Phi_{t-2} - \bar{\Phi})\big) \\ &= \sum_{k=0}^{t-1} \big(\mathcal{T}_\Phi^k \mathcal{T}_h \mathcal{H}_r\big) \vec{r}_{t-1-k} + \sum_{k=0}^{t-1} \big(\mathcal{T}_\Phi^k \mathcal{T}_h \mathcal{H}_\eta\big) \eta_{t-1-k} \end{aligned}$$

with $\Phi_0 = \bar{\Phi}$.

We next show how to express the first two terms in terms of the matrices $\mathcal{M}$ and $\mathcal{Q}$. Start with the term that captures the history of interest rates,

$$\mathcal{D}\left[\sum_{k=0}^{t-1}(\mathcal{T}_\Phi^k\mathcal{T}_h\mathcal{H}_r)\vec{r}_{t-1-k}\right] = \mathcal{Y}_\Phi\begin{pmatrix} I \\ \mathcal{T}_\Phi \\ \mathcal{T}_\Phi^2 \\ \vdots \end{pmatrix}\left[\sum_{k=0}^{t-1}(\mathcal{T}_\Phi^k\mathcal{T}_h\mathcal{H}_r)\vec{r}_{t-1-k}\right]$$

$$= \mathcal{Y}_\Phi\sum_{k=0}^{t-1}\begin{pmatrix} (\mathcal{T}_\Phi^k) \\ (\mathcal{T}_\Phi^{k+1}) \\ (\mathcal{T}_\Phi^{k+2}) \\ \vdots \end{pmatrix}\mathcal{T}_h\mathcal{H}_r\vec{r}_{t-1-k}.$$

To see the connection with the monetary transmission matrix, we split $\mathcal{M}$ into two components, one capturing how the evolution of the state and the other the policy function,

$$\mathcal{M} = \begin{pmatrix} 0 \\ \mathcal{Y}_\Phi\mathcal{T}_h\mathcal{H}_r \\ \mathcal{Y}_\Phi\mathcal{T}_\Phi\mathcal{T}_h\mathcal{H}_r + \begin{bmatrix}0 & \mathcal{Y}_\Phi\mathcal{T}_h\mathcal{H}_r\end{bmatrix} \\ \mathcal{Y}_\Phi\mathcal{T}_\Phi^2\mathcal{T}_h\mathcal{H}_r + \begin{bmatrix}0 & \mathcal{Y}_\Phi\mathcal{T}_\Phi\mathcal{T}_h\mathcal{H}_r\end{bmatrix} + \begin{bmatrix}0 & 0 & \mathcal{Y}_\Phi\mathcal{T}_h\mathcal{H}_r\end{bmatrix} \\ \vdots \end{pmatrix} + \begin{pmatrix} \mathcal{Y}_h\mathcal{H}_r \\ \begin{bmatrix}0 & \mathcal{Y}_h\mathcal{H}_r\end{bmatrix} \\ \begin{bmatrix}0 & 0 & \mathcal{Y}_h\mathcal{H}_r\end{bmatrix} \\ \vdots \end{pmatrix}.$$

For general $s = t + 1$, the term of past real rate expectations can then be split into a component involving interest rate innovations up to time $s - 1$ and one component involving expected interest rates from s onward,

$$\mathcal{D}\left[\sum_{k=0}^{s}(\mathcal{T}_\Phi^k\mathcal{T}_h\mathcal{H}_r)\vec{r}_{t-1-k}\right] = \sum_{k=0}^{s-1}\mathcal{M}_{[1+s-k..,1..s-k]}\left[\mathbb{E}_k\begin{pmatrix} r_k \\ \vdots \\ r_{s-1} \end{pmatrix} - \mathbb{E}_{k-1}\begin{pmatrix} r_k \\ \vdots \\ r_{s-1} \end{pmatrix}\right]$$

$$+ \sum_{k=0}^{s-1}(\mathcal{M}_{[1+s-k..,1+s-k..]} - \mathcal{M}_{[s-k..,s-k..]})\mathbb{E}_k\vec{r}_s.$$

We take a similar approach for expressing the historical contribution of the exogenous states η. It will again be convenient to write the matrix $\mathcal{Q}$ as the sum of the state component and the policy component,

$$\mathcal{Q} = \begin{pmatrix} 0 \\ \mathcal{Y}_\Phi\mathcal{T}_h\mathcal{H}_\eta \\ \mathcal{Y}_\Phi\mathcal{T}_\Phi\mathcal{T}_h\mathcal{H}_\eta + \mathcal{Y}_\Phi\mathcal{T}_h\mathcal{H}_\eta\mathcal{F}_\eta \\ \mathcal{Y}_\Phi\mathcal{T}_\Phi^2\mathcal{T}_h\mathcal{H}_\eta + \mathcal{Y}_\Phi\mathcal{T}_\Phi\mathcal{T}_h\mathcal{H}_\eta\mathcal{F}_\eta + \mathcal{Y}_\Phi\mathcal{T}_h\mathcal{H}_\eta\mathcal{F}_\eta^2 \\ \vdots \end{pmatrix} + \begin{pmatrix} \mathcal{Y}_h\mathcal{H}_\eta \\ \mathcal{Y}_h\mathcal{H}_\eta\mathcal{F}_\eta \\ \mathcal{Y}_h\mathcal{H}_\eta\mathcal{F}_\eta^2 \\ \vdots \end{pmatrix}.$$

Then we can express the historical contribution of the exogenous states η to the states solely in terms of past shocks and the $\mathcal{Q}$ matrix,

$$
\begin{aligned}
&\mathcal{D}\left[\sum_{k=0}^{t-1}(\mathcal{T}_\Phi^k \mathcal{T}_h \mathcal{H}_\eta)\eta_{t-1-k}\right] \\
&= \mathcal{Y}_\Phi \sum_{k=0}^{t-1}\begin{pmatrix}(\mathcal{T}_\Phi^k)\\(\mathcal{T}_\Phi^{k+1})\\(\mathcal{T}_\Phi^{k+2})\\ \vdots\end{pmatrix}\mathcal{T}_h\mathcal{H}_\eta\eta_{t-1-k} = \sum_{k=0}^{t-1}(\mathcal{Q}_{[2+k..,..]} - \mathcal{Q}_{[1+k..,..]}\mathcal{F}_\eta)\eta_{t-1-k} \\
&= \sum_{k=0}^{t-1}\mathcal{Q}_{[2+k..,..]}\eta_{t-1-k} - \sum_{k=0}^{t-1}\mathcal{Q}_{[1+k,..]}\mathcal{F}_\eta\eta_{t-1-k} = -\mathcal{Q}\mathcal{F}_\eta\eta_{t-1} + \sum_{k=0}^{t-1}\mathcal{Q}_{[2+k..,..]}\epsilon^\eta_{t-1-k}.
\end{aligned}
$$

Substituting our solution for the state into the equation for r^* yields

$$
\begin{aligned}
\vec{r}_t^* = &-\mathcal{M}^{-1}\sum_{k=0}^{t-1}\mathcal{M}_{[1+t-k..,1..t-k]}\left[\mathbb{E}_k\begin{pmatrix}r_k\\ \vdots\\ r_{t-1}\end{pmatrix} - \mathbb{E}_{k-1}\begin{pmatrix}r_k\\ \vdots\\ r_{t-1}\end{pmatrix}\right] \\
&- \mathcal{M}^{-1}\sum_{k=0}^{t-1}(\mathcal{M}_{[1+t-k..,1+t-k..]} - \mathcal{M}_{[t-k..,t-k..]})\mathbb{E}_k\vec{r}_t - \mathcal{M}^{-1}\sum_{k=0}^{t-1}\mathcal{Q}_{[1+k,..]}\epsilon^\eta_{t-k}.
\end{aligned}
$$

This equation tells us that r^* is not just a function of the shocks (last term), but it can also vary with how past interest rates were set in the past (first term) and with past expectations of current and future rates (second term).

We next solve out for these expectations of current and future rates by assuming that they are set to close all output gaps from time t onwards, consistent with the definition of r^*. Thus, the expectations of future rates are now superscripted with a star,

$$
\begin{aligned}
\vec{r}_t^* = &-\mathcal{M}^{-1}\sum_{k=0}^{t-1}\mathcal{M}_{[1+t-k..,1..t-k]}\left[\mathbb{E}_k\begin{pmatrix}r_k\\ \vdots\\ r_{t-1}\end{pmatrix} - \mathbb{E}_{k-1}\begin{pmatrix}r_k\\ \vdots\\ r_{t-1}\end{pmatrix}\right] \\
&- \mathcal{M}^{-1}\sum_{k=0}^{t-1}(\mathcal{M}_{[1+t-k..,1+t-k..]} - \mathcal{M}_{[t-k..,t-k..]})\mathbb{E}_k\vec{r}_t^* - \mathcal{M}^{-1}\sum_{k=0}^{t-1}\mathcal{Q}_{[1+k,..]}\epsilon^\eta_{t-k}.
\end{aligned}
$$

Taking expectations of this r^* vector yields

$$
\begin{aligned}
\mathcal{M}_{[1+s..,1+s..]}\mathbb{E}_{t-s}\vec{r}_t^* = &-\sum_{k=0}^{t-s}\mathcal{M}_{[1+t-k..,1..t-k]}\left[\mathbb{E}_k\begin{pmatrix}r_k\\ \vdots\\ r_{t-1}\end{pmatrix} - \mathbb{E}_{k-1}\begin{pmatrix}r_k\\ \vdots\\ r_{t-1}\end{pmatrix}\right] \\
&- \sum_{k=0}^{t-s-1}(\mathcal{M}_{[1+t-k..,1+t-k..]} - \mathcal{M}_{[t-k..,t-k..]})\mathbb{E}_k\vec{r}_t^* - \sum_{k=s}^{t}\mathcal{Q}_{[1+k,..]}\epsilon^\eta_{t-k}.
\end{aligned}
$$

We can now write expectations recursively,

$$\mathbb{E}_{t-s}\vec{r}_t^* = \mathbb{E}_{t-s-1}\vec{r}_t^* - \mathcal{M}_{[1+s..,1+s..]}^{-1}\mathcal{M}_{[1+s..,1..s]}\left[\mathbb{E}_{t-s}\begin{pmatrix} r_{t-s} \\ \vdots \\ r_{t-1}\end{pmatrix} - \mathbb{E}_{t-s-1}\begin{pmatrix} r_{t-s} \\ \vdots \\ r_{t-1}\end{pmatrix}\right]$$
$$- \mathcal{M}_{[1+s..,1+s..]}^{-1}\mathcal{Q}_{[1+s,..]}\epsilon_{t-s}^{\eta}.$$

Repeated substitution of the expectation updating into the r^* equation then yields the formula in the text,

$$\vec{r}_t^* = -\sum_{k=0}^{t-1}\mathcal{M}_{[1+t-k..,1+t-k..]}^{-1}\mathcal{M}_{[1+t-k..,1..t-k]}\left[\mathbb{E}_k\begin{pmatrix} r_k \\ \vdots \\ r_{t-1}\end{pmatrix} - \mathbb{E}_{k-1}\begin{pmatrix} r_k \\ \vdots \\ r_{t-1}\end{pmatrix}\right]$$
$$- \sum_{k=0}^{t}\mathcal{M}_{[1+k..,1+k..]}^{-1}\mathcal{Q}_{[1+k,..]}\epsilon_{t-k}^{\eta}.$$

Extension With More Endogenous Prices. Consider the expanded system:

$$\begin{aligned}
\hat{Y}_t &= \mathcal{Y}(h_t, \Phi_t), \\
h_t &= \mathcal{H}(\vec{r}_t, \vec{w}_t, \eta_t), \\
\Phi_{t+1} &= \mathcal{T}(\Phi_t, h_t), \\
\eta_{t+1} &= \mathcal{F}(\eta_t, \epsilon_{t+1}^{\eta}), \\
0 &= \mathcal{P}(h_t, \Phi_t),
\end{aligned}$$

where w_t is a vector of prices (other than real interest rates) at date t and $\vec{w}_t \equiv (w_t, \mathbb{E}_t w_{t+1}, \dots)'$. The second equation therefore allows for other prices besides interest rates to affect household policy rules. $\mathcal{P}(h_{t+s}, \Phi_{t+s}) = 0$ gives the market clearing conditions for the prices in w_{t+s}. If w_{t+s} is a vector of prices, then $\mathcal{P}$ is a vector-valued function. The prices in w_t can include tax rates and the $\mathcal{P}$ can include government budget constraints or fiscal rules that set the tax rate.

Now let us take $\vec{r}_t$ as given and solve for the resulting $\vec{w}_t$. Proceeding as with the forecast of the output gap we have (to a first-order approximation) the market clearing conditions at $t + s$ are

$$0 = \left[\mathcal{P}_\Phi \sum_{k=0}^{s-1}\left(\mathcal{T}_\Phi^k \mathcal{T}_h \mathcal{H}_r S_{s-k-1}\right) + \mathcal{P}_h \mathcal{H}_r S_s\right]\vec{r}_t + \left[\mathcal{P}_\Phi \sum_{k=0}^{s-1}\left(\mathcal{T}_\Phi^k \mathcal{T}_h \mathcal{H}_w S_{s-k-1}\right) + \mathcal{P}_h \mathcal{H}_w S_s\right]\vec{w}_t$$
$$+ \left[\mathcal{P}_\Phi \sum_{k=0}^{s-1}\left(\mathcal{T}_\Phi^k \mathcal{T}_h \mathcal{H}_\eta \rho_\eta^{s-k-1}\right) + \mathcal{P}_h \mathcal{H}_\eta \rho_\eta^s\right]\eta_t + \mathcal{P}_\Phi \mathcal{T}_\Phi^s(\Phi_t - \bar{\Phi}).$$

Stacking this equation for $s \geq 0$ yields

$$\vec{0} = \mathcal{M}_P\vec{r}_t + \mathcal{N}_P\vec{w}_t + \mathcal{Q}_P\eta_t + \mathcal{D}_P(\Phi_t - \bar{\Phi}).$$

Solve this for $\vec{w}_t$

$$\vec{w}_t = -\mathcal{N}_P^{-1}\big[\mathcal{M}_P\vec{r}_t + \mathcal{Q}_P\eta_t + \mathcal{D}_P(\Phi_t - \bar{\Phi})\big]. \tag{6}$$

Forecasting the output gap as before, we arrive at an analogous expression to our simpler case without endogenous prices:

$$\begin{aligned}\mathbb{E}_t\hat{Y}_{t+s} &= \left[\mathcal{Y}'_\Phi\sum_{k=0}^{s-1}\big(\mathcal{T}'^k_\Phi\mathcal{T}_h\mathcal{H}'_r S_{s-k-1}\big) + \mathcal{Y}_h\mathcal{H}'_r S_s\right]\vec{r}_t \\ &\quad + \left[\mathcal{Y}'_\Phi\sum_{k=0}^{s-1}\big(\mathcal{T}'^k_\Phi\mathcal{T}_h\mathcal{H}'_\eta\mathcal{F}_\eta^{s-k-1}\big) + \mathcal{Y}_h\mathcal{H}'_\eta\mathcal{F}^s_\eta\right]\eta_t + \mathcal{Y}'_\Phi\mathcal{T}'^s_\Phi(\Phi_t - \bar{\Phi}),\end{aligned}$$

where we have redefined the matrices as follows:

$$\begin{aligned}\mathcal{Y}'_\Phi &= \mathcal{Y}_\Phi - \mathcal{Y}_h\mathcal{H}_w\mathcal{N}_P^{-1}\mathcal{D}_P, \\ \mathcal{T}'_\Phi &= \mathcal{T}_\Phi - \mathcal{T}_h\mathcal{H}_w\mathcal{N}_P^{-1}\mathcal{D}_P, \\ \mathcal{H}'_r &= \big[I - \mathcal{H}_w\mathcal{N}_P^{-1}\mathcal{Q}_P\big]\mathcal{H}_r, \\ \mathcal{H}'_\eta &= \big[I - \mathcal{H}_w\mathcal{N}_P^{-1}\mathcal{Q}_P\big]\mathcal{H}_\eta.\end{aligned}$$

Stacking these equations for $s \geq 0$ yields

$$\vec{\hat{Y}}_t = \mathcal{M}\vec{r}_t + \mathcal{Q}\eta_t + \mathcal{D}(\Phi_t - \bar{\Phi}).$$

For the decomposition, we use a similar approach of substituting out for $\vec{w}_t$. We can build the decomposition iteratively

$$\Phi_t = \sum_{k=0}^{t-1}\mathcal{T}'^k_\Phi\mathcal{T}_h\mathcal{H}'_r\vec{r}_{t-1-k} + \sum_{k=0}^{t-1}\mathcal{T}'^k_\Phi\mathcal{T}_h\mathcal{H}'_\eta\eta_{t-1-k}.$$

Since the addition of endogenous prices leads to identical expressions up to a redefinition of the matrices, the same steps as in the simpler case can be followed.

APPENDIX I: MODEL IMPULSE RESPONSE FUNCTIONS

Figure A.4 plots the model impulse response functions.

APPENDIX J: ROBUSTNESS TO AGGREGATE NONLINEARITIES

In this Appendix, we investigate the sensitivity of our results to allowing for nonlinear aggregate dynamics. We do so by conducting a version of our analysis in the fully nonlinear version of our model. Solving the nonlinear model is substantially more difficult than solving the linear model and, moreover, our filtering approach relies on linearity. Therefore, in this robustness analysis we conduct a somewhat simpler exercise and we use the full-information version of the model to ease the computational burden. In particular, we perform our inference procedure on a one-time shock. We proceed in the following steps:

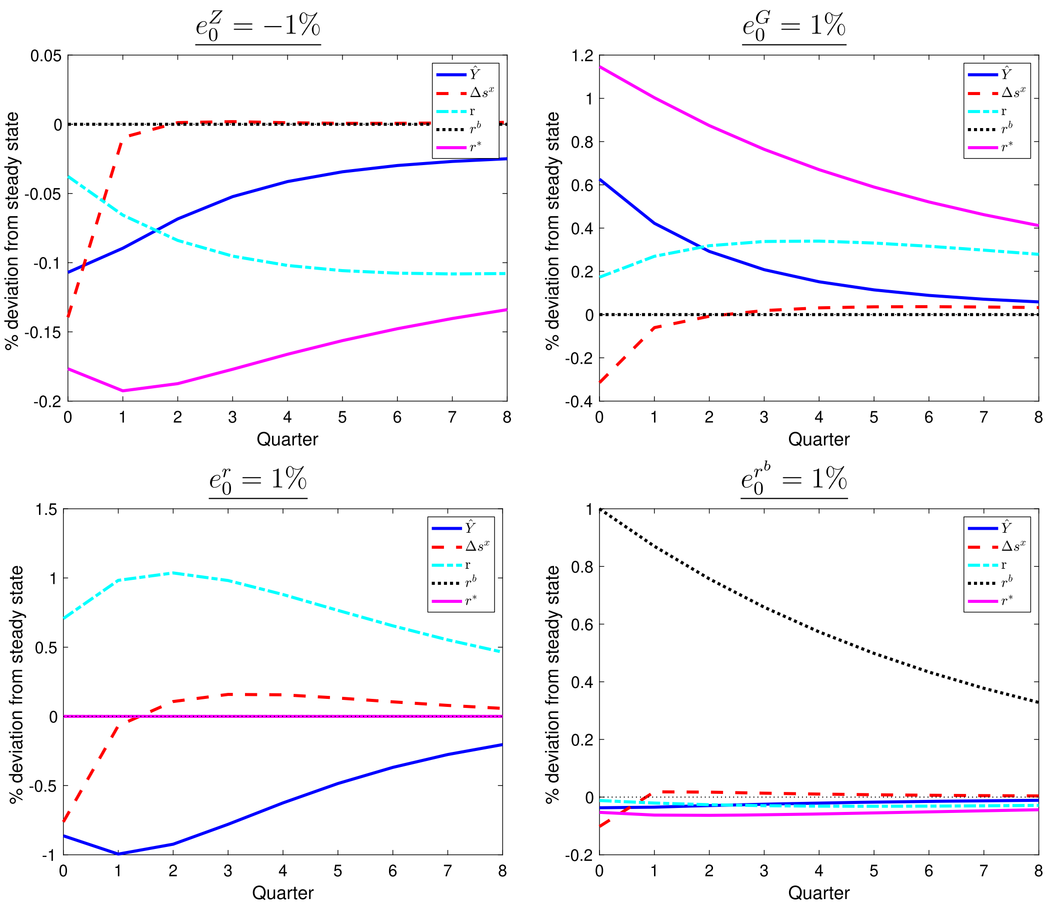


FIGURE A.4.—Impulse response functions for the output gap $\hat{Y}$, the change in the durable expenditure share relative to potential GDP Δs^x, the real interest rate r, the borrowing spread r^b, and the contemporaneous natural rate of interest r^* following a shock to productivity e^Z, nonhousehold demand e^G, the monetary policy rule e^r, and the borrowing spread e^{r^b}.

1. Feed in a one-time permanent reduction of productivity of -16% into the nonlinear model and find the paths for the real interest rate, the relative durable price, and aggregate income that are consistent with market clearing. We obtain a 2.8% output gap and an 78% drop in the endogenous adjustment probability on impact.
2. Use the *linear* model to filter the model generated data and find the implied path of r^* as we do in the main text.
3. Feed the r^* path from the previous step into the nonlinear model and find the paths of the relative durable price and aggregate income that are consistent with market clearing.

Denote the vector of output gaps from step 3 as $\hat{Y}^{\text{NL}}$. If the output gap from step 3 is close to zero, then our procedure for inferring r^* using the linear approximation to the model is accurate because it is indeed the path for interest rates that is needed to close the output gap, which is the definition of r^*. To gauge how important the error is for our calculations, we use the linear model to convert the residual output gap to an adjustment

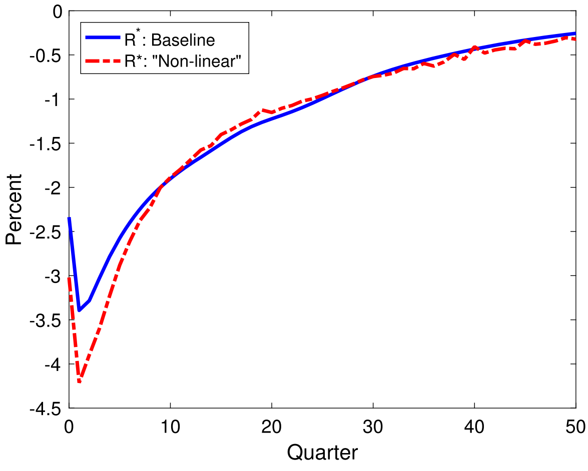


FIGURE A.5.—Path for r^* following a permanent drop in productivity in the full information model. The blue line depicts the r^* path when we use the linear model for filtering the impulse response functions and calculating the corresponding r^*. We feed this path into the nonlinear model, calculate the residual output gap $\hat{Y}^{\mathrm{NL}}$, and convert it into r^* space using $\mathcal{M}^{-1}\hat{Y}^{\mathrm{NL}}$. The red line depicts the r^* from the linear model plus this residual, $\mathcal{M}^{-1}\hat{Y}^{\mathrm{NL}}$.

to the r^* path of $\mathcal{M}^{-1}\hat{Y}^{\mathrm{NL}}$. Premultiplying by $\mathcal{M}^{-1}$ gives the change in real interest rates that would be needed to close the residual output gap.

Figure A.5 plots r^* and $r^* + \mathcal{M}^{-1}\hat{Y}^{\mathrm{NL}}$. Because monetary policy is less powerful in the recession, the linear model initially underestimates how much the real interest rate needs to fall to close the output gap. However, after 10 quarters the two real rate paths follow a very similar pattern. This suggests our baseline analysis likely underestimates the drop in r^* during the Great Recession, but the forecast of persistently low interest rates is robust to state dependence.

There are two reasons that our results are fairly robust to the state dependence implied by the nonlinear model. First, while monetary policy is indeed less powerful during the recession in the nonlinear model, this effect dissipates rather quickly. Second, even if monetary policy is persistently less powerful there are two offsetting effects on our calculation of r^*. When we filter the data, we use the observed movement in r when we infer what shock hit the economy. If we overestimate the power of monetary policy, we overestimate the size of the shock the central bank is reacting to. However, the movement in interest rates that is needed to counter a given shock is underestimated. These two considerations partially offset each other.

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