Econometrica Supplementary Material

SUPPLEMENT TO "A PREFERRED-HABITAT MODEL OF THE TERM STRUCTURE OF INTEREST RATES"

(*Econometrica*, Vol. 89, No. 1, January 2021, 77–112)

DIMITRI VAYANOS
Department of Finance, London School of Economics

JEAN-LUC VILA
Capula Investment Management

NOTE: THIS ONLINE APPENDIX does not include the proof of Theorem 1 and the material in Appendix C. These are included in the Full Appendix, available at http://personal.lse.ac.uk/vayanos/Papers/PHMTSIR_ECMAf.pdf.

APPENDIX A: PROOFS

The proof of Lemma 1 was provided in the text.

PROOF OF PROPOSITION 1: Equations (21) and (22) follow from integrating the linear ODEs (19) and (20) with the initial conditions $A_r(0) = C(0) = 0$. Substituting $A_r(\tau)$ from (21) into (23), we find (25). The left-hand side of (25) is increasing in κ_r^*, is zero for $\kappa_r^* = 0$, and converges to infinity when κ_r^* goes to infinity. The right-hand side of (25) is decreasing in κ_r^*, exceeds $\kappa_r > 0$ for $\kappa_r^* = 0$, and converges to κ_r when κ_r^* goes to zero. Therefore, (25) has a unique solution for κ_r^*, which is positive.

Substituting $C(\tau)$ from (22) into (24), we find

$$\begin{aligned}\kappa_r^*\overline{r}^*&\left[1 + a\sigma_r^2\int_0^\infty \alpha(\tau)\left[\int_0^\tau A_r(u)\,du\right]A_r(\tau)\,d\tau\right]\\ &= \kappa_r\overline{r} + a\sigma_r^2\int_0^\infty \theta_0(\tau)A_r(\tau)\,d\tau\\ &\quad + \frac{a\sigma_r^4}{2}\int_0^\infty \alpha(\tau)\left[\int_0^\tau A_r(u)^2\,du\right]A_r(\tau)\,d\tau. \end{aligned}\tag{A.1}$$

Since

$$\begin{aligned}\kappa_r\overline{r} = \kappa_r^*\overline{r}&\left[1 + a\sigma_r^2\int_0^\infty \alpha(\tau)\left[\int_0^\tau A_r(u)\,du\right]A_r(\tau)\,d\tau\right]\\ &+ \left(\kappa_r - \kappa_r^*\right)\overline{r} - \kappa_r^*\overline{r}a\sigma_r^2\int_0^\infty \alpha(\tau)\left[\int_0^\tau A_r(u)\,du\right]A_r(\tau)\,d\tau\end{aligned}$$

Dimitri Vayanos: d.vayanos@lse.ac.uk
Jean-Luc Vila: JVila@capulaglobal.com

and

$$
\begin{aligned}
&\big(\kappa_r-\kappa_r^*\big)\bar{r}-\kappa_r^*\bar{r}a\sigma_r^2\int_0^\infty \alpha(\tau)\left[\int_0^\tau A_r(u)\,du\right]A_r(\tau)\,d\tau\\
&\quad=-\bar{r}a\sigma_r^2\int_0^\infty \alpha(\tau)A_r(\tau)^2\,d\tau-\kappa_r^*\bar{r}a\sigma_r^2\int_0^\infty \alpha(\tau)\left[\int_0^\tau A_r(u)\,du\right]A_r(\tau)\,d\tau\\
&\quad=-\bar{r}a\sigma_r^2\int_0^\infty \alpha(\tau)\left[A_r(\tau)+\kappa_r^*\int_0^\tau A_r(u)\,du\right]A_r(\tau)\,d\tau\\
&\quad=-\bar{r}a\sigma_r^2\int_0^\infty \alpha(\tau)\tau A_r(\tau)\,d\tau,
\end{aligned}
$$

where the first step follows from (21) and (25), and the third step follows from integrating (19) from zero to τ and using (21) and (25), we can write (A.1) as

$$
\begin{aligned}
&\kappa_r^*\bar{r}^*\left[1+a\sigma_r^2\int_0^\infty \alpha(\tau)\left[\int_0^\tau A_r(u)\,du\right]A_r(\tau)\,d\tau\right]\\
&\quad=\kappa_r^*\bar{r}\left[1+a\sigma_r^2\int_0^\infty \alpha(\tau)\left[\int_0^\tau A_r(u)\,du\right]A_r(\tau)\,d\tau\right]-\bar{r}a\sigma_r^2\int_0^\infty \alpha(\tau)\tau A_r(\tau)\,d\tau\\
&\qquad+a\sigma_r^2\int_0^\infty \theta_0(\tau)A_r(\tau)\,d\tau+\frac{a\sigma_r^4}{2}\int_0^\infty \alpha(\tau)\left[\int_0^\tau A_r(u)^2\,du\right]A_r(\tau)\,d\tau.
\end{aligned}
\tag{A.2}
$$

Equations (21) and (A.2) imply (26). *Q.E.D.*

PROOF OF PROPOSITION 2: Taking expectations conditional on time t in (8), we find

$$
\begin{aligned}
&dE_t(r_{t+\tau})=\kappa_r\big(\bar{r}-E_t(r_{t+\tau})\big)\,d\tau\\
&\quad\Rightarrow\quad E_t(r_{t+\tau})=\big(1-e^{-\kappa_r\tau}\big)\bar{r}+e^{-\kappa_r\tau}r_t.
\end{aligned}
\tag{A.3}
$$

Equation (A.3) implies

$$
\frac{\partial E_t(r_{t+\tau})}{\partial r_t}=e^{-\kappa_r\tau}.
\tag{A.4}
$$

Equation (27) likewise implies

$$
\frac{\partial f_t^{(\tau)}}{\partial r_t}=A_r'(\tau)=e^{-\kappa_r^*\tau},
\tag{A.5}
$$

where the second step follows from (21).

Equation (25) implies that if $a>0$ and $\alpha(\tau)>0$ in a positive-measure subset of $(0,T)$, then $\kappa_r^*>\kappa_r$. Since the right-hand side of (25) increases in a, σ_r^2, and $\alpha(\tau)$, and the difference between the left-hand side and the right-hand side increases in κ_r^*, κ_r^* increases in a, σ_r^2, and $\alpha(\tau)$. *Q.E.D.*

PROOF OF PROPOSITION 3: Equations (1), (2), and (10) imply that the dependent variable in (28) is

$$
\frac{1}{\Delta\tau}\big\{A_r(\tau)r_t+C(\tau)-\big[A_r(\tau-\Delta\tau)r_{t+\Delta\tau}+C(\tau-\Delta\tau)\big]-\big[A_r(\Delta\tau)r_t+C(\Delta\tau)\big]\big\}
$$

and the independent variable is

$$\frac{1}{\Delta\tau}\big\{A_r(\tau)r_t + C(\tau) - \big[A_r(\tau-\Delta\tau)r_t + C(\tau-\Delta\tau)\big] - \big[A_r(\Delta\tau)r_t + C(\Delta\tau)\big]\big\}.$$

Therefore, the FB regression coefficient is

$$\begin{aligned} b_{\mathrm{FB}} &= \frac{\mathbb{C}\mathrm{ov}\big\{\big[A_r(\tau) - A_r(\Delta\tau)\big]r_t - A_r(\tau-\Delta\tau)r_{t+\Delta\tau}, \big[A_r(\tau) - A_r(\tau-\Delta\tau) - A_r(\Delta\tau)\big]r_t\big\}}{\mathbb{V}\mathrm{ar}\big\{\big[A_r(\tau) - A_r(\tau-\Delta\tau) - A_r(\Delta\tau)\big]r_t\big\}} \\ &= \frac{\big[A_r(\tau) - A_r(\Delta\tau)\big]\mathbb{V}\mathrm{ar}(r_t) - A_r(\tau-\Delta\tau)\mathbb{C}\mathrm{ov}(r_{t+\Delta\tau}, r_t)}{\big[A_r(\tau) - A_r(\tau-\Delta\tau) - A_r(\Delta\tau)\big]\mathbb{V}\mathrm{ar}(r_t)}. \end{aligned} \tag{A.6}$$

Since (A.3) implies

$$\mathbb{C}\mathrm{ov}(r_{t+\Delta\tau}, r_t) = \mathbb{V}\mathrm{ar}(r_t)e^{-\kappa_r\Delta\tau}, \tag{A.7}$$

we can write (A.6) as

$$b_{\mathrm{FB}} = \frac{A_r(\tau) - A_r(\tau-\Delta\tau)e^{-\kappa_r\Delta\tau} - A_r(\Delta\tau)}{A_r(\tau) - A_r(\tau-\Delta\tau) - A_r(\Delta\tau)}.$$

Taking the limit $\Delta\tau \to 0$ and noting from (21) that $\frac{A_r(\Delta\tau)}{\Delta\tau} \to 1$, we find

$$b_{\mathrm{FB}} \to \frac{A_r'(\tau) + \kappa_r A_r(\tau) - 1}{A_r'(\tau) - 1} = \frac{(\kappa_r^* - \kappa_r)A_r(\tau)}{\kappa_r^* A_r(\tau)} = \frac{\kappa_r^* - \kappa_r}{\kappa_r^*}, \tag{A.8}$$

where the second step follows from (19) and (25). Since $\kappa_r^* > \kappa_r$ when $a > 0$ and $\alpha(\tau) > 0$ in a positive-measure subset of $(0, T)$, (A.8) implies $b_{\mathrm{FB}} > 0$. Since κ_r^* increases in a, σ_r^2, and $\alpha(\tau)$, (A.8) implies that b_{FB} increases in the same variables.

Equations (1) and (10) imply that the dependent variable in (29) is

$$\frac{A_r(\tau-\Delta\tau)r_{t+\Delta\tau} + C(\tau-\Delta\tau)}{\tau-\Delta\tau} - \frac{A_r(\tau)r_t + C(\tau)}{\tau}$$

and the independent variable is

$$\frac{\Delta\tau}{\tau-\Delta\tau}\left[\frac{A_r(\tau)r_t + C(\tau)}{\tau} - \frac{A_r(\Delta\tau)r_t + C(\Delta\tau)}{\Delta\tau}\right].$$

Therefore, the CS regression coefficient is

$$\begin{aligned} b_{\mathrm{CS}} &= \frac{\mathbb{C}\mathrm{ov}\left\{\frac{A_r(\tau-\Delta\tau)}{\tau-\Delta\tau}r_{t+\Delta\tau} - \frac{A_r(\tau)}{\tau}r_t, \frac{\Delta\tau}{\tau-\Delta\tau}\left[\frac{A_r(\tau)}{\tau} - \frac{A_r(\Delta\tau)}{\Delta\tau}\right]r_t\right\}}{\mathbb{V}\mathrm{ar}\left\{\frac{\Delta\tau}{\tau-\Delta\tau}\left[\frac{A_r(\tau)}{\tau} - \frac{A_r(\Delta\tau)}{\Delta\tau}\right]r_t\right\}} \\ &= \frac{\frac{A_r(\tau-\Delta\tau)}{\tau-\Delta\tau}\mathbb{C}\mathrm{ov}(r_{t+\Delta\tau}, r_t) - \frac{A_r(\tau)}{\tau}\mathbb{V}\mathrm{ar}(r_t)}{\frac{\Delta\tau}{\tau-\Delta\tau}\left[\frac{A_r(\tau)}{\tau} - \frac{A_r(\Delta\tau)}{\Delta\tau}\right]\mathbb{V}\mathrm{ar}(r_t)}. \end{aligned} \tag{A.9}$$

Using (A.7), we can write (A.9) as

$$b_{\mathrm{CS}} = \frac{\dfrac{A_r(\tau-\Delta\tau)}{\tau-\Delta\tau}e^{-\kappa_r\Delta\tau} - \dfrac{A_r(\tau)}{\tau}}{\dfrac{\Delta\tau}{\tau-\Delta\tau}\left[\dfrac{A_r(\tau)}{\tau} - \dfrac{A_r(\Delta\tau)}{\Delta\tau}\right]}.$$

Taking the limit $\Delta\tau \to 0$, we find

$$\begin{aligned} b_{\mathrm{CS}} &\to \frac{\dfrac{A_r(\tau)}{\tau} - \left[A_r'(\tau) + \kappa_r A_r(\tau)\right]}{\dfrac{A_r(\tau)}{\tau} - 1} \\ &= 1 - \frac{A_r'(\tau) + \kappa_r A_r(\tau) - 1}{\dfrac{A_r(\tau)}{\tau} - 1} \\ &= 1 - \frac{\left(\kappa_r^* - \kappa_r\right) A_r(\tau)\tau}{\tau - A_r(\tau)}, \end{aligned} \tag{A.10}$$

where the third step follows from (19) and (25). Since $\kappa_r^* > \kappa_r$ when $a > 0$ and $\alpha(\tau) > 0$ in a positive-measure subset of $(0, T)$, (A.10) implies $b_{\mathrm{CS}} < 1$. Since

$$\frac{A_r(\tau)\tau}{\tau - A_r(\tau)} = \frac{1 - e^{-\kappa_r^*\tau}}{\kappa_r^*\left(1 - \dfrac{1 - e^{-\kappa_r^*\tau}}{\kappa_r^*\tau}\right)},$$

(A.10) implies that b_{CS} increases in τ if the function

$$K(x) \equiv \frac{1 - \dfrac{1 - e^{-x}}{x}}{1 - e^{-x}} = \frac{1}{1 - e^{-x}} - \frac{1}{x}$$

is increasing for $x > 0$. The derivative $K'(x)$ has the same sign as the function

$$\hat{K}(x) \equiv 1 - e^{-x} - xe^{-\frac{x}{2}}.$$

The function $\hat{K}(x)$ is equal to zero for $x = 0$, and its derivative $\hat{K}'(x)$ has the same sign as $e^{-\frac{x}{2}} - 1 + \frac{x}{2}$ which is positive for all x. Therefore, $\hat{K}(x) > 0$ for $x > 0$, and $K(x)$ is increasing. *Q.E.D.*

PROOF OF PROPOSITION 4: The argument in the text shows that $\Delta y_t^{(\tau)} = \kappa_r^* \Delta\bar{r}^* \frac{\int_0^\tau A_r(u)\,du}{\tau}$ and $\Delta\bar{r}^*$ has the same sign as $a\sigma_r^2 \int_0^\infty \Delta\theta_0(\tau) A_r(\tau)\,d\tau$. Hence, when $a > 0$, the change $\Delta\theta_0(\tau)$ raises all yields if $\int_0^\infty \Delta\theta_0(\tau) A_r(\tau)\,d\tau > 0$ and lowers them otherwise. The relative effect across maturities is

$$\frac{\Delta y_t^{(\tau_2)}}{\Delta y_t^{(\tau_1)}} = \frac{\dfrac{\displaystyle\int_0^{\tau_2} A_r(u)\,du}{\tau_2}}{\dfrac{\displaystyle\int_0^{\tau_1} A_r(u)\,du}{\tau_1}}$$

and is independent of $\Delta\theta_0(\tau)$. Since the function $A_r(\tau)$ increases in τ, the function $\frac{\int_0^\tau A_r(u)\,du}{\tau}$ also increases, and hence the relative effect across maturities is larger than 1 for $\tau_1 < \tau_2$. *Q.E.D.*

The proof of Lemma 2 was given in the text.

PROOF OF LEMMA 3: Using the diagonalization

$$M = P^{-1}\,\mathrm{Diag}(\nu_1, \nu_2, \dots, \nu_{K+1})P,$$

where $\mathrm{Diag}(z_1, z_2, \dots, z_N)$ is the $N \times N$ diagonal matrix with elements $(z_1, z_2, \dots, z_N)$, and multiplying the ODE system (36) from the left by P, we can write it as

$$PA'(\tau) + \mathrm{Diag}(\nu_1, \nu_2, \dots, \nu_{K+1})PA(\tau) - P\mathcal{E} = 0. \tag{A.11}$$

Integrating (A.11) with the initial condition $A(0) = 0$ yields

$$PA(\tau) = \mathrm{Diag}\left(\frac{1-e^{-\nu_1\tau}}{\nu_1}, \frac{1-e^{-\nu_2\tau}}{\nu_2}, \dots, \frac{1-e^{-\nu_{K+1}\tau}}{\nu_{K+1}}\right)P\mathcal{E}. \tag{A.12}$$

Using

$$\begin{aligned}
&\mathrm{Diag}\left(\frac{1-e^{-\nu_1\tau}}{\nu_1}, \frac{1-e^{-\nu_2\tau}}{\nu_2}, \dots, \frac{1-e^{-\nu_{K+1}\tau}}{\nu_{K+1}}\right)\\
&\quad = \frac{1-e^{-\nu_1\tau}}{\nu_1}\mathcal{I}_{K+1} + \mathrm{Diag}\left(0, \frac{1-e^{-\nu_2\tau}}{\nu_2} - \frac{1-e^{-\nu_1\tau}}{\nu_1}, \dots, \frac{1-e^{-\nu_{K+1}\tau}}{\nu_{K+1}} - \frac{1-e^{-\nu_1\tau}}{\nu_1}\right),
\end{aligned}$$

where $\mathcal{I}_N$ is the $N \times N$ identity matrix, we can write (A.12) as

$$\begin{aligned}
A(\tau) &= \frac{1-e^{-\nu_1\tau}}{\nu_1}\mathcal{E} + P^{-1}\,\mathrm{Diag}\left(0, \frac{1-e^{-\nu_2\tau}}{\nu_2} - \frac{1-e^{-\nu_1\tau}}{\nu_1}, \dots, \frac{1-e^{-\nu_{K+1}\tau}}{\nu_{K+1}} - \frac{1-e^{-\nu_1\tau}}{\nu_1}\right)P\mathcal{E}\\
\Rightarrow \quad \begin{bmatrix} A_r(\tau) \\ A_{\beta,1}(\tau) \\ \cdots \\ A_{\beta,K}(\tau) \end{bmatrix} &= \frac{1-e^{-\nu_1\tau}}{\nu_1}\begin{bmatrix} 1 \\ 0 \\ \cdots \\ 0 \end{bmatrix}\\
&\quad + P^{-1}\,\mathrm{Diag}\left(0, \frac{1-e^{-\nu_2\tau}}{\nu_2} - \frac{1-e^{-\nu_1\tau}}{\nu_1}, \dots, \frac{1-e^{-\nu_{K+1}\tau}}{\nu_{K+1}} - \frac{1-e^{-\nu_1\tau}}{\nu_1}\right)\\
&\quad \times P\begin{bmatrix} 1 \\ 0 \\ \cdots \\ 0 \end{bmatrix}.
\end{aligned} \tag{A.13}$$

Equation (A.13) implies (39) and (40). Integrating (38) with the initial condition $C(0) = 0$ yields (41). *Q.E.D.*

We next derive the system of equations in the Laplace transforms. We consider the general case where there are K demand factors. We assume $\alpha(\tau) = \alpha e^{-\delta_\alpha \tau}$ and $\theta_k(\tau) = \sum_{n=1}^{N} \theta_{k,n} e^{-\delta_{\theta n}\tau}$, where $N \geq 1$, $(\alpha, \delta_\alpha, \{\theta_{k,n}\}_{k=1,\dots,K,n=1,\dots,N}, \{\delta_{\theta_n}\}_{n=1,\dots,N})$ are scalars and $(\alpha, \delta_\alpha, \{\delta_{\theta_n}\}_{n=1,\dots,N})$ are positive. We set

$$I \equiv \int_0^\infty \alpha(\tau) A(\tau)\, d\tau,$$

$$J \equiv \int_0^\infty \alpha(\tau) A(\tau) A(\tau)^\top\, d\tau,$$

For $n = 1, \dots, N$, we set

$$I_n \equiv \int_0^\infty e^{-\delta_{\theta n}\tau} A(\tau)\, d\tau$$

and denote by Θ_n the $1 \times (K+1)$ vector $(0, \theta_{1,n}, \dots, \theta_{K,n})$. Since the vectors $(I, I_1, \dots, I_N)$ are $(K+1) \times 1$, and since the matrix J is $(K+1) \times (K+1)$ and symmetric, there are a total of

$$K + 1 + \frac{(K+1)(K+2)}{2} + (K+1)N = (K+1)\left(\frac{K}{2} + N + 2\right)$$

distinct elements. These elements are Laplace transforms of the functions $(A_r(\tau), \{A_{\beta,k}(\tau)\}_{k=1,\dots,K})$ and of those functions' pairwise products. Using $(J, \{I_n\}_{n=1,\dots,N}, \{\Theta_n\}_{n=1,\dots,N})$, we can write the matrix M defined in (37) as

$$M \equiv \Gamma^\top - a \int_0^\infty \left(\sum_{n=1}^{N} \Theta_n^\top I_n^\top - J\right) \Sigma\Sigma^\top. \tag{A.14}$$

LEMMA A.1: *Suppose that* $\alpha(\tau) = \alpha e^{-\delta_\alpha \tau}$ *and* $\theta_k(\tau) = \sum_{j=1}^{N} \theta_{k,n} e^{-\delta_{\theta n}\tau}$, *where* $N \geq 1$, $(\alpha, \delta_\alpha, \{\theta_{k,n}\}_{k=1,\dots,K,n=1,\dots,N}, \{\delta_{\theta_n}\}_{n=1,\dots,N})$ *are scalars, and* $(\alpha, \delta_\alpha, \{\delta_{\theta_n}\}_{n=1,\dots,N})$ *are positive. The* $(K+1)(\frac{K}{2} + N + 2)$ *elements of* $(I, J, \{I_n\}_{n=1,\dots,N})$ *solve the system*

$$(\delta_\alpha I_{K+1} + M) I = \frac{\alpha}{\delta_\alpha} \mathcal{E}, \tag{A.15}$$

$$(\delta_{\theta_n} I_{K+1} + M) I_n = \frac{1}{\delta_{\theta_n}} \mathcal{E}, \tag{A.16}$$

for $n = 1, \dots, N$, *and*

$$(\delta_\alpha I_{K+1} + M) J + J M^\top = \mathcal{E} I^\top + I \mathcal{E}^\top. \tag{A.17}$$

PROOF: To derive (A.15), we multiply the ODE system (36) by $\alpha(\tau)$ and integrate from zero to infinity. This yields

$$\int_0^\infty \alpha(\tau) A'(\tau)\, d\tau + MI - \left[\int_0^\infty \alpha(\tau)\, d\tau\right] \mathcal{E} = 0. \tag{A.18}$$

Integration by parts implies

$$\int_0^\infty \alpha(\tau)A'(\tau)\,d\tau = \big[\alpha(\tau)A(\tau)\big]_0^\infty - \int_0^\infty \alpha'(\tau)A(\tau)\,d\tau$$
$$= \lim_{\tau\to\infty}\alpha(\tau)A(\tau) - \alpha(0)A(0) + \delta_\alpha\int_0^\infty \alpha(\tau)A(\tau)\,d\tau$$
$$= \lim_{\tau\to\infty}\alpha(\tau)A(\tau) + \delta_\alpha\int_0^\infty \alpha(\tau)A(\tau)\,d\tau,$$

where the second step follows from $\alpha'(\tau) = -\delta_\alpha\alpha(\tau)$ and the third step follows from $A(0) = 0$. Assuming $\lim_{\tau\to\infty}\alpha(\tau)A(\tau) = 0$, a property that is required for the matrix M to be finite (and that holds for the solution in Theorem 1, as we show at the end of that theorem's proof), we find

$$\int_0^\infty \alpha(\tau)A'(\tau)\,d\tau = \delta_\alpha\int_0^\infty \alpha(\tau)A(\tau)\,d\tau = \delta_\alpha I. \tag{A.19}$$

Using (A.18), (A.19), and $a(\tau) = \alpha e^{-\delta_\alpha\tau}$, we find (A.15).

To derive (A.16), we likewise multiply the ODE system (36) by $e^{-\delta_{\theta n}\tau}$ and integrate from zero to infinity. This yields

$$\int_0^\infty e^{-\delta_{\theta_n}\tau}A'(\tau)\,d\tau + MI_n - \left[\int_0^\infty e^{-\delta_{\theta_n}\tau}\,d\tau\right]\mathcal{E} = 0. \tag{A.20}$$

Integration by parts and a zero limit at infinity imply

$$\int_0^\infty e^{-\delta_{\theta_n}\tau}A'(\tau)\,d\tau = \delta_{\theta_n}\int_0^\infty e^{-\delta_{\theta_n}\tau}A(\tau)\,d\tau = \delta_{\theta_n}I_n. \tag{A.21}$$

Using (A.20) and (A.21), we find (A.16).

To derive (A.17), we multiply the ODE system (36) from the left by $\alpha(\tau)A(\tau)^\top$, add to the resulting $(K+1)\times(K+1)$ matrix its transpose, and integrate from zero to infinity. This yields

$$\int_0^\infty \alpha(\tau)\big[A'(\tau)A(\tau)^\top + A(\tau)A'(\tau)^\top\big]\,d\tau + MJ + JM^\top - \mathcal{E}I^\top - I\mathcal{E}^\top = 0. \tag{A.22}$$

Integration by parts and a zero limit at infinity imply

$$\int_0^\infty \alpha(\tau)\big[A'(\tau)A(\tau)^\top + A(\tau)A'(\tau)^\top\big]\,d\tau$$
$$= \delta_\alpha\int_0^\infty \alpha(\tau)A(\tau)A(\tau)^\top d\tau = \delta_\alpha J. \tag{A.23}$$

Using (A.22) and (A.23), we find (A.17).

The total number of equations is $(K+1)(\frac{K}{2}+N+2)$, which is the same as the number of unknown Laplace transforms: the vector equation (A.15) yields $K+1$ scalar equations, the vector equations (A.16) for $n = 1, \dots, N$ yield $(K+1)N$ scalar equations, and the matrix equation (A.17) yields $\frac{(K+1)(K+2)}{2}$ scalar equations because the matrices in it are symmetric. *Q.E.D.*

PROOF OF THEOREM 1: The theorem specializes Lemma A.1 to the case $K=1$, $N=2$, $\theta_{11}=-\theta_{12}=\theta$, $\delta_{\theta_1}=\delta_\alpha$, $\delta_{\theta_2}=\delta_\theta$, $\Gamma=\text{Diag}(\kappa_r,\kappa_\beta)$, and $\Sigma=\text{Diag}(\sigma_r^2,\sigma_\beta^2)$. Since $K=1$ and $N=2$, there are nine unknown Laplace transforms, which reduce to seven because $\delta_{\theta_1}=\delta_\alpha$ implies $I_1=\frac{I}{\alpha}$. Setting $I\equiv(I_r,I_\beta)^\top$, $I_2\equiv(I_{r,2},I_{\beta,2})^\top$, and

$$J\equiv\begin{bmatrix} I_{r,r} & I_{r,\beta} \\ I_{r,\beta} & I_{\beta,\beta}\end{bmatrix},$$

the seven unknown Laplace transforms are $(I_r,I_\beta,I_{r,2},I_{\beta,2},I_{r,r},I_{r,\beta},I_{\beta,\beta})$. Setting

$$\Delta I_{r,\theta}\equiv\theta\left(\frac{I_r}{\alpha}-I_{r,2}\right)-I_{r,\beta}, \tag{A.24}$$

$$\Delta I_{\beta,\theta}\equiv\theta\left(\frac{I_\beta}{\alpha}-I_{\beta,2}\right)-I_{\beta,\beta}, \tag{A.25}$$

we can write the matrix M given by (A.14) as

$$\begin{bmatrix} \kappa_r+a\sigma_r^2I_{r,r} & a\sigma_\beta^2 I_{r,\beta} \\ -a\sigma_r^2\Delta I_{r,\theta} & \kappa_\beta-a\sigma_\beta^2\Delta I_{\beta,\theta}\end{bmatrix}. \tag{A.26}$$

The vector equation (A.15) yields the two scalar equations

$$\left(\delta_\alpha+\kappa_r+a\sigma_r^2I_{r,r}\right)I_r+a\sigma_\beta^2I_{r,\beta}I_\beta=\frac{\alpha}{\delta_\alpha}, \tag{A.27}$$

$$-a\sigma_r^2\Delta I_{r,\theta}I_r+\left(\delta_\alpha+\kappa_\beta-a\sigma_\beta^2\Delta I_{\beta,\theta}\right)I_\beta=0. \tag{A.28}$$

The vector equation (A.16) yields the two scalar equations

$$\left(\delta_\theta+\kappa_r+a\sigma_r^2I_{r,r}\right)I_{r,2}+a\sigma_\beta^2I_{r,\beta}I_{\beta,2}=\frac{1}{\delta_\theta}, \tag{A.29}$$

$$-a\sigma_r^2\Delta I_{r,\theta}I_{r,2}+\left(\delta_\theta+\kappa_\beta-a\sigma_\beta^2\Delta I_{\beta,\theta}\right)I_{\beta,2}=0. \tag{A.30}$$

The matrix equation (A.17) yields the three scalar equations

$$\left(\frac{\delta_\alpha}{2}+\kappa_r+a\sigma_r^2I_{r,r}\right)I_{r,r}+a\sigma_\beta^2I_{r,\beta}^2=I_r, \tag{A.31}$$

$$\left(\delta_\alpha+\kappa_r+\kappa_\beta+a\sigma_r^2I_{r,r}-a\sigma_\beta^2\Delta I_{\beta,\theta}\right)I_{r,\beta}+a\sigma_\beta^2I_{r,\beta}I_{\beta,\beta}-a\sigma_r^2\Delta I_{r,\theta}I_{r,r}=I_\beta, \tag{A.32}$$

$$-a\sigma_r^2\Delta I_{r,\theta}I_{r,\beta}+\left(\frac{\delta_\alpha}{2}+\kappa_\beta-a\sigma_\beta^2\Delta I_{\beta,\theta}\right)I_{\beta,\beta}=0. \tag{A.33}$$

Equations (A.27)–(A.32) constitute a system of seven equations in the seven unknowns $(I_r,I_\beta,I_{r,r},I_{r,\beta},I_{\beta,\beta},I_{r,2},I_{\beta,2})$. The rest of the proof, which is in the Full Appendix, available at http://personal.lse.ac.uk/vayanos/Papers/PHMTSIR_ECMAf.pdf, shows that this system has a solution. *Q.E.D.*

PROOF OF PROPOSITION 5: Using $K = 1$ and (A.26), we can write the system (36) as

$$A_r'(\tau) + \left(\kappa_r + a\sigma_r^2 I_{r,r}\right)A_r(\tau) + a\sigma_\beta^2 I_{r,\beta}A_\beta(\tau) - 1 = 0, \tag{A.34}$$

$$A_\beta'(\tau) - a\sigma_r^2 \Delta I_{r,\theta}A_r(\tau) + \left(\kappa_\beta - a\sigma_\beta^2 \Delta I_{\beta,\theta}\right)A_\beta(\tau) = 0, \tag{A.35}$$

and the solution to that system, given in Lemma 3, as

$$A_r(\tau) = \frac{1-e^{-\nu_1\tau}}{\nu_1} + \phi_r\left(\frac{1-e^{-\nu_2\tau}}{\nu_2} - \frac{1-e^{-\nu_1\tau}}{\nu_1}\right), \tag{A.36}$$

$$A_\beta(\tau) = \phi_\beta\left(\frac{1-e^{-\nu_2\tau}}{\nu_2} - \frac{1-e^{-\nu_1\tau}}{\nu_1}\right). \tag{A.37}$$

Equations (A.34) and (A.35), together with the initial conditions $A_r(0) = A_\beta(0) = 0$, imply $A_r'(0) = 1$ and $A_\beta'(0) = 0$. Differentiating (A.35) at zero and using $\Delta I_{r,\theta} > 0$, which follows from $M_{2,1} < 0$ and (A.26), we find $A_\beta''(0) > 0$. Hence, $A_r(\tau) > 0$, $A_\beta'(\tau) > 0$, and $A_\beta(\tau) > 0$ for small τ.

Suppose that the two eigenvalues of M are real and without loss of generality set $\nu_1 > \nu_2$. Since the function $(\nu, \tau) \longrightarrow \frac{1-e^{-\nu\tau}}{\nu}$ decreases in ν, the term in parentheses in (A.37) is positive. Since, in addition, $A_\beta(\tau) > 0$ for small τ, $\phi_\beta > 0$, and hence, $A_\beta(\tau) > 0$ for all τ. Since

$$A_\beta'(\tau) = \phi_\beta\left(e^{-\nu_2\tau} - e^{-\nu_1\tau}\right)$$

and $\phi_\beta > 0$, $A_\beta'(\tau) > 0$. Since

$$\begin{aligned}\frac{A_r(\tau)}{A_\beta(\tau)} &= \frac{\dfrac{1-e^{-\nu_1\tau}}{\nu_1}}{\phi_\beta\left(\dfrac{1-e^{-\nu_2\tau}}{\nu_2} - \dfrac{1-e^{-\nu_1\tau}}{\nu_1}\right)} + \frac{\phi_r}{\phi_\beta} \\ &= \frac{1}{\phi_\beta\left(\dfrac{\nu_1}{\nu_2}\dfrac{1-e^{-\nu_2\tau}}{1-e^{-\nu_1\tau}} - 1\right)} + \frac{\phi_r}{\phi_\beta},\end{aligned}$$

and the function $(\nu_1, \nu_2, \tau) \longrightarrow \frac{1-e^{-\nu_2\tau}}{1-e^{-\nu_1\tau}}$ increases in τ because its derivative has the same sign as $\frac{e^{\nu_1\tau}-1}{\nu_1} - \frac{e^{\nu_2\tau}-1}{\nu_2}$, $[\frac{A_r(\tau)}{A_\beta(\tau)}]' < 0$. Since

$$A_r'(\tau) = e^{-\nu_1\tau} + \phi_r\left(e^{-\nu_2\tau} - e^{-\nu_1\tau}\right),$$

the sign of $A_r'(\tau)$ can change at most once. Hence, $A_r'(\tau) > 0$ for $\tau \in (0, \bar{\tau}')$ and $A_r'(\tau) < 0$ for $\tau \in (\bar{\tau}', \infty)$, where $\bar{\tau}'$ is a threshold in $(0, \infty]$. The function $A_r(\tau)$ has the same behavior for a different threshold $\bar{\tau}$.

When $a \approx 0$, $A_r(\tau) > 0$ because Lemma A.2 implies $\phi_r \approx 0$, $\nu_1 \approx \kappa_r > 0$, and $\nu_2 \approx \kappa_\beta > 0$. When $\alpha(\tau) = 0$, $I_{r,r} = I_{r,\beta} = 0$ and, hence, (A.34) implies

$A_r(\tau) = \frac{1-e^{-\kappa_r\tau}}{\kappa_r} > 0$. In both cases, $\bar{\tau} = \infty$. When $a \approx \infty$, Lemma A.2 implies that for τ bounded away from zero,

$$\begin{aligned} A_r(\tau) &\approx \frac{1}{a^{\frac{1}{3}}}\left(\frac{1}{\overline{n}_1} + \overline{c}_r \frac{1-e^{-\overline{\nu}_2\tau}}{\overline{\nu}_2}\right) \\ &= \frac{1}{a^{\frac{1}{3}}\overline{n}_1}\left(1 - \frac{\displaystyle\int_0^\infty \alpha(\tau')\frac{1-e^{-\overline{\nu}_2\tau'}}{\overline{\nu}_2}\,d\tau'}{\displaystyle\int_0^\infty \alpha(\tau')\left(\frac{1-e^{-\overline{\nu}_2\tau'}}{\overline{\nu}_2}\right)^2 d\tau'}\frac{1-e^{-\overline{\nu}_2\tau}}{\overline{\nu}_2}\right) \\ &= \frac{1}{a^{\frac{1}{3}}\overline{n}_1}\frac{\displaystyle\int_0^\infty \alpha(\tau')\left(1-e^{-\overline{\nu}_2\tau'}\right)\left(e^{-\overline{\nu}_2\tau} - e^{-\overline{\nu}_2\tau'}\right)d\tau'}{\displaystyle\int_0^\infty \alpha(\tau')\left(1-e^{-\overline{\nu}_2\tau'}\right)^2 d\tau'}. \end{aligned}$$

Since this is negative for τ close to ∞, $\bar{\tau} < \infty$.

Suppose that the two eigenvalues of M are complex. Since they are conjugates, we set $\nu_1 = \mu + i\xi$ and $\nu_2 = \mu - i\xi$ for real numbers (μ, ξ). Equations (A.36) and (A.37) imply that $(A_r(\tau), A_\beta(\tau))$ takes the form

$$A_r(\tau) = \phi_{r,0} + \phi_{r,1}e^{-\mu\tau}\cos(\xi\tau) + \phi_{r,2}e^{-\mu\tau}\sin(\xi\tau), \tag{A.38}$$

$$A_\beta(\tau) = \phi_{\beta,0} + \phi_{\beta,1}e^{-\mu\tau}\cos(\xi\tau) + \phi_{\beta,2}e^{-\mu\tau}\sin(\xi\tau) \tag{A.39}$$

for real numbers $\{\phi_{j,n}\}_{j=r,\beta,n=0,1,2}$. Since the initial conditions $A_r(0) = A_\beta(0) = 0$ imply $\phi_{j,0} + \phi_{j,1} = 0$ for $j = r, \beta$, condition $A_r'(0) = 1$ implies $-\phi_{r,1}\mu + \phi_{r,2}\xi = 1$, and condition $A_\beta'(0) = 0$ implies $-\phi_{\beta,1}\mu + \phi_{\beta,2}\xi = 0$, we can write (A.38) and (A.39) as

$$A_r(\tau) = \phi_{r,0}\left[1 - \frac{\mu}{\xi}e^{-\mu\tau}\sin(\xi\tau) - e^{-\mu\tau}\cos(\xi\tau)\right] + \frac{1}{\xi}e^{-\mu\tau}\sin(\xi\tau), \tag{A.40}$$

$$A_\beta(\tau) = \phi_{\beta,0}\left[1 - \frac{\mu}{\xi}e^{-\mu\tau}\sin(\xi\tau) - e^{-\mu\tau}\cos(\xi\tau)\right]. \tag{A.41}$$

Differentiating (A.40) and (A.41), we find

$$A_r'(\tau) = \phi_{r,0}\frac{\mu^2+\xi^2}{\xi}e^{-\mu\tau}\sin(\xi\tau) + e^{-\mu\tau}\left[\cos(\xi\tau) - \frac{\mu}{\xi}\sin(\xi\tau)\right], \tag{A.42}$$

$$A_\beta'(\tau) = \phi_{\beta,0}\frac{\mu^2+\xi^2}{\xi}e^{-\mu\tau}\sin(\xi\tau). \tag{A.43}$$

Since $A_\beta'(\tau) > 0$ for small τ, $\phi_{\beta,0} > 0$, and hence, $A_\beta'(\tau) > 0$ for $\tau \in (0, \frac{\pi}{|\xi|})$. The derivative $[\frac{A_r(\tau)}{A_\beta(\tau)}]'$ has the same sign as

$$\begin{aligned} &A_r'(\tau)A_\beta(\tau) - A_r(\tau)A_\beta'(\tau) \\ &\quad = e^{-\mu\tau}\left[\cos(\xi\tau) - \frac{\mu}{\xi}\sin(\xi\tau)\right]\phi_{\beta,0}\left[1 - \frac{\mu}{\xi}e^{-\mu\tau}\sin(\xi\tau) - e^{-\mu\tau}\cos(\xi\tau)\right] \end{aligned}$$

$$- \frac{1}{\xi} e^{-\mu\tau} \sin(\xi\tau) \phi_{\beta,0} \frac{\mu^2+\xi^2}{\xi} e^{-\mu\tau} \sin(\xi\tau)$$

$$= \phi_{\beta,0} e^{-\mu\tau} \left[\cos(\xi\tau) - \frac{\mu}{\xi} \sin(\xi\tau) - e^{-\mu\tau} \right], \tag{A.44}$$

where the second step follows from (A.40)–(A.43) and the third step follows by rearranging. Since $\phi_{\beta,0} > 0$, $[\frac{A_r(\tau)}{A_\beta(\tau)}]'$ is negative if the term in brackets in (A.44) is negative. That term is concave in μ and is maximized for μ given by

$$-\frac{1}{\xi} \sin(\xi\tau) + \tau e^{-\mu\tau} = 0 \quad \Leftrightarrow \quad e^{-\mu\tau} = \frac{\sin(\xi\tau)}{\xi\tau}.$$

The maximum is

$$\cos(\xi\tau) - \frac{\sin(\xi\tau)}{\xi\tau} \left[1 - \log\left(\frac{\sin(\xi\tau)}{\xi\tau} \right) \right] = H(\xi\tau) \frac{\sin(\xi\tau)}{\xi\tau}, \tag{A.45}$$

where

$$H(x) \equiv \frac{x\cos(x)}{\sin(x)} - 1 + \log\left(\frac{\sin(x)}{x} \right).$$

The function $H(x)$ is equal to zero for $x = 0$, and its derivative is

$$H'(x) = -\frac{x}{\sin^2(x)} + \frac{\cos(x)}{\sin(x)} + \frac{\frac{x\cos(x)-\sin(x)}{x^2}}{\frac{\sin(x)}{x}} = -\frac{x^2 - 2x\cos(x)\sin(x) + \sin^2(x)}{x\sin^2(x)}.$$

Since

$$x^2 - 2x\cos(x)\sin(x) + \sin^2(x) > x^2 - 2\left|x\sin(x)\right| + \sin^2(x) = \left(|x| - \left|\sin(x)\right|\right)^2 > 0$$

for $x \neq 0$, $H'(x) > 0$ for $x < 0$ and $H'(x) < 0$ for $x > 0$. Since, in addition, $H(0) = 0$, then $H(x) < 0$. Hence, the maximum (A.45) is negative for $\tau \in (0, \frac{\pi}{|\xi|})$, and so is $[\frac{A_r(\tau)}{A_\beta(\tau)}]'$. This establishes the results in the proposition for $A'_\beta(\tau)$ and $\frac{A_r(\tau)}{A_\beta(\tau)}$, and for the threshold $\hat{\tau} = \frac{\pi}{|\xi|}$. The result for $A_\beta(\tau)$ and for a threshold $\bar{\bar{\tau}} > \hat{\tau}$ follows because $A_\beta(0) = 0$ and $A'_\beta(\tau) > 0$ for $\tau \in (0, \hat{\tau})$ imply $A_\beta(\tau) > 0$ for $\tau \in (0, \hat{\tau}]$.

If $\bar{\bar{\tau}} < \infty$, then $A_\beta(\bar{\bar{\tau}}) = 0$ and $A'_\beta(\bar{\bar{\tau}}) \leq 0$. If $A'_\beta(\bar{\bar{\tau}}) < 0$, then $\Delta I_{r,\theta} > 0$ and (A.35) imply $A_r(\bar{\bar{\tau}}) < 0$. If $A'_\beta(\bar{\bar{\tau}}) = 0$, then $\Delta I_{r,\theta} > 0$ and (A.35) imply $A_r(\bar{\bar{\tau}}) = 0$, and (A.35) implies $A'_r(\bar{\bar{\tau}}) = 1$. Hence, in both cases, $A_r(\tau) < 0$ for τ smaller than and close to $\bar{\bar{\tau}}$. This yields the result in the proposition for $A_r(\tau)$ and for a threshold $\bar{\tau} < \bar{\bar{\tau}}$. *Q.E.D.*

Lemma A.2 derives the asymptotic behavior of $(\nu_1, \nu_2, \phi_r, \phi_\beta)$ when $a \approx 0$ and $a \approx \infty$. To state and prove the lemma, we define the functions

$$F(\nu, \nu') \equiv \int_0^\infty \alpha(\tau) \frac{1-e^{-\nu\tau}}{\nu} \frac{1-e^{-\nu'\tau}}{\nu'} d\tau,$$

$$\hat{F}(\nu, \nu') \equiv F(\nu, \nu') - F(\nu, \nu),$$

$$\hat{\hat{F}}(\nu,\nu') \equiv F(\nu,\nu) + F(\nu',\nu') - 2F(\nu,\nu'),$$

$$G(\nu) \equiv \int_0^\infty \theta(\tau)\frac{1-e^{-\nu\tau}}{\nu}\,d\tau,$$

$$\hat{G}(\nu,\nu') \equiv G(\nu') - G(\nu).$$

We also note that the definitions of $(J, I_{r,r}, I_{r,\beta})$ imply

$$I_{r,r} = \int_0^\infty \alpha(\tau)A_r(\tau)^2\,d\tau, \tag{A.46}$$

$$I_{r,\beta} = \int_0^\infty \alpha(\tau)A_r(\tau)A_\beta(\tau)\,d\tau. \tag{A.47}$$

LEMMA A.2: *Suppose that there is one demand factor, the matrices* (Γ, Σ) *are diagonal, and* $\alpha(\tau)$ *and* $\frac{\theta(\tau)}{\tau}$ *have a positive and a finite limit, respectively, at* $\tau = 0$. *When* $a \approx 0$ *and* $a \approx \infty$, $(\nu_1, \nu_2, \phi_r, \phi_\beta)$ *are real, and their asymptotic behavior is as follows*:

- *When* $a \approx 0$, $(\nu_1, \nu_2, \phi_r, \phi_\beta) \approx (\kappa_r, \kappa_\beta, a^3\underline{c}_r, a\underline{c}_\beta)$, *where*

$$\underline{c}_r = -\frac{\underline{c}_\beta^2\sigma_\beta^2\hat{F}(\kappa_r,\kappa_\beta)}{\kappa_r - \kappa_\beta}, \tag{A.48}$$

$$\underline{c}_\beta = \frac{\sigma_r^2 G(\kappa_r)}{\kappa_r - \kappa_\beta}. \tag{A.49}$$

- *When* $a \approx \infty$, $(\nu_1, \nu_2, \phi_r, \phi_\beta) \approx (a^{\frac{1}{3}}\overline{n}_1, \overline{\nu}_2, a^{-\frac{1}{3}}\overline{c}_r, \overline{\phi}_\beta)$, *where*

$$\overline{n}_1 = \sigma_r^{\frac{2}{3}}\left[\int_0^\infty \alpha(\tau)\,d\tau - \frac{\left[\int_0^\infty \alpha(\tau)\frac{1-e^{-\overline{\nu}_2\tau}}{\overline{\nu}_2}\,d\tau\right]^2}{\int_0^\infty \alpha(\tau)\left(\frac{1-e^{-\overline{\nu}_2\tau}}{\overline{\nu}_2}\right)^2 d\tau}\right]^{\frac{1}{3}} > 0, \tag{A.50}$$

$$\overline{c}_r = -\frac{1}{\overline{n}_1}\frac{\int_0^\infty \alpha(\tau)\frac{1-e^{-\overline{\nu}_2\tau}}{\overline{\nu}_2}\,d\tau}{\int_0^\infty \alpha(\tau)\left(\frac{1-e^{-\overline{\nu}_2\tau}}{\overline{\nu}_2}\right)^2 d\tau} < 0, \tag{A.51}$$

$$\overline{\phi}_\beta = \frac{\int_0^\infty \theta(\tau)\frac{1-e^{-\overline{\nu}_2\tau}}{\overline{\nu}_2}\,d\tau}{\int_0^\infty \alpha(\tau)\left(\frac{1-e^{-\overline{\nu}_2\tau}}{\overline{\nu}_2}\right)^2 d\tau}, \tag{A.52}$$

and $\overline{\nu}_2$ *solves*

$$\frac{\int_0^\infty \theta(\tau)\frac{1-e^{-\nu_2\tau}}{\nu_2}\,d\tau}{\int_0^\infty \theta(\tau)\,d\tau} = \frac{\int_0^\infty \alpha(\tau)\left(\frac{1-e^{-\nu_2\tau}}{\nu_2}\right)^2 d\tau}{\int_0^\infty \alpha(\tau)\frac{1-e^{-\nu_2\tau}}{\nu_2}\,d\tau}. \tag{A.53}$$

PROOF: Substituting (A.36) and (A.37) into (A.34), and identifying terms in $\frac{1-e^{-\nu_1\tau}}{\nu_1}$ and $(\frac{1-e^{-\nu_2\tau}}{\nu_2} - \frac{1-e^{-\nu_1\tau}}{\nu_1})$, we find

$$\phi_r(\nu_1-\nu_2) - \nu_1 + \kappa_r + a\sigma_r^2 I_{r,r} = 0, \tag{A.54}$$

$$-\phi_r\nu_2 + \phi_r\big(\kappa_r + a\sigma_r^2 I_{r,r}\big) + \phi_\beta a\sigma_\beta^2 I_{r,\beta} = 0, \tag{A.55}$$

respectively. Using (A.54), we can write (A.55) as

$$\phi_r(1-\phi_r)(\nu_1-\nu_2) + \phi_\beta a\sigma_\beta^2 I_{r,\beta} = 0. \tag{A.56}$$

Substituting (A.36) and (A.37) into (A.35), and identifying terms, we find

$$\phi_\beta(\nu_1-\nu_2) - a\sigma_r^2\Delta I_{r,\theta} = 0, \tag{A.57}$$

$$-\phi_\beta\nu_2 - \phi_r\Delta I_{r,\theta} + \phi_\beta\big(\kappa_\beta - a\sigma_\beta^2\Delta I_{\beta,\theta}\big) = 0, \tag{A.58}$$

respectively. Using (A.57), we can write (A.58) as

$$-\nu_2 - \phi_r(\nu_1-\nu_2) + \kappa_\beta - a\sigma_\beta^2\Delta I_{\beta,\theta} = 0. \tag{A.59}$$

Equations (A.54), (A.56), (A.57), and (A.59) constitute a system of four equations in the four unknowns $(\nu_1, \nu_2, \phi_r, \phi_\beta)$. Substituting (A.36) and (A.37) into the definitions (A.46), (A.47), (A.24), and (A.25) of $(I_{r,r}, I_{r,\beta}, \Delta I_{r,\theta}, \Delta I_{\beta,\theta})$, we can write that system as

$$\phi_r(\nu_1-\nu_2) - \nu_1 + \kappa_r + a\sigma_r^2\big[F(\nu_1,\nu_1) + 2\phi_r\hat{F}(\nu_1,\nu_2) + \phi_r^2\hat{\hat{F}}(\nu_1,\nu_2)\big] = 0, \tag{A.60}$$

$$\phi_r(1-\phi_r)(\nu_1-\nu_2) + \phi_\beta^2 a\sigma_\beta^2\big[\hat{F}(\nu_1,\nu_2) + \phi_r\hat{\hat{F}}(\nu_1,\nu_2)\big] = 0, \tag{A.61}$$

$$\phi_\beta(\nu_1-\nu_2) - a\sigma_r^2\big[G(\nu_1) + \phi_r\hat{G}(\nu_1,\nu_2) - \phi_\beta\big[\hat{F}(\nu_1,\nu_2) + \gamma_r\hat{\hat{F}}(\nu_1,\nu_2)\big]\big] = 0, \tag{A.62}$$

$$-\nu_2 - \phi_r(\nu_1-\nu_2) + \kappa_\beta - \phi_\beta a\sigma_\beta^2\big[\hat{G}(\nu_1,\nu_2) - \phi_\beta\hat{\hat{F}}(\nu_1,\nu_2)\big] = 0. \tag{A.63}$$

Suppose that $a \approx 0$. Setting $(\phi_r, \phi_\beta) = (a^3c_r, ac_\beta)$, we can write (A.60)–(A.63) as

$$a^3c_r(\nu_1-\nu_2) - \nu_1 + \kappa_r + a\sigma_r^2\big[F(\nu_1,\nu_1) + 2a^3c_r\hat{F}(\nu_1,\nu_2) + a^6c_r^2\hat{\hat{F}}(\nu_1,\nu_2)\big] = 0, \tag{A.64}$$

$$c_r\big(1-a^3c_r\big)(\nu_1-\nu_2) + c_\beta^2\sigma_\beta^2\big[\hat{F}(\nu_1,\nu_2) + a^3c_r\hat{\hat{F}}(\nu_1,\nu_2)\big] = 0, \tag{A.65}$$

$$c_\beta(\nu_1-\nu_2) - \sigma_r^2\big[G(\nu_1) + a^3c_r\hat{G}(\nu_1,\nu_2) - ac_\beta\big[\hat{F}(\nu_1,\nu_2) + a^3c_r\hat{\hat{F}}(\nu_1,\nu_2)\big]\big] = 0, \tag{A.66}$$

$$-\nu_2 - a^3c_r(\nu_1-\nu_2) + \kappa_\beta - a^2c_\beta\sigma_\beta^2\big[\hat{G}(\nu_1,\nu_2) - ac_\beta\hat{\hat{F}}(\nu_1,\nu_2)\big] = 0. \tag{A.67}$$

The asymptotic behavior of $(\nu_1, \nu_2, \phi_r, \phi_\beta)$ is as in the lemma if (A.64)–(A.67) has a nonzero solution $(\nu_1, \nu_2, c_r, c_\beta)$ for $a = 0$. For $a = 0$, (A.64) implies $\nu_1 = \kappa_r$, (A.67) implies $\nu_2 = \kappa_\beta$, (A.66) implies $c_\beta = \underline{c}_\beta$, and (A.65) implies $c_r = \underline{c}_r$.

Suppose that $a \approx \infty$. Setting $(\nu_1, \phi_r) = (a^{\frac{1}{3}} n_1, a^{-\frac{1}{3}} c_r)$, we can write (A.60)–(A.63) as

$$\begin{aligned}
&a^{-\frac{2}{3}} c_r\big(a^{\frac{1}{3}} n_1 - \nu_2\big) - n_1 + a^{-\frac{1}{3}}\kappa_r \\
&\quad + a^{\frac{2}{3}}\sigma_r^2\big[F\big(a^{\frac{1}{3}} n_1, a^{\frac{1}{3}} n_1\big) + 2a^{-\frac{1}{3}} c_r \hat{F}\big(a^{\frac{1}{3}} n_1, \nu_2\big) + a^{-\frac{2}{3}} c_r^2 \hat{\hat{F}}\big(a^{\frac{1}{3}} n_1, \nu_2\big)\big] = 0, \qquad \text{(A.68)}\\
&a^{-1} c_r\big(1 - a^{-\frac{1}{3}} c_r\big)\big(a^{\frac{1}{3}} n_1 - \nu_2\big) \\
&\quad + a^{\frac{1}{3}}\phi_\beta^2\sigma_\beta^2\big[\hat{F}\big(a^{\frac{1}{3}} n_1, \nu_2\big) + a^{-\frac{1}{3}} c_r \hat{\hat{F}}\big(a^{\frac{1}{3}} n_1, \nu_2\big)\big] = 0, \qquad \text{(A.69)}\\
&a^{-\frac{2}{3}}\phi_\beta\big(a^{\frac{1}{3}} n_1 - \nu_2\big) - a^{\frac{1}{3}}\sigma_r^2\big[G\big(a^{\frac{1}{3}} n_1\big) + a^{-\frac{1}{3}} c_r \hat{G}\big(a^{\frac{1}{3}} n_1, \nu_2\big) \\
&\quad - \phi_\beta\big[\hat{F}\big(a^{\frac{1}{3}} n_1, \nu_2\big) + a^{-\frac{1}{3}} c_r \hat{\hat{F}}\big(a^{\frac{1}{3}} n_1, \nu_2\big)\big]\big] = 0, \qquad \text{(A.70)}\\
&a^{-1}\big[-\nu_2 - a^{-\frac{1}{3}} c_r\big(a^{\frac{1}{3}} n_1 - \nu_2\big) + \kappa_\beta\big] \\
&\quad - \phi_\beta\sigma_\beta^2\big[\hat{G}\big(a^{\frac{1}{3}} n_1, \nu_2\big) - \phi_\beta \hat{\hat{F}}\big(a^{\frac{1}{3}} n_1, \nu_2\big)\big] = 0. \qquad \text{(A.71)}
\end{aligned}$$

The asymptotic behavior of $(\nu_1, \nu_2, \phi_r, \phi_\beta)$ is as in the lemma if (A.68)–(A.71) has a nonzero solution $(n_1, \nu_2, c_r, \phi_\beta)$ for $a = \infty$. Noting that

$$\begin{aligned}
\lim_{a\to\infty} a^{\frac{2}{3}} F\big(a^{\frac{1}{3}} n_1, a^{\frac{1}{3}} n_1\big) &= \frac{1}{n_1^2}\int_0^\infty \alpha(\tau)\, d\tau,\\
\lim_{a\to\infty} a^{\frac{1}{3}} F\big(a^{\frac{1}{3}} n_1, \nu_2\big) &= \frac{1}{n_1}\int_0^\infty \alpha(\tau)\frac{1-e^{-\nu_2\tau}}{\nu_2}\, d\tau,\\
\lim_{a\to\infty} a^{\frac{1}{3}} G\big(a^{\frac{1}{3}} n_1\big) &= \frac{1}{n_1}\int_0^\infty \theta(\tau)\, d\tau,
\end{aligned}$$

we can write (A.68)–(A.71) for $a = \infty$ as

$$\begin{aligned}
&n_1 - \sigma_r^2\bigg[\frac{1}{n_1^2}\int_0^\infty \alpha(\tau)\, d\tau + 2c_r\frac{1}{n_1}\int_0^\infty \alpha(\tau)\frac{1-e^{-\nu_2\tau}}{\nu_2}\, d\tau \\
&\quad + c_r^2\int_0^\infty \alpha(\tau)\bigg(\frac{1-e^{-\nu_2\tau}}{\nu_2}\bigg)^2 d\tau\bigg] = 0, \qquad \text{(A.72)}\\
&\frac{1}{n_1}\int_0^\infty \alpha(\tau)\frac{1-e^{-\nu_2\tau}}{\nu_2}\, d\tau + c_r\int_0^\infty \alpha(\tau)\bigg(\frac{1-e^{-\nu_2\tau}}{\nu_2}\bigg)^2 d\tau = 0, \qquad \text{(A.73)}\\
&\frac{1}{n_1}\int_0^\infty \theta(\tau)\, d\tau + c_r\int_0^\infty \theta(\tau)\frac{1-e^{-\nu_2\tau}}{\nu_2}\, d\tau \\
&\quad - \phi_\beta\bigg[\frac{1}{n_1}\int_0^\infty \alpha(\tau)\frac{1-e^{-\nu_2\tau}}{\nu_2}\, d\tau + c_r\int_0^\infty \alpha(\tau)\bigg(\frac{1-e^{-\nu_2\tau}}{\nu_2}\bigg)^2 d\tau\bigg] = 0, \qquad \text{(A.74)}\\
&\int_0^\infty \theta(\tau)\frac{1-e^{-\nu_2\tau}}{\nu_2}\, d\tau - \phi_\beta\int_0^\infty \alpha(\tau)\bigg(\frac{1-e^{-\nu_2\tau}}{\nu_2}\bigg)^2 d\tau = 0. \qquad \text{(A.75)}
\end{aligned}$$

Equations (A.73) and (A.74) imply (A.53). Equation (A.53) has a solution $\bar{\nu}_2$. Indeed, when ν_2 goes to infinity, the left-hand side is

$$\frac{1}{\nu_2}\left[1-\frac{\displaystyle\int_0^\infty \theta(\tau)e^{-\nu_2\tau}\,d\tau}{\displaystyle\int_0^\infty \theta(\tau)\,d\tau}\right]=\frac{1}{\nu_2}\left[1+o\left(\frac{1}{\nu_2}\right)\right]$$

because $\frac{\theta(\tau)}{\tau}$ has a finite limit at zero, and the right-hand side is

$$\frac{1}{\nu_2}\left[1-\frac{\displaystyle\int_0^\infty \alpha(\tau)\big(1-e^{-\nu_2\tau}\big)e^{-\nu_2\tau}\,d\tau}{\displaystyle\int_0^\infty \alpha(\tau)\big(1-e^{-\nu_2\tau}\big)\,d\tau}\right]=\frac{1}{\nu_2}\left[1-\frac{\alpha(0)}{\nu_2\displaystyle\int_0^\infty \alpha(\tau)\,d\tau}+o\left(\frac{1}{\nu_2}\right)\right]$$

because $\alpha(\tau)$ has a positive limit at zero. Hence, the left-hand side exceeds the right-hand side. When $(\alpha(\tau),\{\theta_k(\tau)\}_{k=1,\dots,K})$ becomes zero for τ larger than a finite threshold T, and ν_2 goes to minus infinity, the left-hand side is

$$\frac{e^{-\nu_2 T}}{\nu_2}\frac{\displaystyle\int_0^\infty \theta(\tau)\big[e^{\nu_2 T}-e^{\nu_2(T-\tau)}\big]\,d\tau}{\displaystyle\int_0^\infty \theta(\tau)\,d\tau}=\frac{e^{-\nu_2 T}}{\nu_2^2}\frac{\theta(T)}{\displaystyle\int_0^\infty \theta(\tau)\,d\tau}+o\left(\frac{1}{\nu_2^2}\right)$$

and is smaller than the right-hand side, which is

$$\frac{e^{-\nu_2 T}}{\nu_2}\frac{\displaystyle\int_0^\infty \alpha(\tau)\big[e^{\nu_2 T}-e^{\nu_2(T-\tau)}\big]^2\,d\tau}{\displaystyle\int_0^\infty \alpha(\tau)\big[e^{\nu_2 T}-e^{\nu_2(T-\tau)}\big]\,d\tau}=\frac{e^{-\nu_2 T}}{-2\nu_2}+o\left(\frac{1}{\nu_2}\right).$$

Hence, a solution $\bar{\nu}_2\in(-\infty,\infty)$ to (A.53) exists. When $T=\infty$, $(\alpha(\tau),\theta(\tau))\approx(\alpha e^{-\delta_\alpha\tau},\theta e^{-\delta'_\alpha\tau})$ for τ large and for $0<\delta_\alpha\le\delta'_\alpha$. When ν_2 goes to $-\frac{\delta_\alpha}{2}$, the right-hand side goes to infinity, while the left-hand side remains finite. Hence, a solution $\bar{\nu}_2\in(-\frac{\delta_\alpha}{2},\infty)$ to (A.53) exists.

Using (A.73) to eliminate c_r in (A.72), we find $n_1=\bar{n}_1$. Equations (A.73) and (A.75) imply $c_r=\bar{c}_r$ and $\phi_\beta=\bar{\phi}_\beta$, respectively. The Cauchy–Schwarz inequality implies $\bar{n}_1>0$ and, hence, $\bar{c}_r<0$. *Q.E.D.*

PROOF OF PROPOSITION 6: Proceeding as in the proof of Proposition 3, we find that the FB regression coefficient is

$$\begin{aligned}b_{\mathrm{FB}}&=\frac{N_{\mathrm{FB},r}\,\mathbb{V}\mathrm{ar}(r_t)+N_{\mathrm{FB},\beta}\,\mathbb{V}\mathrm{ar}(\beta_t)}{\big[A_r(\tau)-A_r(\tau-\Delta\tau)-A_r(\Delta\tau)\big]^2\,\mathbb{V}\mathrm{ar}(r_t)+\big[A_\beta(\tau)-A_\beta(\tau-\Delta\tau)\big]^2\,\mathbb{V}\mathrm{ar}(\beta_t)}\\&=\frac{N_{\mathrm{FB},r}\dfrac{\sigma_r^2}{\kappa_r}+N_{\mathrm{FB},\beta}\dfrac{\sigma_\beta^2}{\kappa_\beta}}{\big[A_r(\tau)-A_r(\tau-\Delta\tau)-A_r(\Delta\tau)\big]^2\dfrac{\sigma_r^2}{\kappa_r}+\big[A_\beta(\tau)-A_\beta(\tau-\Delta\tau)\big]^2\dfrac{\sigma_\beta^2}{\kappa_\beta}},\end{aligned}\tag{A.76}$$

where

$$N_{\text{FB},j} = \big[A_j(\tau) - A_j(\tau-\Delta\tau)e^{-\kappa_j\Delta\tau} - A_j(\Delta\tau)\big]\big[A_j(\tau) - A_j(\tau-\Delta\tau) - A_j(\Delta\tau)\big]$$

for $j = r, \beta$. Taking the limit in (A.76) when $\Delta\tau \to 0$, and noting from (A.36) and (A.37) that $\frac{A_r(\Delta\tau)}{\Delta\tau} \to 1$ and $\frac{A_\beta(\Delta\tau)}{\Delta\tau} \to 0$, we find

$$b_{\text{FB}} = \frac{\big[A_r'(\tau) + \kappa_r A_r(\tau) - 1\big]\big[A_r'(\tau) - 1\big]\frac{\sigma_r^2}{\kappa_r} + \big[A_\beta'(\tau) + \kappa_\beta A_\beta(\tau)\big]A_\beta'(\tau)\frac{\sigma_\beta^2}{\kappa_\beta}}{\big[A_r'(\tau) - 1\big]^2\frac{\sigma_r^2}{\kappa_r} + A_\beta'(\tau)^2\frac{\sigma_\beta^2}{\kappa_\beta}}. \tag{A.77}$$

For $\tau < \min\{\bar{\tau}, \hat{\tau}\}$, $A_r(\tau) > 0$, $A_\beta(\tau) > 0$, and $A_\beta'(\tau) > 0$. Moreover, (A.34) implies

$$A_r'(\tau) + \kappa_r A_r(\tau) - 1 = -a\sigma_r^2 I_{r,r} A_r(\tau) - a\sigma_\beta^2 I_{r,\beta} A_\beta(\tau) \le 0, \tag{A.78}$$

$$A_r'(\tau) - 1 = -\big(\kappa_r + a\sigma_r^2 I_{r,r}\big)A_r(\tau) - a\sigma_\beta^2 I_{r,\beta} A_\beta(\tau) < 0, \tag{A.79}$$

where the inequalities follow from $A_r(\tau) > 0$, $A_\beta(\tau) > 0$, $I_{r,r} \ge 0$, and $I_{r,\beta} \ge 0$, which in turn follows from $M_{1,2} \ge 0$ and (A.26). Equations (A.77), $A_\beta(\tau) > 0$, $A_\beta'(\tau) > 0$, (A.78), and (A.79) imply $b_{\text{FB}} > 0$.

When $a \approx 0$, (A.36), (A.37) and $(\nu_1, \nu_2, \phi_r, \phi_\beta) \approx (\kappa_r, \kappa_\beta, a^3\underline{c}_r, a\underline{c}_\beta)$ (Lemma A.2) imply

$$b_{\text{FB}} = \frac{\frac{\nu_1 - \kappa_r}{\nu_1}\big(1 - e^{-\kappa_r\tau}\big)^2\frac{\sigma_r^2}{\kappa_r} + a^2\underline{c}_\beta^2\big[L_\beta'(\tau) + \kappa_\beta L_\beta(\tau)\big]L_\beta'(\tau)\frac{\sigma_\beta^2}{\kappa_\beta}}{\big(1 - e^{-\kappa_r\tau}\big)^2\frac{\sigma_r^2}{\kappa_r} + a^2\underline{c}_\beta^2 L_\beta'(\tau)^2\frac{\sigma_\beta^2}{\kappa_\beta}} + o\big(a^2\big),$$

where

$$L_\beta(\tau) \equiv \frac{1 - e^{-\kappa_\beta\tau}}{\kappa_\beta} - \frac{1 - e^{-\kappa_r\tau}}{\kappa_r}.$$

Since $L_\beta(\tau)L_\beta'(\tau) > 0$, and (A.46) and (A.54) imply

$$\nu_1 - \kappa_r = a\sigma_r^2\int_0^\infty \alpha(\tau)\left(\frac{1 - e^{-\kappa_r\tau}}{\kappa_r}\right)^2 d\tau + o\big(a^2\big), \tag{A.80}$$

then $b_{\text{FB}} > 0$.

When $a \approx \infty$, (A.36), (A.37), and $(\nu_1, \nu_2, \phi_r, \phi_\beta) \approx (a^{\frac{1}{3}}\overline{n}_1, \overline{\nu}_2, a^{-\frac{1}{3}}\overline{c}_r, \overline{\phi}_\beta)$ (Lemma A.2) imply that for τ bounded away from zero,

$$
\begin{aligned}
b_{\mathrm{FB}} &= \frac{\frac{\sigma_r^2}{\kappa_r} + \overline{\phi}_\beta^2 \left(e^{-\overline{\nu}_2\tau} + \kappa_\beta \frac{1 - e^{-\overline{\nu}_2\tau}}{\overline{\nu}_2} \right) e^{-\overline{\nu}_2\tau} \frac{\sigma_\beta^2}{\kappa_\beta}}{\frac{\sigma_r^2}{\kappa_r} + \overline{\phi}_\beta^2 e^{-2\overline{\nu}_2\tau} \frac{\sigma_\beta^2}{\kappa_\beta}} + o(1) \\
&= 1 + \frac{\overline{\phi}_\beta^2 \frac{1 - e^{-\overline{\nu}_2\tau}}{\overline{\nu}_2} e^{-\overline{\nu}_2\tau} \sigma_\beta^2}{\frac{\sigma_r^2}{\kappa_r} + \overline{\phi}_\beta^2 e^{-2\overline{\nu}_2\tau} \frac{\sigma_\beta^2}{\kappa_\beta}} + o(1).
\end{aligned}
\tag{A.81}
$$

Hence, $b_{\mathrm{FB}} > 1$. We next show that b_{FB} increases in τ if (43) holds. Equation (43) implies that the left-hand side of (A.53) exceeds the right-hand side for $\nu_2 = 0$, and, hence, (A.53) has a solution $\overline{\nu}_2 < 0$. We write (A.81) as

$$
b_{\mathrm{FB}} = 1 + \frac{\overline{\phi}_\beta^2 N_{\mathrm{FB}}(\tau) \sigma_\beta^2}{\frac{\sigma_r^2}{\kappa_r} + \overline{\phi}_\beta^2 D_{\mathrm{FB}}(\tau) \frac{\sigma_\beta^2}{\kappa_\beta}} + o(1),
\tag{A.82}
$$

where

$$
\begin{aligned}
N_{\mathrm{FB}}(\tau) &\equiv \frac{e^{2z\tau} - e^{z\tau}}{z}, \\
D_{\mathrm{FB}}(\tau) &\equiv e^{2z\tau},
\end{aligned}
$$

and $z \equiv -\overline{\nu}_2 > 0$, and consider the derivative

$$
\begin{aligned}
&\left[\frac{\overline{\phi}_\beta^2 N_{\mathrm{FB}}(\tau) \sigma_\beta^2}{\frac{\sigma_r^2}{\kappa_r} + \overline{\phi}_\beta^2 D_{\mathrm{FB}}(\tau) \frac{\sigma_\beta^2}{\kappa_\beta}} \right]' \\
&\quad = \frac{\frac{\sigma_r^2}{\kappa_r} \overline{\phi}_\beta^2 \sigma_\beta^2 N'_{\mathrm{FB}}(\tau) + \overline{\phi}_\beta^4 \frac{\sigma_\beta^4}{\kappa_\beta} \left[N'_{\mathrm{FB}}(\tau) D_{\mathrm{FB}}(\tau) - N_{\mathrm{FB}}(\tau) D'_{\mathrm{FB}}(\tau) \right]}{\left[\frac{\sigma_r^2}{\kappa_r} + \overline{\phi}_\beta^2 D_{\mathrm{FB}}(\tau) \frac{\sigma_\beta^2}{\kappa_\beta} \right]^2}.
\end{aligned}
$$

Since

$$
\left[\frac{N_{\mathrm{FB}}(\tau)}{D_{\mathrm{FB}}(\tau)} \right]' = \left[\frac{1 - e^{-z\tau}}{z} \right]' = e^{-z\tau} > 0,
$$

$N'_{\mathrm{FB}}(\tau) D_{\mathrm{FB}}(\tau) - N_{\mathrm{FB}}(\tau) D'_{\mathrm{FB}}(\tau) > 0$. Since, in addition,

$$
N'_{\mathrm{FB}}(\tau) = 2e^{2z\tau} - e^{z\tau} > 0,
$$

b_{FB} increases in τ.

Proceeding as in the proof of Proposition 3, we find that the CS regression coefficient is

$$
\begin{aligned}
b_{\mathrm{CS}} &= \frac{N_{\mathrm{CS},r}\,\mathbb{V}\mathrm{ar}(r_t) + N_{\mathrm{CS},\beta}\,\mathbb{V}\mathrm{ar}(\beta_t)}{\frac{\Delta\tau}{\tau-\Delta\tau}\left\{\left[\frac{A_r(\tau)}{\tau} - \frac{A_r(\Delta\tau)}{\Delta\tau}\right]^2 \mathbb{V}\mathrm{ar}(r_t) + \left[\frac{A_\beta(\tau)}{\tau} - \frac{A_\beta(\Delta\tau)}{\Delta\tau}\right]^2 \mathbb{V}\mathrm{ar}(\beta_t)\right\}} \\
&= \frac{N_{\mathrm{CS},r}\frac{\sigma_r^2}{\kappa_r} + N_{\mathrm{CS},\beta}\frac{\sigma_\beta^2}{\kappa_\beta}}{\frac{\Delta\tau}{\tau-\Delta\tau}\left\{\left[\frac{A_r(\tau)}{\tau} - \frac{A_r(\Delta\tau)}{\Delta\tau}\right]^2 \frac{\sigma_r^2}{\kappa_r} + \left[\frac{A_\beta(\tau)}{\tau} - \frac{A_\beta(\Delta\tau)}{\Delta\tau}\right]^2 \frac{\sigma_\beta^2}{\kappa_\beta}\right\}},
\end{aligned}
\tag{A.83}
$$

where

$$
N_{\mathrm{CS},j} = \left[\frac{A_j(\tau-\Delta\tau)}{\tau-\Delta\tau}e^{-\kappa_j\Delta\tau} - \frac{A_j(\tau)}{\tau}\right]\left[\frac{A_j(\tau)}{\tau} - \frac{A_j(\Delta\tau)}{\Delta\tau}\right]
$$

for $j = r, \beta$. Taking the limit in (A.83) when $\Delta\tau \to 0$, we find

$$
\begin{aligned}
b_{\mathrm{CS}} \to {} & \left(\left[\frac{A_r(\tau)}{\tau} - \left[A_r'(\tau) + \kappa_r A_r(\tau)\right]\right]\left[\frac{A_r(\tau)}{\tau} - 1\right]\frac{\sigma_r^2}{\kappa_r}\right. \\
& \left. + \left[\frac{A_\beta(\tau)}{\tau} - \left[A_\beta'(\tau) + \kappa_\beta A_\beta(\tau)\right]\right]\frac{A_\beta(\tau)}{\tau}\frac{\sigma_\beta^2}{\kappa_\beta}\right) \\
& \Big/ \left(\left[\frac{A_r(\tau)}{\tau} - 1\right]^2\frac{\sigma_r^2}{\kappa_r} + \left[\frac{A_\beta(\tau)}{\tau}\right]^2\frac{\sigma_\beta^2}{\kappa_\beta}\right) \\
= {} & 1 - \left(\left[A_r'(\tau) + \kappa_r A_r(\tau) - 1\right]\left[\frac{A_r(\tau)}{\tau} - 1\right]\frac{\sigma_r^2}{\kappa_r}\right. \\
& \left. + \left[A_\beta'(\tau) + \kappa_\beta A_\beta(\tau)\right]\frac{A_\beta(\tau)}{\tau}\frac{\sigma_\beta^2}{\kappa_\beta}\right) \\
& \Big/ \left(\left[\frac{A_r(\tau)}{\tau} - 1\right]^2\frac{\sigma_r^2}{\kappa_r} + \left[\frac{A_\beta(\tau)}{\tau}\right]^2\frac{\sigma_\beta^2}{\kappa_\beta}\right).
\end{aligned}
\tag{A.84}
$$

For $\tau < \min\{\bar{\tau}, \hat{\tau}\}$, $A_\beta(\tau) > 0$ and $A_\beta'(\tau) > 0$, and (A.78) and (A.79) hold. Equation (A.79) and the initial condition $A_r(0) = 0$ imply $A_r(\tau) - \tau < 0$. Equation (A.9), $A_\beta(\tau) > 0$, $A_\beta'(\tau) > 0$, (A.78), and $A_r(\tau) - \tau < 0$ imply $b_{\mathrm{CS}} < 1$.

When $a \approx 0$, (A.36), (A.37), $(\nu_1, \nu_2, \phi_r, \phi_\beta) \approx (\kappa_r, \kappa_\beta, a^3\underline{c}_r, a\underline{c}_\beta)$ (Lemma A.2), and (A.80) imply

$$
b_{\mathrm{CS}} = 1 - a\frac{\sigma_r^2(1 - e^{-\kappa_r\tau})}{\kappa_r\left(1 - \frac{1 - e^{-\kappa_r\tau}}{\kappa_r\tau}\right)}\int_0^\infty \alpha(\tau)\left(\frac{1 - e^{-\kappa_r\tau}}{\kappa_r}\right)^2 d\tau + o(a).
$$

Hence, b_{CS} is smaller than and close to 1. Moreover, b_{CS} increases in τ because the function $K(x)$ defined in Proposition 3 is increasing for $x > 0$.

When $a \approx \infty$, (A.36), (A.37), and $(\nu_1, \nu_2, \phi_r, \phi_\beta) \approx (a^{\frac{1}{3}}\overline{n}_1, \overline{\nu}_2, a^{-\frac{1}{3}}\overline{c}_r, \overline{\phi}_\beta)$ (Lemma A.2) imply that for τ bounded away from zero,

$$b_{\mathrm{CS}} = 1 - \frac{\frac{\sigma_r^2}{\kappa_r} + \overline{\phi}_\beta^2\left(e^{-\overline{\nu}_2\tau} + \kappa_\beta \frac{1-e^{-\overline{\nu}_2\tau}}{\overline{\nu}_2}\right)\frac{1-e^{-\overline{\nu}_2\tau}}{\overline{\nu}_2\tau}\frac{\sigma_\beta^2}{\kappa_\beta}}{\frac{\sigma_r^2}{\kappa_r} + \overline{\phi}_\beta^2\left(\frac{1-e^{-\overline{\nu}_2\tau}}{\overline{\nu}_2\tau}\right)^2\frac{\sigma_\beta^2}{\kappa_\beta}} + o(1). \tag{A.85}$$

Hence, $b_{\mathrm{CS}} < 1$. We next show that b_{CS} is negative and decreasing in τ if (43) holds. We write (A.85) as

$$b_{\mathrm{CS}} = 1 - \frac{\frac{\sigma_r^2}{\kappa_r} + \overline{\phi}_\beta^2 N_{\mathrm{CS}}(\tau)\frac{\sigma_\beta^2}{\kappa_\beta}}{\frac{\sigma_r^2}{\kappa_r} + \overline{\phi}_\beta^2 D_{\mathrm{CS}}(\tau)\frac{\sigma_\beta^2}{\kappa_\beta}} + o(1), \tag{A.86}$$

where

$$N_{\mathrm{CS}}(\tau) \equiv \left(e^{z\tau} + \kappa_\beta \frac{e^{z\tau}-1}{z}\right)\frac{e^{z\tau}-1}{z\tau}, \qquad D_{\mathrm{CS}}(\tau) \equiv \left(\frac{e^{z\tau}-1}{z\tau}\right)^2,$$

and $z \equiv -\overline{\nu}_2 > 0$. Equation (A.86) implies

$$b_{\mathrm{CS}} = -\frac{\overline{\phi}_\beta^2\left[N_{\mathrm{CS}}(\tau) - D_{\mathrm{CS}}(\tau)\right]\frac{\sigma_\beta^2}{\kappa_\beta}}{\frac{\sigma_r^2}{\kappa_r} + \overline{\phi}_\beta^2 D_{\mathrm{CS}}(\tau)\frac{\sigma_\beta^2}{\kappa_\beta}} + o(1). \tag{A.87}$$

Since

$$\begin{aligned} N_{\mathrm{CS}}(\tau) - D_{\mathrm{CS}}(\tau) &= \left[e^{z\tau} + \left(\kappa_\beta - \frac{1}{\tau}\right)\frac{e^{z\tau}-1}{z}\right]\frac{e^{z\tau}-1}{z\tau} \\ &> \left[e^{z\tau} - \frac{e^{z\tau}-1}{z\tau}\right]\frac{e^{z\tau}-1}{z\tau} = \frac{z\tau e^{z\tau} - e^{z\tau} + 1}{z\tau}\frac{e^{z\tau}-1}{z\tau} \end{aligned}$$

and $xe^x - e^x + 1 > 0$ for all x, (A.87) implies $b_{\mathrm{CS}} < 0$. Consider next the derivative

$$\begin{aligned} &\left[\frac{\frac{\sigma_r^2}{\kappa_r} + \overline{\phi}_\beta^2 N_{\mathrm{CS}}(\tau)\frac{\sigma_\beta^2}{\kappa_\beta}}{\frac{\sigma_r^2}{\kappa_r} + \overline{\phi}_\beta^2 D_{\mathrm{CS}}(\tau)\frac{\sigma_\beta^2}{\kappa_\beta}}\right]' \\ &= \frac{\frac{\sigma_r^2}{\kappa_r}\overline{\phi}_\beta^2\frac{\sigma_\beta^2}{\kappa_\beta}\left[N'_{\mathrm{CS}}(\tau) - D'_{\mathrm{CS}}(\tau)\right] + \overline{\phi}_\beta^4\frac{\sigma_\beta^4}{\kappa_\beta^2}\left[N'_{\mathrm{CS}}(\tau)D_{\mathrm{CS}}(\tau) - N_{\mathrm{CS}}(\tau)D'_{\mathrm{CS}}(\tau)\right]}{\left[\frac{\sigma_r^2}{\kappa_r} + \overline{\phi}_\beta^2 D_{\mathrm{CS}}(\tau)\frac{\sigma_\beta^2}{\kappa_\beta}\right]^2}. \end{aligned}$$

Since

$$
\begin{aligned}
N'_{\mathrm{CS}}(\tau) - D'_{\mathrm{CS}}(\tau) &= \left[ze^{z\tau} + \left(\kappa_\beta - \frac{1}{\tau}\right)e^{z\tau} + \frac{e^{z\tau}-1}{z\tau^2}\right]\frac{e^{z\tau}-1}{z\tau} \\
&\quad + \left[e^{z\tau} + \left(\kappa_\beta - \frac{1}{\tau}\right)\frac{e^{z\tau}-1}{z}\right]\frac{z^2\tau e^{z\tau} - z\big(e^{z\tau}-1\big)}{z^2\tau^2} \\
&> \frac{z^2\tau^2 e^{z\tau} - z\tau e^{z\tau} + e^{z\tau} - 1}{z\tau^2}\frac{e^{z\tau}-1}{z\tau} + \frac{z\tau e^{z\tau} - e^{z\tau} + 1}{z\tau}\frac{z\tau e^{z\tau} - e^{z\tau} + 1}{z\tau^2}
\end{aligned}
$$

and $x^2e^x - xe^x + e^x - 1 > 0$ for all x, $N'_{\mathrm{CS}}(\tau) - D'_{\mathrm{CS}}(\tau) > 0$. Since

$$
\left[\frac{N_{\mathrm{CS}}(\tau)}{D_{\mathrm{CS}}(\tau)}\right]' = \left[\frac{z\tau e^{z\tau}}{e^{z\tau}-1}\right]' = ze^{z\tau}\frac{(1+z\tau)\big(e^{z\tau}-1\big) - z\tau e^{z\tau}}{\big(e^{z\tau}-1\big)^2} = ze^{z\tau}\frac{e^{z\tau}-1-z\tau}{\big(e^{z\tau}-1\big)^2}
$$

and $e^x - 1 - x > 0$ for all x, $N'_{\mathrm{CS}}(\tau)D_{\mathrm{CS}}(\tau) - N_{\mathrm{CS}}(\tau)D'_{\mathrm{CS}}(\tau) > 0$. Hence, b_{CS} decreases in τ. *Q.E.D.*

PROOF OF PROPOSITION 7: Substituting $C(\tau)$ from (41) into (42), using $\Gamma = \mathrm{Diag}(\kappa_r, \kappa_\beta)$ and $\Sigma = \mathrm{Diag}(\sigma_r^2, \sigma_\beta^2)$, and dropping the subscript 1 from functions of the single demand factor, we find

$$
\begin{aligned}
\chi_r = \kappa_r\bar{r} + a\sigma_r^2\bigg[&\int_0^\infty \theta_0(\tau)A_r(\tau)\,d\tau \\
&- \chi_r\int_0^\infty \alpha(\tau)\left(\int_0^\tau A_r(u)\,du\right)A_r(\tau)\,d\tau \\
&- \chi_\beta\int_0^\infty \alpha(\tau)\left(\int_0^\tau A_\beta(u)\,du\right)A_r(\tau)\,d\tau \\
&+ \frac{\sigma_r^2}{2}\int_0^\infty \alpha(\tau)\left(\int_0^\tau A_r(u)^2\,du\right)A_r(\tau)\,d\tau \\
&+ \frac{\sigma_\beta^2}{2}\int_0^\infty \alpha(\tau)\left(\int_0^\tau A_\beta(u)^2\,du\right)A_r(\tau)\,d\tau\bigg],
\end{aligned}
\tag{A.88}
$$

$$
\begin{aligned}
\chi_\beta = a\sigma_\beta^2\bigg[&\int_0^\infty \theta_0(\tau)A_\beta(\tau)\,d\tau \\
&- \chi_r\int_0^\infty \alpha(\tau)\left(\int_0^\tau A_r(u)\,du\right)A_\beta(\tau)\,d\tau \\
&- \chi_\beta\int_0^\infty \alpha(\tau)\left(\int_0^\tau A_\beta(u)\,du\right)A_\beta(\tau)\,d\tau \\
&+ \frac{\sigma_r^2}{2}\int_0^\infty \alpha(\tau)\left(\int_0^\tau A_r(u)^2\,du\right)A_\beta(\tau)\,d\tau \\
&+ \frac{\sigma_\beta^2}{2}\int_0^\infty \alpha(\tau)\left(\int_0^\tau A_\beta(u)^2\,du\right)A_\beta(\tau)\,d\tau\bigg].
\end{aligned}
\tag{A.89}
$$

The system of (A.88) and (A.89) is linear in (χ_r, χ_β) and its solution is

$$\begin{aligned}\chi_r = \frac{1}{D}\Bigg\{&\left[\kappa_r\bar{r} + a\sigma_r^2\int_0^\infty \theta_0(\tau)A_r(\tau)\,d\tau + C_r\right]\\ &\times\left[1 + a\sigma_\beta^2\int_0^\infty \alpha(\tau)\left(\int_0^\tau A_\beta(u)\,du\right)A_\beta(\tau)\,d\tau\right]\\ &-\left[a\sigma_\beta^2\int_0^\infty \theta_0(\tau)A_\beta(\tau)\,d\tau + C_\beta\right]\\ &\times\left[a\sigma_r^2\int_0^\infty \alpha(\tau)\left(\int_0^\tau A_\beta(u)\,du\right)A_r(\tau)\,d\tau\right]\Bigg\},\end{aligned} \tag{A.90}$$

$$\begin{aligned}\chi_\beta = \frac{1}{D}\Bigg\{&\left[a\sigma_\beta^2\int_0^\infty \theta_0(\tau)A_\beta(\tau)\,d\tau + C_\beta\right]\\ &\times\left[1 + a\sigma_r^2\int_0^\infty \alpha(\tau)\left(\int_0^\tau A_r(u)\,du\right)A_r(\tau)\,d\tau\right]\\ &-\left[\kappa_r\bar{r} + a\sigma_r^2\int_0^\infty \theta_0(\tau)A_r(\tau)\,d\tau + C_r\right]\\ &\times\left[a\sigma_\beta^2\int_0^\infty \alpha(\tau)\left(\int_0^\tau A_r(u)\,du\right)A_\beta(\tau)\,d\tau\right]\Bigg\},\end{aligned} \tag{A.91}$$

where

$$\begin{aligned}D \equiv &\left[1 + a\sigma_r^2\int_0^\infty \alpha(\tau)\left(\int_0^\tau A_r(u)\,du\right)A_r(\tau)\,d\tau\right]\\ &\times\left[1 + a\sigma_\beta^2\int_0^\infty \alpha(\tau)\left(\int_0^\tau A_\beta(u)\,du\right)A_\beta(\tau)\,d\tau\right]\\ &-\left[a\sigma_r^2\int_0^\infty \alpha(\tau)\left(\int_0^\tau A_\beta(u)\,du\right)A_r(\tau)\,d\tau\right]\\ &\times\left[a\sigma_\beta^2\int_0^\infty \alpha(\tau)\left(\int_0^\tau A_r(u)\,du\right)A_\beta(\tau)\,d\tau\right]\end{aligned}$$

and

$$\begin{aligned}C_j \equiv &\frac{a\sigma_j^2\sigma_r^2}{2}\int_0^\infty \alpha(\tau)\left(\int_0^\tau A_r(u)^2\,du\right)A_j(\tau)\,d\tau\\ &+\frac{a\sigma_j^2\sigma_\beta^2}{2}\int_0^\infty \alpha(\tau)\left(\int_0^\tau A_\beta(u)^2\,du\right)A_j(\tau)\,d\tau\end{aligned}$$

for $j = r, \beta$. The effect of a change in the demand intercept from $\theta_0(\tau)$ to $\theta_0(\tau) + \Delta\theta_0(\tau)$ on the yield $y_t^{(\tau)}$ for maturity τ is $\Delta y_t^{(\tau)} \equiv \frac{\Delta C(\tau)}{\tau}$, which from (41), (A.90), and (A.91) is

$$\begin{aligned}\Delta y_t^{(\tau)} = \frac{1}{D}\Bigg\{&\left[a\sigma_r^2\int_0^\infty \Delta\theta_0(\tau)A_r(\tau)\,d\tau\right]\\ &\times\left[1 + a\sigma_\beta^2\int_0^\infty \alpha(\tau)\left(\int_0^\tau A_\beta(u)\,du\right)A_\beta(\tau)\,d\tau\right]\end{aligned}$$

$$
\begin{aligned}
&-\left[a\sigma_\beta^2\int_0^\infty \Delta\theta_0(\tau)A_\beta(\tau)\,d\tau\right]\left[a\sigma_r^2\int_0^\infty \alpha(\tau)\left(\int_0^\tau A_\beta(u)\,du\right)A_r(\tau)\,d\tau\right]\Bigg\}\\
&\times\frac{\int_0^\tau A_r(u)\,du}{\tau}\\
&+\frac{1}{D}\Bigg\{\left[a\sigma_\beta^2\int_0^\infty \Delta\theta_0(\tau)A_\beta(\tau)\,d\tau\right]\\
&\times\left[1+a\sigma_r^2\int_0^\infty \alpha(\tau)\left(\int_0^\tau A_r(u)\,du\right)A_r(\tau)\,d\tau\right]\\
&-\left[a\sigma_r^2\int_0^\infty \Delta\theta_0(\tau)A_r(\tau)\,d\tau\right]\left[a\sigma_\beta^2\int_0^\infty \alpha(\tau)\left(\int_0^\tau A_r(u)\,du\right)A_\beta(\tau)\,d\tau\right]\Bigg\}\\
&\times\frac{\int_0^\tau A_\beta(u)\,du}{\tau}.
\end{aligned}
\tag{A.92}
$$

Hence, the change $\Delta\theta_0(\tau)$ affects yields only through $\int_0^\infty \Delta\theta_0(\tau)A_r(\tau)\,d\tau$ and $\int_0^\infty \Delta\theta_0(\tau)A_\beta(\tau)\,d\tau$.

When the change $\Delta\theta_0(\tau)$ is a Dirac function with point mass at τ^*,

$$
\int_0^\infty \Delta\theta_0(\tau)A_j(\tau)\,d\tau = A_j(\tau^*)
$$

for $j=r,\beta$, and (A.92) becomes

$$
\Delta y_{t,\tau^*}^{(\tau)} = \frac{1}{D}\left[\Lambda_r(\tau^*)\frac{\int_0^\tau A_r(u)\,du}{\tau}+\Lambda_\beta(\tau^*)\frac{\int_0^\tau A_\beta(u)\,du}{\tau}\right],
\tag{A.93}
$$

where

$$
\begin{aligned}
\Lambda_r(\tau^*) &\equiv a\sigma_r^2 A_r(\tau^*)\left[1+a\sigma_\beta^2\int_0^\infty \alpha(\tau)\left(\int_0^\tau A_\beta(u)\,du\right)A_\beta(\tau)\,d\tau\right]\\
&\quad - a\sigma_\beta^2 A_\beta(\tau^*)\left[a\sigma_r^2\int_0^\infty \alpha(\tau)\left(\int_0^\tau A_\beta(u)\,du\right)A_r(\tau)\,d\tau\right],\\
\Lambda_\beta(\tau^*) &\equiv a\sigma_\beta^2 A_\beta(\tau^*)\left[1+a\sigma_r^2\int_0^\infty \alpha(\tau)\left(\int_0^\tau A_r(u)\,du\right)A_r(\tau)\,d\tau\right]\\
&\quad - a\sigma_r^2 A_r(\tau^*)\left[a\sigma_\beta^2\int_0^\infty \alpha(\tau)\left(\int_0^\tau A_r(u)\,du\right)A_\beta(\tau)\,d\tau\right].
\end{aligned}
$$

Using (A.93), we can write (44) in the equivalent form

$$\begin{aligned}
&\left[\Lambda_r(\tau_1)\frac{\int_0^{\tau_1} A_r(u)\,du}{\tau_1}+\Lambda_\beta(\tau_1)\frac{\int_0^{\tau_1} A_\beta(u)\,du}{\tau_1}\right]\\
&\quad\times\left[\Lambda_r(\tau_2)\frac{\int_0^{\tau_2} A_r(u)\,du}{\tau_2}+\Lambda_\beta(\tau_2)\frac{\int_0^{\tau_2} A_\beta(u)\,du}{\tau_2}\right]\\
&\quad>\left[\Lambda_r(\tau_1)\frac{\int_0^{\tau_2} A_r(u)\,du}{\tau_2}+\Lambda_\beta(\tau_1)\frac{\int_0^{\tau_2} A_\beta(u)\,du}{\tau_2}\right]\\
&\quad\times\left[\Lambda_r(\tau_2)\frac{\int_0^{\tau_1} A_r(u)\,du}{\tau_1}+\Lambda_\beta(\tau_2)\frac{\int_0^{\tau_1} A_\beta(u)\,du}{\tau_1}\right]\\
&\quad\Leftrightarrow\left[\Lambda_r(\tau_1)\Lambda_\beta(\tau_2)-\Lambda_r(\tau_2)\Lambda_\beta(\tau_1)\right]\\
&\quad\times\left[\frac{\int_0^{\tau_1} A_r(u)\,du}{\tau_1}\frac{\int_0^{\tau_2} A_\beta(u)\,du}{\tau_2}-\frac{\int_0^{\tau_2} A_r(u)\,du}{\tau_2}\frac{\int_0^{\tau_1} A_\beta(u)\,du}{\tau_1}\right]>0.
\end{aligned}\tag{A.94}$$

To show that (A.94) holds, we show that each of the two terms in brackets is positive. The second term is positive because it has the same sign as

$$\begin{aligned}
&\int_0^{\tau_1} A_r(u)\,du\int_0^{\tau_2} A_\beta(u)\,du-\int_0^{\tau_2} A_r(u)\,du\int_0^{\tau_1} A_\beta(u)\,du\\
&\quad=\int_0^{\tau_1} A_r(u)\,du\int_{\tau_1}^{\tau_2} A_\beta(u)\,du\\
&\qquad-\int_{\tau_1}^{\tau_2} A_r(u)\,du\int_0^{\tau_1} A_\beta(u)\,du\\
&\quad>\int_0^{\tau_1}\left[A_\beta(u)\frac{A_r(\tau_1)}{A_\beta(\tau_1)}\right]du\int_{\tau_1}^{\tau_2} A_\beta(u)\,du\\
&\qquad-\int_{\tau_1}^{\tau_2}\left[A_\beta(u)\frac{A_r(\tau_1)}{A_\beta(\tau_1)}\right]du\int_0^{\tau_1} A_\beta(u)\,du\\
&\quad=0,
\end{aligned}$$

where the second step follows because $A_\beta(\tau)>0$ and $[\frac{A_r(\tau)}{A_\beta(\tau)}]'<0$ for $\tau\in(0,\hat{\tau})$. The first term is equal to

$$\left[A_r(\tau_1)A_\beta(\tau_2)-A_r(\tau_2)A_\beta(\tau_1)\right]D,$$

and is positive if $D > 0$, since $A_\beta(\tau) > 0$ and $[\frac{A_r(\tau)}{A_\beta(\tau)}]' < 0$ for $\tau \in (0, \hat{\tau})$. Integration by parts implies that for $j = r, \beta$,

$$\begin{aligned}
&\int_0^\infty \alpha(\tau)\left(\int_0^\tau A_j(u)\,du\right)A_j(\tau)\,d\tau \\
&\quad = \left[\alpha(\tau)\left(\int_0^\tau A_j(u)\,du\right)^2\right]_0^\infty + \int_0^\infty \left(\int_0^\tau A_j(u)\,du\right)^2 d\hat{\alpha}(\tau) \\
&\qquad - \int_0^\infty \alpha(\tau)\left(\int_0^\tau A_j(u)\,du\right)A_j(\tau)\,d\tau, \qquad \text{(A.95)}
\end{aligned}$$

where $d\hat{\alpha}(\tau)$ denotes the measure generated by the nondecreasing function $-\alpha(\tau)$ (which is possibly discontinuous at a finite threshold T). Since

$$\left[\alpha(\tau)\left(\int_0^\tau A_j(u)\,du\right)^2\right]_0^\infty = \lim_{\tau\to\infty}\left[\alpha(\tau)\left(\int_0^\tau A_j(u)\,du\right)^2\right] = 0,$$

where the second step follows because M is finite, (A.95) implies

$$\int_0^\infty \alpha(\tau)\left(\int_0^\tau A_j(u)\,du\right)A_j(\tau)\,d\tau = \frac{\int_0^\infty \left(\int_0^\tau A_j(u)\,du\right)^2 d\hat{\alpha}(\tau)}{2} \geq 0. \qquad \text{(A.96)}$$

Likewise,

$$\begin{aligned}
&\int_0^\infty \alpha(\tau)\left(\int_0^\tau A_r(u)\,du\right)A_\beta(\tau)\,d\tau + \int_0^\infty \alpha(\tau)\left(\int_0^\tau A_\beta(u)\,du\right)A_r(\tau)\,d\tau \\
&\quad = 2\left[\alpha(\tau)\left(\int_0^\tau A_r(u)\,du\right)\left(\int_0^\tau A_\beta(u)\,du\right)\right]_0^\infty \\
&\qquad + 2\int_0^\infty \left(\int_0^\tau A_r(u)\,du\right)\left(\int_0^\tau A_\beta(u)\,du\right)d\hat{\alpha}(\tau) \\
&\qquad - \int_0^\infty \alpha(\tau)\left(\int_0^\tau A_r(u)\,du\right)A_\beta(\tau)\,d\tau - \int_0^\infty \alpha(\tau)\left(\int_0^\tau A_\beta(u)\,du\right)A_r(\tau)\,d\tau \\
&\quad \Rightarrow \int_0^\infty \alpha(\tau)\left(\int_0^\tau A_r(u)\,du\right)A_\beta(\tau)\,d\tau + \int_0^\infty \alpha(\tau)\left(\int_0^\tau A_\beta(u)\,du\right)A_r(\tau)\,d\tau \\
&\quad = \int_0^\infty \left(\int_0^\tau A_r(u)\,du\right)\left(\int_0^\tau A_\beta(u)\,du\right)d\hat{\alpha}(\tau)
\end{aligned}$$

and, hence,

$$\begin{aligned}
&\left[\int_0^\infty \alpha(\tau)\left(\int_0^\tau A_r(u)\,du\right)A_\beta(\tau)\,d\tau\right]\left[\int_0^\infty \alpha(\tau)\left(\int_0^\tau A_\beta(u)\,du\right)A_r(\tau)\,d\tau\right] \\
&\qquad \leq \frac{\left[\int_0^\infty \left(\int_0^\tau A_r(u)\,du\right)\left(\int_0^\tau A_\beta(u)\,du\right)d\hat{\alpha}(\tau)\right]^2}{4}. \qquad \text{(A.97)}
\end{aligned}$$

Equations (A.96) and (A.97) imply that $D > 0$ if

$$\left[\int_0^\infty \left(\int_0^\tau A_r(u)\,du\right)^2 d\hat{\alpha}(\tau)\right]\left[\int_0^\infty \left(\int_0^\tau A_\beta(u)\,du\right)^2 d\hat{\alpha}(\tau)\right]$$
$$\geq \left[\int_0^\infty \left(\int_0^\tau A_r(u)\,du\right)\left(\int_0^\tau A_\beta(u)\,du\right) d\hat{\alpha}(\tau)\right]^2,$$

which holds because of the Cauchy–Schwarz inequality. *Q.E.D.*

APPENDIX B: DEMAND OF PREFERRED-HABITAT INVESTORS

There are overlapping generations of preferred-habitat investors living for a period of length $T < \infty$ and arbitrageurs living for a period of length dt. Thus, at each point in time there is a continuum of investor generations and one arbitrageur generation. Arbitrageurs and investors receive endowment W at the beginning of their life and consume at the end of their life. Arbitrageurs use their endowment to buy bonds. Investors use their endowment to buy bonds and to invest in a private opportunity ("real estate") that pays at the end of their life. To ensure that the slope of the investors' demand for bonds is finite, we require that substitution between bonds and the private opportunity is imperfect. We model imperfect substitution by assuming that bonds pay in a good 1 ("money") and the private opportunity pays in a different good 2 ("real estate services"). The endowment W is in good 1. Arbitrageurs and investors can use good 1 to invest in bonds and in the private opportunity.

Consider the optimization problem of an investor n born at time 0. We denote by $\hat{Z}_{n,t}^{(\tau)}$ the number of units of the bond with maturity τ that the investor holds at time $t \in [0, T]$, where one unit of the bond is an investment in the bond with face value 1. We denote by $W_{n,t}$ the value of the investor's bond portfolio at time t and denote by $dc_{n,t}$ the investment in the private opportunity between t and $t + dt$, both expressed in units of good 1. We denote by $(\hat{W}_{n,t}, d\hat{c}_{n,t})$ the counterparts of $(W_{n,t}, dc_{n,t})$ when expressed in units of the bond maturing at time T:

$$\hat{W}_{n,t} \equiv \frac{W_{n,t}}{P_t^{(T-t)}},$$

$$d\hat{c}_{n,t} \equiv \frac{dc_{n,t}}{P_t^{(T-t)}}.$$

We finally denote by $\hat{\beta}_{n,t}^{(T-t)} > 0$ the number of units of good 2 that an investment of one unit of good 1 at time t yields at time T. The investor's budget constraint is

$$d\hat{W}_{n,t} = \int_0^T \hat{Z}_{n,t}^{(\tau)}\, d\left(\frac{P_t^{(\tau)}}{P_t^{(T-t)}}\right) d\tau - d\hat{c}_{n,t}. \tag{B.1}$$

The investor's utility at time T is

$$u(C_T) + \int_0^T \hat{\beta}_{n,t}^{(T-t)} P_t^{(T-t)}\, d\hat{c}_{n,t} \tag{B.2}$$

and it consists of two parts: a utility $u(C_T)$ that is an increasing and concave function of the consumption C_T of good 1 at time T, and a utility $\int_0^T \hat{\beta}_{n,t}^{(T-t)} P_t^{(T-t)}\, d\hat{c}_{n,t}$ that is equal to the consumption of good 2 at time T and is derived from the accumulated investment in the private opportunity between times 0 and T. The marginal utility $u'(C_T)$ converges to infinity when C_T goes to a lower bound $\underline{C}$ and converges to zero when C_T goes to infinity. The investor has max-min preferences. At each time $t \in [0, T]$, the investor chooses $(\hat{Z}_{n,t}^{(\tau)}, \hat{c}_{n,t})$ to maximize the minimum of (B.2) over sample paths of $q_t = (r_t, \beta_{1,t}, \dots, \beta_{K,t})^\top$ and $\hat{\beta}_{n,t}^{(T-t)}$, subject to the budget constraint (B.1) and the terminal condition $C_T = \hat{W}_T$.

PROPOSITION B.1: *Assume that Σ has full rank, $K \geq 1$, $\hat{\beta}_{n,t}^{(T-t)}$ is an invertible function of $(\beta_{1,t}, \dots, \beta_{K,t})^\top$, and the term structure involves no arbitrage* (*i.e.*, (34) *holds*). *At time t, the investor holds only the bond maturing at time T and no other bonds. The number $\hat{Z}_{n,t}^{(T-t)}$ of units of the bond held by the investor solves*

$$u'\big(\hat{Z}_{n,t}^{(T-t)}\big) = P_t^{(T-t)} \hat{\beta}_{n,t}^{(T-t)}. \tag{B.3}$$

PROOF: Defining $(\mu_{\hat{Z},n,t}, \sigma_{\hat{Z},n,t})$ by

$$\int_0^T \hat{Z}_{n,t}^{(\tau)} \, d\left(\frac{P_t^{(\tau)}}{P_t^{(T-t)}}\right) d\tau \equiv \mu_{\hat{Z},n,t} \, dt + \sigma_{\hat{Z},n,t} \, dB_t,$$

where $dB_t = (dB_{r,t}, dB_{\beta,1,t}, \dots, dB_{\beta,K,t})^\top$, we write the budget constraint (B.1) as

$$d\hat{W}_{n,t} = \mu_{\hat{Z},n,t} \, dt + \sigma_{\hat{Z},n,t} \, dB_t - d\hat{c}_{n,t}. \tag{B.4}$$

Integrating (B.4) from 0 to T and using the terminal condition $C_T = \hat{W}_T$, we write the investor's optimization problem at $t = 0$ as

$$\begin{aligned} &\max_{\hat{Z}_{n,t}^{(\tau)}, \hat{c}_{n,t}} \min_{q_t, \hat{\beta}_{n,t}^{(T-t)}} \left[u\left(\hat{W}_0 + \int_0^T \mu_{\hat{Z},n,t} \, dt + \int_0^T \sigma_{\hat{Z},n,t} \, dB_t - \Delta\hat{c}_{0,n} - \int_0^T d\hat{c}_{n,t} \right) \right. \\ &\qquad \left. + \hat{\beta}_{0,n}^{(T)} P_0^{(T)} \Delta\hat{c}_{n,t} + \int_0^T \hat{\beta}_{n,t}^{(T-t)} P_t^{(T-t)} \, d\hat{c}_{n,t} \right], \end{aligned} \tag{B.5}$$

where we allow for the possibility that $\hat{c}_t$ has a discrete change $\Delta\hat{c}_{n,0}$ at $t = 0$. Since Σ has full rank and $K \geq 1$, r_t is not perfectly correlated with $(\beta_{1,t}, \dots, \beta_{K,t})$. Since, in addition, $\hat{\beta}_{n,t}^{(T-t)}$ is an invertible function of $(\beta_{1,t}, \dots, \beta_{K,t})$, sample paths of q_t and $\hat{\beta}_{n,t}^{(T-t)}$ exist such that $\hat{\beta}_{n,t}^{(T-t)} P_t^{(T-t)} = u'(\hat{W}_0 - \Delta\hat{c}_0)$ for $t > \epsilon$ and for any $\epsilon > 0$. Hence, the minimum in (B.5) is smaller than

$$\begin{aligned} &\min_{q_t, \hat{\beta}_{n,t}^{(T-t)}} \left[u\left(\hat{W}_0 + \int_0^T \mu_{\hat{Z},n,t} \, dt + \int_0^T \sigma_{\hat{Z},n,t} \, dB_t - \Delta\hat{c}_{0,n} - \int_0^T d\hat{c}_{n,t} \right) \right. \\ &\qquad \left. + \hat{\beta}_{0,n}^{(T)} P_0^{(T)} \Delta\hat{c}_{0,n} + u'(\hat{W}_0 - \Delta\hat{c}_0) \int_0^T d\hat{c}_{n,t} \right], \end{aligned}$$

which in turn is smaller than

$$\begin{aligned} &\min_{q_t, \hat{\beta}_{n,t}^{(T-t)}} \left[u(\hat{W}_0 - \Delta\hat{c}_0) + u'(\hat{W}_0 - \Delta\hat{c}_0) \left(\int_0^T \mu_{\hat{Z},n,t} \, dt + \int_0^T \sigma_{\hat{Z},n,t} \, dB_t \right) \right. \\ &\qquad \left. + \hat{\beta}_{0,n}^{(T)} P_0^{(T)} \Delta\hat{c}_{0,n} \right] \end{aligned} \tag{B.6}$$

because u is concave. If $\sigma_{\hat{Z},n,t} \neq 0$ for any interval in $(0, T)$, then the minimum in (B.6) is minus infinity because the Brownian motion has infinite variation. Therefore, $\sigma_{\hat{Z},n,t} = 0$, that is, the investor holds the bond maturing at time T and zero units of all other bonds. Since absence of arbitrage requires $\mu_{\hat{Z},n,t} = 0$, (B.6) is smaller than

$$u(\hat{W}_0 - \Delta\hat{c}_0) + \hat{\beta}_{0,n}^{(T)} P_0^{(T)} \Delta\hat{c}_{0,n}$$

and, hence,

$$\begin{aligned}
&\max_{\hat{Z}_{n,t}^{(\tau)}, \hat{c}_{n,t}} \min_{q_t, \hat{\beta}_{n,t}^{(T-t)}} \biggl[u\biggl(\hat{W}_0 + \int_0^T \mu_{\hat{Z},n,t}\, dt + \int_0^T \sigma_{\hat{Z},n,t}\, dB_t - \Delta\hat{c}_{0,n} - \int_0^T d\hat{c}_{n,t} \biggr) \\
&\qquad + \hat{\beta}_{0,n}^{(T)} P_0^{(T)} \Delta\hat{c}_{n,t} + \int_0^T \hat{\beta}_{n,t}^{(T-t)} P_t^{(T-t)}\, d\hat{c}_{n,t} \biggr] \\
&\quad \leq \max_{\Delta\hat{c}_{0,n}} \bigl[u(\hat{W}_0 - \Delta\hat{c}_0) + \hat{\beta}_{0,n}^{(T)} P_0^{(T)} \Delta\hat{c}_{0,n} \bigr]. \qquad \text{(B.7)}
\end{aligned}$$

Setting $\hat{Z}_{n,t}^{(\tau)} = 0$ for $t \geq 0$ and $\tau \neq T - t$, and $d\hat{c}_{n,t} = 0$ for $t > 0$ in (B.5), we find that (B.7) holds also in the reverse sense and is, therefore, an equality. The optimal $\Delta\hat{c}_{0,n}$ thus satisfies

$$u'(\hat{W}_0 - \Delta\hat{c}_{0,n}) = \hat{\beta}_{0,n}^{(T)} P_0^{(T)}. \qquad \text{(B.8)}$$

Since $\hat{W}_0 - \Delta\hat{c}_{0,n}$ represents units of the bond maturing at time T that the investor holds at time 0, (B.8) yields (B.3) for $t = 0$. The same argument yields (B.3) for $t > 0$. *Q.E.D.*

Proposition B.1 implies that preferred-habitat investors demand only the bond whose maturity coincides with the time when they consume. To ensure that the demand by preferred-habitat investors takes the specific functional form (5)–(7), we assume specific functions for the utility u and the return $\hat{\beta}_{n,t}^{(\tau)}$ on the private opportunity.

Suppose $\underline{C} = -\infty$, $u(C_T) = -e^{-C_T}$, and $\hat{\beta}_{n,t}^{(\tau)} = e^{\beta_t^{(\tau)}}$, where $\beta_t^{(\tau)}$ is given by (6) and (7). Proposition B.1 implies that the number $\hat{Z}_{n,t}^{(T-t)}$ of units of the bond maturing at time T and held at time t by an investor born at time 0 is given by

$$e^{-\hat{Z}_{n,t}^{(T-t)}} = P_t^{(T-t)} \hat{\beta}_{n,t}^{(T-t)} \quad \Leftrightarrow \quad \hat{Z}_{n,t}^{(T-t)} = -\log\bigl(P_t^{(T-t)}\bigr) - \beta_t^{(T-t)}.$$

This coincides with the demand (5)–(7) with $\alpha(\tau) = 1$, except that (5)–(7) concern the present value of the bond rather than its face value, that is, the units of the bond. To derive the demand (5)–(7) expressed in present-value terms, we modify the assumed functions for u and $\hat{\beta}_{n,t}^{(\tau)}$. We can obtain the demand (5)–(7) for a set of values of q_t whose probability can be made arbitrarily close to 1.

Suppose that there are two types of preferred-habitat investors born at each time t in equal measure. For type 1 investors, $\underline{C} = 0$, $u(C_{t+T}) = \log(C_{t+T})$, and $\hat{\beta}_{n,t'}^{(T+t-t')} = -\frac{1}{\min\{\beta_{t'}^{(T+t-t')}, -\epsilon\}}$, where $\beta_t^{(\tau)}$ is given by (6) and (7), and ϵ is positive and small. For type 2 investors, $\underline{C} = -\infty$ and $\hat{\beta}_{n,t'}^{(T+t-t')} = 1$. To define $u(C_{t+T})$ for type 2 investors, we start with the function

$$N(x) \equiv -\frac{\log(x)}{x},$$

defined for $x > 0$. The function $N(x)$ converges to infinity when x goes to zero and converges to zero when x goes to infinity. It decreases for $x \in (0, e)$ and increases for $x \in (e, T)$. Its minimum value, obtained for $x = e$, is $-\frac{1}{e}$. We take x to represent marginal utility $u'(C_{t+T})$ and take $N(x)$ to represent C_{t+T}. This defines $u(C_{t+T})$ for $C_{t+T} > -\frac{1}{e}$ and $u'(C_{t+T}) \in (0, e)$. To define $u(C_{t+T})$ for $C_{t+T} < -\frac{1}{e}$ and $u'(C_{t+T}) > e$, we extend $u'(C_{t+T})$ as a linear function of C_t. (Other extensions are possible as well.) We set the derivative of the linear function so that $u'(C_{t+T})$ is continuously differentiable at the extension point, and we take the extension point to be $u'(C_{t+T}) = e(1-\epsilon)$ (rather than $u'(C_{t+T}) = e$) so that the derivative is finite. We thus set

$$u'(C_{t+T}) = N^{-1}(C_{t+T}) \quad \text{for } C_{t+T} \geq N[e(1-\epsilon)],$$
$$u'(C_{t+T}) = e(1-\epsilon) - \frac{e^2(1-\epsilon)^2}{\log(1-\epsilon)}\big[C_{t+T} - N\big[e(1-\epsilon)\big]\big] \quad \text{for } C_{t+T} < N\big[e(1-\epsilon)\big].$$

Since $u'(C_{t+T})$ is positive and decreasing, $u(C_{t+T})$ is increasing and concave.

Proposition B.1 implies that the number $\hat{Z}_{n,t}^{(T-t)}$ of units of the bond maturing at time T and held at time t by a type 1 investor born at time 0 is given by

$$\frac{1}{\hat{Z}_{n,t}^{(T-t)}} = P_t^{(T-t)}\hat{\beta}_{n,t}^{(T-t)}.$$

This yields the demand

$$P_t^{(T-t)}\hat{Z}_{n,t}^{(T-t)} = \frac{1}{\hat{\beta}_{n,t}^{(T-t)}} = -\beta_t^{(T-t)},$$

expressed in present-value terms, when $\beta_t^{(T-t)} < -\epsilon$. Proposition B.1 implies that the number $\hat{Z}_{n,t}^{(T-t)}$ of units of the bond maturing at time T and held at time t by a type 2 investor born at time 0 is given by

$$N^{-1}\big(\hat{Z}_{n,t}^{(T-t)}\big) = P_t^{(T-t)}$$

when $P_t^{(T-t)} < e(1-\epsilon)$. This yields the demand

$$P_t^{(T-t)}\hat{Z}_{n,t}^{(T-t)} = P_t^{(T-t)}N\big(P_t^{(T-t)}\big) = -\log\big(P_t^{(T-t)}\big),$$

expressed in present-value terms. The aggregate demand, expressed in present-value terms, across type 1 and type 2 investors when $\beta_t^{(T-t)} < -\epsilon$ and $P_t^{(T-t)} < e(1-\epsilon)$ is

$$-\log\big(P_t^{(T-t)}\big) - \beta_t^{(T-t)}$$

and coincides with the demand (5)–(7) with $\alpha(\tau) = 1$. Condition $\beta_t^{(T-t)} < -\epsilon$ requires that the demand intercept in (5) is negative (smaller than $-\epsilon$). Condition $P_t^{(T-t)} < e(1-\epsilon)$ requires that zero-coupon bonds trade below $e(1-\epsilon)$ and, hence, below par value. The probability of the set of values of q_t such that the two conditions hold simultaneously can be made arbitrarily close to 1 if $\bar{r}$ is sufficiently large and $\theta_0(\tau)$ is sufficiently small.

Proposition B.1 and the subsequent analysis require $K \geq 1$. To extend them to $K = 0$, we assume that $\hat{\beta}_{n,t}^{(T-t)}$ is equal to a deterministic function of $T-t$ plus random noise that is independent across investors n in the same generation. Because of the random noise,

$\hat{\beta}_{n,t}^{(T-t)}$ is not perfectly correlated with r_t, and the proof of Proposition B.1 goes through. Because the random noise is independent across investors in the same generation, $\hat{\beta}_{n,t}^{(T-t)}$ averages to a deterministic function of $T-t$.

APPENDIX C: CALIBRATION

The material in Appendix C is included in the Full Appendix, available at http://personal.lse.ac.uk/vayanos/Papers/PHMTSIR_ECMAf.pdf.

Co-editor Giovanni L. Violante handled this manuscript.

Manuscript received 30 June, 2019; final version accepted 6 September, 2020; available online 18 September, 2020.