Econometrica Supplementary Material

SUPPLEMENT TO "CASCADING FAILURES IN PRODUCTION NETWORKS"

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APPENDIX A: ADDITIONAL FIGURE

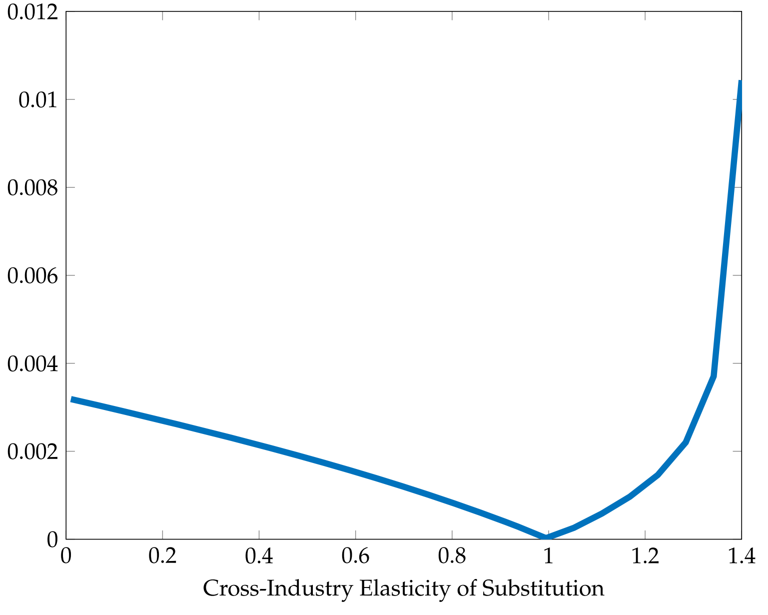

FIGURE 4.—The standard deviation of the sales-share weighted average of log net entry as a function of σ with intensive margin shocks only z_k^w. The cross-elasticity of substitution is chosen to match the observed volatility of 1% reported by Broda and Weinstein (2010).

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APPENDIX B: PROOFS

For the proofs, the object v^s, where v is a vector and s is a scalar, should be interpreted as the vector v raised to the power of s element-wise.

LEMMA 4: *Demand for the output of firm i in industry k is*

$$\begin{aligned} y(k,i) &= c(k,i) + \sum_l \int_0^{M_l} x(l,j,k,i)\,\mathrm{d}j, \\ &= \beta_k \left(\frac{p(k,i)}{p_k}\right)^{-\varepsilon_k} M_k^{-\varphi_k\varepsilon_k} \left(\frac{p_k}{p_c}\right)^{-\sigma} C\left(\frac{\overline{C}}{\overline{y}_k}\right)^{\sigma-1} \\ &\quad + \sum_l M_l \omega_{lk} \left(\frac{p(k,i)}{p_k}\right)^{-\varepsilon_k} M_k^{-\varphi_k\varepsilon_k} \left(\frac{p_k}{\lambda_l}\right)^{-\sigma} y(l,j)\left(\frac{\overline{y}_l}{\overline{y}_k}\right)^{\sigma-1}, \end{aligned}$$

where λ_l is the marginal cost of the firms in industry l.

PROOF: Cost minimization by each firm implies firm j in industry l's demand for inputs from firm i in industry k is given by

$$x(l,j,k,i) = \omega_{lk} \left(\frac{p(k,i)}{p_k}\right)^{-\varepsilon_k} M_k^{-\varphi_k\varepsilon_k} \left(\frac{p_k}{\lambda_l}\right)^{-\sigma} y(l,j)\left(\frac{\overline{y}_l}{\overline{y}_k}\right)^{\sigma-1},$$

where λ_l is the marginal cost of firms in industry l,

$$\lambda_k = \frac{1}{\overline{y}_k}\left(\alpha_k z_k^{\sigma-1}(\bar{l}w)^{1-\sigma} + \sum_l \omega_{kl}(\overline{y}_l p_l)^{1-\sigma}\right)^{\frac{1}{1-\sigma}},$$

and p_k is the price index for industry k,

$$p_k = \left(M_k^{-\varphi_k\varepsilon_k} \int_0^{M_k} p(k,i)^{1-\varepsilon_k}\,\mathrm{d}i\right)^{\frac{1}{1-\varepsilon_k}}.$$

Household demand for goods from firm i in industry k is

$$c(k,i) = \beta_k \left(\frac{p(k,i)}{p_k}\right)^{-\varepsilon_k} M_k^{-\varphi_k\varepsilon_k} \left(\frac{p_k}{P_c}\right)^{-\sigma} C\left(\frac{\overline{C}}{\overline{y}_k}\right)^{\sigma-1}.$$

Adding the household and firms' demands together gives the result, assuming the symmetric equilibrium. *Q.E.D.*

PROOF OF LEMMA 1: *Part* (a). By Lemma 4,

$$\begin{aligned} y(k,i)/\overline{y}_k^{1-\sigma} &= \beta_k M_k^{-\varphi_k\varepsilon_k} p(k,i)^{-\varepsilon_k} p_k^{\varepsilon_k-\sigma} p_c^{\sigma} C/\overline{C}^{1-\sigma} \\ &\quad + \sum_l M_l \omega_{lk} M_k^{-\varphi_k\varepsilon_k} p(k,i)^{-\varepsilon_k} p_k^{\varepsilon_k-\sigma} \lambda_l^{\sigma} y(l,j)/\overline{y}_l^{1-\sigma}. \end{aligned}$$

In equilibrium, we can substitute $p_k = M_k^{\frac{1-\varphi_k\varepsilon_k}{1-\varepsilon_k}} p(k,i)$ to get

$$M^{\frac{1-\varphi_k\varepsilon_k}{\varepsilon_k-1}\varepsilon_k-\varphi_k\varepsilon_k} p_k^\sigma y(k,i)/\overline{y}_k^{1-\sigma} = \beta_k p_c^\sigma C/\overline{C}^{1-\sigma} + \sum_l M_l \lambda_l^\sigma \omega_{lk} y(l,j)/\overline{y}_l^{1-\sigma}. \tag{25}$$

Observe that, in equilibrium,

$$y_k = \left(M_k^{-\varphi_k}\int_0^{M_k} y(k,i)^{\frac{\varepsilon_k-1}{\varepsilon_k}}\,\mathrm{d}i\right)^{\frac{\varepsilon_k}{\varepsilon_k-1}} = M_k^{\frac{1-\varphi_k}{\varepsilon_k-1}\varepsilon_k} y(k,i).$$

Furthermore,

$$\frac{1-\varphi_k\varepsilon_k}{\varepsilon_k-1}\varepsilon + \varphi_k\varepsilon_k = \frac{1-\varphi_k}{1-\varepsilon_k}\varepsilon_k.$$

Therefore, we can rewrite (25) as

$$p_k^\sigma y_k/\overline{y}_k^{1-\sigma} = \beta_k P_c^\sigma C/\overline{C}^{1-\sigma} + \sum_l M_l \omega_{lk}\lambda_l^\sigma y(l,j)/\overline{y}_l^{1-\sigma}.$$

Now substitute in

$$\lambda_l = \mu_l^{-1} p(l,i) = \mu_l^{-1} p_l M_l^{\frac{1-\varphi_l\varepsilon_l}{\varepsilon_l-1}}$$

to get

$$p_k^\sigma y_k/\overline{y}_k^{1-\sigma} = \beta_k P_c^\sigma C/\overline{C}^{1-\sigma} + \sum \omega_{lk} M_l^{1+\frac{1-\varphi_l\varepsilon_l}{\varepsilon_l-1}\sigma - \frac{1-\varphi_l\varepsilon_l}{\varepsilon_l}\varepsilon_l} \mu_l^{-\sigma} p_l^\sigma y_l/\overline{y}_l^{1-\sigma}.$$

Finally, note that

$$1 + \frac{1-\varphi_l\varepsilon_l}{\varepsilon_l-1}\sigma - \frac{1-\varphi_l\varepsilon_l}{\varepsilon_l}\varepsilon_l = \frac{\sigma-1}{\varepsilon_l-1}(1-\varphi_l\varepsilon_l).$$

Recall that $\tilde{M}$ is defined to be the diagonal matrix whose kth diagonal element is $M_k^{\frac{1-\varphi_k\varepsilon_k}{\varepsilon_k-1}}$, and μ is the diagonal matrix whose kth element is industry k's markup. So, denote $s_k = p_k^\sigma y_k/\overline{y}_k^{1-\sigma}$. This means that we can write

$$s' = \beta' P_c^\sigma C + s'\tilde{M}^{\sigma-1}\mu^{-\sigma}\Omega.$$

Rewrite this to get

$$\begin{aligned} s' &= \beta'\left(I - \tilde{M}^{\sigma-1}\mu^{-\sigma}\Omega\right)^{-1} P_c^\sigma C \\ &= \tilde{\beta}' P_c^\sigma C. \end{aligned}$$

Part (b). By definition,

$$p(k,i) = \mu_k\lambda_k,$$

where λ_k is marginal cost for firms in industry k. Substituting this into the definition of λ_k, we get

$$p(k,i) = \frac{\mu_k}{\overline{y}_k}\left(\alpha_k z_k^{\sigma-1}(\bar{l}w)^{1-\sigma} + \sum_l \omega_{kl}(\overline{y}_l p_l)^{1-\sigma}\right)^{\frac{1}{1-\sigma}}.$$

Note that, in the symmetric equilibrium,

$$p_k = \mu_k M_k^{\frac{1-\varphi_k\varepsilon_k}{1-\varepsilon_k}}\lambda_k,$$

so

$$\mu_k^{\sigma-1} M_k^{\frac{1-\varphi_k\varepsilon_k}{\varepsilon_k-1}(1-\sigma)}(p_k\overline{y}_k)^{1-\sigma} = \alpha_k\left(z_k^w\right)^{\sigma-1}(w\bar{l})^{1-\sigma} + \sum_l \omega_{kl}(p_l\overline{y}_l)^{1-\sigma}.$$

Let P be the vector $p_k\overline{y}_k$ and let $P^{1-\sigma}$ represent element-wise exponentiation. Then

$$\mu^{\sigma-1}\tilde{M}^{1-\sigma}P^{1-\sigma} = \left(\alpha \circ z^{\sigma-1}\right)(w\bar{l})^{1-\sigma} + \Omega P^{1-\sigma}.$$

Rearrange this to get

$$P^{1-\sigma} = \left(I - \mu^{1-\sigma}\tilde{M}^{\sigma-1}\Omega\right)^{-1}\mu^{1-\sigma}\tilde{M}^{\sigma-1}\left(\alpha \circ z^{\sigma-1}\right)(w\bar{l})^{1-\sigma} = \tilde{\alpha}(w\bar{l})^{1-\sigma}. \qquad \textit{Q.E.D.}$$

PROOF OF LEMMA 2: Note that the profits of firm i in industry k are

$$\pi(k,i) = p(k,i)y(k,i) - \lambda_k y(k,i) - wf_k.$$

This is equivalent to

$$\pi(k,i) = p(k,i)y(k,i) - \frac{1}{\mu_k}p(k,i)y(k,i) - wf_k$$
$$= \frac{\mu_k - 1}{\mu_k}p(k,i)y(k,i) - wf_k.$$

Since all active firms in industry k are identical, this is

$$\pi(k,i) = \frac{\mu_k - 1}{\mu_k}\frac{1}{M_k}p_k y_k - wf_k.$$

By Lemma 1,

$$p_k y_k = p_k^{\sigma} y_k p^{1-\sigma} = \tilde{\beta}_k\tilde{\alpha}_k P_c^{\sigma} C w^{1-\sigma},$$

and so

$$\pi(k,i) = \frac{\mu_k - 1}{\mu_k}\frac{1}{M_k}\tilde{\beta}_k\tilde{\alpha}_k P_c^{\sigma} C w^{1-\sigma} - wf_k. \qquad \textit{Q.E.D.}$$

PROPOSITION 4: *Consider two industries k and l such that*

$$\omega_{jk} = \omega_{jl} \quad (j = 1, \ldots, N), \qquad \beta_k = \beta_l, \qquad \mu_k = \mu_l, \qquad M_k = M_l;$$

then $\tilde{\beta}_k = \tilde{\beta}_l$. In other words, if two industries have the same immediate customer base, their supplier-centralities are the same.

PROOF: Let $A = \mu^{-\sigma}\tilde{M}^{1-\sigma}\Omega$. Note that, by assumption, $A_{jk} = A_{jl}$ for every j. Let $A^{(n)}_{ji}$ denote the jith entry of A^n. We wish to show, by induction, that

$$A^{(n)}_{jk} = A^{(n)}_{jl} \quad \big(n \in \{1, 2, \dots\}\big).$$

To that end, fix $n \in \mathbb{N}$ and assume that

$$A^{(n-1)}_{jk} = A^{(n-1)}_{jl}.$$

Then, it follows that

$$\begin{aligned} A^{(n)}_{jk} &= \sum_i A_{ji} A^{(n-1)}_{ik}, \\ &= \sum_i A_{ji} A^{(n-1)}_{il} \\ &= A^{(n)}_{jl}. \end{aligned}$$

Since the induction assumption holds for $n = 1$ by assumption, it follows that

$$A^{(n)}_{jk} = A^{(n)}_{jl} \quad \big(n \in \{1, 2, \dots\}\big).$$

Finally, observe that

$$\begin{aligned} \tilde{\beta}_k &= \sum_{n=0}^{\infty} \beta_j A^{(n)}_{jk} \\ &= \beta_k + \sum_{n=1}^{\infty} \beta_j A^{(n)}_{jk}, \end{aligned}$$

which we have shown is equal to

$$= \beta_k + \sum_{n=1}^{\infty} \beta_j A^{(n)}_{jl},$$

which, since $\beta_k = \beta_l$, equals

$$\begin{aligned} &= \sum_{n=0}^{\infty} \beta_j A^{(n)}_{jl} \\ &= \tilde{\beta}_l. \end{aligned}$$

Q.E.D.

PROPOSITION 5: *Consider two industries k and l such that*

$$\omega_{kj} = \omega_{lj} \quad (j = 1, \dots, N), \qquad \alpha_k = \alpha_l, \qquad \mu_k = \mu_l, \qquad M_k = M_l;$$

then $\tilde{\alpha}_k = \tilde{\alpha}_l$. In other words, if two industries have the same immediate supplier base, their consumer-centralities are the same.

PROOF: Similar to the proof of Proposition 4. *Q.E.D.*

LEMMA 5: *Let e_k denote the kth standard basis vector. Then, in equilibrium, we have*

$$\left(\frac{\mathrm{d}\log(M\mathbf{1})}{\mathrm{d}\log(z_k^m)}\right) = (I-\Lambda)^{-1}\left(e_k - \mathbf{1}\frac{\mathrm{d}P_c^\sigma C/\mathrm{d}z_k^m}{P_c^\sigma C}\right).$$

PROOF: To cut down on notation, I take derivatives with respect to overhead costs f rather than management productivity z^m; since they are reciprocals of one another, the elasticity with respect to one is negative the elasticity with respect to the other. Observe that, with the Dixit–Stiglitz structure,

$$\log(M\mathbf{1}) = \log(\tilde{\beta}) + \log(\tilde{\alpha}) + \log(P_c^\sigma C)\mathbf{1} - \log(f) - \log(\varepsilon).$$

Hence,

$$\frac{\mathrm{d}\log(M\mathbf{1})}{\mathrm{d}\log(f_i)} = \Lambda\frac{\mathrm{d}\log(M\mathbf{1})}{\mathrm{d}\log(f_i)} + \frac{\mathrm{d}\log(P_c^\sigma C)}{\mathrm{d}\log(f_i)}\mathbf{1} - e_i.$$

Rearrange this to get

$$\frac{\mathrm{d}\log(M\mathbf{1})}{\mathrm{d}\log(f_i)} = (I-\Lambda)^{-1}\left(\frac{\mathrm{d}\log(P_c^\sigma C)}{\mathrm{d}\log(f_i)}\mathbf{1} - e_i\right).$$

Q.E.D.

PROOF OF PROPOSITION 1: Recall that

$$\log(C) = (\beta'\tilde{\alpha})^{\frac{1}{\sigma-1}},$$

whence, by Lemma 5,

$$\frac{\mathrm{d}\log(C)}{\mathrm{d}\log(f_i)} = \frac{1}{\sigma-1}\frac{1}{\beta'\tilde{\alpha}}\beta'\frac{\mathrm{d}\tilde{\alpha}}{\mathrm{d}\log(M\mathbf{1})}\frac{\mathrm{d}\log(M\mathbf{1})}{\mathrm{d}\log(f_i)}.$$

Since

$$\frac{\mathrm{d}\log(P_c^\sigma C)}{\mathrm{d}\log(f_i)} = -(\sigma-1)\frac{\mathrm{d}\log(C)}{\mathrm{d}\log(f_i)},$$

we can write

$$\begin{aligned}
\frac{\mathrm{d}\log(C)}{\mathrm{d}\log(f_i)} &= \frac{1}{\sigma-1}\frac{1}{\beta'\tilde{\alpha}}\beta'\Psi_2(I-\Lambda)^{-1}\left(\frac{\mathrm{d}\log(P_c^\sigma C)}{\mathrm{d}\log(f_i)}\mathbf{1} - e_i\right) \\
&= -\frac{1}{\sigma-1}\frac{1}{\beta'\tilde{\alpha}}\beta'\Psi_2(I-\Lambda)^{-1}\left(e_i + (1-\sigma)\frac{\mathrm{d}\log(C)}{\mathrm{d}\log(f_i)}\mathbf{1}\right) \\
&= -\frac{1}{1+\frac{1}{\beta'\tilde{\alpha}}\beta'\Psi_2(I-\Lambda)^{-1}\mathbf{1}}\frac{1}{\sigma-1}\frac{1}{\beta'\tilde{\alpha}}\beta'\Psi_2(I-\Lambda)^{-1}e_i.
\end{aligned}$$

Q.E.D.

PROOF OF PROPOSITION 2: Differentiate the zero profit condition to get

$$\frac{\mathrm{d}\log(M)}{\mathrm{d}\log(z^w)} = \frac{\partial\log(\tilde{\beta})}{\partial\log(M)}\frac{\mathrm{d}\log(M)}{\mathrm{d}\log(z^w)} + \frac{\partial\log(\tilde{\alpha})}{\partial\log(M)}\frac{\mathrm{d}\log(M)}{\mathrm{d}\log(z^w)} + \frac{\partial\log(\tilde{\alpha})}{\partial\log(z^w)} + \frac{\mathrm{d}\log(P_c^\sigma C)}{\mathrm{d}\log(z^w)}.$$

Rearrange this as

$$\frac{\mathrm{d}\log(M)}{\mathrm{d}\log(z_i^w)} = (I-\Lambda)^{-1}\big((\sigma-1)\operatorname{diag}(\tilde{\alpha})^{-1}\Psi_d e_i \alpha_i (z_i^w)^{\sigma-1} - (\sigma-1)\mathrm{d}\log(C)/\mathrm{d}\log(z_i^w)\big). \tag{26}$$

This shows that when $\sigma = 1$, intensive margin shocks have no effect on the equilibrium mass of products:

$$\begin{aligned}\frac{\mathrm{d}\log(C)}{\mathrm{d}\log(z_i^w)} &= \frac{1}{\sigma-1}\frac{\mathrm{d}\log(\beta'\tilde{\alpha})}{\mathrm{d}\log(z_i^w)} \\ &= \frac{1}{\sigma-1}\frac{1}{\beta'\tilde{\alpha}}\beta'\left(\frac{\partial\tilde{\alpha}}{\partial\log M}\frac{\mathrm{d}\log(M)}{\mathrm{d}\log(z_i^w)} + \frac{\partial\tilde{\alpha}}{\partial\log(z_i^w)}\right);\end{aligned}$$

use (26) to get

$$\begin{aligned}&= \frac{1}{\sigma-1}\frac{1}{\beta'\tilde{\alpha}}\beta'\big(\Psi_2(I-\Lambda)^{-1}\big((\sigma-1)\operatorname{diag}(\tilde{\alpha})^{-1}\Psi_d e_i\alpha_i(z_i^w)^{\sigma-1} \\ &\quad - (\sigma-1)\mathrm{d}\log(C)/\mathrm{d}\log(z_i^w)\big) + \Psi_d e_i \alpha_i (z_i^w)^{\sigma-1}(\sigma-1)\big).\end{aligned}$$

Rearrange this to get the desired result. *Q.E.D.*

PROOF OF LEMMA 3: First, start with $\tilde{\beta}$. Recall that

$$\tilde{\beta}' = \beta'(I-S\Omega)^{-1},$$

where S is a diagonal matrix. So

$$\begin{aligned}\frac{\partial\tilde{\beta}'}{\partial M_i} &= -\beta'\Psi_s\frac{\partial(I-S\Omega)}{\partial M_i}\Psi_s \\ &= \beta'\Psi_s\frac{\partial S}{\partial M_i}S^{-1}S\Omega\Psi_s \\ &= \beta'\Psi_s\frac{\partial\log S}{\partial M_i}(\Psi_s - I) \\ &= \tilde{\beta}'\frac{\partial\log S}{\partial M_i}(\Psi_s - I).\end{aligned}$$

So

$$\frac{\mathrm{d}\tilde{\beta}_k}{\mathrm{d}M_i} = \tilde{\beta}_i\frac{\partial\log S_i}{\mathrm{d}M_i}\big(\Psi_{ik}^s - \mathbf{1}(i=k)\big).$$

Assuming the Dixit–Stiglitz structure, so that $S = \tilde{M}^{\sigma-1}\mu^{-\sigma}$, means

$$\tilde{\beta}' = \beta'\big(I - \tilde{M}^{\sigma-1}\mu^{-\sigma}\Omega\big)^{-1}.$$

So

$$
\begin{aligned}
\frac{\partial \tilde{\beta}'}{\partial \log(M_i)} &= M_i \frac{\partial \tilde{\beta}'}{\partial M_i} \\
&= -M_i \beta' \Psi_s \frac{\partial\left(I - \tilde{M}^{\sigma-1}\mu^{-\sigma}\Omega\right)}{\partial M_i} \Psi_s \\
&= M_i \beta' \Psi_s \frac{\partial \tilde{M}^{\sigma-1}}{\partial M_i} \mu^{-\sigma} \Omega \Psi_s \\
&= M_i \beta' \Psi_s \frac{\partial \tilde{M}^{\sigma-1}}{\partial M_i} \tilde{M}^{\sigma-1} \tilde{M}^{1-\sigma} \mu^{-\sigma} \Omega \Psi_s \\
&= M_i \beta' \Psi_s \frac{\partial \tilde{M}^{\sigma-1}}{\partial M_i} \tilde{M}^{\sigma-1} (\Psi_s - I).
\end{aligned}
$$

The kth element of this vector is

$$
\frac{\partial \tilde{\beta}_k}{\partial \log(M_i)} = \frac{\sigma - 1}{\varepsilon_i - 1} \tilde{\beta}_i (\Psi_s - I) e_k.
$$

Putting this all into a matrix gives

$$
\frac{\partial \tilde{\beta}'}{\partial \log(M\mathbf{1})} = \mathrm{diag}(\tilde{\beta}) \mathrm{diag}\left(\frac{\sigma - 1}{\varepsilon - 1}\right) (\Psi_s - I).
$$

Now, we turn to $\tilde{\alpha}$. Observe that we can write $\tilde{\alpha}$ as

$$
\tilde{\alpha} = (I - D\Omega)^{-1} D\alpha
$$

for some diagonal matrix D. So

$$
\begin{aligned}
\frac{\partial \tilde{\alpha}}{\partial M_i} &= (I - D\Omega)^{-1} \frac{\partial D}{\partial M_i} \Omega (I - D\Omega)^{-1} D\alpha + (I - D\Omega)^{-1} \frac{\partial D}{\partial M_i} \alpha \\
&= (I - D\Omega)^{-1} D D^{-1} \frac{\partial D}{\partial M_i} D^{-1} D\Omega (I - D\Omega)^{-1} D\alpha + (I - D\Omega)^{-1} \frac{\partial D}{\partial M_i} \alpha \\
&= \Psi_d D^{-1} \frac{\partial D}{\partial M_i} D^{-1} \left(\Psi_d D^{-1} I\right) D\alpha + \Psi_d D^{-1} \frac{\partial D}{\partial M_i} \alpha \\
&= \Psi_d \frac{\partial \log D}{\partial M_i} D^{-1} \tilde{\alpha}.
\end{aligned}
$$

So

$$
\frac{\partial \tilde{\alpha}_k}{\partial M_i} = \tilde{\alpha}_i \frac{\partial \log D_i}{\partial M_i} \Psi_{ki}^d \frac{1}{D_i}.
$$

Assuming the Dixit–Stiglitz structure, so that $D = \tilde{M}^{\sigma-1}\mu^{-\sigma}$, means

$$
\tilde{\alpha} = \left(I - \tilde{M}^{\sigma-1}\mu^{1-\sigma}\Omega\right)^{-1} \tilde{M}^{\sigma-1}\mu^{1-\sigma}\alpha.
$$

To simplify the notation, for this proof, let

$$B = \left(I - \tilde{M}^{\sigma-1}\mu^{1-\sigma}\Omega\right)^{-1}.$$

So

$$\begin{aligned}
\frac{1}{M_i}\frac{\partial\tilde{\alpha}}{\partial\log(M_i)} &= \frac{\partial\tilde{\alpha}}{\partial M_i} \\
&= B\frac{\partial\tilde{M}^{\sigma-1}}{\partial M_i}\mu^{1-\sigma}\alpha + B\frac{\partial\left(\tilde{M}^{\sigma-1}\right)}{\partial M_i}\left(\tilde{M}^{\sigma-1}\right)^{-1}\left(\tilde{M}^{\sigma-1}\right)\mu^{1-\sigma}\Omega B\left(\tilde{M}^{\sigma-1}\right)\mu^{1-\sigma}\alpha \\
&= B\frac{\partial\tilde{M}^{\sigma-1}}{\partial M_i}\mu^{1-\sigma}\alpha + B\frac{\partial\left(\tilde{M}^{\sigma-1}\right)}{\partial M_i}\left(\tilde{M}^{\sigma-1}\right)^{-1}(B-I)\left(\tilde{M}^{\sigma-1}\right)\mu^{1-\sigma}\alpha \\
&= B\frac{\partial\left(\tilde{M}^{\sigma-1}\right)}{\partial M_i}\left(\tilde{M}^{\sigma-1}\right)^{-1}B\left(\tilde{M}^{\sigma-1}\right)\mu^{1-\sigma}\alpha \\
&= B\frac{\partial\left(\tilde{M}^{\sigma-1}\right)}{\partial M_i}\left(\tilde{M}^{\sigma-1}\right)^{-1}\tilde{\alpha} \\
&= \Psi_d(\tilde{M})^{1-\sigma}\mu^{\sigma-1}\frac{\partial\left(\tilde{M}^{\sigma-1}\right)}{\partial M_i}\left(\tilde{M}^{\sigma-1}\right)^{-1}\tilde{\alpha}.
\end{aligned}$$

The kth element of this vector is

$$\frac{\partial\tilde{\alpha}_k}{\partial\log(M_i)} = \left(\frac{\sigma-1}{\varepsilon_i-1}\right)\left(\frac{\varepsilon_i}{\varepsilon_i-1}\right)^{\sigma-1}e_k'\Psi_d e_i\tilde{\alpha}_i\frac{1}{\tilde{M}^{\sigma_i-1}}.$$

Putting this all into a matrix gives

$$\frac{\partial\tilde{\alpha}}{\partial\log(M\mathbf{1})} = \Psi_d\mathrm{diag}(\tilde{\alpha})\mu^{\sigma-1}\tilde{M}^{1-\sigma}\mathrm{diag}\left(\frac{\sigma-1}{\varepsilon-1}\right).$$

Q.E.D.

PROOF OF EXAMPLE 1: Using the zero profit condition and Lemma 2, we have that

$$M_1 = \frac{1}{3}\frac{\tilde{M}_1^{\sigma-1}\mu_1^{1-\sigma}\tilde{M}_3^{\sigma-1}\mu_3^{1-\sigma}}{f_1\varepsilon_1}P_c^{\sigma}C, \tag{27}$$

$$M_3 = \frac{1}{3}\frac{\left(\tilde{M}_1^{\sigma-1}\mu_1^{-\sigma}+1\right)\tilde{M}_3^{\sigma-1}\mu_3^{1-\sigma}}{f_3\varepsilon_3}P_c^{\sigma}C. \tag{28}$$

The sales of industry 2 are given by

$$p_2y_2 = \frac{1}{3}\tilde{M}_3^{\sigma-1}\mu_3^{1-\sigma}P_c^{\sigma}C = \frac{1}{3}\tilde{M}_3^{\sigma-1}\mu_3^{1-\sigma}C^{1-\sigma}.$$

This implies that

$$\frac{\mathrm{d}\log(p_2y_2)}{\mathrm{d}\log(f_1)} = (\sigma-1)\left(\frac{1}{1-\varepsilon_3}\frac{\mathrm{d}\log(M_3)}{\mathrm{d}\log(f_1)} - \frac{\mathrm{d}\log C}{\mathrm{d}\log(f_1)}\right),$$

so the sales of A_2 are negatively affected, if

$$\frac{1}{\varepsilon_3 - 1}\frac{\mathrm{d}\log(M_3)}{\mathrm{d}\log(f_1)} < \frac{\mathrm{d}\log C}{\mathrm{d}\log(f_1)}.$$

Note that

$$\begin{aligned}
\frac{\mathrm{d}\log(M_1)}{\mathrm{d}\log(f_1)} &= \frac{\sigma - 1}{\varepsilon_1 - 1}\frac{\mathrm{d}\log(M_1)}{\mathrm{d}\log(f_1)} + \frac{\sigma - 1}{\varepsilon_3 - 1}\frac{\mathrm{d}\log(M_3)}{\mathrm{d}\log(f_1)} + (1-\sigma)\frac{\mathrm{d}\log(C)}{\mathrm{d}\log(f_1)} - 1,\\
\frac{\mathrm{d}\log(M_3)}{\mathrm{d}\log(f_1)} &= \frac{1}{3\tilde{\beta}_3}\mu^{-\sigma}\tilde{M}_1^{\sigma-1}\frac{\sigma - 1}{\varepsilon_1 - 1}\frac{\mathrm{d}\log(M_1)}{\mathrm{d}\log(f_1)} + \frac{\sigma - 1}{\varepsilon_3 - 1}\frac{\mathrm{d}\log(M_3)}{\mathrm{d}\log(f_1)} + (1-\sigma)\frac{\mathrm{d}\log(C)}{\mathrm{d}\log(f_1)},
\end{aligned}$$

and

$$\begin{aligned}
\frac{\mathrm{d}\log(C)}{\mathrm{d}\log(f_1)} = \frac{1}{\beta'\tilde{\alpha}}\frac{1}{3}\Bigg[&\frac{1}{\varepsilon_3 - 1}\tilde{M}_3^{\sigma-1}\mu_3^{1-\sigma}\big(1 + \tilde{M}_1^{\sigma-1}\mu_1^{1-\sigma}\big)\frac{\mathrm{d}\log(M_3)}{\mathrm{d}\log(f_1)}\\
&+ \tilde{M}_1^{\sigma-1}\mu_1^{1-\sigma}\tilde{M}_3^{\sigma-1}\mu_3^{1-\sigma}\frac{\mathrm{d}\log(M_1)}{\mathrm{d}\log(f_1)}\Bigg].
\end{aligned}$$

These three equations are collectively a linear system, so denote

$$X = \begin{bmatrix} \frac{\mathrm{d}\log(M_1)}{\mathrm{d}\log(f_1)} \\ \frac{\mathrm{d}\log(M_3)}{\mathrm{d}\log(f_1)} \\ \frac{\mathrm{d}\log(C)}{\mathrm{d}\log(f_1)} \end{bmatrix},$$

and write

$$X = AX - e_1,$$

where e_1 is the first column of the identity matrix and

$$A = \begin{bmatrix} \frac{\sigma - 1}{\varepsilon_1 - 1} & \frac{\sigma - 1}{\varepsilon_3 - 1} & 1-\sigma \\ \frac{\sigma - 1}{\varepsilon_1 - 1}\frac{1}{3\tilde{\beta}_3}\mu^{1-\sigma}\tilde{M}_1^{\sigma-1} & \frac{\sigma - 1}{\varepsilon_3 - 1} & 1-\sigma \\ \frac{\tilde{\alpha}_1}{3\beta'\tilde{\alpha}(\varepsilon_1 - 1)} & \frac{\tilde{\alpha}_1 + \tilde{\alpha}_3}{3\beta'\tilde{\alpha}(\varepsilon_3 - 1)} & 1 \end{bmatrix}.$$

Then we can write

$$X = -(I - A)^{-1}e_1.$$

In order to solve this, we only need to know the first column of $(I - A)^{-1}$, which can be solved for by hand. So,

$$X = \frac{1}{\det A}\begin{bmatrix} \dfrac{\varepsilon_3 - \sigma}{\varepsilon_3 - 1} + (\sigma - 1)\dfrac{\tilde{\alpha}_1 + \tilde{\alpha}_3}{3\beta'\tilde{\alpha}(\varepsilon_3 - 1)} \\ \dfrac{\sigma - 1}{\varepsilon_1 - 1}\left(\dfrac{\mu_1^{1-\sigma}\tilde{M}_1^{\sigma-1}}{3\tilde{\beta}_3} - \dfrac{\tilde{\alpha}_1}{3\beta\tilde{\alpha}}\right) \\ \dfrac{\sigma - 1}{\varepsilon_1 - 1}\dfrac{\mu_1^{1-\sigma}\tilde{M}_1^{\sigma-1}}{3\tilde{\beta}_3}\dfrac{\tilde{\alpha}_1 + \tilde{\alpha}_3}{3\beta\tilde{\alpha}(\varepsilon_3 - 1)} + \dfrac{\varepsilon_3 - \sigma}{\varepsilon_3 - 1}\dfrac{\tilde{\alpha}_1}{3\beta'\tilde{\alpha}(\varepsilon_1 - 1)} \end{bmatrix},$$

where we know that $\det A < 0$, because $\mathrm{d}\log(M_1)/\mathrm{d}\log(f_1) < 0$. Finally, note that

$$\frac{\mathrm{d}\log(p_2 y_2)}{\mathrm{d}\log(f_1)} = (\sigma - 1)\left(\frac{1}{1 - \varepsilon_3}\frac{\mathrm{d}\log(M_3)}{\mathrm{d}\log(f_1)} - \frac{\mathrm{d}\log C}{\mathrm{d}\log(f_1)}\right),$$

which is a linear combination of the values of X. So, $\frac{\mathrm{d}\log(p_2 y_2)}{\mathrm{d}\log(f_1)} < 0$ if and only if

$$\frac{1}{\varepsilon_3 - 1}\frac{\mathrm{d}\log(M_3)}{\mathrm{d}\log(f_1)} < \frac{\mathrm{d}\log(C)}{\mathrm{d}\log(f_1)}.$$

In other words, we need

$$\frac{\sigma - 1}{\varepsilon_1 - 1}\left(\frac{\mu_1^{1-\sigma}\tilde{M}_1^{\sigma-1}}{3\tilde{\beta}_3} - \frac{\tilde{\alpha}_1}{3\beta\tilde{\alpha}}\right) > \frac{\sigma - 1}{\varepsilon_1 - 1}\frac{\mu_1^{1-\sigma}\tilde{M}_1^{\sigma-1}}{3\tilde{\beta}_3}\frac{\tilde{\alpha}_1 + \tilde{\alpha}_3}{3\beta\tilde{\alpha}} + (\varepsilon_3 - \sigma)\frac{\tilde{\alpha}_1}{3\beta'\tilde{\alpha}(\varepsilon_1 - 1)}.$$

Rearrange this to get

$$\varepsilon_3 < (\sigma - 1)\left(\frac{\mu_1^{1-\sigma}\tilde{M}_1^{\sigma-1}\beta'\tilde{\alpha}}{\tilde{\beta}_3\tilde{\alpha}_1} - 1\right) - \frac{\mu_1^{1-\sigma}\tilde{M}^{\sigma-1} + 1}{\tilde{\beta}_3}(\sigma - 1) + \sigma.$$

Simplify this to get

$$\varepsilon_3 < (\sigma - 1)\left(\frac{\mu_1^{1-\sigma}\tilde{M}_1^{\sigma-1}}{\tilde{\beta}_3}\left(\frac{\beta'\tilde{\alpha}}{\tilde{\alpha}_1} - 1\right) - 1\right) + \sigma = \varepsilon_3^*.$$

Q.E.D.

PROOF OF PROPOSITION 3: The sales of industry l are given by

$$p_l y_l = \tilde{\beta}_l \times \tilde{\alpha}_l \times (w/P_c)^{1-\sigma} \times P_c C.$$

By our normalization, $P_c C = 1$. Therefore,

$$\frac{\mathrm{d}\log(p_l y_l)}{\mathrm{d}\log(z_k^w)} = \frac{\mathrm{d}\log(\tilde{\alpha}_l)}{\mathrm{d}\log(z_k^w)} + (1 - \sigma)\frac{\mathrm{d}\log(w/P_c)}{\mathrm{d}\log(z_k^w)}.$$

And,

$$\begin{aligned}\frac{\mathrm{d}\log(\tilde{\alpha}_l)}{\mathrm{d}\log(z_k^w)} &= \frac{\mathrm{d}}{\mathrm{d}\log(z_k^w)}\left(e_l'\Psi_d\left(\alpha \circ z^{\sigma-1}\right)\right) \\ &= (\sigma - 1)e_l'\Psi_d e_k'\alpha_k\left(z_k^w\right)^{\sigma-1}.\end{aligned}$$

Q.E.D.

LEMMA 6: *When the elasticity of substitution is equal to* 1, *and markups are constant*,

$$\log\big(C\big(z^w, z^m\big)\big) = \beta'(I-\Omega)^{-1}\left(\alpha \circ \log\big(z^w\big) + \frac{1}{\varepsilon - 1} \circ \log\big(z^m\big)\right) + \text{const},$$

so that

$$\tilde{v} = \beta'(I-\Omega)^{-1} \circ \frac{1}{\varepsilon - 1}.$$

Furthermore, *the equilibrium mass of entrants is given by*

$$M_k = \frac{\tilde{\beta}_k z_k^m}{f_k \varepsilon_k}.$$

PROOF: Note that real GDP can be written as

$$C = \frac{P_c C}{P_c} = \frac{wl + \pi}{P_c},$$

where π is total profits. By free entry, profits are zero in equilibrium. Normalize $w = 1$. Then

$$\log(C) = -\log(P_c).$$

The marginal costs of firms in industry k are given by

$$\lambda_k = \left(\frac{\alpha_k z_k^w}{w}\right)^{-\alpha_k} \prod_l \left(\frac{\omega_{kl}}{p_l}\right)^{-\omega_{kl}}.$$

Substitute

$$\lambda_k = M_k^{\frac{1}{\varepsilon_k - 1}} \frac{\varepsilon_k - 1}{\varepsilon_k} p_k,$$

and let P denote the vector of industry prices. Then, in equilibrium,

$$\log(P) = (I-\Omega)^{-1}\big(-\alpha \circ \log\big(z^w\big) + \log(\mu \mathbf{1}) - \log(\tilde{M}\mathbf{1})\big).$$

Free entry implies that

$$\tilde{M}_k = \left(\frac{\tilde{\beta}_k P_c C z_k^m}{f_k \varepsilon_k}\right)^{\frac{1}{\varepsilon_k - 1}}.$$

Substitute this into the previous expression and combine it with the fact that

$$\log(P_c) = \beta' \log(P),$$

to get

$$\log(C) = -\log(P_c) = \beta'(I-\Omega)^{-1}\left(\alpha \circ \log\big(z^w\big) + \frac{1}{\varepsilon - 1} \circ \log\big(z^m\big)\right) + \text{const},$$

where

$$\text{const} = -\beta'(I-\Omega)^{-1}\left(\log(\mu\mathbf{1}) - \frac{1}{\varepsilon - 1}\circ\big[\log(\tilde{\beta}) - \log(f) - \log(\varepsilon)\big]\right).$$

In the above expression, $1/(\varepsilon - 1)$ is the vector of $1/(\varepsilon_k - 1)$. *Q.E.D.*

PROOF OF EXAMPLE 2: Apply Lemma 6, since in the vertical economy σ can be set to 1 without loss of generality. The formula then follows from the fact that

$$\beta'(I-\Omega)^{-1} = \mathbf{1}.$$

On the other hand, sales share is given by

$$\tilde{\beta} = \beta'\big(I - \mu^{-1}\Omega\big)^{-1}e_k = \prod_{i=1}^{k-1}\mu_i^{-1}.$$

Q.E.D.

PROPOSITION 6: $\mathrm{d}\log C/\mathrm{d}\log z_k^m$ *goes to zero as the* k*th industry becomes perfectly competitive*:

$$\tilde{v}_k \to 0 \quad \textit{as } \varepsilon_k \to \infty.$$

PROOF: We simply take a limit of the expression in Proposition 1. Namely,

$$\tilde{v}_k = \frac{1}{1 + \frac{1}{\beta'\tilde{\alpha}}\beta'\Psi_2(I-\Lambda)^{-1}\mathbf{1}}\frac{\sigma - 1}{\beta'\tilde{\alpha}}\beta'\Psi_2(I-\Lambda)^{-1}e_k.$$

Proceed in steps. First, we show that $(I-\Lambda)^{-1}e_k \to e_k$. Lemma 3 implies that, in the limit, $\Psi_2 e_k = \Psi_1 e_k = \mathbf{0}$. This means that $\Lambda e_k = \mathbf{0}$ in the limit. This means

$$e_k = \Lambda e_k + e_k,$$

whence $(I-\Lambda)^{-1}e_k = e_k$. Substituting this into the formula for $\tilde{v}_k$ gives

$$\tilde{v}_k = \frac{1}{1 + \frac{1}{\beta'\tilde{\alpha}}\beta'\Psi_2(I-\Lambda)^{-1}\mathbf{1}}\frac{\sigma - 1}{\beta'\tilde{\alpha}}\beta'\Psi_2 e_k.$$

Since $\Psi_2 e_k = \mathbf{0}$, we have that $\tilde{v}_k = 0$ in the limit. *Q.E.D.*

APPENDIX C: VARIABLE MARKUPS

There are different ways to get variable markups in this context. The simplest is to assume Cournot or oligopolistic competition with a finite number of firms in each industry, but ignore the integer constraints on entry so that we can take derivatives. Alternatively, one can suppose that each industry contains atomistic incumbents, with an infinitesimal periphery on the margin. As more firms enter, the large firms lose market power, and therefore industry-level markups decline continuously as a function of entry. Finally, one could model within-industry demand as having variable elasticity of substitution so that the elasticity of substitution is increasing in the mass of products, but maintain constant elasticity of substitution across industries. This would give rise to declining markups due to demand-side pricing complementarities as in Krugman (1979) and Zhelobodko et al. (2012).

Two simple examples, when we ignore the integer constraints on M_k, are

(a) *Oligopolistic Competition*: Each firm chooses its price to maximize its profit, taking as given the economy-wide price levels and demand;

(b) *Cournot in Quantities*: Each firm chooses its quantity to maximize its profit, taking as given the economy-wide price levels and demand, and there is no product differentiation.

Under condition (a), the markup function μ is given by

$$\mu_k = \frac{\varepsilon_k M_k - (\varepsilon_k - \sigma)}{(\varepsilon_k - 1)M_k - (\varepsilon_k - \sigma)}.$$

Under condition (b), the markup function μ is given by

$$\mu_k = \frac{\sigma M_k}{\sigma M_k - 1}.$$

APPENDIX D: DETAILS OF CALIBRATION

This appendix was written with help from my research assistant Tiancheng Sun.

D.1. *Introduction*

To calibrate the production system, we use Input Output Table (Commodity by Industry) from BEA to give us the industry level expenditure shares W, labor shares a, and profit shares π. The profit of each industry (gross of entry costs) is set to be value-added minus compensation of employees, net taxes, and a depreciation rate of 10%. If this yields a negative number, it is instead taken to be zero. Expenditures on production workers is taken to be value-added minus π. We also need Use Table before Redefinitions (Commodity by Industry) from BEA to provide information about final expenditure shares b of the whole economy. Finally, the across-industry substitutability σ is assumed to be some known constant.

In calibration, we normalize TFP of firms, industry level price p, wage w, total labor force $l + f' \operatorname{diag}(M_l)\mathbf{1}$, mass of variety M, and aggregate price level p_c all to be 1, and assume that $[W, a, \pi, b]$ we observed in data correspond to steady-state equilibrium. We are now ready to calibrate the production system.

First, we can use profit shares to back out markup: $\mu_l = \frac{\varepsilon_l}{\varepsilon_l - 1}$, $\varepsilon_l = \frac{1}{\pi_l}$, where μ_l is markup of firm in industry l, and ε_l is the within-industry elasticity. The distribution of ε_l is winsorized at the the first percentile.

The production functions of firms are given by matrix Ω_{firm}, which governs the expenditure share table in equilibrium, and vector α_{firm}, which governs labor shares in equilibrium. Using cost minimization condition in equilibrium, we have:

$$\Omega_{\text{firm}} = \operatorname{diag}\bigl(\mu_l^{\sigma-1}\bigr)W, \qquad \alpha_{\text{firm}} = \operatorname{diag}\bigl(\mu_l^{\sigma-1}\bigr)a.$$

The utility function of the representative agent is given by vector $\beta = b$, which reflects the utility maximization condition in equilibrium.

The gross output vector in calibration can be given by $y' = b'(I - \operatorname{diag}(\mu_l^{-1})W)^{-1}c$, where c is real GDP and is 1 in calibration, because in our model c equals total labor force times real wage. The logic behind the above equation is similar to the use of Leontief inverse to back gross output once net output is known.

Finally, we use free-entry condition in equilibrium to calibrate fixed costs of each industry:

$$f = y - \operatorname{diag}(\mu_l^{-1})y.$$

In sum, we can use $[W, a, \pi, b, \sigma]$ to calibrate parameters $[\Omega_{\text{firm}}, \alpha_{\text{firm}}, \mu, \beta, f]$.

In the following, Section D.2 talks about data cleaning, and Section D.3 provides the proofs of the equations used in calibration by solving cost minimization problem for firms.

D.2. *The Input-Output Data and the Use Table Data*

From BEA, we can find input-output data. The data has a format like below.

Commodity / Industry		Oilseed farming	Grain farming		Secondary smelting and alloying of aluminum		Private households		Federal electric utilities	Other federal government enterprises	State and local general government	State and local government passenger transit	State and local government electric utilities	Other state and local government enterprises
Code	Commodity Description	1111A0	1111B0	...	331314	...	814000	...	S00101	S00102	S00700	S00201	S00202	S00203
1111A0	Oilseed farming	0.1100620												
1111B0	Grain farming	0.0000000												
813B00	Civic, social, professional, and similar organizations	0.0040300												
814000	Private households	0.0000000												
S00500	Federal general government (defense)	0.0000000												
S00600	Federal general government (nondefense)	0.0000000												
491000	Postal service	0.0001610												
S00102	Other federal government enterprises	0.0000000												
S00700	State and local general government	0.0000000												
S00203	Other state and local government enterprises	0.0000000												
S00401	Scrap	0.0000000												
S00402	Used and secondhand goods	0.0000000												
S00300	Noncomparable imports	0.0026792												
S00900	Rest of the world adjustment	0.0000000												
V00100	Compensation of employees	0.0050909												
V00200	Taxes on production and imports, less subsidies	0.0176097												
V00300	Gross operating surplus	0.4970980												
T011	**Total Direct Requirements**	**1.0000000**												

The column of the table describes the direct expenditure shares on each industry in order to produce a certain commodity. The sum is always equal to 1.

Several things need to be adjusted before we can use the input-output table to do the calibration.

First, there are a total of 389 industry NAICS codes from 1111A0 to S00203; however, four industries that do show up at the output side do not show up at the input side and should therefore be deleted from the column data. These four industries are 331314, S00101, S00201, and S00202.

Second, 814000, which represents private households sector, can also be deleted both from the column data and from the row data, as no industry requires 814000 as its input and all the direct requirements of 814000 go to "Compensation of employees".

Third, S00401 "Scrap," S00402 "Used and Second hand goods," and S00900 "Rest of the world adjustment" shall be deleted from the row data as they do not belong to those 389 industry codes and are quantitatively ignorable.

Fourth, S00300 "Noncomparable Imports" is important but does not belong to those 389 industry codes. In addition, we are looking at a closed economy. Thus, it is deleted from the row data.

Fifth, the sum of V00200 "Taxes on production and imports," V00300 "Gross operating surplus," and V00100 "Compensation of employees" is value-added. To get the profit share, we subtract net taxes, compensation of employees, and 10% depreciation, replacing negative numbers by zero. Value-added minus the profit share is the gross labor share of production workers.

After the above adjustment, the table we now have contains 385 rows which include 384 industries and gross labor share, and 384 columns which are those 384 industries. We renormalize the matrix to make the sum of the column to be 1 to back out the expenditure shares on intermediary inputs and labor.

Finally, we transpose the renormalized matrix to get $[W, a]$, where $W = [w_{li}]$ is a square matrix with w_{li} denoting industry l's expenditure share on industry i; and a is a vector with a_l denoting industry l's expenditure share on labor.

For the Use Table Data, we look at the sector "total final uses" only to get the vector b. We have relevant data from F01000 to F10N00. However, we ignore investment in inventories, exports, and imports to avoid negative numbers. This is also consistent with our model assumption which looks only at a closed economy and views the data as in steady-state equilibrium. The NAICS codes in Use Table Data also need to be cleaned to match the above cleaned Input-Output data. Only 384 industries shall be kept.

D.3. *Calibration of CES Production Function With TFP Normalized to 1*

PROPOSITION 6: *If the following conditions hold*:

(1) *The CES production function is*

$$y = \left(\sum_{i=1}^{n} \omega_i^{\frac{1}{\sigma}} x_i^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}.$$

(2) *The firm who operates with this production function is a price taker in the input markets where the prices for inputs are strictly positive* $\forall i: \ p_i > 0$,
then we have the following conclusions:

(1) *The marginal cost is* $\lambda = (\sum_{i=1}^{n} \omega_i p_i^{1-\sigma})^{\frac{1}{1-\sigma}}$.

(2) *The expenditure share of industry i in equilibrium is*

$$\forall i: w_i = \frac{\omega_i p_i^{1-\sigma}}{\sum_{i=1}^{n} \omega_i p_i^{1-\sigma}}.$$

(3) *The distribution parameters can be calibrated by*

$$\forall i: \omega_i = w_i^{\sigma} \left(\frac{x_i}{y} \right)^{1-\sigma}.$$

PROOF: We solve the cost minimization problem for the firm:

$$\min : \sum_{i=1}^{n} p_i x_i$$

$$\text{s.t.} \left(\sum_{i=1}^{n} \omega_i^{\frac{1}{\sigma}} x_i^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} \geq y.$$

The first-order condition gives

$$\forall i: p_i = \lambda \omega_i^{\frac{1}{\sigma}} \left(\frac{y}{x_i} \right)^{\frac{1}{\sigma}},$$

$$y = \left(\sum_{i=1}^{n} \omega_i^{\frac{1}{\sigma}} x_i^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}.$$

Because the production function is constant return to scale, we have that the marginal cost λ is given by

$$\lambda y = \sum_{i=1}^{n} p_i x_i,$$

$$\lambda y = \sum_{i=1}^{n} p_i \left(\frac{\lambda}{p_i} \right)^{\sigma} \omega_i y,$$

$$\lambda^{1-\sigma} = \sum_{i=1}^{n} \omega_i p_i^{1-\sigma},$$

$$\lambda = \left(\sum_{i=1}^{n} \omega_i p_i^{1-\sigma} \right)^{\frac{1}{1-\sigma}}.$$

Let the expenditure share of factor i be defined as $w_i \triangleq \frac{p_i x_i}{\sum_{i=1}^{n} p_i x_i} = \frac{p_i x_i}{\lambda y}$; then

$$\omega_i^{\frac{1}{\sigma}} = \frac{p_i}{\lambda} \left(\frac{x_i}{y} \right)^{\frac{1}{\sigma}} = \left(\frac{p_i x_i}{\lambda y} \right) \left(\frac{x_i}{y} \right)^{\frac{1-\sigma}{\sigma}} = w_i \left(\frac{x_i}{y} \right)^{\frac{1-\sigma}{\sigma}}.$$

Thus,

$$\omega_i = w_i^{\sigma} \left(\frac{x_i}{y} \right)^{1-\sigma}.$$

Equivalently, we can write

$$\omega_i^{\frac{1}{\sigma}} = \frac{p_i}{\lambda} \left(\frac{x_i}{y} \right)^{\frac{1}{\sigma}} = \left(\frac{p_i}{\lambda} \right)^{\frac{\sigma-1}{\sigma}} \left(\frac{p_i x_i}{\lambda y} \right)^{\frac{1}{\sigma}} = \left(\frac{p_i}{\lambda} \right)^{\frac{\sigma-1}{\sigma}} w_i^{\frac{1}{\sigma}},$$

$$\omega_i = \left(\frac{p_i}{\lambda} \right)^{\sigma-1} w_i,$$

$$\forall i: w_i = \frac{\omega_i p_i^{1-\sigma}}{\sum_{i=1}^{n} \omega_i p_i^{1-\sigma}}.$$

Q.E.D.

PROPOSITION 7: *If each firm in the economy has a CES production function of the form stated in Proposition* 1, *and the assumptions in Proposition* 1 *hold*; *in addition*:

(1) *Each firm in the same industry is identical with the same production function and market power.*

(2) *Goods and services within industry l can be aggregated by*

$$x_l = \left(M_l^{-\varphi_l} \sum_{j=1}^{N_l} x_l(j)^{\frac{\varepsilon_l - 1}{\varepsilon_l}} \Delta_l \right)^{\frac{\varepsilon_l}{\varepsilon_l - 1}}.$$

(3) *Industry level prices are all normalized to* 1: $\forall l: \ p_l = 1$.

then:

(1) *The distribution parameters of the production function of firm f in industry l can be calibrated by*

$$\forall i : \omega_i^{l,f} = w_i^l \big(\mu_l M_l^{\frac{1-\varphi_l \varepsilon_l}{1-\varepsilon_l}} \big)^{\sigma - 1}.$$

(2) *The distribution parameters of the aggregate industry production function in industry l can be calibrated by*

$$\forall i : \omega_i^l = w_i^l \mu_l^{\sigma - 1}.$$

PROOF: Consider firm f in industry l. As every firm in the industry is assumed to be identical, we have $w_i^{l,f} \equiv w_i^l$ in equilibrium and $p^{l,f} = p_l M_l^{\frac{1-\varphi_l \varepsilon_l}{\varepsilon_l - 1}}$. The marginal cost shall be related to the firm level price by the markup μ_l:

$$\lambda^{l,f} \mu_l = p^{l,f} = p_l M_l^{\frac{1-\varphi_l \varepsilon_l}{\varepsilon_l - 1}}.$$

In addition, $\forall l: \ p_l = 1$:

$$\begin{aligned}
\omega_i^{l,f} &= \big(w_i^{l,f} \big)^{\sigma} \left(\frac{x_i^{l,f}}{y^{l,f}} \right)^{1-\sigma} \\
&= \big(w_i^l \big)^{\sigma} \left(\frac{p_i x_i^{l,f}}{\lambda^{l,f} y^{l,f}} \right)^{1-\sigma} \big(\lambda^{l,f} \big)^{1-\sigma} \\
&= w_i^l \big(\lambda^{l,f} \big)^{1-\sigma} \\
&= w_i^l \big(\mu_l M_l^{\frac{1-\varphi_l \varepsilon_l}{1-\varepsilon_l}} \big)^{\sigma-1}.
\end{aligned}$$

The industry level distribution parameter can be shown by the same argument to be given by

$$\omega_i^l = w_i^l \big(\lambda^l \big)^{1-\sigma}.$$

λ^l is related to $\lambda^{l,f}$ in the following way:

$$\lambda^l = M_l^{\frac{\varepsilon_l - \varphi_l \varepsilon_l}{1-\varepsilon_l}} \lambda^{l,f} M_l = \mu_l^{-1} p_l = \mu_l^{-1}.$$

This is because to produce 1 unit of aggregate industry good requires $M_l^{\frac{\varepsilon_l - \varphi_l \varepsilon_l}{1-\varepsilon_l}}$ units of goods produced by each firm in the industry, and there are M_l firms each of which has a marginal cost of $\lambda^{l,f}$.

Therefore, ω_i^l can further be written as

$$\omega_i^l = w_i^l(\mu_l^{-1})^{1-\sigma} = w_i^l\mu_l^{\sigma-1}. \qquad Q.E.D.$$

PROPOSITION 8: *For a system of CES production functions with identical firms in each industry which satisfy the assumptions in Proposition 7, and the representative consumer has a CES preference,*

$$U = \left(\sum_{i=1}^{n} \beta_i^{\frac{1}{\sigma}} x_i^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}},$$

we have:

(1)

$$\begin{aligned} y' \operatorname{diag}(p_l^{\sigma}) &= \beta' p_c^{\sigma} c + y'_{\text{firm}} \operatorname{diag}(M_l \lambda_{l,f}^{\sigma})\Omega_{\text{firm}} \\ &= \beta' p_c^{\sigma} c + y' \operatorname{diag}\big(M_l^{\frac{1-\varphi_l\varepsilon_l}{1-\varepsilon_l}} \lambda_{l,f}^{\sigma}\big)\Omega_{\text{firm}} \\ &= \beta' p_c^{\sigma} c + y' \operatorname{diag}\big(M_l^{\frac{1-\varphi_l\varepsilon_l}{1-\varepsilon_l}} \big(\mu_l^{-1} p_l M_l^{\frac{1-\varphi_l\varepsilon_l}{\varepsilon_l-1}}\big)^{\sigma}\big)\Omega_{\text{firm}} \\ &= \beta' p_c^{\sigma} c + y' \operatorname{diag}(p_l^{\sigma}) \operatorname{diag}\big(\big(M_l^{\frac{1-\varphi_l\varepsilon_l}{1-\varepsilon_l}}\big)^{1-\sigma} \mu_l^{-\sigma}\big)\Omega_{\text{firm}}. \end{aligned}$$

(2)

$$y' \operatorname{diag}(p_l^{\sigma}) = \beta' p_c^{\sigma} c + y' \operatorname{diag}(\lambda_l^{\sigma})\Omega = \beta' p_c^{\sigma} c + y' \operatorname{diag}(p_l^{\sigma}) \operatorname{diag}(\mu_l^{-\sigma})\Omega.$$

(3)

$$\lambda_{\text{firm}}^{1-\sigma} = \Omega_{\text{firm}} p^{1-\sigma} + \alpha_{\text{firm}} w^{1-\sigma} = \operatorname{diag}\big(\mu_l M_l^{\frac{1-\varphi_l\varepsilon_l}{1-\varepsilon_l}}\big)^{\sigma-1} p^{1-\sigma};$$

thus

$$p^{1-\sigma} = \operatorname{diag}\big(\mu_l M_l^{\frac{1-\varphi_l\varepsilon_l}{1-\varepsilon_l}}\big)^{1-\sigma} \Omega_{\text{firm}} p^{1-\sigma} + \operatorname{diag}\big(\mu_l M_l^{\frac{1-\varphi_l\varepsilon_l}{1-\varepsilon_l}}\big)^{1-\sigma} \alpha_{\text{firm}} w^{1-\sigma}.$$

(4)

$$\lambda^{1-\sigma} = \Omega p^{1-\sigma} + \alpha w^{1-\sigma} = \operatorname{diag}(\mu_l)^{\sigma-1} p^{1-\sigma};$$

thus

$$p^{1-\sigma} = \operatorname{diag}(\mu_l)^{1-\sigma} \Omega p^{1-\sigma} + \operatorname{diag}(\mu_l)^{1-\sigma} \alpha w^{1-\sigma}.$$

(5)

$$\tilde{\beta}' = \beta'\big(I - \operatorname{diag}\big(\big(M_l^{\frac{1-\varphi_l\varepsilon_l}{1-\varepsilon_l}}\big)^{1-\sigma} \mu_l^{-\sigma}\big)\Omega_{\text{firm}}\big)^{-1}.$$

(6)

$$\tilde{\beta}' = \beta'\big(I - \operatorname{diag}(\mu_l^{-\sigma})\Omega\big)^{-1}.$$

(7)

$$\tilde{\alpha} = \left(I - \operatorname{diag}\left(\mu_l M_l^{\frac{1-\varphi_l \varepsilon_l}{1-\varepsilon_l}}\right)^{1-\sigma} \Omega_{\text{firm}}\right)^{-1} \operatorname{diag}\left(\mu_l M_l^{\frac{1-\varphi_l \varepsilon_l}{1-\varepsilon_l}}\right)^{1-\sigma} \alpha_{\text{firm}}.$$

(8)

$$\tilde{\alpha} = \left(I - \operatorname{diag}(\mu_l)^{1-\sigma} \Omega\right)^{-1} \operatorname{diag}(\mu_l)^{1-\sigma} \alpha.$$

(9)

$$\tilde{\beta}' p_c^{\sigma} c \tilde{\alpha} = p' y.$$

(10)

$$\begin{aligned} p_c c &= wl + p' \operatorname{diag}\left(1 - \mu_l^{-1}\right) y \\ &= wl + wf' \operatorname{diag}(M_l)\mathbf{1} \quad (\textit{free entry and resource constraint}). \end{aligned}$$

PROPOSITION 9: *If we normalize industry level price p, wage w, total labor force $l + f' \operatorname{diag}(M_l)\mathbf{1}$, mass of variety M, and aggregate price level p_c all to be 1, then in calibration where we assume that our observations of various expenditure shares and income shares correspond to steady-state equilibrium of the model*:

(1)

$$\begin{aligned} \Omega_{\text{firm}} &= \operatorname{diag}\left(\left(\mu_l M_l^{\frac{1-\varphi_l \varepsilon_l}{1-\varepsilon_l}}\right)^{\sigma-1}\right) W = \operatorname{diag}\left(\mu_l^{\sigma-1}\right) W = \Omega, \\ \alpha_{\text{firm}} &= \operatorname{diag}\left(\left(\mu_l M_l^{\frac{1-\varphi_l \varepsilon_l}{1-\varepsilon_l}}\right)^{\sigma-1}\right) a = \operatorname{diag}\left(\mu_l^{\sigma-1}\right) a = \alpha. \end{aligned}$$

(2)

$$\beta = b \quad (\textit{final expenditure shares}).$$

(3)

$$y' = b' c + y' \operatorname{diag}\left(\mu_l^{-1}\right) W;$$

or

$$y' = b' \left(I - \operatorname{diag}\left(\mu_l^{-1}\right) W\right)^{-1} c \quad (\textit{Gross Output Vector}).$$

(4)

$$\mathbf{1} = W\mathbf{1} + a.$$

(5)

$$\operatorname{diag}(M_l) f = f = y - \operatorname{diag}\left(\mu_l^{-1}\right) y \quad (\textit{fixed cost}).$$

(6)

$$c = p_c c = wl + wf' \operatorname{diag}(M_l)\mathbf{1} = 1 \quad (\textit{real GDP}).$$

(7)

$$\begin{aligned} y'\mathbf{1} = p'y &= \tilde{\beta}' p_c^{\sigma} c\tilde{\alpha} \\ &= \tilde{\beta}'\tilde{\alpha} \\ &= b'\big(I - \operatorname{diag}\big(\mu_l^{-1}\big)W\big)^{-1}(I - W)^{-1}a \\ &= b'\big(I - \operatorname{diag}\big(\mu_l^{-1}\big)W\big)^{-1}\mathbf{1} \quad (\textit{Gross Output}). \end{aligned}$$

(8)

$$\tilde{\alpha} = (I - W)^{-1}a = \mathbf{1}.$$

(9)

$$\tilde{\beta}' = b'\big(I - \operatorname{diag}\big(\mu_l^{-1}\big)W\big)^{-1} = y'$$

or

$$b' = y'\big(I - \operatorname{diag}\big(\mu_l^{-1}\big)W\big).$$

(10)

$$\begin{aligned} &f' \operatorname{diag}(M_l)\mathbf{1} \\ &\quad = y'\big(I - \operatorname{diag}\big(\mu_l^{-1}\big)\big)\mathbf{1} \\ &\quad = b'\big(I - \operatorname{diag}\big(\mu_l^{-1}\big)W\big)^{-1}\big(I - \operatorname{diag}\big(\mu_l^{-1}\big)\big)\mathbf{1} \quad (\textit{Labor Used as a Fixed Factor}). \end{aligned}$$

(11)

$$\mu_l = \frac{\varepsilon_l}{\varepsilon_l - 1}, \qquad \varepsilon_l = \frac{1}{\pi_l},$$

π_l is the profit share observed in industry l.

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