Econometrica Supplementary Material

SUPPLEMENT TO "QUANTIFYING CONFIDENCE"

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S.1. ADDITIONAL MATERIAL FOR SECTIONS 3 AND 4

IN THIS APPENDIX, we provide a few results that complement the analysis in Sections 3 and 4.

Heterogeneous versus Common Priors (Continued)

In Section 3.3 of the main text, we established an observational equivalence between a special case of our heterogeneous-prior model and a common-prior variant featuring idiosyncratic uncertainty. We now elaborate on the bounds that this mapping can impose on the magnitude and the persistence of the ξ_t shock in our setting.

Suppose that we had data that allowed the estimation of the AR(1) process described in condition (3.17). Suppose next that we possessed information on the value for $\tilde{\sigma}_a$, perhaps from microeconomic observations. This information could be used in Proposition 2 to derive the bounds on (φ, ψ) and then, using Corollary 1, to get bounds on (ρ_ξ, σ_ξ).

Figure 7 depicts these bounds. To construct this figure, we let $\nu = 0.5$ and $\tilde{\sigma}_a = 0.2$. The latter value is based on the observation that $\tilde{\sigma}_a$ determines the uncertainty that islands face about their terms of trade (demand for their products), and may thus be proxied by the idiosyncratic risk that the typical firm faces about its productivity and sales.[1] In the left panel of the figure, we plot the set of the (φ, ψ) pairs that satisfy the bounds in Proposition 2, under the assumed value for $\tilde{\sigma}_a$. Using Corollary 1, we can translate this set into corresponding values for (ρ_ξ, σ_ξ).

In the right panel of the figure, we plot a more useful transformation of this set: instead of measuring σ_ξ on the vertical axis, we measure the corresponding value of σ_y, where σ_y henceforth stands for the standard deviation of the business-cycle component of output (i.e., of output bandpass filtered over 6–32 quarters) that is accounted by the confidence shock. Finally, the dot indicates the values of φ (in the left panel) and of σ_y (in the right panel) that obtain when we fix $\rho_\xi = 0.75$ and calibrate the volatilities of the confidence shock and the technology shock in the model so as to match the volatilities of aggregate output and employment in the data. The figure shows that under a plausible value for

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[1]Empirical estimates of the volatility of firm-level productivity suggest setting $\tilde{\sigma}_a$ between 0.2 and 0.43 (Abraham and White (2006), Foster, Haltiwanger, and Syverson (2008)). In a similar setting as ours, Huo and Takayama (2015) used a value of 0.14.

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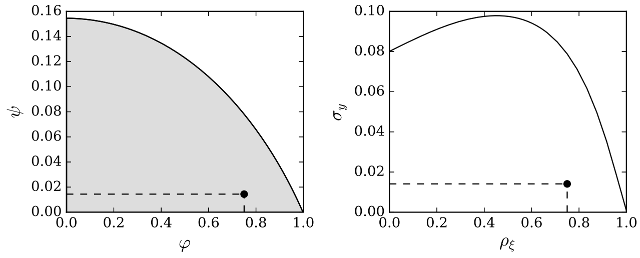

FIGURE 7.—The bounds on persistence and volatility.

$\tilde{\sigma}_a$, the range of values for σ_ξ that would be consistent with the restrictions imposed by a common-prior specification is very large.

Note that the relevant bound remains large even for lower values of $\tilde{\sigma}_a$, say 4%. Such a value would not appear implausibly large even if we confined first-order uncertainty to concern aggregate fundamentals. For instance, this value is only about twice as large as the standard deviation of the quarterly innovations in the aggregate Solow residual. Furthermore, as the behavior in the richer models used in the quantitative exercises in this paper is forward-looking, it seems more appropriate to think about a present-value measure of the uncertainty in fundamentals, as opposed to merely the quarter-by-quarter changes. Therefore, even though we cannot extend the results of this subsection to such richer models, we feel confident that our quantitative findings are consistent with realistic common-prior models. The recent work of Huo and Takayama (2015) seems to corroborate this conjecture. That said, there is no reason to view our approach *exclusively* as a proxy for incomplete information and rational confusion.

The Confidence Shock in the Baseline New Keynesian Model

In Section 4 of the main text, we compared the co-movement patterns generated by the confidence shock to those of a few alternative shocks within the context of the baseline RBC model. We now extend the comparison to the baseline New Keynesian model. The latter is obtained from the former by adding monopoly power, sticky prices, and a Taylor rule for monetary policy.

Table V revisits the exercise conducted in Table I. The preferences, the technology, and the confidence shock remain as before; the monopoly distortion is offset by a subsidy; the Calvo parameter is set to 0.75; and the Taylor rule is specified as $R_t = \phi_\pi \pi_t$ with $\phi_\pi = 1.5$.

The following key findings emerge. First, the good and superior to other shocks empirical performance of the confidence shock survives as we move from the RBC model to the New Keynesian model. Second, with the exception of the monetary shock, none of the competing shocks is able to generate realistic co-movement patterns in the relevant quantities. Finally, the similarity between the real effects of the confidence shock and those of the monetary shock provide further justification for our claim that the confidence plays a similar role in the RBC framework as demand shocks

Belief-Driven Wedges

In this section, we derive the predictions of our theories about the wedges. We consider both the overall wedges between the marginal rates of substitution and the corresponding

TABLE V

CONDITIONAL CO-MOVEMENTS (6–32 QUARTERS)[a]

	Filtering		Our Mechanism		Alternative Mechanisms				
	(a)	(b)	(c)	(d)	I Shock	C Shock	News	E Shock	M Shock
σ_n/σ_y	0.87	1.07	1.43	1.43	1.44	1.44	0.87	0.70	1.44
σ_c/σ_y	0.56	0.52	0.25	0.22	0.44	1.01	0.16	0.19	0.13
σ_i/σ_y	3.54	3.65	3.92	4.10	6.09	8.26	4.80	4.30	4.52
$\sigma_{y/n}/\sigma_y$	0.40	0.63	0.44	0.44	0.47	0.49	0.18	0.33	0.45
$\text{corr}(c, y)$	0.86	0.85	0.85	0.76	−0.83	−0.94	−0.16	0.60	0.38
$\text{corr}(i, y)$	0.94	0.95	0.99	0.99	0.99	0.99	0.99	0.99	0.99
$\text{corr}(n, y)$	0.91	0.82	0.99	1.00	0.99	0.98	0.99	0.99	1.00
$\text{corr}(c, n)$	0.86	0.75	0.81	0.70	−0.90	−0.99	−0.25	0.47	0.31
$\text{corr}(i, n)$	0.85	0.81	0.99	1.00	1.00	1.00	1.00	1.00	1.00
$\text{corr}(c, i)$	0.75	0.79	0.78	0.67	−0.90	−0.98	−0.27	0.50	0.29
$\text{corr}(y, y/n)$	−0.03	0.21	−0.96	−0.96	−0.90	−0.86	0.76	0.94	−0.97
$\text{corr}(n, y/n)$	−0.43	−0.40	−0.98	−0.98	−0.95	−0.93	0.67	0.87	−0.99
σ_π/σ_y	0.16	0.42	–	0.07	0.22	0.04	0.07	0.03	0.10
σ_R/σ_y	0.24	0.72	–	0.10	0.34	0.06	0.10	0.05	0.02
$\text{corr}(y, \pi)$	0.21	−0.90	–	0.96	0.99	0.42	0.37	0.84	0.99
$\text{corr}(y, R)$	0.38	−0.81	–	0.96	0.99	0.42	0.37	0.84	−0.38

[a]Columns (a) and (b) refer to the residuals that obtain, respectively, from the projection of the data on current and past TFP and from the removal of the technology shock identified in the same way as in Galí (1999). Column (c) refers to the predictions of our baseline model and column (d) to those of its New Keynesian variant. All other columns refer to alternative New Keynesian models.

marginal rates of transformation, and their decomposition in household- and firm-side wedges.

Let us fill in the details. First, denote with $MRSN_t \equiv \nu N_t + \gamma C_t$ the measured marginal rate of intra-temporal substitution between leisure and consumption; with $MRSC_{t,t+1} \equiv \gamma(C_{t+1} - C_t)$ the measured marginal rate of inter-temporal substitution in consumption; with $MPL_t \equiv Y_t - N_t$ the measured marginal product of labor; and with $MPK_t = Y_t - K_t$ the measured marginal product of capital. Next, define the wedges τ_t^{nh}, τ_t^{kh}, τ_t^{nf}, and τ_t^{kf} so that the following conditions hold:

$$MRSN_t = w_t - \tau_t^{nh}, \qquad \mathbb{E}_t[MRSC_{t,t+1}] = \big(1 - \beta(1-\delta)\big)\big(R_t - \tau_t^{kh}\big), \tag{S.1}$$

$$MPL_t = w_t + \tau_t^{nf}, \qquad \mathbb{E}_t[MPK_{t+1}] = R_t + \tau_t^{kf}. \tag{S.2}$$

This means that τ_t^{nh} and τ_t^{kh} can be interpreted as taxes paid by the household on labor income and on the return to savings, while τ_t^{nf} and τ_t^{kf} can be interpreted as taxes paid by the firm on the use of labor and capital. We finally measure the total labor wedge by $\tau^n \equiv \tau_t^{nh} + \tau^{nf}$ and the total capital wedge by $\tau^k \equiv \tau_t^{kh} + \tau^{kf}$.

When the data are generated by the plain-vanilla RBC model, all the wedges are zero. At the other extreme, the wedges can be arbitrary stochastic processes if the data are generated by a medium-scale model that lets each of the optimality conditions of the RBC model be perturbed by a different shock. Our model is in between these two extremes, arguably closer to the plain-vanilla RBC model than to DSGE models such as Smets and Wouters (2007): the wedges differ from zero but they are all linear functions of the underlying confidence shock.

Furthermore, as shown next, $\xi_t > 0$ maps to $\tau_t^{nh} > 0$, $\tau_t^{kh} > 0$, $\tau_t^{nf} < 0$, and $\tau_t^{kf} < 0$. That is, whenever there is a boost in confidence, it is *as if* the household faces a positive tax on its supply of labor and savings, while the firm faces a positive subsidy on its use of labor and capital services. The first property reflects the excessive optimism that the households have about their income during a confidence-driven boom; the second property reflects the excessive optimism that the firms have about the demand for their product and their terms of trade. Finally, the combination of these forces gives rise to a pro-cyclical labor wedge and a counter-cyclical capital wedge, in line with the U.S. data.

The Labor Wedge on the Household Side

Consider first τ_t^{nh}, which is defined as the equivalent of a labor-income tax faced by the as-if representative household:

$$\tau_t^{nh} \equiv w_t - MRSN_t = w_t - (C_t + \nu N_t).$$

In the equilibrium of our model, the household of every island i equates the local wage to the local expectation of its marginal rate of substitution between consumption and leisure:

$$w_{i,t} = \mathbb{E}_{it}[c_{i,t}] - \nu n_{i,t}.$$

In addition, the realized outcomes satisfy $w_{i,t} = w_t$, $n_{i,t} = N_t$, and $c_{i,t} = C_t$ for all i. It follows that

$$\tau_t^{nh} = \mathbb{E}_{it}[c_{i,t}] - C_t = \mathbb{E}_{it}[c_{i,t}] - c_{it} \quad \forall i,$$

which reveals that τ_t^{nh} captures the excessive optimism (during a boom) or pessimism (during a recession) of the households about their own consumption. Condition (3.13) in the main text, together with the fact that $k_{it} = K_t$ for all i, implies that $c_{it} = \Gamma_K^c K_t + \Gamma_z^c z_{it} + \Gamma_{\bar{z}}^c \bar{z}_t + \Gamma_a^c A_t + \Gamma_\xi^c \xi_t$ and therefore $\mathbb{E}_{it}[c_{it}] = \Gamma_K^c K_t + (\Gamma_z^c + \Gamma_{\bar{z}}^c + \Gamma_a^c) A_t + (\Gamma_\xi^c + \Gamma_{\bar{z}}^c)\xi_t$. Realized consumption, on the other hand, is given by $C_t = \Gamma_K^c K_t + (\Gamma_z^c + \Gamma_{\bar{z}}^c + \Gamma_a^c) A_t + \Gamma_\xi^c \xi_t$. Combining, we infer that

$$\tau_t^{nh} = \Gamma_{\bar{z}}^c \xi_t.$$

The adopted parameterization implies $\tau_t^{nh} = 0.0152\xi_t$.

The Labor Wedge on the Firm Side

Consider next τ_t^{nf}, which is defined as the equivalent of a payroll tax faced by the as-if representative firm:

$$\tau_t^{nf} \equiv MPL_t - w_t = (Y_t - N_t) - w_t.$$

In the equilibrium of our model, the firm of every island i equates the local wage to the local expectation of the marginal revenue product of labor:

$$w_{i,t} = \mathbb{E}_{it}[MRPL_{it}] = \mathbb{E}_{it}[p_{it} + y_{it} - n_{it}] = \mathbb{E}_{it}[Y_t] - n_{it} \quad \forall i.$$

In addition, the realized outcomes satisfy $w_{it} = w_t$ and $n_{it} = N_t$ for all i. It follows that

$$\tau_t^{nf} = Y_t - \mathbb{E}_{it}[Y_t] \quad \forall i,$$

which reveals that τ_t^{nf} captures the excessive optimism or pessimism of the firms about aggregate income and the resulting demand for the local good. Using conditions (3.12) and (3.15) from the main text along with the production function, we have that $Y_t =$

$\Gamma^y_K K_t + \Gamma^y_a A_t + \Gamma^y_{\bar z}\bar z_t + \Gamma^y_\xi \xi_t$ and therefore $\mathbb{E}_{it}[Y_t] = \Gamma^y_K K_t + (\Gamma^y_a + \Gamma^y_{\bar z})A_t + (\Gamma^y_{\bar z} + \Gamma^y_\xi)\xi_t$. It follows that

$$\tau^{nf}_t \equiv -\Gamma^y_{\bar z}\xi_t.$$

For our parameterization, we have $\tau^{nf}_t = -0.2548\xi_t$.

The Capital Wedge on the Firm Side

Consider now τ^{kf}_t, which is defined as the equivalent of an investment tax faced by the as-if representative firm:

$$\tau^{kf}_t \equiv \mathbb{E}_t[MPK_{t+1}] - R_t = \mathbb{E}_t[Y_{t+1}] - K_{t+1} - R_t,$$

where $\mathbb{E}_t$ is the rational (or objective) expectation operator. In the equilibrium of our model,

$$R_t = \mathbb{E}'_{it}[MRPK_{i,t+1}] = \mathbb{E}'_{it}[p_{i,t+1} + y_{i,t+1} - k_{i,t+1}] = \mathbb{E}'_{it}[Y_{t+1}] - k_{i,t+1} \quad \forall i,$$

where $\mathbb{E}_{i,t}$ is the subjective expectation operator in the morning of period t. It follows that

$$\tau^{kf}_t = \mathbb{E}_t[Y_{t+1}] - \mathbb{E}'_{it}[Y_{t+1}] \quad \forall i,$$

which reveals that τ^{kf}_t captures the excessive optimism or pessimism of the firms about aggregate income and demand next period. Using similar steps as before, we can show that

$$\tau^{kf}_t = -\Gamma^y_{\bar z}\rho\xi_t,$$

where $\Gamma^y_{\bar z}$ is the elasticity of the realized income of each island with respect to the realized average signal. For our calibration, we have $\tau^{kf}_t = -0.1911\xi_t$.

The Savings Wedge on the Household Side

Consider τ^{kh}_t, which is defined as the tax on the returns to savings faced by the as-if representative household:

$$\tau^{kh}_t \equiv R_t - \frac{1}{1-\beta(1-\delta)}\mathbb{E}_t[MRSC_{t,t+1}] = R_t + \frac{\gamma}{1-\beta(1-\delta)}\mathbb{E}_t[C_{t+1} - C_t].$$

In the equilibrium of our model,

$$R_t = \frac{1}{1-\beta(1-\delta)}\mathbb{E}'_{it}[MRSC_{i,t,t+1}] = \frac{\gamma}{1-\beta(1-\delta)}\mathbb{E}'_{it}[c_{i,t+1} - c_t],$$

where $\mathbb{E}'_{it}$ is the subjective operator in the afternoon of period t. It follows that

$$\begin{aligned}\tau^{kh}_t &= \frac{\gamma}{1-\beta(1-\delta)}\big(\mathbb{E}'_{it}[c_{it+1}] - \mathbb{E}_t[C_{t+1}]\big)\\ &= \frac{\gamma}{1-\beta(1-\delta)}\big(\mathbb{E}'_{it}[c_{it+1}] - \mathbb{E}_t[c_{i,t+1}]\big),\end{aligned}$$

which reveals that τ_t^{kh} captures the excessive optimism or pessimism of the households about their future consumption. From the policy rules for individual and aggregate consumption:

$$\mathbb{E}'_{it}[c_{it+1}] = \Gamma_K^c K_{t+1} + \left(\Gamma_z^c + \Gamma_{\bar{z}}^c + \Gamma_a^c\right) A_t + \left(\Gamma_{\bar{z}}^c + \Gamma_\xi^c\right)\rho\xi_t,$$
$$\mathbb{E}_t[C_{t+1}] = \Gamma_K^c K_{t+1} + \left(\Gamma_z^c + \Gamma_{\bar{z}}^c + \Gamma_a^c\right) A_t + \Gamma_\xi^c \rho\xi_t.$$

Combining, we infer that

$$\tau_t^{kh} = \frac{\Gamma_{\bar{z}}^c \rho}{1-\beta(1-\delta)}\xi_t.$$

For our parameterization, we have $\tau_t^{kh} = 0.3277\xi_t$.

The Total Wedges in the Model

Combining the above results, we conclude that the total labor wedge in the calibrated version of our baseline model is given by $\tau_t^n = \tau_t^{nh} + \tau_t^{nf} = -0.2396\xi_t$, whereas the total capital wedge is given by $\tau_t^k = \tau_t^{kh} + \tau_t^{kf} = 0.1366\xi_t$. That is, the labor wedge is negatively correlated with the confidence shock, and therefore counter-cyclical, while the capital wedge is positively correlated with the confidence shock, and therefore pro-cyclical.

In the main text, we claimed that both of these predictions are driven by the fact that the ξ_t shock shifts the perceptions of short-run returns without moving much the perceptions of permanent income. Let us now explain why this is the case. As noted above, our model predicts that the wedges for firms and households move in opposite directions. Furthermore, the pro-cyclicality of τ_t^{nh} is tied to the effect of the confidence shock on perceived permanent income, while the counter-cyclicality of $\tau_t^n f$ is tied to the effect on the perceived marginal return to labor. For the reasons already explained, the latter effect dominates the former. Consequently, the overall labor wedge, τ_t^n, is predicted to be counter-cyclical. The opposite is true for the capital wedge, τ_t^k. To see why, note first that the Euler condition equates expected consumption growth with a quantity that is equal to unity plus the expected return to capital. Note next that, while the variation in τ_t^{kf} is of similar magnitude to the variation in τ_t^{nf}, it represents a small component in the aforementioned quantity, and is therefore overwhelmed by the variation in τ_t^{kh}, which captures the household's optimism and pessimism about future consumption. It follows that τ_t^k shares the cyclical properties of τ_t^{kh}, that is, the total capital wedge is pro-cyclical.

Estimation of Wedges in the Data

We now turn attention to the estimation of the wedges in U.S. data. This is done in a similar fashion as in Chari, Kehoe, and McGrattan (2007).

The estimation is based on the baseline RBC model, augmented with ad hoc stochastic processes for the following four wedges: an efficiency wedge, τ_t^e, a labor wedge, τ_t^n, a capital wedge, τ_t^k, and a resource wedge, τ_t^g. Accordingly, the system to be estimated is the following:

$$\nu N_t + C_t = Y_t - N_t - \tau_t^n, \tag{S.3}$$

$$\mathbb{E}_t[C_{t+1}] - C_t = \left(1-\beta(1-\delta)\right)\left(\mathbb{E}_t[Y_{t+1} - K_{t+1}] - \tau_t^k\right), \tag{S.4}$$

$$Y_t + (1-\delta)K_t = C_t + K_{t+1} + \tau_t^g, \tag{S.5}$$

$$Y_t = \tau_t^e + \alpha K_t + (1-\alpha)N_t. \tag{S.6}$$

We set the structural parameters ν, α, β, and δ to the values chosen in our baseline calibration. As in Chari, Kehoe, and McGrattan (2007), we assume that the vector $T_t = (\tau_t^e, \tau_t^n, \tau_t^k, \tau_t^g)'$ follows a VAR(1) process of the form

$$T_t = \Phi T_{t-1} + \mathcal{E}_t,$$

where Φ is a matrix, $\mathcal{E}_t = (\varepsilon_t^e, \varepsilon_t^n, \varepsilon_t^k, \varepsilon_t^g)'$ is normally distributed with $\mathbb{E}(\mathcal{E}_t) = 0$ and $\mathbb{E}(\mathcal{E}_t\mathcal{E}_t') = \Omega\Omega'$, and Ω is a lower-triangular matrix. We finally estimate the matrices Φ and Ω using data on GDP, investment, hours, and the difference between GDP and the sum of investment and consumption, over the period 1960Q1–2007Q4. The estimation yields

$$\Phi = \begin{pmatrix} 0.6537 & 0.1184 & 0.2268 & 0.0049 \\ -0.2487 & 1.0716 & 0.1605 & 0.0089 \\ -0.2808 & 0.0883 & 1.1620 & 0.0068 \\ 0.2017 & -0.1390 & -0.1741 & 0.9829 \end{pmatrix} \quad \text{and}$$

$$\Omega = \begin{pmatrix} 0.6148 & 0.0000 & 0.0000 & 0.0000 \\ 0.2580 & 0.8828 & 0.0000 & 0.0000 \\ 0.6261 & -0.3505 & 0.1793 & 0.0000 \\ 0.2492 & 0.2278 & 0.4964 & 1.5210 \end{pmatrix},$$

and results to the moments reported in Table VI. We thus see that the labor wedge is counter-cyclical and the capital wedge pro-cyclical, just as predicted by our theory.

S.2. DATA

In this appendix, we describe the data we use in this paper to obtain the various business-cycle moments and to estimate the models considered in Section 5.

Table VII summarizes the data, all of which are from FRED, the Economic Database of the Federal Reserve Bank of Saint-Louis. GDP, Y, is measured by the seasonally adjusted GDP. Consumption, C, is measured by the sum of personal consumption expenditures in nondurables goods (CND) and services (CS). Investment, I, is measured by the sum of personal consumption expenditures on durables goods (CD), fixed private investment (FPI), and changes in inventories (DI). Government Spending, G, is measured by government consumption expenditures (GCE). Hours worked, N, are measured by hours of all persons in the non-farm business sector. Labor productivity, Y/N, is measured by real output per hour of all persons in the non-farm business sector. The inflation rate, π, is the log-change in the implicit GDP deflator. The nominal interest rate, R, is the effective federal funds rate measured on a quarterly basis. Given that the effective federal funds rate is available at the monthly frequency, we use the average over the quarter (denoted

TABLE VI

WEDGES IN THE DATA

	Efficiency	Labor	Capital
Standard deviation	0.86	1.40	1.04
Correlation with output	0.78	−0.57	0.91

TABLE VII

DESCRIPTION OF THE DATA

Data	Formula
GDP	`Y=GDP/(GDPDEF×CNP16OV)`
Consumption	`C=(CND+CS)/(GDPDEF×CNP16OV)`
Investment	`I=(CD+FPI+DI)/(GDPDEF×CNP16OV)`
Government Spending	`G=GCE/(GDPDEF×CNP16OV)`
Hours Worked	`H=HOANBS/CNP16OV`
Labor Productivity	`GDP/H`
Inflation Rate	$\pi = \log(\mathrm{GDPDEF}) - \log(\mathrm{GDPDEF})_{-1}$
Nominal Interest Rate	`R=FEDFUNDS/4`

Mnemonic	Source
GDP	http://research.stlouisfed.org/fred2/series/GDP
CND	http://research.stlouisfed.org/fred2/series/PCND
CD	http://research.stlouisfed.org/fred2/series/PCEDG
CS	http://research.stlouisfed.org/fred2/series/PCESV
FPI	http://research.stlouisfed.org/fred2/series/FPI
DI	http://research.stlouisfed.org/fred2/series/CBI
GCE	http://research.stlouisfed.org/fred2/series/GCE
HOANBS	http://research.stlouisfed.org/fred2/series/HOANBS
GDPDEF	http://research.stlouisfed.org/fred2/series/GDPDEF
FEDFUNDS	http://research.stlouisfed.org/fred2/series/FEDFUNDS
CNP16OV	http://research.stlouisfed.org/fred2/series/CNP16OV

FEDFUNDS). Finally, when relevant, Total Factor Productivity (TFP) is measured as in Fernald (2014), which adjusts for utilization.

The sample ranges from the first quarter of 1960 to the last quarter of 2007. We dropped the post-2007 data because the models we study are not designed to deal with the financial phenomena that appear to have played a more crucial role in the recent recession as opposed to earlier times. All quantities are expressed in real, per capita terms—that is, deflated by the implicit GDP deflator (GDPDEF) and by the civilian non-institutional population (CNP16OV). Because the latter is reported monthly, we used the last month of each quarter as the quarterly observation.

S.3. ADDITIONAL MATERIAL FOR SECTION 5

This appendix contains additional material regarding the two estimated models in Section 5. Table VIII reports the priors and the posteriors of the estimated parameters. Figures 8 and 9 report the IRFs of our estimated models with respect to all the structural shocks. Tables IX and X report the estimated contribution of the shocks to, respectively, the variances and the co-variances of the key variables at business-cycle frequencies. The confidence shock is omitted here, because its contributions were reported in the main text.

S.4. ESTIMATING ϱ AND σ_ξ

In the main text, we noted that the data considered in Section 5 do not allow us to identify separately the standard deviation of the confidence shock and the degree of strategic complementary. Nevertheless, this may be achieved if the data set were augmented to

TABLE VIII
ESTIMATED PARAMETERS

	Priors			Posteriors			
				Flexible Price Model		Sticky Price Model	
	Shape	Mean	Std. Dev.	Median	90%HPDI	Median	90%HPDI
ν	G	0.500	0.200	0.456	[0.226, 0.814]	0.282	[0.161, 0.429]
α	B	0.300	0.150	0.261	[0.234, 0.286]	0.255	[0.229, 0.280]
ψ	B	0.500	0.200	0.576	[0.255, 0.856]	0.500	[0.315, 0.708]
φ	G	2.000	1.000	3.370	[2.026, 5.346]	3.312	[1.917, 5.394]
b	B	0.500	0.200	0.860	[0.809, 0.899]	0.758	[0.649, 0.836]
χ	B	0.660	0.100	–	–	0.732	[0.673, 0.782]
κ_R	B	0.600	0.200	–	–	0.198	[0.072, 0.371]
κ_π	N	1.700	0.300	–	–	2.271	[1.901, 2.660]
κ_y	N	0.125	0.050	–	–	0.121	[0.052, 0.199]
ρ_a	B	0.500	0.200	0.394	[0.126, 0.747]	0.412	[0.115, 0.846]
ρ_n	B	0.500	0.200	0.309	[0.113, 0.545]	0.224	[0.075, 0.428]
ρ_i	B	0.500	0.200	0.365	[0.136, 0.626]	0.374	[0.155, 0.604]
ρ_c	B	0.500	0.200	0.477	[0.175, 0.786]	0.888	[0.802, 0.964]
ρ_g	B	0.500	0.200	0.787	[0.588, 0.921]	0.786	[0.632, 0.902]
ρ_m	B	0.500	0.200	–	–	0.647	[0.471, 0.753]
ρ_ξ	B	0.500	0.200	0.620	[0.369, 0.804]	0.833	[0.717, 0.911]
σ_a^{P}	IG	1.000	4.000	0.396	[0.270, 0.565]	0.406	[0.278, 0.569]
σ_a^{T}	IG	1.000	4.000	0.338	[0.239, 0.489]	0.347	[0.244, 0.498]
σ_n	IG	1.000	4.000	0.376	[0.266, 0.521]	0.378	[0.263, 0.520]
σ_i^{P}	IG	1.000	4.000	0.845	[0.358, 2.252]	0.610	[0.321, 1.306]
σ_i^{T}	IG	1.000	4.000	5.961	[2.046, 11.657]	5.805	[2.839, 11.029]
σ_c	IG	1.000	4.000	0.658	[0.327, 2.676]	0.357	[0.244, 0.564]
σ_g	IG	1.000	4.000	1.675	[1.387, 2.072]	1.705	[1.431, 2.076]
σ_m	IG	1.000	4.000	–	–	0.313	[0.256, 0.388]
σ_ξ	IG	1.000	4.000	1.798	[1.208, 2.839]	0.613	[0.348, 1.194]

[a]B, G, IG, N stand, respectively, for Beta, Gamma, Inverse Gamma, and Normal distribution.

include data on expectations. In this appendix, we elaborate on these points and describe the "augmented estimation" that motivates the value of ϱ used in Section 5.

To illustrate the main identification issue, consider again the example studied in Sections 3.2 and 3.3. From conditions (3.17) and (3.18), we see that the volatility of the non-fundamental (confidence-driven) innovations in output is given by

$$\mathrm{Var}\big(Y_t - Y_t^* | \text{history}\big) = \psi^2 = \frac{\omega^2}{(1-\omega)^4}\sigma_\xi^2, \tag{S.7}$$

where Y_t^* is the fundamental (TFP-driven) component, σ_ξ is the standard deviation of the confidence shock, and ω is the degree of strategic complementarity. Under the assumption, made in the baseline model, that the CES aggregator across the islands is Cobb–Douglas, ϱ is unity. Relaxing this assumption gives ω as a monotone function of ϱ. From condition (6.20), it is then evident that exactly the same non-fundamental volatility in output can be accounted for by a continuum of values for the pair (ϱ, σ_ξ). This is the crux of the identification issue faced in Section 5: the models of that section are more complicated, something that hinders analytical results, but the issue remains the same.

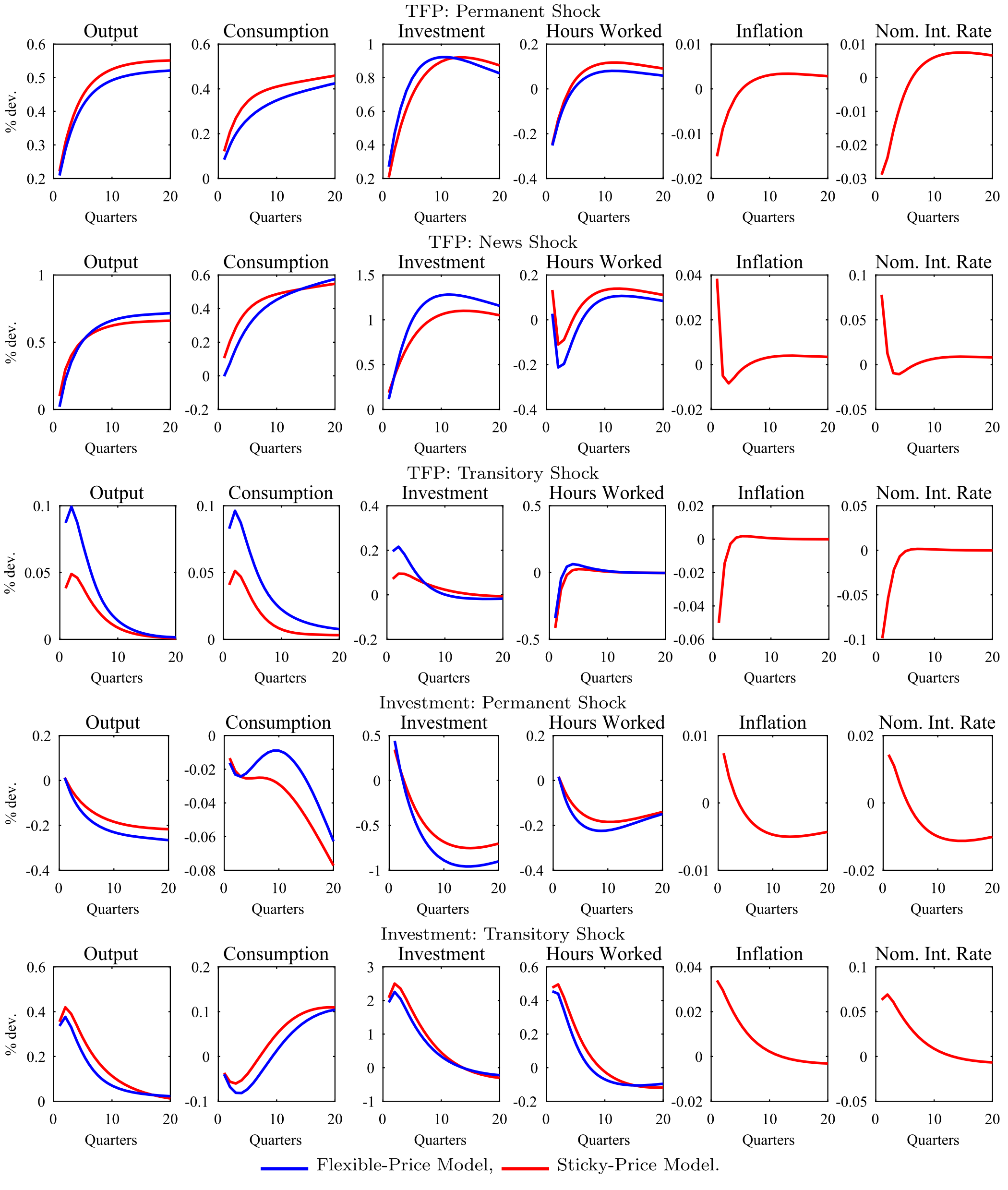

FIGURE 8.—Theoretical IRFs, Part I.

To illustrate how data on expectations could possibly aid identification, aggregate condition (3.11) to obtain the following equation:

$$N_t - N_t^* = \omega \cdot \bar{\mathbb{E}}_t[N_t - N_t^*],$$

where N_t^* denotes the fundamental component of employment. This condition reveals how expectations of employment (or some other variable) together with a measure of its

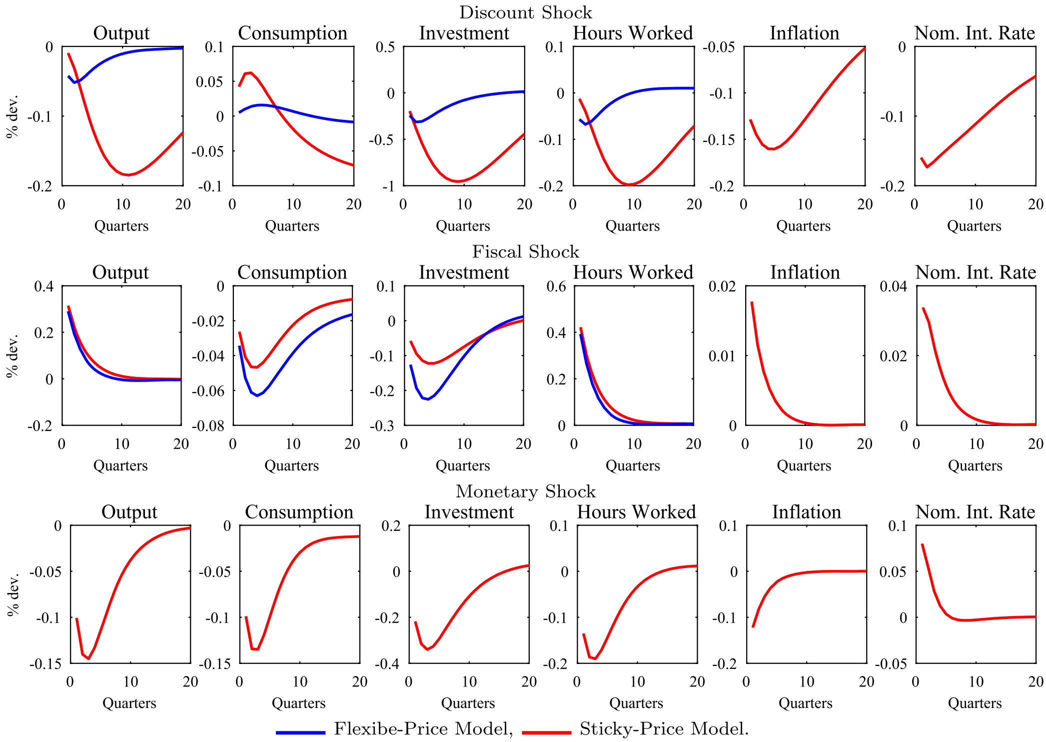

FIGURE 9.—Theoretical IRFs, Part II.

"fundamental" component can be used to identify the degree of strategic complementarity, and therefore ϱ.

The procedure, though, is fraught with difficulties. Unlike the example discussed above, the models of Section 5 have expectations mattering through multiple horizons and multiple channels. It is not clear how to combine these expectations into a single measure, or how to map the theoretical objects to the available empirical measures. For instance, the University of Michigan Index of Consumer Sentiment, which is known to forecast future employment and output, is constructed on the basis of answers to qualitative questions that do not have an immediate counterpart in the theory.

These challenges, in combination with the desire to stay as close as possible to standard practice, account for our choice to estimate the models of Section 5 on the macroeconomic data alone. Note, though, that this choice does not matter for the estimated contribution of the confidence shock to the business cycle. Fixing the value of ϱ or allowing it to be estimated freely makes little difference for the shock's estimated contribution to the variances and covariances of the macroeconomic quantities.

Does the lack of identification of σ_ξ and ϱ pose a problem for our assertion that confidence shocks are a major driver of the business cycle? We think it does not. Not being able to rule out values of σ_ξ that seem implausibly high relative to the innovations in aggregate TFP and other fundamentals only implies that a narrow interpretation of the confidence shock as capturing mis-coordination and higher-order uncertainty may be tenuous. But it allows our shock to proxy for alternative kinds of waves of optimism and pessimism, for instance, irrational beliefs.

TABLE IX

CONTRIBUTION OF SHOCKS TO VOLATILITIES (6–32 QUARTERS)

	Y	C	I	N	π	R
Permanent TFP Shock						
Flexible Price	9.05	8.25	3.13	2.91	–	–
Sticky Price	9.66	12.50	1.99	2.94	0.34	1.02
In the absence of belief shock						
Flexible Price	16.45	28.30	4.50	1.30	–	–
Sticky Price	19.84	23.23	6.05	5.20	0.11	0.16
Transitory TFP Shock						
Flexible Price	0.95	2.14	0.40	2.05	–	–
Sticky Price	0.22	0.52	0.08	2.86	1.69	5.26
In the absence of belief shock						
Flexible Price	12.98	3.75	24.49	5.10	–	–
Sticky Price	0.17	0.39	0.07	7.38	0.03	0.79
Permanent Investment Shock						
Flexible Price	1.81	0.30	4.30	2.15	–	–
Sticky Price	0.98	0.09	2.34	1.13	0.13	0.38
In the absence of belief shock						
Flexible Price	49.70	39.12	26.04	78.87	–	–
Sticky Price	1.17	0.44	4.05	2.48	0.12	0.12
Transitory Investment Shock						
Flexible Price	12.94	3.90	43.52	15.16	–	–
Sticky Price	15.48	2.43	49.34	17.20	2.42	6.93
In the absence of belief shock						
Flexible Price	2.71	2.58	13.80	4.50	–	–
Sticky Price	18.58	1.38	62.23	40.70	0.16	1.59

	Y	C	I	N	π	R
News Shock						
Flexible Price	15.92	14.13	5.67	3.89	–	–
Sticky Price	13.72	16.51	2.67	1.40	0.64	1.90
In the absence of belief shock						
Flexible Price	13.37	22.39	5.52	3.06	–	–
Sticky Price	47.24	67.06	14.49	12.17	3.16	12.27
Discount Shock						
Flexible Price	0.27	0.10	0.89	0.35	–	–
Sticky Price	1.52	1.40	3.97	1.71	63.25	37.75
In the absence of belief shock						
Flexible Price	2.53	2.41	12.83	4.21	–	–
Sticky Price	8.28	3.45	12.84	20.72	96.38	55.48
Fiscal Shock						
Flexible Price	4.34	0.98	0.50	5.17	–	–
Sticky Price	4.98	0.48	0.12	5.94	0.40	1.19
In the absence of belief shock						
Flexible Price	2.26	1.45	12.81	2.97	–	–
Sticky Price	3.61	0.59	0.07	8.62	0.03	0.22
Monetary Policy Shock						
Flexible Price	–	–	–	–	–	–
Sticky Price	2.16	4.10	0.99	2.68	19.50	4.73
In the absence of belief shock						
Flexible Price	–	–	–	–	–	–
Sticky Price	1.10	3.46	0.20	2.73	0.02	29.37

TABLE X

CONTRIBUTION OF SHOCKS TO CO-MOVEMENTS (6–32 QUARTERS)

	Covariance					
	(Y,N)	(Y,I)	(Y,C)	(N,I)	(N,C)	(I,C)
Permanent TFP Shock						
Flexible Price	0.21	5.91	10.62	0.74	1.99	8.86
Sticky Price	0.86	4.82	13.99	1.24	2.46	10.03
In the absence of belief shock						
Flexible Price	4.60	11.52	26.05	3.04	9.24	23.50
Sticky Price	5.36	11.87	27.87	5.63	8.14	31.12
Transitory TFP Shock						
Flexible Price	−0.39	0.69	1.80	−0.34	−0.56	1.61
Sticky Price	−0.42	0.15	0.44	−0.19	−1.02	0.40
In the absence of belief shock						
Flexible Price	8.97	24.10	8.36	16.84	6.45	20.31
Sticky Price	−1.14	0.12	0.33	−0.25	−5.15	0.42
Permanent Investment Shock						
Flexible Price	2.38	2.75	−0.17	2.98	−0.02	−1.18
Sticky Price	1.23	1.57	−0.00	1.60	0.06	−0.39
In the absence of belief shock						
Flexible Price	70.47	43.08	52.57	57.90	88.89	52.91
Sticky Price	2.58	2.76	−0.39	4.10	−2.16	−1.82
Transitory Investment Shock						
Flexible Price	16.64	26.60	−7.33	30.25	−7.99	−19.91
Sticky Price	18.91	31.96	−7.28	34.04	−9.72	−21.39
In the absence of belief shock						
Flexible Price	3.92	8.25	−3.04	11.86	−5.02	−12.64
Sticky Price	41.57	43.07	−3.41	66.09	−18.16	−19.19

	Covariance					
	(Y,N)	(Y,I)	(Y,C)	(N,I)	(N,C)	(I,C)
News Shock						
Flexible Price	0.10	10.67	18.07	0.40	3.47	15.08
Sticky Price	0.09	6.80	19.28	0.58	0.46	13.40
In the absence of belief shock						
Flexible Price	5.44	10.27	20.70	6.11	8.03	18.60
Sticky Price	20.27	28.84	68.05	2.85	83.48	65.41
Discount Shock						
Flexible Price	0.37	0.54	−0.14	0.67	−0.17	−0.41
Sticky Price	1.90	2.63	0.16	3.02	−0.11	−1.46
In the absence of belief shock						
Flexible Price	3.68	7.67	−2.82	11.10	−4.68	−11.73
Sticky Price	20.08	13.04	6.75	21.11	30.17	21.36
Fiscal Shock						
Flexible Price	5.81	−0.90	−1.34	−1.10	−1.79	1.23
Sticky Price	6.47	−0.45	−1.34	−0.51	−1.91	0.49
In the absence of belief shock						
Flexible Price	2.92	−4.89	−1.82	−6.85	−2.93	9.05
Sticky Price	8.60	−0.22	−1.74	−0.36	−7.46	0.47
Monetary Policy Shock						
Flexible Price	–	–	–	–	–	–
Sticky Price	2.86	1.68	3.80	1.93	5.52	4.04
In the absence of belief shock						
Flexible Price	–	–	–	–	–	–
Sticky Price	2.67	0.51	2.54	0.83	11.15	2.22

Notwithstanding our preference for a broad interpretation of the confidence shock, we now describe an exercise that supports the more narrow interpretation and justifies the value of ϱ used in Section 5. Consider either one of the models of Section 5 and construct an "augmented" model by adding the following equation, for some $k \geq 0$:

$$\text{MCSI}_t = \lambda \bar{\mathbb{E}}_t[N_{t+k}] + \eta_t, \tag{S.8}$$

where N_{t+k} is aggregate employment k periods ahead, λ is a scalar, and η_t is a random variable, that is orthogonal to the confidence shock and other structural shocks, and that follows an AR(1) process $\eta_t = \rho_\eta \eta_{t-1} + \varepsilon_t^\eta$ where $\rho_\eta \in [0, 1)$ and $\varepsilon_t^\eta \rightsquigarrow \mathcal{N}(0, \sigma_\eta^2)$. We take MCSI_t as the theoretical counterpart of the University of Michigan Consumer Sentiment Index; η_t as measurement error, or as a crude proxy for misspecification in the "true" relation between the theory and the aforementioned index;[2] and λ as a scaling parameter.

Now let θ be the vector that collects all the parameters of the original model, inclusive of σ_ξ and ϱ. The parameters of the augmented model are given by the union of θ and $(\lambda, \sigma_\eta, \rho_\eta)$. Trying to estimate all the parameters jointly creates a new problem that prevents the MCMC from converging properly. This seems to be due to the fact that the same covariation between the sentiment index and the macroeconomic variables can be captured with different combinations of the scaling parameter λ, the volatility of the measurement error, and the degree of strategic complementarity. To cut the Gordian knot, we chose to impose an ad hoc identification restriction that requires the augmented model to produce a particular value for the share of the variance in MCSI_t that is accounted for by the measurement error η_t. This is equivalent to imposing one's prior on the noise-to-signal ratio in the sentiment index.

More specifically, for any $k \geq 0$, there exists a function v_k such that

$$\text{Var}\big(\bar{\mathbb{E}}_t[N_{t+k}]\big) = v_k(\theta).$$

This function is generated by the same system of equations as the one that pins down the equilibrium outcomes and is not affected by the addition of equation (S.8). It follows that the relative contribution of the measurement error in the theoretical counterpart of the sentiment index is given by

$$\frac{\text{Var}(\eta_t)}{\text{Var}(\text{MCSI}_t)} = \mathcal{M}\big(\lambda, \theta'\big) \equiv \frac{\sigma_\eta^2}{(1-\rho_\eta)\lambda^2 v_k(\theta) + \sigma_\eta^2},$$

where $\theta' \equiv (\theta, \sigma_\eta, \rho_\eta)$. For any θ' and any target $me \in (0, 1)$ for the contribution of the measurement error, solving the equation $\mathcal{M}(\lambda, \theta') = me$ gives the value of λ that is consistent with that target. Fixing a value for me is therefore equivalent to adding an identification restriction on the parameters of the augmented model; in that case, the MCMC converges properly and θ' is well identified for any given me. Our strategy is therefore to select various values for me and to estimate θ' on the data used in Section 5 together with the time series of the aforementioned index.

The results from the "augmented" estimation are reported in Table XI. Let us focus on the flexible-price model and consider two values for me, the share of the measurement error, and three values for k, the horizon of the expectations that show up in condition (S.8).

[2]In the data, the correlation of the Consumer Sentiment Index with hours worked attains a maximal value of about 0.7 when the former leads the latter by 3 quarters. To some extent, this corroborates the specification assumed above and suggests $k = 3$ as a possible benchmark.

TABLE XI

ESTIMATING BOTH σ_ξ AND ϱ

		Estimated Parameters			Variance Contribution of Confidence Shock			
k	me	ϱ	σ_ξ	σ_ξ^2/σ_a^2	Y	C	I	N
Flexible-Price Model								
0	0.25	0.70	0.64	0.99	54.26	54.45	50.29	84.56
	0.50	0.72	0.57	0.78	54.41	52.35	52.01	85.19
4	0.25	0.70	0.43	0.40	45.80	24.40	47.79	84.01
	0.50	0.70	0.45	0.44	46.26	29.53	49.25	84.78
20	0.25	0.79	0.44	0.41	58.09	46.31	61.04	85.33
	0.50	0.77	0.45	0.47	55.60	44.49	58.58	84.75
Sticky-Price Model								
0	0.25	0.60	0.58	0.27	12.37	13.52	10.04	21.69
	0.50	0.72	0.36	0.32	53.99	50.74	51.06	71.50
4	0.25	0.69	0.36	0.30	45.15	41.93	39.87	64.26
	0.50	0.72	0.36	0.30	46.12	43.28	43.05	67.51
20	0.25	0.65	0.38	0.33	50.80	39.60	46.87	70.36
	0.50	0.69	0.38	0.33	49.45	41.89	46.34	69.84

For each k and me (first two columns), the table reports the estimated values for ϱ and σ_ξ (next two columns), the ratio of the estimated σ_ξ to the estimated σ_a (fifth column), and the estimated contributions of the confidence shock to the business-cycle volatilities of output, consumption, investment, and hours (last four columns). The findings suggest a value of ϱ in the neighborhood of 0.75, which in turn motivates the value used in Section 5. Furthermore, the estimated σ_ξ is smaller than the estimated σ_a, allowing for a narrow interpretation of the confidence shock. Finally, the estimated contribution of the confidence shock to the business cycle is of the same magnitude as the one estimated in Section 5. There are, however, two notable differences: the confidence shock now explains a larger share of the volatility in hours and a smaller in consumption.

We find the results of this exercise useful even if they do not constitute proof that ϱ and σ_ξ lie in those ranges.

S.5. LOG-LINEAR SOLUTION

In this appendix, we explain how to augment a large class of DSGE models with our proposed type of higher-order belief dynamics and how to obtain the solution of the augmented model as a simple transformation of the solution of the original model.

A Prelude

Before considering the general case, it is instructive to review the linearized version of our baseline model.

The log-linearized equilibrium conditions of the model are given by (3.6)–(3.10) in Section 3 and have a familiar interpretation. The only novelty is the presence of two distinct expectation operators $\mathbb{E}_{it}$ and $\mathbb{E}'_{it}$, which denote local expectations in stage 1 and stage 2 of period t, respectively. The difference between these two expectation operators derives from the fact that islands form beliefs about one another's signals and thereby about Y_t in stage 1 on the basis of their misspecified priors, but observe the true state of nature and the true realized Y_t in stage 2. Under the supply first timing protocol, the first expectation shows up in the optimality condition for labor, while the second shows up in the optimality condition for consumption/saving.

The following points are worth emphasizing. The aggregate-level variables are, of course, obtained from averaging the individual-level variables across all islands. In equilibrium, the *realized* values of the aggregate variables coincide with the realized values of the corresponding individual variables; for example, $y_{it} = Y_t$ for all i, all t, and all realizations of uncertainty. This is because all islands receive the same signals and the same fundamentals. However, this does not mean that one can just replace the island-specific variables in the above conditions with the aggregate ones, or vice versa. Even though the "objective truth" is that all islands receive the same signals, in stage 1 of each period each island believes that the signals of other islands can differ from its own signal. Accordingly, each island reasons that y_{it} can differ from Y_t, even when all other islands follow the same strategy as itself and receive the same TFP shock.

Keeping track of this delicate difference between the realizations and the beliefs of different variables is key to obtaining the solution to the model. Our method deals with this delicate matter by (i) using appropriate notation to distinguish the signal received by each agent/island from either the average signal in the population or the true underlying shock to fundamentals; and (ii) choosing appropriate state spaces for both the individual policy rules and the aggregate ones.

In what follows, we first set up the general class of log-linear DSGE models that our solution method handles. We next introduce a class of linear policy rules, which describe the behavior of each agent as a function of his information set. Assuming that all other islands follow such policy rules, we can use the equilibrium conditions of the model to obtain the policy rules that are optimal for the individual island; that is, we can characterize the best responses of the model. Since the policy rules are linear, they are parameterized by a collection of coefficients (matrices), and the aforementioned best responses reduce to a system of equations in these coefficients. The solution to this system gives the equilibrium of the model.

A "Generic" DSGE Model

We henceforth consider an economy whose equilibrium is represented by the following linear dynamic system:

$$
\begin{aligned}
M_{yy}y_{it} &= M_{yx}x^b_{it} + M_{yX}X^b_t + M_{yY}\mathbb{E}_{it}Y_t + M_{yf}\mathbb{E}_{it}x^f_{it} + M_{yF}\mathbb{E}_{it}X^f_t + M_{ys}z_{it}, \\
M_{xx0}x^b_{it+1} &= M_{xx1}x^b_{it} + M_{xX1}X^b_t + M_{xy1}y_{it} + M_{xY1}Y_t + M_{xf1}x^f_{it} + M_{xF1}X^f_t + M_{xs1}s_t, \\
M_{ff0}\mathbb{E}'_{it}x^f_{it+1} &= M_{fF0}\mathbb{E}'_{it}X^f_{t+1} + M_{ff1}x^f_{it} + M_{fF1}X^f_t + M_{fx0}x^b_{it+1} + M_{fx1}x^b_{it} + M_{fX1}X^b_t \\
&\quad + M_{fy0}\mathbb{E}'_{it}y_{it+1} + M_{fY0}\mathbb{E}'_{it}Y_{t+1} + M_{fy1}y_{it} + M_{fY1}Y_t + M_{fs0}\mathbb{E}'_{it}s_{t+1} + M_{fs1}s_t, \\
s_t &= Rs_{t-1} + \varepsilon_t, \\
\xi_t &= Q\xi_{t-1} + \nu_t.
\end{aligned}
$$

This system is a generalization of the one we obtained in our baseline RBC model. Here, x^b, x^f, y, s, and ξ are allowed to be vectors; x^b collects backward-looking variables (such as capital in our model); x^f collects forward-looking variables that are chosen in stage 2 of each period (such as consumption and investment in our model); y collects the variables that are instead chosen in stage 1 (such as employment in our model); s collects the shocks to payoff (such as technology); and finally, ξ is meant to capture the shocks to higher-order beliefs. X^b, X^f, and Y correspond to the aggregate versions of, respectively, x^b, x^f, and y.

Beliefs

We assume that, as of stage 2, the realizations of s_t, of all the signals, and of all the stage-1 choices become commonly known, which implies that y_{it}, x^f_{it}, x^b_{it+1} and Y_t, X^f_t, X^b_{t+1} are also commonly known in equilibrium. Furthermore, the actual realizations of the signals satisfy $z_{it} = s_t$ for all t and all i. However, the agents have misspecified belief in stage 1. In particular, for all i, all $j \neq i$, all t, and all states of nature, agent i's beliefs during stage 1 satisfy

$$\mathbb{E}_{it}[s_t] = z_{it},$$
$$\mathbb{E}_{it}[\mathbb{E}_{jt} s_t] = \mathbb{E}_{it}[z_{jt}] = z_{it} + \Delta \xi_t,$$

where z_{it} is the signal received by agent i, ξ_t is the higher-order belief shocks, and Δ is a loading matrix. We next let $\bar{z}_t$ denote the average signal in the economy and note that the "truth" is that $z_{it} = \bar{z}_t = s_t$. Yet, this truth is publicly revealed only in stage 2 of period t. In stage 1, instead, each island believes, incorrectly, that

$$\mathbb{E}_{it} \bar{z}_t = z_{it} + \Delta \xi_t.$$

Note next that the stage-1 variables, y_{it}, can depend on the local signal z_{it}, along with the commonly observed belief shock ξ_t and the backward-looking (predetermined) state variables x^b_{it} and X^b_t, but cannot depend on either the aggregate signal $\bar{z}_t$ or the underlying fundamental s_t, because these variables are not known in stage 1. By contrast, the stage-2 decisions depend on the entire triplet $(z_{it}, \bar{z}_t, s_t)$. As already mentioned, the truth is that these three variables coincide. Nevertheless, the islands believe in stage 1 that the average signal can differ from either their own signal or the actual fundamental. Accordingly, it is important to write stage-2 strategies as functions of the three conceptually distinct objects in $(z_{it}, \bar{z}_t, s_t)$ in order to specify the appropriate equilibrium beliefs in stage 1. (Note that this is equivalent to expressing the stage-2 strategies as functions of the realized values of the stage-1 variables y and Y, which is the approach we took in the characterization of the recursive equilibrium in Section 3.) In what follows, we show how this belief structure facilitates a tractable solution of the aforementioned general DSGE model.

Preview of Key Result

To preview the key result, let us first consider the underlying "belief-free" model, that is, of the complete-information, representative-agent, counterpart of the model we are studying. The equilibrium system is given by the following:

$$Y_t = M_X X^b_t + M_{EY} Y_t + M_F X^f_t + M_s s_t,$$
$$X^b_{t+1} = N_X X^b_t + N_Y Y_t + N_F X^f_t + N_s s_t,$$
$$(P_{f0} - P_{F0}) \mathbb{E}_t X^f_{t+1} = P_{F1} X^f_t + P_{Y0} \mathbb{E}_t Y_{t+1} + P_X X^b_t + P_{Y1} Y_t + P_s s_t,$$
$$s_t = R s_{t-1} + \varepsilon_t,$$
$$\xi_t = Q \xi_{t-1} + \nu_t.$$

(This system can be obtained from the one we introduced before once we impose the restriction that all period-t variables are commonly known in period t, which means that $\mathbb{E}'_{it}[x_t] = \mathbb{E}_{it}[x_t] = x_t$ for any variable x.) It is well known how to obtain the policy rules of such a representative-agent model. Our goal in this appendix is to show how the policy rules of the belief-augmented model that we described above can be obtained as a simple, tractable transformation of the policy rules of the representative-agent benchmark.

In particular, we will show that the policy rules for our general DSGE economy are as follows:

$$\mathbf{X}_t = \Theta_X X_t^b + \Theta_s s_t + \Theta_\xi \xi_t,$$

where $\mathbf{X}_t = (Y_t, X_t^f, X_{t+1}^b)$ collects all the variables, Θ_X and Θ_s are the same matrices as those that appear in the solution of the underlying belief-free model, and Θ_ξ is a new matrix, which encapsulates the effects of higher-order beliefs.

The Model, Restated

To ease subsequent algebraic manipulations, we henceforth restate the model as follows:

$$\begin{aligned} y_{it} = {} & M_x\big(x_{it}^b - X_t^b\big) + M_X X_t^b + M_{EY}\mathbb{E}_{it} Y_t \\ & + M_f \mathbb{E}_{it}\big(x_{it}^f - X_t^f\big) + M_F \mathbb{E}_{it} X_t^f + M_s z_{it}, \end{aligned} \tag{S.9}$$

$$\begin{aligned} x_{it+1}^b = {} & N_x\big(x_{it}^b - X_t^b\big) + N_X X_t^b + N_y(y_{it} - Y_t) + N_Y Y_t \\ & + N_f\big(x_{it}^f - X_t^f\big) + N_F X_t^f + N_s s_t, \end{aligned} \tag{S.10}$$

$$\begin{aligned} P_{f0}\mathbb{E}'_{it} x_{it+1}^f = {} & P_{f1}\big(x_{it}^f - X_t^f\big) + P_{F0}\mathbb{E}'_{it} X_{t+1}^f + P_{F1} X_t^f + P_x\big(x_{it}^b - X_t^b\big) + P_X X_t^b \\ & + P_{y0}\big(\mathbb{E}'_{it} y_{it+1} - \mathbb{E}'_{it} Y_{t+1}\big) + P_{Y0}\mathbb{E}'_{it} Y_{t+1} \\ & + P_{y1}(y_{it} - Y_t) + P_{Y1} Y_t + P_s s_t, \end{aligned} \tag{S.11}$$

where

$$\begin{aligned}
& M_x = M_{yy}^{-1} M_{yx}, \qquad M_X = M_{yy}^{-1}(M_{yx} + M_{yX}), \qquad M_{EY} = M_{yy}^{-1} M_{yY}, \\
& M_f = M_{yy}^{-1} M_{yf}, \qquad M_F = M_{yy}^{-1}(M_{yf} + M_{yF}), \qquad M_s = M_{yy}^{-1} M_{ys}, \\
& N_x = M_{xx0}^{-1} M_{xx1}, \qquad N_X = M_{xx0}^{-1}(M_{xx1} + M_{xX1}), \\
& N_y = M_{xx0}^{-1} M_{xy1}, \qquad N_Y = M_{xx0}^{-1}(M_{xy1} + M_{xY1}), \\
& N_f = M_{xx0}^{-1} M_{xf1}, \qquad N_F = M_{xx0}^{-1}(M_{xf1} + M_{xF1}), \qquad N_s = M_{xx0}^{-1} M_{xs1}, \\
& P_{f0} = M_{ff0}, \qquad P_{f1} = M_{ff1} + M_{fx0} N_f, \qquad P_{F0} = M_{fF0}, \\
& P_{F1} = M_{fF1} + M_{ff1} + M_{fx0} N_F, \\
& P_x = M_{fx1} + M_{fx0} N_x, \qquad P_X = M_{fX1} + M_{fx1} + M_{fx0} N_X, \\
& P_{y0} = M_{fy0}, \qquad P_{Y0} = M_{fY0} + M_{fy0}, \qquad P_{y1} = M_{fy1} + M_{fx0} N_y, \\
& P_Y = M_{fy1} + M_{fY1} + M_{fx0} N_Y, \\
& P_s = M_{fs0} R + M_{fs1} + M_{fx0} N_s.
\end{aligned}$$

Proposed Policy Rules

We propose that the equilibrium policy rules take the following form:

$$y_{it} = \Lambda_x\left(x_{it}^b - X_t^b\right) + \Lambda_X X_t^b + \Lambda_z z_{it} + \Lambda_\xi \xi_t, \tag{S.12}$$

$$x_{it}^f = \Gamma_x\left(x_{it}^b - X_t^b\right) + \Gamma_X X_t^b + \Gamma_z z_{it} + \Gamma_{\bar{z}} \bar{z}_t + \Gamma_s s_t + \Gamma_\xi \xi_t, \tag{S.13}$$

where the Λ's and Γ's are coefficients (matrices), whose equilibrium values are to be obtained in the sequel. Following our earlier discussion, note that the stage-2 policy rules are allowed to depend on the triplet $(z_{it}, \bar{z}_t, s_t)$, while the stage-1 policy rules are restricted to depend only on the local signal z_{it}. It is also useful to note that we would obtain the same solution if we were to represent the stage-2 policy rules as functions of y_{it} and Y_t in place of, respectively, z_{it} and $\bar{z}_t$: the latter two variables enter the equilibrium conditions that determine the stage-2 decisions, namely, conditions (S.10) and (S.11), *only* through the realized values of the stage-1 outcomes y_{it} and Y_t.

Obtaining the Solution

We obtain the solution in three steps. In step 1, we start by characterizing the equilibrium determination of the stage-1 policy rules, taking as given the stage-2 rules. Formally, we fix an arbitrary rule in (S.13); we assume that all islands believe that the stage-2 variables are determined according to this rule; and we then look for the particular rule in (S.12) that solves the fixed-point relation between y_{it} and Y_t described in (S.9) under this assumption. This step, which we can think of as the "static" component of the equilibrium, gives us a mapping from Γ matrices to the Λ matrices. In step 2, we obtain a converse mapping by characterizing the policy rules for the forward-looking variables that solve conditions (S.10) and (S.11) under the assumption that the stage-1 outcomes are determined according to an arbitrary rule in (S.13). We can think of this step as solving for the "dynamic" component of the equilibrium. In step 3, we use the fixed point between these two mappings to obtain the overall solution to the model.

Step 1

As noted above, we start by studying the equilibrium determination of the stage-1 policy rules, taking as given the stage-2 policy rules.

Thus suppose that all islands follow a policy rule as in (S.13) and consider the beliefs that a given island i forms, under this assumption, about the stage-2 variables x_{it}^f and X_t^f. From (S.13), we have

$$x_{it}^f = \Gamma_x\left(x_{it}^b - X_t^b\right) + \Gamma_X X_t^b + \Gamma_z z_{it} + \Gamma_{\bar{z}} \bar{z}_t + \Gamma_s s_t + \Gamma_\xi \xi_t,$$

$$X_t^f = \Gamma_X X_t^b + (\Gamma_z + \Gamma_{\bar{z}})\bar{z}_t + \Gamma_s s_t + \Gamma_\xi \xi_t.$$

Along with the fact that $\mathbb{E}_{it}[s_t] = z_{it}$ and $\mathbb{E}_{it}[\bar{z}_t] = z_{it} + \Delta\xi_t$, the above gives

$$\mathbb{E}_{it} x_{it}^f = \Gamma_x\left(x_{it}^b - X_t^b\right) + \Gamma_X X_t^b + (\Gamma_z + \Gamma_{\bar{z}} + \Gamma_s) z_{it} + (\Gamma_\xi + \Gamma_{\bar{z}}\Delta)\xi_t,$$

$$\mathbb{E}_{it} X_t^f = \Gamma_X X_t^b + (\Gamma_z + \Gamma_{\bar{z}} + \Gamma_s) z_{it} + \left(\Gamma_\xi + (\Gamma_z + \Gamma_{\bar{z}})\Delta\right)\xi_t,$$

which also implies that

$$x^f_{it} - X^f_t = \Gamma_x\left(x^b_{it} - X^b_t\right) + \Gamma_z(z_{it} - \bar{z}_t),$$
$$\mathbb{E}_{it}\left(x^f_{it} - X^f_t\right) = \Gamma_x\left(x^b_{it} - X^b_t\right) - \Gamma_z\Delta\xi_t.$$

Plugging the above in (S.9), the equilibrium equation for y_{it}, we get

$$\begin{aligned} y_{it} &= M_x\left(x^b_{it} - X^b_t\right) + M_X X^b_t + M_{EY}\mathbb{E}_{it}Y_t + M_f\mathbb{E}_{it}\left(x^f_{it} - X^f_t\right) + M_F\mathbb{E}_{it}X^f_t + M_s z_{it} \\ &= M_x\left(x^b_{it} - X^b_t\right) + M_X X^b_t + M_{EY}\mathbb{E}_{it}Y_t + M_f\left[\Gamma_x\left(x^b_{it} - X^b_t\right) - \Gamma_z\Delta\xi_t\right] \\ &\quad + M_F\left[\Gamma_X X^b_t + (\Gamma_z + \Gamma_{\bar{z}} + \Gamma_s)z_{it} + \left(\Gamma_\xi + (\Gamma_z + \Gamma_{\bar{z}})\Delta\right)\xi_t\right] + M_s z_{it}. \end{aligned}$$

Equivalently,

$$\begin{aligned} y_{it} &= (M_x + M_f\Gamma_x)\left(x^b_{it} - X^b_t\right) + (M_X + M_F\Gamma_X)X^b_t + M_{EY}\mathbb{E}_{it}Y_t \\ &\quad + \left(M_s + M_F(\Gamma_z + \Gamma_{\bar{z}} + \Gamma_s)\right)z_{it} \\ &\quad + \left(M_F\Gamma_\xi + M_F\Gamma_{\bar{z}}\Delta + (M_F - M_f)\Gamma_z\Delta\right)\xi_t. \end{aligned} \tag{S.14}$$

Note that the above represents a *static* fixed-point relation between y_{it} and Y_t. This relation is itself determined by the Γ matrices (i.e., by the presumed policy rule for the stage-2 variables). Notwithstanding this fact, we now focus on the solution of this static fixed point.

Thus suppose that this solution takes the form of a policy rule as in (S.12). If all other islands follow this rule, then at the aggregate we have

$$Y_t = \Lambda_X X^b_t + \Lambda_z\bar{z}_t + \Lambda_\xi\xi_t,$$

and therefore the stage-1 forecast of island i about Y_t is given by

$$\mathbb{E}_{it}Y_t = \Lambda_X X^b_t + \Lambda_z z_{it} + (\Lambda_\xi + \Lambda_z\Delta)\xi_t.$$

Plugging this into (S.14), we obtain the following best response for island i:

$$\begin{aligned} y_{it} &= (M_x + M_f\Gamma_x)\left(x^b_{it} - X^b_t\right) + (M_X + M_F\Gamma_X)X^b_t \\ &\quad + M_{EY}\left(\Lambda_X X^b_t + \Lambda_z z_{it} + (\Lambda_\xi + \Lambda_z\Delta)\xi_t\right) \\ &\quad + \left(M_s + M_F(\Gamma_z + \Gamma_{\bar{z}} + \Gamma_s)\right)z_{it} + \left(M_F(\Gamma_\xi + \Gamma_{\bar{z}}\Delta) + (M_F - M_f)\Gamma_z\Delta\right)\xi_t. \end{aligned}$$

For this to be consistent with our guess in (S.12), we must have

$$\Lambda_x = M_x + M_f\Gamma_x, \tag{S.15}$$

$$\Lambda_X = (I - M_{EY})^{-1}(M_X + M_F\Gamma_X), \tag{S.16}$$

$$\Lambda_z = (I - M_{EY})^{-1}\left[M_s + M_F(\Gamma_z + \Gamma_{\bar{z}} + \Gamma_s)\right], \tag{S.17}$$

$$\Lambda_\xi = (I - M_{EY})^{-1}\left\{M_F(\Gamma_\xi + \Gamma_{\bar{z}}\Delta) + (M_F - M_f)\Gamma_z\Delta + M_{EY}\Lambda_z\Delta\right\}. \tag{S.18}$$

This completes the first step of our solution strategy: we have characterized the "static" component of the equilibrium and have thus obtained the Λ coefficients as functions of primitives and of the Γ coefficients.

Step 2

We now proceed with the second step, which is to characterize the equilibrium behavior in stage 2, taking as given the behavior in stage 1.

Recall that, once agents enter stage 2, they observe the true current values of the triplet $(z_{it}, \bar{z}_t, s_t)$ along with the realized values of the past stage-1 outcomes, y_{it} and Y_t. Furthermore, in equilibrium this implies common certainty of current choices, namely, of the variables x^f_{it} and X^f_t, and thereby also of the variables x^b_{it+1} and X^b_{t+1}. Nevertheless, agents face uncertainty about the next-period realizations of the aforementioned triplet and of the corresponding endogenous variables. In what follows, we thus take special care in characterizing the beliefs that agents form about the relevant future outcomes.

Consider first an agent's beliefs about the aggregate next-period stage-1 variables:

$$Y_{t+1} = \Lambda_X X^b_{t+1} + \Lambda_z \bar{z}_{t+1} + \Lambda_\xi \xi_{t+1},$$

$$\mathbb{E}_{it+1} Y_{t+1} = \Lambda_X X^b_{t+1} + \Lambda_z z_{it+1} + (\Lambda_\xi + \Lambda_z \Delta)\xi_{t+1},$$

$$\mathbb{E}'_{it} Y_{t+1} = \Lambda_X X^b_{t+1} + \Lambda_z R s_t + (\Lambda_\xi + \Lambda_z \Delta) Q \xi_t.$$

Consider next his beliefs about his own next-period stage-1 variables:

$$y_{it+1} = \Lambda_x \left(x^b_{it+1} - X^b_{t+1}\right) + \Lambda_X X^b_{t+1} + \Lambda_z z_{it+1} + \Lambda_\xi \xi_{t+1},$$

$$\mathbb{E}'_{it} y_{it+1} = \Lambda_x \left(x^b_{it+1} - X^b_{t+1}\right) + \Lambda_X X^b_{t+1} + \Lambda_z R s_t + \Lambda_\xi Q \xi_t.$$

It follows that

$$\mathbb{E}'_{it}(y_{it+1} - Y_{t+1}) = \Lambda_x \left(x^b_{it+1} - X^b_{t+1}\right) - \Lambda_z \Delta Q \xi_t.$$

Consider now his beliefs about his own next-period forward variables:

$$x^f_{it+1} = \Gamma_x \left(x^b_{it+1} - X^b_{t+1}\right) + \Gamma_X X^b_{t+1} + \Gamma_z z_{it+1} + \Gamma_{\bar{z}} \bar{z}_{t+1} + \Gamma_s s_{t+1} + \Gamma_\xi \xi_{t+1},$$

$$\mathbb{E}_{it+1} x^f_{it+1} = \Gamma_x \left(x^b_{it+1} - X^b_{t+1}\right) + \Gamma_X X^b_{t+1} + (\Gamma_z + \Gamma_{\bar{z}} + \Gamma_s) z_{it+1} + (\Gamma_\xi + \Gamma_{\bar{z}} \Delta)\xi_{t+1},$$

$$\mathbb{E}'_{it} x^f_{it+1} = \Gamma_x \left(x^b_{it+1} - X^b_{t+1}\right) + \Gamma_X X^b_{t+1} + (\Gamma_z + \Gamma_{\bar{z}} + \Gamma_s) R s_t + (\Gamma_\xi + \Gamma_{\bar{z}} \Delta) Q \xi_t.$$

For the aggregate next-period forward variables, we have

$$\mathbb{E}_{it+1} X^f_{t+1} = \Gamma_X X^b_{t+1} + (\Gamma_z + \Gamma_{\bar{z}} + \Gamma_s) z_{it+1} + \left(\Gamma_\xi + (\Gamma_z + \Gamma_{\bar{z}})\Delta\right)\xi_{t+1},$$

$$\mathbb{E}'_{it} X^f_{t+1} = \Gamma_X X^b_{t+1} + (\Gamma_z + \Gamma_{\bar{z}} + \Gamma_s) R s_t + \left(\Gamma_\xi + (\Gamma_z + \Gamma_{\bar{z}})\Delta\right) Q \xi_t,$$

and therefore

$$\mathbb{E}'_{it}\left(x^f_{it+1} - X^f_{t+1}\right) = \Gamma_x \left(x^b_{it+1} - X^b_{t+1}\right) - \Gamma_z \Delta Q \xi_t.$$

Next, note that our guesses for the policy rules imply the following properties for the current-period variables:

$$y_{it} - Y_t = \Lambda_x \left(x^b_{it} - X^b_t\right) + \Lambda_z (z_{it} - \bar{z}_t),$$

$$x^f_{it} - X^f_t = \Gamma_x \left(x^b_{it} - X^b_t\right) + \Gamma_z (z_{it} - \bar{z}_t),$$

$$Y_t = \Lambda_X X_t^b + \Lambda_z \bar{z}_t + \Lambda_\xi \xi_t,$$
$$X_t^f = \Gamma_X X_t^b + (\Gamma_z + \Gamma_{\bar{z}})\bar{z}_t + \Gamma_s s_t + \Gamma_\xi \xi_t.$$

Plugging these results in the law of motion of backward variables, we get

$$\begin{aligned} x_{it+1}^b &= N_x\big(x_{it}^b - X_t^b\big) + N_X X_t^b + N_y(y_{it} - Y_t) + N_Y Y_t + N_f\big(x_{it}^f - X_t^f\big) + N_F X_t^f + N_s s_t \\ &= N_x\big(x_{it}^b - X_t^b\big) + N_X X_t^b + N_y\big\{\Lambda_x\big(x_{it}^b - X_t^b\big) + \Lambda_z(z_{it} - \bar{z}_t)\big\} \\ &\quad + N_Y\big\{\Lambda_X X_t^b + \Lambda_z \bar{z}_t + \Lambda_\xi \xi_t\big\} \\ &\quad + N_f\big\{\Gamma_x\big(x_{it}^b - X_t^b\big) + \Gamma_z(z_{it} - \bar{z}_t)\big\} \\ &\quad + N_F\big\{\Gamma_X X_t^b + (\Gamma_z + \Gamma_{\bar{z}})\bar{z}_t + \Gamma_s s_t + \Gamma_\xi \xi_t\big\} + N_s s_t. \end{aligned}$$

Equivalently,

$$x_{it+1}^b = \Omega_x\big(x_{it}^b - X_t^b\big) + \Omega_X X_t^b + \Omega_z z_{it} + \Omega_{\bar{z}} \bar{z} + \Omega_s s_t + \Omega_\xi \xi_t,$$

and hence

$$X_{t+1}^b = \Omega_X X_t^b + (\Omega_z + \Omega_{\bar{z}})\bar{z}_t + \Omega_s s_t + \Omega_\xi \xi_t,$$
$$x_{it+1}^b - X_{t+1}^b = \Omega_x\big(x_{it}^b - X_t^b\big) + \Omega_z(z_{it} - \bar{z}_t),$$

where

$$\begin{aligned} &\Omega_x = N_x + N_y \Lambda_x + N_f \Gamma_x, \qquad \Omega_z = N_y \Lambda_z + N_f \Gamma_z, \\ &\Omega_X = N_X + N_Y \Lambda_X + N_F \Gamma_X, \qquad \Omega_{\bar{z}} = (N_Y - N_y)\Lambda_z + (N_F - N_f)\Gamma_z + N_F \Gamma_{\bar{z}}, \\ &\Omega_s = N_s + N_F \Gamma_s, \qquad \Omega_\xi = N_Y \Lambda_\xi + N_F \Gamma_\xi. \end{aligned}$$

It follows that

$$\begin{aligned} \mathbb{E}_{it}' x_{it+1}^f &= \Gamma_x\big(x_{it+1}^b - X_{t+1}^b\big) + \Gamma_X X_{t+1}^b + (\Gamma_z + \Gamma_{\bar{z}} + \Gamma_s) R s_t + (\Gamma_\xi + \Gamma_{\bar{z}}\Delta) Q \xi_t \\ &= \Gamma_x\big\{\Omega_x\big(x_{it}^b - X_t^b\big) + \Omega_z(z_{it} - \bar{z}_t)\big\} + \Gamma_X\big\{\Omega_X X_t^b + (\Omega_z + \Omega_{\bar{z}})\bar{z}_t + \Omega_s s_t + \Omega_\xi \xi_t\big\} \\ &\quad + (\Gamma_z + \Gamma_{\bar{z}} + \Gamma_s) R s_t + (\Gamma_\xi + \Gamma_{\bar{z}}\Delta) Q \xi_t, \end{aligned}$$

or equivalently,

$$\mathbb{E}_{it}' x_{it+1}^f = \Phi_x\big(x_{it}^b - X_t^b\big) + \Phi_X X_t^b + \Phi_z z_{it} + \Phi_{\bar{z}} \bar{z}_t + \Phi_s s_t + \Phi_\xi \xi_t, \tag{S.19}$$

where

$$\begin{aligned} &\Phi_x = \Gamma_x \Omega_x, \qquad \Phi_z = \Gamma_x \Omega_z, \qquad \Phi_s = \Gamma_X \Omega_s + (\Gamma_z + \Gamma_{\bar{z}} + \Gamma_s) R, \\ &\Phi_X = \Gamma_X, \Omega_X \qquad \Phi_{\bar{z}} = (\Gamma_X - \Gamma_x)\Omega_z + \Gamma_X \Omega_{\bar{z}}, \qquad \Phi_\xi = \Gamma_X \Omega_\xi + (\Gamma_\xi + \Gamma_{\bar{z}}\Delta) Q. \end{aligned}$$

Similarly, the expectation of the corresponding aggregate variable is given by

$$\mathbb{E}_{it}' X_{t+1}^f = \Phi_X X_t^b + \Phi_z z_{it} + \Phi_{\bar{z}} \bar{z}_t + \Phi_s s_t + (\Phi_\xi + \Gamma_z \Delta Q)\xi_t. \tag{S.20}$$

With the above steps, we have calculated all the objects that enter the Euler condition (S.11). We can thus proceed to characterize the fixed-point relation that pins down the solution for the stage-2 policy rule.

To ease the exposition, let us repeat the Euler condition (S.11) below:

$$\begin{aligned} P_{f0}\mathbb{E}'_{it}x^f_{it+1} =& P_{f1}\left(x^f_{it}-X^f_t\right)+P_{F0}\mathbb{E}'_{it}X^f_{t+1}+P_{F1}X^f_t+P_x\left(x^b_{it}-X^b_t\right)+P_XX^b_t \\ &+P_{y0}\left(\mathbb{E}'_{it}y_{it+1}-\mathbb{E}'_{it}Y_{t+1}\right)+P_{Y0}\mathbb{E}'_{it}Y_{t+1}+P_{y1}(y_{it}-Y_t)+P_{Y1}Y_t+P_ss_t. \end{aligned}$$

Use now (S.19) to write the left-hand side of the Euler condition as

$$P_{f0}\mathbb{E}'_{it}x^f_{it+1}=P_{f0}\left\{\Phi_x\left(x^b_{it}-X^b_t\right)+\Phi_XX^b_t+\Phi_zz_{it}+\Phi_{\bar z}\bar z_t+\Phi_ss_t+\Phi_\xi\xi_t\right\}.$$

Next, use our preceding results to replace all the expectations that show up in the right-hand side of the Euler condition, as well as the stage-1 outcomes. This gives

$$\begin{aligned} P_{f0}\mathbb{E}'_{it}x^f_{it+1}=& P_{f1}\left\{\Gamma_x\left(x^b_{it}-X^b_t\right)+\Gamma_z(z_{it}-\bar z_t)\right\} \\ &+P_{F0}\left\{\Phi_XX^b_t+(\Phi_z+\Phi_{\bar z})\bar z_t+\Phi_ss_t+(\Phi_\xi+\Gamma_z\Delta Q)\xi_t\right\} \\ &+P_{F1}\left\{\Gamma_XX^b_t+(\Gamma_z+\Gamma_{\bar z})\bar z_t+\Gamma_ss_t+\Gamma_\xi\xi_t\right\}+P_x\left\{x^b_{it}-X^b_t\right\} \\ &+P_XX^b_t+P_{y0}\left\{\Lambda_x\left(\Omega_x\left(x^b_{it}-X^b_t\right)+\Omega_z(z_{it}-\bar z_t)\right)-\Lambda_z\Delta Q\xi_t\right\} \\ &+P_{Y0}\left\{\Lambda_X\left(\Omega_XX^b_t+(\Omega_z+\Omega_{\bar z})\bar z_t+\Omega_ss_t+\Omega_\xi\xi_t\right)\right. \\ &\left.+\Lambda_zRs_t+(\Lambda_\xi+\Lambda_z\Delta)Q\xi_t\right\} \\ &+P_{y1}\left\{\Lambda_x\left(x^b_{it}-X^b_t\right)+\Lambda_z(z_{it}-\bar z_t)\right\}+P_{Y1}\left\{\Lambda_XX^b_t+\Lambda_z\bar z_t+\Lambda_\xi\xi_t\right\}+P_ss_t. \end{aligned}$$

For our guess to be correct, the above two expressions must coincide in all states of nature, and the following must therefore be true:

$$P_{f0}\Phi_x=P_x+P_{f1}\Gamma_x+P_{y0}\Lambda_x\Omega_x+P_{y1}\Lambda_x, \tag{S.21}$$

$$(P_{f0}-P_{F0})\Phi_X=P_{F1}\Gamma_X+P_X+P_{Y0}\Lambda_X\Omega_X+P_{Y1}\Lambda_X, \tag{S.22}$$

$$P_{f0}\Phi_z=P_{f1}\Gamma_z+P_{y0}\Lambda_x\Omega_z+P_{y1}\Lambda_z, \tag{S.23}$$

$$\begin{aligned}(P_{f0}-P_{F0})\Phi_{\bar z}=& P_{F0}\Phi_z+(P_{F1}-P_{f1})\Gamma_z+P_{F1}\Gamma_{\bar z} \\ &+P_{Y0}\Lambda_X(\Omega_z+\Omega_{\bar z})-P_{y0}\Lambda_x\Omega_z+(P_{Y1}-P_{y1})\Lambda_z,\end{aligned} \tag{S.24}$$

$$(P_{f0}-P_{F0})\Phi_s=P_{F1}\Gamma_s+P_{Y0}(\Lambda_X\Omega_s+\Lambda_zR)+P_s, \tag{S.25}$$

$$\begin{aligned}(P_{f0}-P_{F0})\Phi_\xi=& P_{F0}\Gamma_z\Delta Q+P_{F1}\Gamma_\xi+P_{Y0}\{\Lambda_X\Omega_\xi+\Lambda_\xi Q\} \\ &+(P_{Y0}-P_{y0})\Lambda_z\Delta Q+P_{Y1}\Lambda_\xi.\end{aligned} \tag{S.26}$$

Recall that the Φ and Ω matrices are themselves transformations of the Γ and Λ matrices. Therefore, the above system is effectively a system of equations in Γ and Λ matrices. This completes step 2.

Step 3

Steps 1 and 2 resulted in two systems of equations in the Λ and Γ matrices, namely, system (S.15)–(S.18) and system (S.21)–(S.26). We now look at the joint solution of these two systems, which completes our guess-and-verify strategy and gives the sought-after equilibrium policy rules.

First, let us write the solution of the underlying representative-agent model as

$$Y_t = \Lambda_X^* X_t^b + \Lambda_s^* s_t \quad \text{and} \quad X_t^f = \Gamma_X^* X_t^b + \Gamma_s^* s_t.$$

It is straightforward to check that the solution to the beliefs-augmented model satisfies the following:

$$\Lambda_X = \Lambda_X^*, \qquad \Lambda_z = \Lambda_s^*, \qquad \Gamma_X = \Gamma_X^*, \quad \text{and} \quad \Gamma_z + \Gamma_{\bar{z}} + \Gamma_s = \Gamma_s^*.$$

That is, the solution for the matrices Λ_X, Λ_z, and Γ_X, and for the sum $\bar{\Gamma}_s \equiv \Gamma_z + \Gamma_{\bar{z}} + \Gamma_s$, can readily be obtained from the solution of the underlying representative-agent model.

With the sum $\bar{\Gamma}_s \equiv \Gamma_z + \Gamma_{\bar{z}} + \Gamma_s$ determined as above, we can next obtain each of its three components as follows. First, Γ_s can be obtained from (S.25):

$$(P_{f0} - P_{F0})\Phi_s = P_{F1}\Gamma_s + P_{Y0}(\Lambda_X \Omega_s + \Lambda_z R) + P_s.$$

Plugging the definition of Φ_s and Ω_s in the above, we have

$$\underbrace{-\big\{\big((P_{F0} - P_{f0})\Gamma_X + P_{Y0}\Lambda_X\big)N_F + P_{F1}\big\}}_{A_S}\Gamma_s$$
$$= \underbrace{P_s + P_{Y0}(\Lambda_z R + \Lambda_X N_s) + (P_{F0} - P_{f0})(\bar{\Gamma}_s R + \Gamma_X N_s)}_{B_S},$$

and therefore $\Gamma_s = A_S^{-1} B_S$. Next, Γ_z can be obtained from (S.23). Plugging the definition of Φ_z and Ω_z in this condition, we have

$$\underbrace{\big((P_{f0}\Gamma_x - P_{y0}\Lambda_x)N_f - P_{f1}\big)}_{A_Z}\Gamma_z = \underbrace{P_{y1}\Lambda_z - (P_{f0}\Gamma_x - P_{y0}\Lambda_x)N_y\Lambda_z}_{B_Z},$$

and therefore $\Gamma_z = A_Z^{-1} B_Z$. Finally, we obtain $\Gamma_{\bar{z}}$ simply from the fact that $\Gamma_{\bar{z}} = \bar{\Gamma}_s - \Gamma_z - \Gamma_s$.

Consider now the matrices Λ_x and Γ_x. These are readily obtained from (S.15) and (S.21) once we replace the already-obtained results. It is also straightforward to check that these matrices correspond to the solution of the version of the model that shuts down all kinds of uncertainty but allows for heterogeneity in the backward-looking state variables ("wealth").

To complete our solution, what remains is to determine the matrices Γ_ξ and Λ_ξ. These matrices solve conditions (S.18) and (S.26), which we repeat below:

$$\Lambda_\xi = (I - M_{EY})^{-1}\big\{M_F(\Gamma_\xi + \Gamma_{\bar{z}}\Delta) + (M_F - M_f)\Gamma_z\Delta + M_{EY}\Lambda_z\Delta\big\},$$
$$(P_{f0} - P_{F0})\Phi_\xi = P_{F0}\Gamma_z\Delta Q + P_{F1}\Gamma_\xi + P_{Y0}\{\Lambda_X\Omega_\xi + \Lambda_\xi Q\} + (P_{Y0} - P_{y0})\Lambda_z\Delta Q + P_{Y1}\Lambda_\xi.$$

Let us use the first condition to substitute away Λ_ξ from the second, and then the facts that

$$\Omega_\xi = N_Y\Lambda_\xi + N_F\Gamma_\xi,$$
$$\Phi_\xi = \Gamma_X(N_Y\Lambda_\xi + N_F\Gamma_\xi) + (\Gamma_\xi + \Gamma_{\bar{z}}\Delta)Q$$

to substitute away also Ω_ξ and Φ_ξ. We then obtain a single equation in Γ_ξ, namely,

$$B\Gamma_\xi + A\Gamma_\xi Q + C = 0,$$

where

$$\begin{aligned}
A &\equiv (P_{F0} - P_{f0}) + P_{Y0}(I - M_{EY})^{-1}M_F, \\
B &\equiv \big((P_{F0} - P_{f0})\Gamma_X N_Y + P_{Y0}\Lambda_X N_Y + P_{Y1}\big)(I - M_{EY})^{-1}M_F \\
&\quad + (P_{F0} - P_{f0})\Gamma_X N_F + P_{F1} + P_{Y0}\Lambda_X N_F, \\
C &\equiv \big(P_{F0}\Gamma_z \Delta Q + (P_{Y0} - P_{y0})\Lambda_z \\
&\quad + (P_{F0} - P_{f0})\Gamma_{\bar{z}} + P_{Y0}(I - M_{EY})^{-1}\big[M_F\Gamma_{\bar{z}} + (M_F - M_f)\Gamma_z + M_{EY}\Lambda_z\big]\big)\Delta Q \\
&\quad + \big((P_{F0} - P_{f0})\Gamma_X N_Y + P_{Y0}\Lambda_X N_Y + P_{Y1}\big)(I - M_{EY})^{-1} \\
&\quad \times \big[M_F\Gamma_{\bar{z}} + (M_F - M_f)\Gamma_z + M_{EY}\Lambda_z\big]\Delta.
\end{aligned}$$

Note that A, B, and C are determined by primitives, plus some of the coefficients that we have also characterized. The above equation therefore gives us the unique solution for the matrix Γ_ξ as a function of the primitives of the model. Λ_ξ is then readily obtained from (S.18). This completes the solution.

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