6 Online appendix

General framework for the proofs of Propositions 1–3

Letting $\overline{u}(c_1, c_2) = u(c_1, c_2) + v(c_1, c_2)$, the first-order conditions for the competitive consumer's maximization problem are given by

$$(1+\tau_i)\overline{u}_1(c_1, c_2) = r_2\overline{u}_2(c_1, c_2) \text{ and } (1+\tau_i)v_1(\tilde{c}_1, \tilde{c}_2) = r_2 v_2(\tilde{c}_1, \tilde{c}_2),$$

where

$$c_1 = r_1 k_1 + w_1 + s - (1+\tau_i)k_2, \quad c_2 = r_2 k_2 + w_2,$$
$$\tilde{c}_1 = r_1 k_1 + w_1 + s - (1+\tau_i)\tilde{k}_2, \text{ and } \tilde{c}_2 = r_2\tilde{k}_2 + w_2.$$

Using the first-order conditions of the consumer, it is easy to show that $\bar{k}_2 > \bar{\tilde{k}}_2$ and $\overline{u}_1(c_1, c_2) - v_1(\tilde{c}_1, \tilde{c}_2) = u_1(c_1, c_2) + v_1(c_1, c_2) - v_1(\tilde{c}_1, \tilde{c}_2) > 0$. We will use these below. The value function of the representative agent is given by

$$U(\bar{k}_1, P, \tau_i) = \overline{u}(r_1\bar{k}_1 + w_1 - \bar{k}_2, r_2 k_2 + w_2) - v(r_1\bar{k}_1 + w_1 + \tau_i(\bar{k}_2 - \bar{\tilde{k}}_2) - \bar{\tilde{k}}_2, r_2\bar{\tilde{k}}_2 + w_2).$$

Differentiating the value function with respect to τ_i and using the consumer's first-order conditions, we obtain

$$\frac{dU}{d\tau_i} = \bar{u}_1(c_1, c_2)\tau_i\frac{d\bar{k}_2}{d\tau_i} + \bar{u}_2(c_1, c_2)\left(\frac{dr_2}{d\tau_i}\bar{k}_2 + \frac{dw_2}{d\tau_i}\right)$$
$$-v_1(\tilde{c}_1, \tilde{c}_2)\left\{\bar{k}_2 - \bar{\tilde{k}}_2 + \tau_i\frac{d\bar{k}_2}{d\tau_i}\right\} - v_2(\tilde{c}_1, \tilde{c}_2)\left(\frac{dr_2}{d\tau_i}\bar{\tilde{k}}_2 + \frac{dw_2}{d\tau_i}\right).$$

Proof of Proposition 1: In partial equilibrium, $\frac{dr_2}{d\tau_i} = 0$ and $\frac{dw_2}{d\tau_i} = 0$. Therefore, we obtain

$$\frac{dU}{d\tau_i} = (\overline{u}_1(c_1, c_2) - v_1(\tilde{c}_1, \tilde{c}_2))\tau_i\frac{d\bar{k}_2}{d\tau_i} - v_1(\tilde{c}_1, \tilde{c}_2)\{\bar{k}_2 - \bar{\tilde{k}}_2\}.$$

Since $\bar{k}_2 > \bar{\tilde{k}}_2$ and $\overline{u}_1(c_1, c_2) - v_1(\tilde{c}_1, \tilde{c}_2) > 0$, then $\frac{dU}{d\tau_i} < 0$ for all $\tau_i \geq 0$. Therefore, the optimal tax rate has to be negative.

Proof of Proposition 3: In this case, $\frac{dr_2}{d\tau_i}\bar{k}_2 + \frac{dw_2}{d\tau_i} = w'(\bar{k}_2) + r'(\bar{k}_2)\bar{k}_2 = 0$ and $\frac{dr_2}{d\tau_i}\bar{\tilde{k}}_2 + \frac{dw_2}{d\tau_i} = r'(\bar{k}_2)(\bar{\tilde{k}}_2 - \bar{k}_2)\frac{d\bar{k}_2}{d\tau_i}$. Using these relations,

$$\begin{aligned}\frac{dU}{d\tau_i} &= (\overline{u}_1(c_1, c_2) - v_1(\tilde{c}_1, \tilde{c}_2))\tau_i\frac{d\bar{k}_2}{d\tau_i} - v_1(\tilde{c}_1, \tilde{c}_2)\{\bar{k}_2 - \bar{\tilde{k}}_2\} + v_2(\tilde{c}_1, \tilde{c}_2)(\bar{k}_2 - \bar{\tilde{k}}_2)\frac{dr(\bar{k}_2)}{d\tau_i} \\ &= (\overline{u}_1(c_1, c_2) - v_1(\tilde{c}_1, \tilde{c}_2))\tau_i\frac{d\bar{k}_2}{d\tau_i} + v_1(\tilde{c}_1, \tilde{c}_2)\{\bar{k}_2 - \bar{\tilde{k}}_2\}\left\{\widetilde{MRS}\frac{dr(\bar{k}_2)}{d\tau_i} - 1\right\},\end{aligned}$$

where $\widetilde{MRS} = \frac{v_2(\tilde{c}_1,\tilde{c}_2)}{v_1(\tilde{c}_1,\tilde{c}_2)}$. Taking the derivative of the first-order condition for the actual choice with respect to τ_i, we can show that $\frac{d\bar{k}_2}{d\tau_i} < 0$. We will show that $1 - \widetilde{MRS}\frac{dr(\bar{k}_2)}{d\tau_i} > 0$. This implies that $\frac{dU}{d\tau_i} < 0$ for all $\tau_i \geq 0$. Thus, the optimal tax is negative, i.e., $\tau_i < 0$. To show this note that, in equilibrium, $r(\bar{k}_2) \times MRS = r(\bar{k}_2) \times \widetilde{MRS} = 1 + \tau_i$, where $MRS = \frac{\overline{u}_2(c_1,c_2)}{\overline{u}_1(c_1,c_2)}$. Therefore, it is enough to show that $1 - MRS\frac{dr(\bar{k}_2)}{d\tau_i} > 0$. Taking the derivative of $r(\bar{k}_2) \times MRS = 1 + \tau_i$ with respect to τ_i, we obtain $1 - MRS\frac{dr(\bar{k}_2)}{d\tau_i} = \frac{dMRS}{d\tau_i} r(\bar{k}_2)$. Given that $MRS = \frac{\overline{u}_2(c_1,c_2)}{\overline{u}_1(c_1,c_2)}$ and $\frac{d\bar{k}_2}{d\tau_i} < 0$, it is then clear that $\frac{dMRS}{d\tau_i} > 0$.

Proof of Proposition 2: In this case, $\frac{dw_2}{d\tau_i} = 0$, $\bar{k}_1 = 0$, $\bar{k}_2 = 0$, $c_1 = w_1$, and $c_2 = w_2$. Given these, we obtain

$$\begin{aligned}\frac{dU}{d\tau_i} &= \overline{u}_1(c_1,c_2)\tau_i\frac{d\bar{k}_2}{d\tau_i} - v_1(\tilde{c}_1,\tilde{c}_2)\left\{-\bar{\bar{k}}_2 + \tau_i\frac{d\bar{k}_2}{d\tau_i}\right\} - v_2(\tilde{c}_1,\tilde{c}_2)\frac{dr_2}{d\tau_i}\bar{\bar{k}}_2 \\ &= (\overline{u}_1(c_1,c_2) - v_1(\tilde{c}_1,\tilde{c}_2))\tau_i\frac{d\bar{k}_2}{d\tau_i} + v_1(\tilde{c}_1,\tilde{c}_2)\bar{\bar{k}}_2\left(1 - \widetilde{MRS}\frac{dr_2}{d\tau_i}\right).\end{aligned}$$

The key difference between the previous case and this one is that the consumer consumes his endowment, i.e., $MRS = \frac{\bar{u}_2(w_1,w_2)}{\bar{u}_1(w_1,w_2)}$. Therefore, $\frac{dMRS}{d\tau_i} = 0$, which implies that $1 - \widetilde{MRS}\frac{dr_2}{d\tau_i} = 0$. Second, $\frac{d\bar{k}_2}{d\tau_i} = 0$. Thus, we obtain that $\frac{dU}{d\tau_i} = 0$ independent of τ_i, which implies that the consumer is indifferent to any τ_i.

Proof of Proposition 4: See the proof to Proposition 8, which studies a T-period economy with logarithmic utility.

Proof of Proposition 5: The problem of the consumer can be written as

$$U(k_1,\overline{k}_1,\tau_i) = \max_{c_1,c_2}(1+\gamma)\frac{c_1^{1-\sigma}}{1-\sigma} + \delta(1+\beta\gamma)\frac{c_2^{1-\sigma}}{1-\sigma} - \gamma\left[\max_{\tilde{c}_1,\tilde{c}_2}\frac{\tilde{c}_1^{1-\sigma}}{1-\sigma} + \delta\beta\frac{\tilde{c}_2^{1-\sigma}}{1-\sigma}\right]$$

s.t.

$$c_1 + \frac{c_2}{r(\overline{k}_2)}(1+\tau_i) = r(\overline{k}_1)k_1 + w(\overline{k}_1) + s + \frac{w(\overline{k}_2)}{r(\overline{k}_2)}(1+\tau_i) = Y.$$

The first-order conditions are

$$c_1^{-\sigma} = \frac{\delta(1+\beta\gamma)}{1+\gamma}\underbrace{\frac{r(\overline{k}_2)}{1+\tau_i}}_{m(\overline{k}_2,\tau_i)}c_2^{-\sigma} \text{ and } \tilde{c}_1^{-\sigma} = \delta\beta\frac{r(\overline{k}_2)}{1+\tau_i}\tilde{c}_2^{-\sigma}.$$

This implies

$$c_1 = \frac{Y}{1+\left[\frac{\delta(1+\beta\gamma)}{1+\gamma}\right]^{1/\sigma}\left[m\left(\overline{k}_2,\tau_i\right)\right]^{(1-\sigma)/\sigma}} \text{ and } c_2 = \left[\frac{\delta(1+\beta\gamma)}{1+\gamma}m\left(\overline{k}_2,\tau_i\right)\right]^{1/\sigma}c_1$$

$$\tilde{c}_1 = \frac{Y}{1+[\delta\beta]^{1/\sigma}\left[m\left(\overline{k}_2,\tau_i\right)\right]^{(1-\sigma)/\sigma}}, \text{ and } \tilde{c}_2 = \left[\delta\beta m\left(\overline{k}_2,\tau_i\right)\right]^{1/\sigma}\tilde{c}_1.$$

From these expressions we obtain

$$\frac{\tilde{c}_1}{c_1} = \frac{1+\left[\frac{\delta(1+\beta\gamma)}{1+\gamma}\right]^{1/\sigma}\left[m\left(\overline{k}_2,\tau_i\right)\right]^{(1-\sigma)/\sigma}}{1+[\delta\beta]^{1/\sigma}\left[m\left(\overline{k}_2,\tau_i\right)\right]^{(1-\sigma)/\sigma}} = x_1$$

and

$$\frac{\tilde{c}_2}{c_2} = \left[\frac{\beta(1+\gamma)}{1+\beta\gamma}\right]^{1/\sigma}\frac{\tilde{c}_1}{c_1} = x_2 = \left[\frac{\beta(1+\gamma)}{1+\beta\gamma}\right]^{1/\sigma} x_1.$$

Then we can write the objective function of the government, inserting the expressions above, as

$$\begin{aligned} U(\overline{k}_1,\overline{k}_1,\tau_i) &= (1+\gamma)\frac{c_1^{1-\sigma}}{1-\sigma} + \delta(1+\beta\gamma)\frac{c_2^{1-\sigma}}{1-\sigma} - \gamma\left[\frac{\tilde{c}_1^{1-\sigma}}{1-\sigma} + \delta(1+\beta\gamma)\frac{\tilde{c}_2^{1-\sigma}}{1-\sigma}\right] \\ &= \frac{c_1^{1-\sigma}}{1-\sigma} + \delta\frac{c_2^{1-\sigma}}{1-\sigma} + \gamma\left[\left(1-x_1^{1-\sigma}\right)\frac{c_1^{1-\sigma}}{1-\sigma} + \left(1-x_2^{1-\sigma}\right)\delta\beta\frac{c_2^{1-\sigma}}{1-\sigma}\right], \end{aligned}$$

where

$$c_1 = (1-d)\,\overline{k}_1 + f(\overline{k}_1) - \overline{k}_2 \text{ and } c_2 = (1-d)\,\overline{k}_2 + f(\overline{k}_2).$$

Taking the derivative of the objective function with respect to τ_i and inserting $\frac{dx_2}{d\tau_i} = \left[\frac{\beta(1+\gamma)}{1+\beta\gamma}\right]^{1/\sigma}\frac{dx_1}{d\tau_i}$ we obtain

$$\frac{d}{d\tau_i}U(\overline{k}_1,\overline{k}_1,\tau_i) = \left[-c_1^{-\sigma} + \delta r(\overline{k}_2)c_2^{-\sigma}\right]\frac{d\overline{k}_2}{d\tau_i}$$

$$+\gamma\left[-\left(1-x_1^{1-\sigma}\right)c_1^{-\sigma} + \left(1-x_2^{1-\sigma}\right)\delta\beta r(\overline{k}_2)c_2^{-\sigma}\right]\frac{d\overline{k}_2}{d\tau_i} - \gamma\left[x_1^{-\sigma}c_1^{1-\sigma} + \delta\beta\left[\frac{\beta(1+\gamma)}{1+\beta\gamma}\right]^{1/\sigma}x_2^{-\sigma}c_2^{1-\sigma}\right]\frac{dx_1}{d\tau_i}.$$

Let τ_i^* be the tax rate that maximizes the commitment utility. Then τ_i^* will generate the following condition:

$$c_1^{-\sigma} = \delta r(\overline{k}_2)c_2^{-\sigma}.$$

Using the first-order condition $c_1^{-\sigma} = \frac{\delta(1+\beta\gamma)}{1+\gamma}m\left(\overline{k}_2,\tau_i\right)c_2^{-\sigma}$, this implies

$$\frac{(1+\beta\gamma)}{1+\gamma}m\left(\overline{k}_2,\tau_i^*\right) = r(\overline{k}_2).$$

It is easy to see that $\frac{d}{d\tau_i}U(\overline{k}_1,\overline{k}_1,\tau_i^*) = 0$ at $\sigma = 1$. Thus the subsidy that maximizes utility under logarithmic utility is the same as the subsidy that maximizes the commitment utility.

We now will characterize the condition under which the following holds:

$$\frac{d}{d\tau_i}U(\overline{k}_1,\overline{k}_1,\tau_i^*) < 0 \text{ for } \sigma > 1$$

so that for $\sigma > 1$ the optimal subsidy is larger than the optimal subsidy that maximizes commitment utility. To do that we take the derivative of $\frac{d}{d\tau_i}U(\bar{k}_1, \bar{k}_1, \tau_i^*)$ with respect to σ and evaluate at $\sigma = 1$. If the derivative is negative at $\sigma = 1$, then $\frac{d}{d\tau_i}U(\bar{k}_1, \bar{k}_1, \tau_i^*) < 0$ for σ marginally above $\sigma = 1$. If the derivative is positive at $\sigma = 1$, then $\frac{d}{d\tau_i}U(\bar{k}_1, \bar{k}_1, \tau_i^*) > 0$ for σ marginally above $\sigma = 1$.

First, for later use, we compute the following objects:

$$\frac{dx_1}{d\tau_i} = \frac{1-\sigma}{\sigma}\left[m\left(\bar{k}_2, \tau_i\right)\right]^{\frac{1-2\sigma}{\sigma}} \frac{\left[\frac{\delta(1+\beta\gamma)}{1+\gamma}\right]^{1/\sigma} - [\delta\beta]^{1/\sigma}}{\left[1 + [\delta\beta]^{1/\sigma}\left[m\left(\bar{k}_2, \tau_i\right)\right]^{(1-\sigma)/\sigma}\right]^2} \frac{dm\left(\bar{k}_2, \tau_i\right)}{d\tau_i} = \frac{1-\sigma}{\sigma} H_1 \frac{dm\left(\bar{k}_2, \tau_i\right)}{d\tau_i},$$

$$\frac{dx_2}{d\tau_i} = \left[\frac{\beta(1+\gamma)}{1+\beta\gamma}\right]^{1/\sigma}\frac{dx_1}{d\tau_i}, \text{ and } H_1(\sigma = 1) = \left[m\left(\bar{k}_2, \tau_i\right)\right]^{-1}\frac{\delta(1-\beta)}{(1+\gamma)\left[1+\delta\beta\right]^2}.$$

Second, to find $\frac{d\bar{k}_2}{d\tau_i}$, take the derivative of the expression $c_1^{-\sigma} = \frac{\delta(1+\beta\gamma)}{1+\gamma}m\left(\bar{k}_2, \tau_i\right)c_2^{-\sigma}$ with respect to τ_i to obtain

$$\frac{d\bar{k}_2}{d\tau_i} = \frac{\frac{\delta(1+\beta\gamma)}{1+\gamma}c_2^{-\sigma}}{\left[\sigma c_1^{-\sigma-1} + \sigma c_2^{-\sigma-1}\frac{\delta(1+\beta\gamma)}{1+\gamma}m\left(\bar{k}_2, \tau_i\right)r\left(\bar{k}_2\right)\right]}\frac{dm\left(\bar{k}_2, \tau_i\right)}{d\tau_i} = H_2\frac{dm\left(\bar{k}_2, \tau_i\right)}{d\tau_i}.$$

We know that $\frac{d\bar{k}_2}{d\tau_i} < 0$, and thus $\frac{dm\left(\bar{k}_2, \tau_i\right)}{d\tau_i} < 0$ too. Moreover $H_2(\sigma = 1) = \frac{(1+\beta\gamma)}{1+\gamma}\frac{c_1}{\sigma r(\bar{k}_2)[1+1/\delta]}$.

At $\sigma = 1$, we have that

$$x_1 = \frac{1+\gamma+\delta\left(1+\beta\gamma\right)}{(1+\gamma)(1+\delta\beta)} \text{ and } x_2 = \frac{\beta}{1+\beta\gamma}\frac{1+\gamma+\delta\left(1+\beta\gamma\right)}{(1+\delta\beta)}.$$

Using the expressions above we can write $\frac{d}{d\tau_i}U(\bar{k}_1, \bar{k}_1, \tau_i^*)$ as

$$\begin{aligned}\frac{d}{d\tau_i}U(\bar{k}_1, \bar{k}_1, \tau_i^*) &= \gamma\underbrace{\left[-\left(1-x_1^{1-\sigma}\right) + \left(1-x_2^{1-\sigma}\right)\beta\right]}_{K_{11}}\underbrace{H_2 c_1^{-\sigma}\frac{dm\left(\bar{k}_2, \tau_i\right)}{d\tau_i}}_{K_{12}} \\ &\quad -\gamma\underbrace{\frac{1-\sigma}{\sigma}}_{K_{21}}\underbrace{\left[x_1^{-\sigma} + \delta\beta\left[\frac{\beta(1+\gamma)}{1+\beta\gamma}\right]^{1/\sigma}x_2^{-\sigma}\left[\delta r(\bar{k}_2)\right]^{(1-\sigma)/\sigma}\right]H_1 c_1^{1-\sigma}\frac{dm\left(\bar{k}_2, \tau_i\right)}{d\tau_i}}_{K_{22}}.\end{aligned}$$

Take the derivative of $\frac{d}{d\tau_i}U(\bar{k}_1, \bar{k}_1, \tau_i^*)$ with respect to σ to obtain

$$\frac{d}{d\sigma}\left[\frac{d}{d\tau_i}U(\bar{k}_1, \bar{k}_1, \tau_i^*)\right]\frac{1}{\gamma} = K_{11}\frac{dK_{12}}{d\sigma} + K_{12}\frac{dK_{11}}{d\sigma} - K_{21}\frac{dK_{22}}{d\sigma} - K_{22}\frac{dK_{21}}{d\sigma}.$$

If we evaluate this expression at $\sigma = 1$ we obtain

$$\frac{d}{d\sigma}\left[\frac{d}{d\tau_i}U(\bar{k}_1, \bar{k}_1, \tau_i^*)\right]\frac{1}{\gamma} = K_{12}\frac{dK_{11}}{d\sigma} - K_{22}\frac{dK_{21}}{d\sigma}$$

$$= \frac{(1+\beta\gamma)}{1+\gamma}\frac{\delta}{\sigma r(\overline{k}_2)\left[1+\delta\right]}\frac{dm\left(\overline{k}_2,\tau_i\right)}{d\tau_i}\frac{dK_{11}}{d\sigma}-\frac{\delta(1-\beta)}{1+\gamma+\delta\left(1+\beta\gamma\right)}\left[m\left(\overline{k}_2,\tau_i\right)\right]^{-1}\frac{dm\left(\overline{k}_2,\tau_i\right)}{d\tau_i}\frac{dK_{21}}{d\sigma},$$

where $\frac{dK_{21}}{d\sigma}=-\frac{1}{\sigma^2}$ and $\frac{dK_{11}}{d\sigma}=\beta\log(x_2)-\log(x_1)$. Evaluating at $\sigma=1$ and inserting $m\left(\overline{k}_2,\tau_i\right)=\frac{1+\gamma}{1+\beta\gamma}r(\overline{k}_2)$ we obtain

$$\begin{aligned}&\frac{d}{d\sigma}\left[\frac{d}{d\tau_i}U(\overline{k}_1,\overline{k}_1,\tau_i^*)\right]\frac{1}{\gamma}\\ =\ &\frac{dm\left(\overline{k}_2,\tau_i\right)}{d\tau_i}\frac{\delta\left(1+\beta\gamma\right)}{(1+\gamma)r(\overline{k}_2)}\left[\begin{array}{c}\left(\beta\log\left[\frac{\beta}{1+\beta\gamma}\frac{1+\gamma+\delta(1+\beta\gamma)}{(1+\delta\beta)}\right]-\log\left[\frac{1+\gamma+\delta(1+\beta\gamma)}{(1+\gamma)(1+\delta\beta)}\right]\right)\frac{1}{[1+\delta]}\\+\frac{(1-\beta)}{1+\gamma+\delta(1+\beta\gamma)}\end{array}\right].\end{aligned}$$

Since $\frac{dm\left(\overline{k}_2,\tau_i\right)}{d\tau_i}<0$, if $\left(\beta\log\left[\frac{\beta}{1+\beta\gamma}\frac{1+\gamma+\delta(1+\beta\gamma)}{(1+\delta\beta)}\right]-\log\left[\frac{1+\gamma+\delta(1+\beta\gamma)}{(1+\gamma)(1+\delta\beta)}\right]\right)\frac{1}{[1+\delta]}+\frac{(1-\beta)}{1+\gamma+\delta(1+\beta\gamma)}>0$, then $\frac{d}{d\sigma}\left[\frac{d}{d\tau_i}U(\overline{k}_1,\overline{k}_1,\tau_i^*)\right]\frac{1}{\gamma}<0$ at $\sigma=1$. Therefore, it is optimal to increase the subsidy for $\sigma>1$ if this condition above holds.

To show that it holds, let $\varphi(\beta,\gamma,\delta)=\beta\log\left(\frac{\beta(1+\gamma+\delta(1+\beta\gamma))}{(1+\beta\gamma)(1+\delta\beta)}\right)-\log\left(\frac{1+\gamma+\delta(1+\beta\gamma)}{(1+\gamma)(1+\delta\beta)}\right)+\frac{(1-\beta)(1+\delta)}{1+\gamma+\delta(1+\beta\gamma)}$. First, it is easy to show that $\lim_{\gamma\to\infty}\varphi(\beta,\gamma,\delta)=0$. Second, we will show that $\frac{d\varphi(\beta,\gamma,\delta)}{d\gamma}<0$ for all $\beta,\delta<1$, which implies that $\varphi(\beta,\gamma,\delta)>0$ for all finite $\gamma>0$ and $\beta,\delta<1$.

$$\begin{aligned}\frac{d\varphi(\beta,\gamma,\delta)}{d\gamma} &= \beta\frac{\frac{(1+\delta\beta)(1+\beta\gamma)-\beta(1+\gamma+\delta(1+\beta\gamma))}{(1+\beta\gamma)^2}}{\frac{1+\gamma+\delta(1+\beta\gamma)}{1+\beta\gamma}}-\frac{\frac{(1+\delta\beta)(1+\gamma)-(1+\gamma+\delta(1+\beta\gamma))}{(1+\gamma)^2}}{\frac{1+\gamma+\delta(1+\beta\gamma)}{1+\gamma}}-\frac{(1-\beta)(1+\delta)(1+\delta\beta)}{(1+\gamma+\delta(1+\beta\gamma))^2}\\ &= \frac{1-\beta}{1+\gamma+\delta(1+\beta\gamma)}\left\{\frac{\beta}{1+\beta\gamma}+\frac{\delta}{1+\gamma}-\frac{(1+\delta)(1+\delta\beta)}{1+\gamma+\delta(1+\beta\gamma)}\right\}\\ &= \frac{1-\beta}{1+\gamma+\delta(1+\beta\gamma)}\left\{\frac{\beta+\beta\gamma+\delta+\delta\beta\gamma}{1+\gamma+\beta\gamma+\beta\gamma^2}-\frac{1+\delta+\delta\beta+\delta^2\beta}{1+\gamma+\delta+\delta\beta\gamma}\right\}.\end{aligned}$$

The numerator of the term in curly paranthesis is

$$\begin{aligned}&= [\beta+\beta\gamma+\delta+\delta\beta\gamma]+[\beta\gamma+\beta\gamma^2+\delta\gamma+\delta\beta\gamma^2]+[\delta\beta+\delta\beta\gamma+\delta^2+\delta^2\beta\gamma]\\ &+ [\delta\beta^2\gamma+\delta\beta^2\gamma^2+\delta^2\beta\gamma+\delta^2\beta^2\gamma^2]\\ &- [1+\delta+\delta\beta+\delta^2\beta]-[\gamma+\delta\gamma+\delta\beta\gamma+\delta^2\beta\gamma]-[\beta\gamma+\delta\beta\gamma+\delta\beta^2\gamma+\delta^2\beta^2\gamma]\\ &- [\beta\gamma^2+\delta\beta\gamma^2+\delta\beta^2\gamma^2+\delta^2\beta^2\gamma^2]\\ &= \beta+\beta\gamma+\delta^2+\delta^2\beta\gamma-1-\delta^2\beta-\gamma-\delta^2\beta^2\gamma\\ &= (\beta-1)+\delta^2(1-\beta)+\gamma(\beta-1)+\delta^2\beta\gamma(1-\beta)\\ &= (1-\beta)[\delta^2+\delta^2\beta\gamma-1-\gamma]\\ &= (1-\beta)[\delta^2(1+\beta\gamma)-(1+\gamma)].\end{aligned}$$

Using this expression in $\frac{d\varphi(\beta,\gamma,\delta)}{d\gamma}$, we obtain

$$\frac{d\varphi(\beta,\gamma,\delta)}{d\gamma} = \frac{(1-\beta)^2}{(1+\gamma+\delta(1+\beta\gamma))^2}\frac{\delta^2(1+\beta\gamma)-(1+\gamma)}{(1+\gamma)(1+\beta\gamma)}.$$

Note that $\delta^2(1+\beta\gamma) < 1+\gamma$ for all $\delta, \beta < 1$. As a result, $\frac{d\varphi(\beta,\gamma,\delta)}{d\gamma} < 0$ for all $\delta, \beta < 1$.

Next, we will show that $\frac{d}{d\sigma}\left(\frac{c_2(\tau_i(\sigma))}{c_1(\tau_i(\sigma))}\right)|_{\sigma=1} > 0$. For this purpose let $k_2(\tau_i(\sigma))$ be the competitive-equilibrium savings associated with the optimal tax policy $\tau_i(\sigma)$ and $k_2^c(\sigma)$ be the commitment savings for a given σ. We will show that $\frac{d}{d\sigma}(k_2(\tau_i(\sigma)) - k_2^c(\sigma))|_{\sigma=1} > 0$. Thus, the competitive-equilibrium savings under the optimal policy is higher than commitment savings when σ is marginally higher than 1. To see this, first consider the consumer's optimality conditions under commitment and in competitive equilibrium.

$$\begin{aligned} \delta f'(k_2^c(\sigma))\left(\frac{y_1 - k_2^c(\sigma)}{f(k_2^c(\sigma))}\right)^{\sigma} &= 1 && \text{under commitment,} \\ \frac{\delta(1+\beta\gamma)}{(1+\gamma)(1+\tau_i(\sigma))} f'(k_2(\tau_i(\sigma)))\left(\frac{y_1 - k_2(\tau_i(\sigma))}{f(k_2(\tau_i(\sigma)))}\right)^{\sigma} &= 1 && \text{in competitive equilibrium.} \end{aligned}$$

We can rewrite this problem as

$$\begin{aligned} F(k_2^c(\sigma),\sigma) &= 1 && \text{under commitment,} \\ \frac{(1+\beta\gamma)}{(1+\gamma)(1+\tau_i(\sigma))} F(k_2^c(\sigma),\sigma) &= 1 && \text{in competitive equilibrium,} \end{aligned}$$

where $F(k_2,\sigma) = \delta f'(k_2)\left(\frac{y_1-k_2}{f(k_2)}\right)^{\sigma}$. We know that

1. $k_2^c(1) = k_2(\tau_i(1))$
2. $\frac{(1+\beta\gamma)}{(1+\gamma)(1+\tau_i(\sigma))} = 1$
3. $\tau_i'(1) < 0$.

Next, take the derivative of the commitment and competitive equilibrium optimality condition with respect to σ to obtain

$$\begin{aligned} \left.\frac{dk_2^c(\sigma)}{d\sigma}\right|_{\sigma=1} &= -\frac{F_2(k_2^c(1),1)}{F_1(k_2^c(1),1)} \\ \left.\frac{dk_2(\tau_i(\sigma))}{d\sigma}\right|_{\sigma=1} &= -\frac{F_2(k_2(\tau_i(1)),1)}{F_1(k_2(\tau_i(1)),1)} + \frac{\tau_i'(1)F(k_2(\tau_i(1)),1)}{(1+\tau_i(1))F_1(k_2(\tau_i(1)),1)}. \end{aligned}$$

Since $k_2^c(1) = k_2(\tau_i(1))$, taking the difference yields

$$\frac{d}{d\sigma}(k_2(\tau_i(\sigma)) - k_2^c(\sigma))|_{\sigma=1} = \frac{\tau_i'(1)F(k_2(\tau_i(1)),1)}{(1+\tau_i(1))F_1(k_2(\tau_i(1)),1)}.$$

Note that $\tau_i'(1) < 0$ and it is easy to see that $F_1(k_2(\tau_i(1)),1) < 0$. As a result, $\frac{d}{d\sigma}(k_2(\tau_i(\sigma)) - k_2^c(\sigma))|_{\sigma=1} > 0$. This directly implies that $\frac{d}{d\sigma}\left(\frac{c_2(\tau_i(\sigma))}{c_1(\tau_i(\sigma))}\right)|_{\sigma=1} > 0$.

Proof of Proposition 7: To prove this proposition, we solve the consumer's problem backwards, find her optimal consumption choices, and use those decision rules to obtain her value function.

Problem at time $T-1$: The consumer's problem reads

$$\max_{c_{T-1},c_T}(1+\gamma)\log(c_{T-1})+\delta(1+\beta\gamma)\log(c_T)-\gamma\max_{\tilde{c}_{T-1},\tilde{c}_T}\log(\tilde{c}_{T-1})+\delta\beta\ \log(\tilde{c}_T)$$

subject to the budget constraints

$$c_{T-1}+(1+\tau_{i,T-1})k_T=r(\bar{k}_{T-1})k_T+w(k_T)+s_T \text{ and } c_T=Y_T=r(\bar{k}_T)k_T+w(\bar{k}_T).$$

The rest-of-lifetime budget constraint is thus

$$c_{T-1}+c_T\frac{1+\tau_{i,T-1}}{r(\bar{k}_T)}=r(\bar{k}_{T-1})k_{T-1}+w(\bar{k}_{T-1})+s_{T-1}+w(\bar{k}_T)\frac{1+\tau_{i,T-1}}{r(\bar{k}_T)}=Y_{T-1}.$$

The first-order condition is $\frac{1}{c_{T-1}}=\frac{\delta(1+\beta\gamma)}{1+\gamma}\frac{r(\bar{k}_T)}{1+\tau_{i,T-1}}\frac{1}{c_T}$. Inserting c_T into the rest-of-lifetime budget constraint, we obtain

$$c_{T-1}=\frac{1+\gamma}{1+\gamma+\delta(1+\beta\gamma)}Y_{T-1} \text{ and } c_T=\frac{\delta(1+\beta\gamma)}{1+\gamma+\delta(1+\beta\gamma)}\frac{r(\bar{k}_T)}{1+\tau_{i,T-1}}Y_{T-1}.$$

This implies

$$\tilde{c}_{T-1}=\frac{1}{1+\delta\beta}Y_{T-1} \text{ and } \tilde{c}_T=\frac{\delta\beta}{1+\delta\beta}\frac{r(\bar{k}_T)}{1+\tau_{i,T-1}}Y_{T-1}.$$

Notice that the c and the $\tilde{c}$ are constant multiples of each other. As a result, the value function becomes

$$U_{T-1}(k_{T-1},\bar{k}_{T-1},\tau)=\log(c_{T-1})+\delta\log(c_T)+\text{a constant}.$$

Now rewrite the value function in period $T-1$ to be used in the problem of the consumer in period $T-2$ by inserting the consumption allocations as functions of Y_{T-1}. This delivers

$$U_{T-1}(k_{T-1},\bar{k}_{T-1},\tau)=(1+\delta)\log(Y_{T-1})+\delta\log\left(r(\bar{k}_T)/(1+\tau_{i,T-1})\right)+\text{a constant}.$$

Problem at time $T-2$: Using the $T-2$ budget constraint and the rest-of-lifetime budget constraint at time $T-1$ for the consumer, we obtain the rest-of-lifetime budget constraint at time $T-2$ as

$$c_{T-2}+\frac{1+\tau_{i,T-2}}{r(\bar{k}_{T-1})}Y_{T-1}=Y_{T-2}$$

$$=r(\bar{k}_{T-2})k_{T-2}+w(\bar{k}_{T-2})+s_{T-2}+\frac{w(\bar{k}_{T-1})+s_{T-1}}{r(\bar{k}_{T-1})}(1+\tau_{i,T-2})+\frac{w(\bar{k}_T)}{r(\bar{k}_{T-1})r(\bar{k}_T)}(1+\tau_{i,T-2})(1+\tau_{i,T-1}).$$

The objective of the government is to maximize

$$\max_{c_{T-2},Y_{T-1}}(1+\gamma)\log(c_{T-2})+\delta(1+\beta\gamma)\left[(1+\delta)\log(Y_{T-1})+\delta\log\left(\frac{r(\bar{k}_T)}{1+\tau_{i,T-1}}\right)+\text{a constant}\right]$$

$$-\gamma \max_{\tilde{c}_{T-2},\tilde{Y}_{T-1}} \log(\tilde{c}_{T-2}) + \delta\beta\left[(1+\delta)\log(\tilde{Y}_{T-1}) + \delta \log\left(\frac{r(\bar{k}_T)}{1+\tau_{i,T-1}}\right) + \text{a constant.}\right]$$

The first-order condition is

$$\frac{1}{c_{T-2}} = \frac{\delta(1+\delta)(1+\beta\gamma)}{1+\gamma}\frac{r(\bar{k}_{T-1})}{1+\tau_{i,T-2}}\frac{1}{Y_{T-1}}.$$

Using the budget constraint, we obtain

$$c_{T-2} = \frac{1+\gamma}{1+\gamma+\delta(1+\delta)(1+\beta\gamma)}Y_{T-2} \text{ and } Y_{T-1} = \frac{\delta(1+\delta)(1+\beta\gamma)}{1+\gamma+\delta(1+\delta)(1+\beta\gamma)}\frac{r(\bar{k}_{T-1})}{1+\tau_{i,T-2}}Y_{T-2}.$$

Inserting Y_{T-1} in terms of c_{T-1} into the consumer's problem, we obtain the following Euler equation:

$$\frac{1}{c_{T-2}} = \frac{\delta(1+\delta)(1+\beta\gamma)}{1+\gamma+\delta(1+\beta\gamma)}\frac{r(\bar{k}_{T-1})}{1+\tau_{i,T-2}}\frac{1}{c_{T-1}}.$$

The temptation allocations are given by

$$\tilde{c}_{T-2} = \frac{1}{1+\delta\beta(1+\delta)}Y_{T-2} \text{ and } \tilde{Y}_{T-1} = \frac{\delta\beta(1+\delta)}{1+\delta\beta(1+\delta)}\frac{r(\bar{k}_{T-1})}{1+\tau_{i,T-2}}Y_{T-2}.$$

The objective function of the government is

$$U_{T-2}(k_{T-2},\bar{k}_{T-2},\tau_i) = \log(c_{T-2}) + \delta(1+\delta)\log(Y_{T-1}) + \delta^2 \log\left(\frac{r(\bar{k}_T)}{1+\tau_{i,T-1}}\right) + \text{a constant.}$$

Since c_{T-1} is a multiple of Y_{T-1} and c_T is a multiple of $\left(\frac{r(\bar{k}_T)}{1+\tau_{i,T-1}}\right)Y_{T-1}$, inserting those we obtain

$$U_{T-2}(k_{T-2},\bar{k}_{T-2},\tau_i) = \log(c_{T-2}) + \delta\log(c_{T-1}) + \delta^2\log(c_T) + \text{a constant.}$$

Problem at $T-3$: The first-order condition for the consumer is

$$\frac{1}{c_{T-3}} = \frac{\delta(1+\delta+\delta^2)(1+\beta\gamma)}{1+\gamma}\frac{r(\bar{k}_{T-2})}{1+\tau_{i,T-3}}\frac{1}{Y_{T-2}} = \frac{\delta(1+\delta+\delta^2)(1+\beta\gamma)}{1+\gamma+\delta(1+\delta)(1+\beta\gamma)}\frac{r(\bar{k}_{T-2})}{1+\tau_{i,T-3}}\frac{1}{c_{T-2}}.$$

$$U_{T-3}(k_{T-2},\bar{k}_{T-3},\tau_i) = \log(c_{T-3}) + \delta\log(c_{T-2}) + \delta^2\log(c_{T-1}) + \log(c_T) + \text{a constant.}$$

Continuing this procedure backwards completes the proof.

Proof of Proposition 9: We will solve the problem of the consumer and find tax rates that implement the commitment allocation. Proposition 6 implies that the problem of a consumer at age t is given by

$$\max_{c_t,Y_{t+1}} \frac{c_t^{1-\sigma}}{1-\sigma} + \delta\beta U_{t+1}(Y_{t+1})$$

subject to

$$c_t + \frac{1+\tau_{i,t}}{r_{t+1}} Y_{t+1} = Y_t$$

where

$$U_t(Y_t) = \frac{c_t^{1-\sigma}}{1-\sigma} + \delta\beta U_{t+1}(Y_{t+1}).$$

We guess and verify that $U_{t+1}(Y_t) = b_t \frac{Y_t^{1-\sigma}}{1-\sigma}$, where $b_T = 1$. The optimality condition for the consumer is given by

$$c_t^{-\sigma} = \delta\beta b_{t+1} \frac{r_{t+1}}{1+\tau_{i,t}} Y_{t+1}^{-\sigma}.$$

Inserting this into the budget constraint, we obtain

$$\begin{aligned} c_t &= \frac{Y_t}{1 + (\delta\beta b_{t+1})^{1/\sigma} \left(\frac{r_{t+1}}{1+\tau_{i,t}}\right)^{(1-\sigma)/\sigma}} \\ Y_{t+1} &= \frac{\left(\delta\beta b_{t+1} \frac{r_{t+1}}{1+\tau_{i,t}}\right)^{1/\sigma} Y_t}{1 + (\delta\beta b_{t+1})^{1/\sigma} \left(\frac{r_{t+1}}{1+\tau_{i,t}}\right)^{(1-\sigma)/\sigma}}. \end{aligned}$$

Using these decision rules, we obtain

$$b_t = \frac{1 + \frac{1}{\beta}(\delta\beta b_{t+1})^{1/(\sigma)} \left(\frac{r_{t+1}}{1+\tau_{i,t}}\right)^{(1-\sigma)/\sigma}}{\left(1 + (\delta\beta b_{t+1})^{1/(\sigma)} \left(\frac{r_{t+1}}{1+\tau_{i,t}}\right)^{(1-\sigma)/\sigma}\right)^{1-\sigma}}.$$

Note that the optimality condition for the consumer can be written as

$$c_t^{-\sigma} = \delta r_{t+1} \frac{\beta b_{t+1}}{1+\tau_{i,t}} \left(1 + (\delta\beta b_{t+2})^{1/\sigma} \left(\frac{r_{t+2}}{1+\tau_{i,t+1}}\right)^{(1-\sigma)/\sigma}\right)^{-\sigma} c_{t+1}^{-\sigma}.$$

Inserting b_{t+1}

$$c_t^{-\sigma} = \delta r_{t+1} \frac{\beta}{1+\tau_{i,t}} \frac{1 + \frac{1}{\beta}(\delta\beta b_{t+2})^{1/\sigma} \left(\frac{r_{t+2}}{1+\tau_{i,t+1}}\right)^{(1-\sigma)/\sigma}}{1 + (\delta\beta b_{t+2})^{1/\sigma} \left(\frac{r_{t+2}}{1+\tau_{i,t+1}}\right)^{(1-\sigma)/\sigma}} c_{t+1}^{-\sigma}.$$

To implement the commitment allocation, the government should set

$$\frac{\beta}{1+\tau_{i,t}} \frac{1 + \frac{1}{\beta}(\delta\beta b_{t+2})^{1/\sigma} \left(\frac{r_{t+2}}{1+\tau_{i,t+1}}\right)^{(1-\sigma)/\sigma}}{1 + (\delta\beta b_{t+2})^{1/\sigma} \left(\frac{r_{t+2}}{1+\tau_{i,t+1}}\right)^{(1-\sigma)/\sigma}} = 1,$$

where r_t for all t is the equilibrium interest rate that arises under commitment, i.e. $r_t = r(\bar{k}_t)$.

The recursive formulas for b_t and $\tau_{i,t}$ jointly determine the sequence of optimal tax rates. We solve these formulas backwards noting that $b_T = 1$ and $b_{T+1} = 0$. Thus, $\tau_{i,T-1} = \beta - 1$ and

$b_{T-1} = \frac{1+\delta^{1/\sigma} r_T^{(1-\sigma)/\sigma}}{(1+\beta\delta^{1/\sigma} r_T^{(1-\sigma)/\sigma})^{1-\sigma}}$. Continuing backwards, we obtain $\tau_{i,T-2} = \frac{\beta-1}{1+\beta\delta^{1/\sigma} r_T^{(1-\sigma)/\sigma}}$, $b_{T-2} = \frac{1+\delta^{1/\sigma} r_{T-1}^{(1-\sigma)/\sigma}(1+\delta^{1/\sigma} r_T^{(1-\sigma)/\sigma})}{(1+\beta\delta^{1/\sigma} r_{T-1}^{(1-\sigma)/\sigma}(1+\delta^{1/\sigma} r_T^{(1-\sigma)/\sigma}))^{1-\sigma}}$,

$$\tau_{i,T-3} = \frac{\beta - 1}{1 + \beta(\delta^{1/\sigma} r_{T-1}^{(1-\sigma)/\sigma} + \delta^{2/\sigma} r_{T-1}^{(1-\sigma)/\sigma} r_T^{(1-\sigma)/\sigma})},$$

and

$$\tau_{i,T-4} = \frac{\beta - 1}{1 + \beta(\delta^{1/\sigma} r_{T-2}^{(1-\sigma)/\sigma} + \delta^{2/\sigma} r_{T-2}^{(1-\sigma)/\sigma} r_{T-1}^{(1-\sigma)/\sigma} + \delta^{3/\sigma} r_{T-2}^{(1-\sigma)/\sigma} r_{T-1}^{(1-\sigma)/\sigma} r_T^{(1-\sigma)/\sigma})}.$$

One can notice the pattern in the expressions above, which implies the optimal tax for period t is given by:

$$\tau_{i,t} = \frac{\beta - 1}{1 + \beta \sum_{m=t+2}^{T} \left\{ \left(\delta^{1/\sigma}\right)^{m-(t+1)} \prod_{n=t+2}^{m} r(\bar{k}_n)^{(1-\sigma)/\sigma} \right\}}.$$

We can also show that as $T \to \infty$, the optimal tax rate converges to a negative value. To see this, let $\{c_t^c\}_{t=0}^{\infty}$ be the consumption sequence associated with the commitment solution. Inserting the commitment Euler equation $\frac{c_{t+1}^c}{c_t^c} = (\delta r_{t+1})^{1/\sigma}$ into the tax expression, we obtain

$$\tau_{i,t} = \frac{\beta - 1}{1 + \frac{\beta}{c_{t+1}^c} \left[\frac{c_{t+2}^c}{r_{t+2}} + \frac{c_{t+3}^c}{r_{t+2} r_{t+3}} + ... + \frac{c_T^c}{r_{t+2} r_{t+3} ... r_T} \right]}.$$

Note that

$$c_{t+1}^c + \frac{c_{t+2}^c}{r_{t+2}} + \frac{c_{t+3}^c}{r_{t+2} r_{t+3}} + ... + \frac{c_T^c}{r_{t+2} r_{t+3} ... r_T} = Y_{t+1}^c,$$

where Y_t^c is the lifetime income at time t associated with the commitment solution. Thus, the optimal tax rate can be written as

$$\tau_{i,t} = \frac{(\beta - 1)\frac{c_{t+1}^c}{Y_{t+1}^c}}{(1 - \beta)\frac{c_{t+1}^c}{Y_{t+1}^c} + \beta}.$$

Note that since $c_{t+1}^c / Y_{t+1}^c > 0$ for any t and T, we obtain that $\tau_{i,t} < 0$ for all t. Moreover, since the equilibrium allocation under the optimal tax sequence is the same as the allocation associated with the commitment solution and self-control cost is zero, the optimal tax policy delivers first best welfare.