Econometrica Supplementary Material

SUPPLEMENT TO "EQUILIBRIUM LABOR TURNOVER, FIRM GROWTH, AND UNEMPLOYMENT"

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S1. PROOFS

PROOF OF LEMMA 1: CONSIDER ANY TWO FIRMS with $x > x' \in [0, 1]$ and so $v(x, \cdot) > v(x', \cdot) > 0$. If $w' = w(x', \theta, G)$ solves (6) for firm x', then optimality implies

$$w' + \lambda(\cdot)\big[1 - F\big(W\big(w'\cdot\big)\big)\big]v\big(x', \cdot\big) \leq \omega + \lambda(\cdot)\big[1 - F\big(W(\omega, \cdot)\big)\big]v\big(x', \cdot\big) \tag{S1}$$

for all $\omega < w'$ satisfying $W(\omega, \theta, G) \geq V_u(\theta, G)$. As this inequality implies $[1 - F(W(w'\cdot))] < [1 - F(W(\omega, \cdot))]$ for all such ω, then $v(x', \cdot) < v(x, \cdot)$ and (S1) further imply

$$\begin{aligned} &w' + \lambda(\cdot)\big[1 - F\big(W\big(w'\cdot\big)\big)\big]v(x, \cdot) \\ &\quad < \omega + \lambda(\cdot)\big[1 - F\big(W(\omega, \cdot)\big)\big]v(x, \theta, G) \end{aligned}$$

(with strict inequality) for all such ω. Thus if wage w' is optimal for firm x', firm $x > x'$ sets a no lower wage. This completes the proof of Lemma 1. *Q.E.D.*

PROOF OF LEMMA 2: We consider each part in turn.

(i) *The distribution of posted wages is continuous (no mass points) and has connected support.*

The proofs are by contradiction. Suppose there is a mass of firms that optimally post wage w''. Equation (6) implies a firm in this mass point is strictly better off by paying a marginally higher wage $w' > w''$, as this causes its quit rate to fall by a discrete amount. Wage w'' is therefore not optimal, which is the required contradiction.

Suppose the support is not connected; that is, there exist two equilibrium wages w', w'' with $w' > w''$, where no mass points imply $F(W(w', \cdot), \cdot) = F(W(w'', \cdot), \cdot)$. Equation (6) implies that announcing w' is not optimal, which is the required contradiction.

(ii) *Equilibrium wage strategies $w(x, \theta, G)$ are strictly increasing in $x \in [0, 1]$, where the lowest wage paid* is $w(0, \theta, G) = R(\theta, G) = b$.

Distribution function $G(\cdot)$ must have a connected support (the startup entry distribution Γ_0 is uniform and so is connected). Hence equilibrium wage strategies must be strictly increasing in x because there can be no mass points.

We next prove $w(0, \theta, G) = R(\theta, G)$ using a contradiction argument. First note that posting $w(0, \theta, G) < R(\theta, G)$ cannot be optimal since all

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workers quit into unemployment, which yields zero profit. Suppose instead $w(0, \theta, G) > R(\theta, G)$. No mass points in $F(\cdot)$ and (6) imply posting wage $w' = R(\theta, G)$ strictly dominates posting wage $w(0, \theta, G) > R(\theta, G)$, which contradicts $w(0, \theta, G)$ an equilibrium wage offer.

We now show $w(0, \theta, G) = b$. Let $\underline{w}(\theta, G) = w(0, \theta, G)$ denote the lowest wage paid in the market. As $x = 0$ is an absorbing state, then, conditional on survival, this firm forever posts wage $\underline{w}(\theta, G)$. Thus the value of being employed at firm $x = 0$, denoted $\underline{W}(\theta, G)$, is given by

$$\text{(S2)}\qquad \begin{aligned} r\underline{W}(\theta, G) &= \underline{w}(\theta, G) + \delta(\theta)\big[V_u(\cdot) - \underline{W}\big] \\ &\quad + \lambda(\cdot)\int_{\underline{W}}^{\overline{W}} \big[W' - \underline{W}\big]\, dF\big(W', \cdot\big) \\ &\quad + \alpha \int_{\underline{\theta}}^{\overline{\theta}} \big[\underline{W}\big(\theta', \cdot\big) - \underline{W}(\theta, \cdot)\big]\, dH\big(\theta' | \theta\big) + \frac{\partial \underline{W}}{\partial t}, \end{aligned}$$

where the term $\partial \underline{W}/\partial t$ describes the expected capital gain through the dynamic evolution of G.

The flow value of being unemployed and choosing home production is given by

$$\text{(S3)}\qquad \begin{aligned} rV_u &= b + \lambda(\cdot)\int_{\underline{W}}^{\overline{W}} \big[W' - V_u(\cdot)\big]\, dF\big(W', \cdot\big) \\ &\quad + \alpha \int_{\underline{\theta}}^{\overline{\theta}} \big[V_u\big(\theta', \cdot\big) - V_u(\theta, \cdot)\big]\, dH\big(\theta' | \theta\big) + \frac{\partial V_u}{\partial t}, \end{aligned}$$

while free entry into entrepreneurship implies $V_u(\cdot)$ is also given by

$$\begin{aligned} rV_u &= \lambda(\cdot)\int_{\underline{W}}^{\overline{W}} \big[W' - V_u(\cdot), 0\big]\, dF\big(W', \cdot\big) \\ &\quad + \alpha \int_{\underline{\theta}}^{\overline{\theta}} \big[V_u\big(\theta', \cdot\big) - V_u(\theta, \cdot)\big]\, dH\big(\theta' | \theta\big) \\ &\quad + \frac{\mu}{E}\int_0^1 \big[v(x, \theta, G] + W\big(w(x, \cdot), \theta, G\big) - V_u(\cdot)\big]\, dx + \frac{\partial V_u}{\partial t}, \end{aligned}$$

where at rate μ/E, the entrepreneur creates a new startup company, which, with one employee, generates expected profit $v(x, \theta, G)$ that is sold to outside investors for its value, and he/she becomes the first employee with value $W(w', \theta, G)$ on equilibrium wage $w' = w(x, \cdot)$. Thus free entry implies

$$\text{(S4)}\qquad E(\theta, G) = \frac{\mu}{b}\int_0^1 \big[v(x, \cdot) + W\big(w(x), \cdot\big) - V_u(\cdot)\big]\, d\Gamma_0(x),$$

where it is assumed that μ/b is sufficiently small that $E < U$ along the equilibrium path. As the definition of the reservation wage implies $\underline{W}(\theta, G) = V_u(\theta, G)$, (S2) and (S3) now imply $\underline{w}(\theta, G) = b$.

(iii) *Given any job offer* (w', θ, G), *each employee believes* $x = \widehat{x}(w', \theta, G)$, *where* $\widehat{x} \in [0, 1]$ *solves*

$$
\begin{aligned}
&w(\widehat{x}, \theta, G) = w' \quad \text{when} \quad w' \in \big[b, w(1, \theta, G)\big],\\
&\widehat{x} = 0 \quad \text{when} \quad w' < b,\\
&\widehat{x} = 1 \quad \text{when} \quad w' > w(1, \theta, G).
\end{aligned}
$$

It follows directly, as wages are fully revealing, that beliefs must be consistent with Bayes rule and that beliefs are monotonic;

(iv) *That any employee on wage* $w' \geq b$ *quits if and only if the outside offer* $w'' \geq w'$ *was established in the text.*

(v) *That any employee on wage* $w' < b$ *quits into unemployment follows since workers believe the firm's state* $\widehat{x} = 0$ *and that the firm will forever post wage* $w = b$ *in the future, and so given* $w' < b$, *it is better to be unemployed.*

This completes the proof of Lemma 2. *Q.E.D.*

PROOF OF PROPOSITION 1: We first show that (12) is necessary. Equation (11) implies the firm's optimal wage w satisfies the necessary first order condition

$$
1 - v(x, \cdot)\frac{h(\widehat{x}, \cdot)G'(\widehat{x})}{G(\widehat{x})}\frac{\partial \widehat{x}}{\partial w} = 0, \tag{S5}
$$

where belief $\widehat{x}(w, \cdot)$ solves $w = w(\widehat{x}, \cdot)$. As Lemma 2 implies $\partial\widehat{x}/\partial w = [1/\frac{\partial w}{\partial x}]$, (S5) implies that (12) is a necessary condition for equilibrium.

To show that (12) is sufficient, let $w(\cdot, \theta, G)$ denote the solution to the initial value problem defined in Proposition 1. As $G(0) = U > 0$, this solution is continuous and strictly increasing in x.

Now consider any firm $x \in (0, 1]$ and let

$$
C(w, \theta, G) = w + v(x, \theta, G)\int_{\widehat{x}(w,\theta,G)}^{1}\frac{h(z, \theta, G)\,dG(z)}{G(z)}
$$

describe the minimand in (11). If the firm sets a lower wage $w' = w(x', \cdot) < w$ with $x' \in [0, x)$, its employees believe $\widehat{x} = x' < x$. Hence

$$
\frac{\partial C}{\partial w}\big(w', \theta, G\big) = 1 - v(x, \theta, G)\frac{h\big(x', \theta, G\big)dG\big(x'\big)}{G\big(x'\big)}\frac{\partial \widehat{x}}{\partial w'}
$$

for such w'. But (S5) implies

$$
1 - v\big(x', \cdot\big)\frac{h\big(x', \theta, G\big)G'\big(x'\big)}{G\big(x'\big)}\frac{\partial \widehat{x}}{\partial w'} = 0
$$

at x' and combining yields

$$\frac{\partial C}{\partial w'} = 1 - \frac{v(x, \theta, G)}{v(x', \theta, G)} < 0$$

because values $v(\cdot)$ are strictly increasing in x. Thus for $w' < w(x, \theta, G)$, an increase in w' strictly decreases $C(\cdot)$. The same argument establishes that increasing w' when $x' \in (x, 1]$ strictly increases $C(\cdot)$. Finally note for wages $w' > w(1, \theta, G)$, the worker's belief is fixed at $\widehat{x} = 1$ and so higher wages strictly increase C, while wage $w' < b$ does not satisfy the constraint $W \geq V_u$. Hence given all other firms offer wages according to Proposition 1, the cost minimizing wage for any firm $x \in [0, 1]$ is to offer $w = w(x, \theta, G)$. This completes the proof of Proposition 1. *Q.E.D.*

S2. A [PARTIALLY POOLING] STATIONARY BAYESIAN EQUILIBRIA WITH MASS POINTS AND NON-MONOTONE BELIEFS

We construct a steady state example with $\alpha, \gamma = 0$ (no shocks) and $\mu < \delta$, and homogenous firms $p(x) = \overline{p}$. Equilibrium implies that all firms make the same profit $v(x) = \overline{v}$ and so hire at the same rate $\overline{h}$, where $c'(\overline{h}) = \overline{v}/\overline{p}$. With monotone beliefs, Proposition 1 establishes the equilibrium wage equation

$$w(x) = b + \overline{h}\overline{v} \log\big[G(x)/U\big].$$

We construct a stationary Bayesian equilibrium with a mass point as follows. Fix an $x^c \in (0, 1)$ and define $\overline{w} \equiv w(x^c) = b + \overline{h}\overline{v} \log[G(x^c)/U]$. Consider the set of equilibrium wage strategies

$$w^e(x) = w(x) \quad \text{for} \quad x \in \big[0, x^c\big),$$
$$w^e(x) = \overline{w} \quad \text{for} \quad x \in \big[x^c, 1\big];$$

that is, mass $1 - x^c$ of firms announce the same wage $\overline{w} = w(x^c)$. Each firm's steady state quit rate is then

$$\widehat{q}(x) = \int_x^1 \frac{h(z, \theta, G)}{G(z)}\, dG(z) = -\overline{h} \log G(x) \quad \text{for} \quad x \in \big[0, x^c\big),$$
$$\widehat{q}(x) = -\overline{h} \log G\big(x^c\big) \quad \text{for} \quad x \in \big[x^c, 1\big],$$

since workers employed by firms in the mass point quit when indifferent. Steady state turnover arguments imply, for any $x \leq x^c$, that $G(x)$ must satisfy

$$\delta\big[1 - G(x)\big] = \mu[1 - x] + \widehat{q}(x) G(x)$$

and so $G(x)$ is uniquely determined by the implicit function

$$G(x)\big[\delta - \overline{h}\log G(x)\big] = \delta - \mu[1-x] \quad \text{for} \quad x \le x^c. \tag{S6}$$

It is easy to show that $x < 1$ implies $G(x) < 1$. Putting $x = 0$ in (3) implies that $\overline{v} > 0$ satisfies

$$(r+\delta)\overline{v} = \overline{p} - b - \overline{p}c(\overline{h}) + \overline{h}\overline{v}[1 + \log U],$$

with steady state unemployment $U = G(0) > 0$ given by the implicit function

$$U[\delta - \overline{h}\log U] = \delta - \mu.$$

In any such equilibrium, all firms $x \in [0, 1]$ make the same profit $\overline{v}$, but all firms with $x \ge x^c$ post the same wage $\overline{w}$ and have the same quit rate $\widehat{q}(x^c) > 0$. This describes a stationary Bayesian equilibrium with the following beliefs:

Non-Monotone Beliefs: Given any job offer w', each employee believes $x = \widehat{x}(w')$, where $\widehat{x}$ solves

$$\begin{aligned}
&w^e(\widehat{x}) = w' \quad \text{when} \quad w' \in [b, \overline{w}),\\
&\widehat{x} \sim U\big[x^c, 1\big] \quad \text{when} \quad w' = \overline{w},\\
&\widehat{x} = 0 \quad \text{when} \quad w' > \overline{w},\\
&\widehat{x} = 0 \quad \text{when} \quad w' < b.
\end{aligned}$$

Should any firm in the mass point $x \in [x^c, 1]$ deviate to wage $w' > \overline{w}$, these beliefs imply workers expect wage $w = b$ in the entire future, which increases their quit rate to $\widehat{q}(0) > \widehat{q}(x^c)$. Equation (6) thus implies any such wage deviation is strictly profit reducing. As, by construction, all wages $w' \in [b, \overline{w}]$ generate equal value (while $w' < b$ generates zero profit because all quit into unemployment), a stationary Bayesian equilibrium exists with a mass point of firms offering $\overline{w}$.

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