

SUPPLEMENT TO “LIQUIDITY IN ASSET MARKETS
WITH SEARCH FRICTIONS”

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APPENDIX B: GENERALIZED PREFERENCE SHOCKS

IN THIS APPENDIX we generalize the stochastic process for preference shocks as follows. As before, each investor receives a preference shock with Poisson arrival rate δ , and this process is independent across investors. But now we let $\Pi = [\pi_{ij}]$ denote an $I \times I$ matrix and assume that conditional on receiving a preference shock, an investor with preference type i draws preference type j with probability $\pi_{ij} > 0$, with $\sum_{j=1}^I \pi_{ij} = 1$ for all $i \in \mathbb{X}$. The formulation studied in the body of the paper corresponds to the i.i.d. case, $\pi_{ij} = \pi_j$ for all i .

Equilibrium

The investor’s value function $V_i(a, t)$ still satisfies (1) and the dealer’s value function is unchanged. The bargaining outcome is also unchanged, so $V_i(a, t)$ also satisfies (5). The following lemma generalizes Lemma 1.

LEMMA 5: *An investor with preference type i and asset holdings a who readjusts his asset position at time t solves*

$$(39) \quad \max_{a' \geq 0} [\bar{u}_i(a') - q(t)a'],$$

where

$$(40) \quad \bar{u}_i(a) = \sum_{k=0}^{\infty} \sum_{j=1}^I \mu_k \pi_{ij}^{(k)} u_j(a) \quad \text{for } i = 1, \dots, I,$$

$$(41) \quad q(t) = (r + \kappa) \left[p(t) - \kappa \int_0^{\infty} e^{-(r+\kappa)s} p(t+s) ds \right],$$

$\Pi^k = [\pi_{ij}^{(k)}]$ for $k \geq 1$, $\pi_{ij}^{(0)} = \mathbb{I}_{\{j=i\}}$, and $\mu_k = \left(\frac{r+\kappa}{r+\kappa+\delta}\right) \left(\frac{\delta}{r+\kappa+\delta}\right)^k$.

PROOF: As before, $V_i(a, t)$ satisfies (25), so the problem of an investor with preference shock i who gains access to the market at time t is given by (27) with $\bar{U}_i(a)$ as in (26). Notice that (29) is unchanged, so we only have to calculate $\bar{U}_i(a)$. Equation (26) can be written as

$$(r + \kappa + \delta) \bar{U}_i(a) = u_i(a) + \delta \sum_{j=1}^I \pi_{ij} \bar{U}_j(a) \quad \text{for } i = 1, \dots, I$$

or, equivalently,

$$(42) \quad \left(\mathbf{I} - \frac{\delta}{r + \kappa + \delta} \Pi \right) \bar{\mathbf{u}} = \frac{r + \kappa}{r + \kappa + \delta} \mathbf{u},$$

where \mathbf{I} is the $I \times I$ identity matrix, and $\bar{\mathbf{u}}$ and \mathbf{u} are $I \times 1$ vectors with i th entry $\bar{u}_i(a) \equiv (r + \kappa) \bar{U}_i(a)$ and $u_i(a)$, respectively. Since $\lim_{k \rightarrow \infty} \left(\frac{\delta}{r + \kappa + \delta} \Pi \right)^k = 0$, $\left(\mathbf{I} - \frac{\delta}{r + \kappa + \delta} \Pi \right)^{-1}$ exists, $\sum_{k=0}^{\infty} \left(\frac{\delta}{r + \kappa + \delta} \Pi \right)^k$ converges, and $\left(\mathbf{I} - \frac{\delta}{r + \kappa + \delta} \Pi \right)^{-1} = \sum_{k=0}^{\infty} \left(\frac{\delta}{r + \kappa + \delta} \Pi \right)^k$. Thus

$$\bar{\mathbf{u}} = \sum_{k=0}^{\infty} \left(\frac{\delta}{r + \kappa + \delta} \Pi \right)^k \frac{r + \kappa}{r + \kappa + \delta} \mathbf{u},$$

which can be written as in (40). Substitute $\bar{U}_i(a) = \bar{u}_i(a)/(r + \kappa)$ and (29) into (27), and multiply through by $(r + \kappa)$ to obtain the formulation of the investor's problem stated in the lemma. *Q.E.D.*

Intuitively, $\bar{u}_i(a)/(r + \kappa)$ is the expected discounted utility to an investor with preference type i from holding a until the next (*effective*) time when he readjusts his holdings. We can write

$$\bar{u}_i(a) = \sum_{k=0}^{\infty} \mu_k \bar{u}_i^{(k)}(a),$$

where μ_k is the probability the investor receives k preference shocks before his next effective contact with a dealer, and

$$(43) \quad \bar{u}_i^{(k)}(a) \equiv \sum_{j=1}^I \pi_{ij}^{(k)} u_j(a)$$

is his expected utility conditional on preference type i and conditional on his receiving k preference shocks over that time period. With this generalized expression for $\bar{u}_i(a)$, a choice of asset holdings, $a_i(t)$, still satisfies (9), and Lemma 2 and (11) remain unchanged.

The law of motion for the measure of investors with preference type i is

$$\dot{n}_i(t) = \delta \sum_{j=1}^I \pi_{ji} n_j(t) - \delta n_i(t),$$

which implies $\mathbf{n}(t) = \mathbf{n}(0)e^{\delta(\mathbf{I}-1)t}$, where \mathbf{I} is the $I \times I$ identity matrix and $\mathbf{n}(t)$ denotes the $1 \times I$ vector with i th element $n_i(t)$. Thus

$$(44) \quad n_i(t) = \sum_{j=1}^I \rho_{ji}(t)n_j(0),$$

where $\rho_{ji}(t)$ denotes the j th element of the matrix $e^{\delta(\mathbf{I}-1)t}$ and represents the transition probability for an investor from preference type j to preference type i in a period of length t . The clearing condition in the interdealer market is still (13), but with $n_i(t)$ given by (44). With this, it is straightforward to show that Lemma 3 generalizes as follows.

LEMMA 6: *The measure of investors across individual states at time t satisfies (15) for all $(\mathcal{A}, \mathcal{I}) \in \Sigma$, where*

$$(45) \quad n_{ji}^0(\mathcal{A}, t) = e^{-\alpha t} \rho_{ji}(t) H_0(\mathcal{A}, \{j\}),$$

$$(46) \quad n_{ji}(\tau, t) = \alpha e^{-\alpha \tau} \rho_{ji}(\tau) n_j(t - \tau).$$

An equilibrium is a time path $\langle \{a_i(t)\}, q(t), p(t), \{\phi_i(a, t)\}, H_t \rangle$ that satisfies (9) (with $\bar{u}_i(a)$ given by (40)), (14), (11), (13) (with $n_i(t)$ given by (44)), and (15) (with $n_{ji}^0(\mathcal{A}, t)$ and $n_{ji}(\tau, t)$ given by (45) and (46), respectively). The proof of Proposition 1 can be immediately extended to show that there exists a unique equilibrium. In the limiting case $\alpha \rightarrow \infty$, we have $\bar{u}_i(a) \rightarrow u_i(a)$ (from (42)) and $u'_i[a_i(t)] \leq q(t) = rp(t) - \dot{p}(t)$ for all i (from (8) and (9)). Also, $q(t) \rightarrow q^*(t)$, where $q^*(t)$ solves $\sum_{i \in \mathcal{I}_t^+} n_i(t) u_i^{-1}[q^*(t)] = A$ and $\mathcal{I}_t^+ = \{i \in \mathbb{X} : a_i(t) > 0\}$ (from (13)), and $\phi_i(a, t) \rightarrow 0$ for all a, i , and t (from (11)). Finally, $\alpha \rightarrow \infty$ implies that every investor holds his desired asset position at all times. Thus, as before, the equilibrium fees, asset price, and distribution of asset holdings converge to their Walrasian counterparts as frictions vanish.

Efficiency

The planner's problem is

$$\max_{\{a_i(t)\}} \int_0^\infty \frac{\alpha}{r + \alpha} \sum_{i=1}^I \hat{u}_i[a_i(t)] n_i(t) e^{-rt} dt$$

subject to $\sum_{i=1}^I n_i(t) a_i(t) \leq A$, where $n_i(t)$ is given by (44) and $\hat{u}_i(a) = \sum_{k=0}^\infty \hat{\mu}_k \bar{u}_i^{(k)}$, with $\hat{\mu}_k = \left(\frac{r+\alpha}{r+\alpha+\delta}\right) \left(\frac{\delta}{r+\alpha+\delta}\right)^k$. The first-order necessary and sufficient conditions are (a) $\hat{u}'_i[a_i(t)] \leq \lambda(t)$ for $i = 1, \dots, I$ (with “=” if $a_i(t) > 0$), where $\lambda(t)$ is the multiplier on the resource constraint, and (b) $\sum_{i=1}^I n_i(t) \times a_i^*[\lambda(t)] = A$, where $a_i^*[\lambda(t)]$ is the $a_i(t)$ that satisfies (a). Notice that $\hat{\mu}_k = \mu_k$,

and hence $\hat{u}_i = \bar{u}_i$ if and only if $\eta = 0$. Hence, if we set $q(t) = \lambda(t)$ we find that the competitive allocation $\{a_i(t)\}$ coincides with the efficient allocation $\{a_i^*(t)\}$ if and only if $\eta = 0$.

Steady State

Our assumptions ensure that there exists a unique row vector $\pi^* = [\pi_i^*]$ such that $\pi^*(\Pi - \mathbf{I}) = \mathbf{0}$ with $\sum_{i=1}^I \pi_i^* = 1$ and that $\lim_{t \rightarrow \infty} \rho_{ji}(t) = \pi_i^*$. Hence, (44) implies $\lim_{t \rightarrow \infty} n_i(t) = \pi_i^*$ for all i . The generalization of the second part of Proposition 1 is straightforward. The equilibrium allocations and prices $\{\{a_i(t)\}, q(t), p(t), \{\phi_i(a, t)\}, H_t\}$ converge to the unique steady-state allocations and prices $\{\{a_i\}, q, p, \{\phi_i(a)\}, H\}$ that satisfy $p = q/r$, $\bar{u}'_i(a_i) \leq q$ (“=” if $a_i > 0$, with \bar{u}_i as in (40)), $\sum_{i=1}^I \pi_i^* a_i = A$, $\phi_i(a)$ as in (20), and $\lim_{t \rightarrow \infty} H_t(\mathcal{A}, \mathcal{T}) = H(\mathcal{A}, \mathcal{T})$, where $H(\{a_j\}, \{i\}) = \pi_j^* \int_0^\infty \alpha e^{-\alpha \tau} \rho_{ji}(\tau) d\tau$ and $H(\mathcal{A}, \mathcal{T}) = 0$ for all $(\mathcal{A}, \mathcal{T}) \in \Sigma$ such that $\bigcup_{j=1}^I \{a_j\} \cap \mathcal{A} = \emptyset$.

Asset Positions, Prices, and Trade Volume

Focus on the steady state and assume $u'_i(0) = \infty$ and $u'_i(\infty) = 0$ for each i . An investor’s asset choice satisfies

$$(47) \quad \sum_{k=0}^{\infty} \mu_k \bar{u}_i^{(k)'}(a_i) = rp.$$

As before, when an investor with preference type i chooses his asset holdings, he evaluates his expected marginal utility from holding the asset until the next trading time. If he is hit by k preference shocks over the holding period, his expected marginal utility from a_i is $\bar{u}_i^{(k)'}(a_i)$. Since the number of preference shocks he experiences is random, the investor also takes expectations over $\bar{u}_i^{(k)'}(a_i)$ using the (discounting-adjusted) probability distribution of preference shocks, $\{\mu_k\}_{k=0}^{\infty}$.

Let $a_i = g_i(\kappa; p)$ denote the choice of asset holdings characterized by (47). Then

$$(48) \quad \frac{\partial g_i(\kappa; p)}{\partial \kappa} = \frac{\sum_{k=0}^{\infty} \left(\frac{\delta}{r + \kappa} - k \right) \mu_k \bar{u}_i^{(k)'}(a_i)}{-\bar{u}_i''(a_i)(r + \kappa + \delta)},$$

which generalizes (23), has the sign of the numerator. From (47), notice that κ only affects the probability distribution $\{\mu_k\}$; intuitively, a marginal increase in κ increases the probability of k preference shocks for $k < \frac{\delta}{r + \kappa}$ and decreases it for $k > \frac{\delta}{r + \kappa}$. This means that an increase in κ induces the investor to put more

weight on $\bar{u}_i^{(k)}$'s with smaller k . If shocks are i.i.d. as in the body of the paper (i.e., $\pi_{ij} = \pi_j$ for all i), then $\bar{u}_i^{(0)'}(a_i) = u_i'(a_i)$ and $\bar{u}_i^{(k)'}(a_i) = \sum_{j=1}^I \pi_j u_j'(a_i)$ for all $k \geq 1$, so in terms of preference shocks over the holding period, there are just two relevant events: either none hits or at least one hits. An increase in κ raises the probability of the former and reduces the probability of the latter, so it makes an investor with preference type i choose a larger asset position if and only if $u_i'(a_i) > \sum_{j=1}^I \pi_j u_j'(a_i)$. Analogously, according to (48), in this more general formulation an investor with preference type i increases his asset demand in response to an increase in κ if and only if $u_i'(a_i) > \sum_{k=1}^{\infty} (\frac{\delta}{r+\kappa} - k) \times \mu_{k-1} \bar{u}_i^{(k)'}(a_i)$. Since this condition may seem intricate, we provide simpler conditions for some special cases.

PROPOSITION 6: (i) *Suppose the sequence $\{\bar{u}_i^{(k)'}(a_i)\}_{k=0}^{\infty}$ is monotone in k . Then $\partial g_i(\kappa; p)/\partial \kappa > 0$ if and only if*

$$(49) \quad u_i'(a_i) > \sum_{j=1}^I \pi_j^* u_j'(a_i).$$

(ii) *Consider the frictionless limit, $\kappa \rightarrow \infty$. Then*

$$\frac{\partial g_i(\kappa; p)}{\partial \left(\frac{1}{r+\kappa} \right)} > 0$$

if and only if

$$(50) \quad u_i'(a_i) < \sum_{j=1}^I \pi_{ij} u_j'(a_i).$$

(iii) *Consider the case $I = 2$. Then for $i, j \in \{1, 2\}$ (with $j \neq i$),*

$$(51) \quad \bar{u}_i(a) = \frac{r + \kappa + \delta \pi_{ji}}{r + \kappa + \delta(\pi_{12} + \pi_{21})} u_i(a) + \frac{\delta \pi_{ij}}{r + \kappa + \delta(\pi_{12} + \pi_{21})} u_j(a)$$

and $\partial g_i(\kappa; p)/\partial \kappa > 0$ if and only if $u_i'(a_i) > u_j'(a_i)$.

PROOF: (i) From (48), $\partial g_i(\kappa; p)/\partial \kappa$ has the sign of $\sum_{k=0}^{\infty} (\frac{\delta}{r+\kappa} - k) \mu_k \bar{u}_i^{(k)'}(a_i)$, so we sign the latter. Let $\bar{\mathbb{Z}} = \mathbb{Z} \cap (-\infty, \frac{\delta}{r+\kappa})$, where \mathbb{Z} denotes the set of integers, and define $\bar{k} = \max_{k \in \bar{\mathbb{Z}}} k$. Suppose that (49) holds. Then $\{\bar{u}_i^{(k)'}\}_{k=0}^{\infty}$ is a decreasing sequence with $\bar{u}_i^{(0)'}(a_i) = u_i'(a_i) > \sum_{j=1}^I \pi_j^* u_j'(a_i) = \lim_{k \rightarrow \infty} \bar{u}_i^{(k)'}(a_i)$. Since $(\frac{\delta}{r+\kappa} - k) \mu_k > 0$ for $k < \bar{k} + 1$ and $(\frac{\delta}{r+\kappa} - k) \mu_k \leq 0$ for $k \geq \bar{k} + 1$, the fact

that $\bar{u}_i^{(k+1)'} \leq \bar{u}_i^{(k)'}$ for all k implies

$$\begin{aligned} & \sum_{k=0}^{\bar{k}} \left(\frac{\delta}{r+\kappa} - k \right) \mu_k \bar{u}_i^{(\bar{k})'}(a_i) + \sum_{k=\bar{k}+1}^{\infty} \left(\frac{\delta}{r+\kappa} - k \right) \mu_k \bar{u}_i^{(\bar{k}+1)'}(a_i) \\ & \leq \sum_{k=0}^{\infty} \left(\frac{\delta}{r+\kappa} - k \right) \mu_k \bar{u}_i^{(k)'}(a_i). \end{aligned}$$

Since $\sum_{k=0}^{\infty} \left(\frac{\delta}{r+\kappa} - k \right) \mu_k = 0$, the above inequality can be written as

$$(52) \quad \begin{aligned} 0 & \leq [\bar{u}_i^{(\bar{k})'}(a_i) - \bar{u}_i^{(\bar{k}+1)'}(a_i)] \sum_{k=0}^{\bar{k}} \left(\frac{\delta}{r+\kappa} - k \right) \mu_k \\ & \leq \sum_{k=0}^{\infty} \left(\frac{\delta}{r+\kappa} - k \right) \mu_k \bar{u}_i^{(k)'}(a_i). \end{aligned}$$

If $\bar{u}_i^{(\bar{k}+1)'}(a_i) < \bar{u}_i^{(\bar{k})'}(a_i)$, then the first inequality in (52) is strict. Alternatively, if $\bar{u}_i^{(\bar{k}+1)'}(a_i) = \bar{u}_i^{(\bar{k})'}(a_i)$, then the second inequality is strict, since $\bar{u}_i^{(0)'}(a_i) > \lim_{k \rightarrow \infty} \bar{u}_i^{(k)'}(a_i)$, which implies that

$$\sum_{k=0}^{\bar{k}} \left(\frac{\delta}{r+\kappa} - k \right) \mu_k \bar{u}_i^{(\bar{k})'}(a_i) < \sum_{k=0}^{\bar{k}} \left(\frac{\delta}{r+\kappa} - k \right) \mu_k \bar{u}_i^{(k)'}(a_i)$$

or

$$\sum_{k=\bar{k}+1}^{\infty} \left(\frac{\delta}{r+\kappa} - k \right) \mu_k \bar{u}_i^{(\bar{k}+1)'}(a_i) < \sum_{k=\bar{k}+1}^{\infty} \left(\frac{\delta}{r+\kappa} - k \right) \mu_k \bar{u}_i^{(k)'}(a_i)$$

must hold. In any case, $\partial g_i(\kappa; p) / \partial \kappa > 0$ follows. Conversely, suppose that

$$\sum_{k=0}^{\infty} \left(\frac{\delta}{r+\kappa} - k \right) \mu_k \bar{u}_i^{(k)'}(a_i) > 0,$$

but (49) does not hold, that is, $u'_i(a_i) \leq \sum_{j=1}^l \pi_j^* u'_j(a_i)$. Then $\{\bar{u}_i^{(k)'}\}_{k=0}^{\infty}$ is an increasing sequence and

$$\begin{aligned} & \sum_{k=0}^{\infty} \left(\frac{\delta}{r+\kappa} - k \right) \mu_k \bar{u}_i^{(k)'}(a_i) \\ & \leq \sum_{k=0}^{\bar{k}} \left(\frac{\delta}{r+\kappa} - k \right) \mu_k \bar{u}_i^{(\bar{k})'}(a_i) + \sum_{k=\bar{k}+1}^{\infty} \left(\frac{\delta}{r+\kappa} - k \right) \mu_k \bar{u}_i^{(\bar{k}+1)'}(a_i). \end{aligned}$$

This leads to

$$\begin{aligned} & \sum_{k=0}^{\infty} \left(\frac{\delta}{r + \kappa} - k \right) \mu_k \bar{u}_i^{(k)'}(a_i) \\ & \leq [\bar{u}_i^{(\bar{k})'}(a_i) - \bar{u}_i^{(\bar{k}+1)'}(a_i)] \sum_{k=0}^{\bar{k}} \left(\frac{\delta}{r + \kappa} - k \right) \mu_k \leq 0, \end{aligned}$$

a contradiction.

(ii) Let $\varkappa = (r + \kappa)^{-1}$ and differentiate (47) with respect to \varkappa (with p given) to find

$$\begin{aligned} & \frac{\partial g_i \left(\frac{1}{\varkappa} - r; p \right)}{\partial \varkappa} \\ & = \frac{\frac{1}{(1 + \delta \varkappa) \varkappa} \sum_{k=0}^{\infty} (k - \delta \varkappa) \left(\frac{1}{1 + \delta \varkappa} \right) \left(\frac{\delta \varkappa}{1 + \delta \varkappa} \right)^k \bar{u}_i^{(k)'}(a_i)}{-\bar{u}_i''(a_i)}. \end{aligned}$$

The numerator can be written as

$$\delta \left(\frac{1}{1 + \delta \varkappa} \right)^2 \left[\frac{1 - \delta \varkappa}{1 + \delta \varkappa} \bar{u}_i^{(1)'}(a_i) - \bar{u}_i^{(0)'}(a_i) + O(\varkappa) \right],$$

where $O(\varkappa) = \sum_{k=2}^{\infty} (k - \delta \varkappa) \left(\frac{1}{1 + \delta \varkappa} \right)^k (\delta \varkappa)^{k-1} \bar{u}_i^{(k)'}(a_i)$. Since $\lim_{\varkappa \rightarrow 0} O(\varkappa) = 0$, we have

$$\lim_{\varkappa \rightarrow 0} \frac{\partial g_i \left(\frac{1}{\varkappa} - r; p \right)}{\partial \varkappa} = \frac{\delta [\bar{u}_i^{(1)'}(a_i) - \bar{u}_i^{(0)'}(a_i)]}{-\bar{u}_i''(a_i)}.$$

Finally, $\bar{u}_i^{(0)'}(a_i) = u_i'(a_i)$ and $\bar{u}_i^{(1)'}(a_i) = \sum_{j=1}^I \pi_{ij} u_j'(a_i)$ imply that

$$\lim_{\kappa \rightarrow \infty} \frac{\partial g_i(\kappa; p)}{\partial \left(\frac{1}{r + \kappa} \right)} > 0$$

if and only if (50) holds.

(iii) Let $I = 2$. For $i = 1$, (40) reduces to

$$\begin{aligned} \bar{u}_1(a) & = \left(\frac{r + \kappa}{r + \kappa + \delta} \right) \sum_{k=0}^{\infty} \left(\frac{\delta}{r + \kappa + \delta} \right)^k \\ & \quad \times [\pi_{11}^{(k)} u_1(a) + (1 - \pi_{11}^{(k)}) u_2(a)], \end{aligned}$$

where

$$\pi_{11}^{(k)} = \frac{\pi_{21}}{\pi_{12} + \pi_{21}} + \frac{\pi_{12}}{\pi_{12} + \pi_{21}} (1 - \pi_{12} - \pi_{21})^k,$$

since $\pi_{12} + \pi_{21} > 0$. Collect terms to arrive at (51) for $i = 1$. The expression for $i = 2$ is obtained similarly. The first-order condition (47) specializes to

$$\frac{r + \kappa + \delta\pi_{ji}}{r + \kappa + \delta(\pi_{12} + \pi_{21})} u'_i(a_i) + \frac{\delta\pi_{ij}}{r + \kappa + \delta(\pi_{12} + \pi_{21})} u'_j(a_i) = rp.$$

This can be differentiated with respect to κ (for fixed p) to obtain

$$\frac{\partial g_i(\kappa; p)}{\partial \kappa} = \frac{\delta\pi_{ij}[u'_i(a_i) - u'_j(a_i)]}{-\bar{u}'_i(a_i)[r + \kappa + \delta(\pi_{12} + \pi_{21})]^2}.$$

This concludes the proof. Q.E.D.

For the i.i.d. case analyzed in the body of the paper, we found that if trading frictions decrease, an investor increases his asset holdings if his current marginal valuation exceeds his expected marginal valuation over the expected holding period (condition (23)). Proposition 6 extends this result and shows that the key insight does not rely on the preference shocks being i.i.d. For the case of multiplicative preference shocks we analyzed in Section 4, for example, we have $\bar{u}_i(a) = \bar{\varepsilon}_i u(a)$, with

$$(53) \quad \bar{\varepsilon}_i = \sum_{k=0}^{\infty} \mu_k \bar{\varepsilon}_i^{(k)}$$

and $\bar{\varepsilon}_i^{(k)} = \sum_{j=1}^I \pi_{ij}^{(k)} \varepsilon_j$. Note that $\lim_{k \rightarrow \infty} \bar{\varepsilon}_i^{(k)} = \sum_{j=1}^I \pi_j^* \varepsilon_j \equiv \bar{\varepsilon}$. Part (i) of Proposition 6 establishes that if this convergence is monotonic for i , then an investor with preference type i increases his asset holdings if and only if $\varepsilon_i > \bar{\varepsilon}$. This is essentially the same condition we derived in the i.i.d. case where $\pi_{ij}^{(k)} = \pi_j^*$ for all i and all $k \geq 1$. For this multiplicative case, the condition in part (ii) of the proposition reduces to $\varepsilon_i > \bar{\varepsilon}_i^{(1)}$, and if we let $\delta_{ij} \equiv \delta\pi_{ij}$ for $i \neq j$ and $\delta_{ii} \equiv \delta(1 - \pi_{ii})$, it can be written as

$$(54) \quad \varepsilon_i > \frac{\sum_{j \neq i} \delta_{ij} \varepsilon_j}{\sum_{j \neq i} \delta_{ij}}.$$

Proposition 6 parallels Proposition 2 in Gârleanu (2009). Notation aside, (54) is identical to the condition in part (i) of his Proposition 2. The monotonicity condition in part (ii) of his proposition plays the role of the monotonicity

condition in part (i) of ours. The two-valuation case in part (iii) of his proposition parallels part (iii) in ours.

An implication of the i.i.d. case that does not generalize is that if $\varepsilon_i < \varepsilon_j$ and the agent with preference type i increases his asset holdings in response to an increase in κ , then so does the agent with preference type j .²³ The robust insight instead is that an investor whose current marginal valuation is large—in the sense that it exceeds his expected marginal valuation over the expected holding period—increases his asset holdings if κ increases.

The following proposition characterizes the equilibrium price for a particular class of utility functions and generalizes the discussion that followed Proposition 5. Just as in the i.i.d. case, this price is independent of frictions as summarized by κ if the individual asset demand is linear in the idiosyncratic valuation (as is the case with logarithmic preferences).

PROPOSITION 7: *Let $u_i(a) = \varepsilon_i a^{1-\sigma} / (1 - \sigma)$ with $\sigma > 0$. Then*

$$p = \frac{\left(\sum_{i=1}^I \pi_i^* \bar{\varepsilon}_i^{1/\sigma} \right)^\sigma}{rA^\sigma},$$

where $\bar{\varepsilon}_i = \sum_{k=0}^{\infty} \sum_{j=1}^I \mu_k \pi_{ij}^{(k)} \varepsilon_j$. If $u_i(a) = \varepsilon_i \ln a$, then

$$p = \frac{\sum_{j=1}^I \pi_j^* \varepsilon_j}{rA}.$$

PROOF: Since $u_i(a) = \varepsilon_i u(a)$, we have $\bar{u}_i(a) = \bar{\varepsilon}_i u(a)$ with $\bar{\varepsilon}_i$ given by (53), so (47) becomes $\bar{\varepsilon}_i u'(a_i) = rp$. The parametric assumption implies $a_i = (\bar{\varepsilon}_i / (rp))^{1/\sigma}$ so the steady-state market-clearing condition, $\sum_{i=1}^I \pi_i^* a_i = A$, yields the first expression for p . For $\sigma = 1$, $p = (rA)^{-1} \sum_{i=1}^I \pi_i^* \bar{\varepsilon}_i$, where

$$\sum_{i=1}^I \pi_i^* \bar{\varepsilon}_i = \sum_{j=1}^I \sum_{k=0}^{\infty} \sum_{i=1}^I \pi_i^* \pi_{ij}^{(k)} \mu_k \varepsilon_j = \sum_{j=1}^I \pi_j^* \varepsilon_j. \quad Q.E.D.$$

As in the i.i.d. case, it is difficult to sign the general equilibrium effects of α and η on trade volume in general. We provide analytical results for three cases.

²³For example, with a more general process for preference shocks it is possible to have a parametrization $\{\varepsilon_i, \pi_{ij}\}_{i,j=1}^I$ with $\sum_{k=1}^I \pi_{ik} \varepsilon_k < \varepsilon_i < \varepsilon_j < \sum_{k=1}^I \pi_{jk} \varepsilon_k$, which according to part (ii) of Proposition 6 implies that, near the frictionless limit, the high valuation investor (the one with preference type ε_j) will reduce his asset holdings and the low valuation investor will increase his asset holdings if κ increases.

The first has $I = 2$ and a general preference specification, the second considers a market close to the frictionless limit, and the third considers a market with severe trading frictions.

PROPOSITION 8: (i) Let $u_i(a) = \varepsilon_i a^{1-\sigma}/(1-\sigma)$ with $\sigma > 0$ and assume that $I = 2$. Trade volume increases with κ .

(ii) Let $u_i(a) = \varepsilon_i \ln a$ and suppose that $\bar{\varepsilon}_j > \bar{\varepsilon}_i$ implies $\varepsilon_j - \sum_{k=1}^I \pi_{jk} \varepsilon_k > \varepsilon_i - \sum_{k=1}^I \pi_{ik} \varepsilon_k$ for all $i, j \in \mathbb{X}^2$. Trade volume decreases with η in the frictionless limit (as $\kappa \rightarrow \infty$).

(iii) Trade volume approaches zero as $r + \kappa \rightarrow 0$.

PROOF: (i) With $I = 2$,

$$n_{12} = n_{21} = \frac{\delta \pi_{12} \pi_{21}}{[\alpha + \delta(\pi_{12} + \pi_{21})](\pi_{12} + \pi_{21})},$$

so trade volume is

$$\mathcal{V} = \frac{\alpha \delta \pi_{12} \pi_{21}}{[\alpha + \delta(\pi_{12} + \pi_{21})](\pi_{12} + \pi_{21})} (a_2 - a_1).$$

The preference specification together with (51) implies $a_i = (\bar{\varepsilon}_i/(rp))^{1/\sigma}$ for $i = 1, 2$, where

$$\bar{\varepsilon}_1 = \frac{r + \kappa + \delta \pi_{21}}{r + \kappa + \delta(\pi_{12} + \pi_{21})} \varepsilon_1 + \frac{\delta \pi_{12}}{r + \kappa + \delta(\pi_{12} + \pi_{21})} \varepsilon_2$$

and

$$\bar{\varepsilon}_2 = \frac{r + \kappa + \delta \pi_{12}}{r + \kappa + \delta(\pi_{12} + \pi_{21})} \varepsilon_2 + \frac{\delta \pi_{21}}{r + \kappa + \delta(\pi_{12} + \pi_{21})} \varepsilon_1.$$

Since $rp = (\pi_1^* \bar{\varepsilon}_1^{1/\sigma} + \pi_2^* \bar{\varepsilon}_2^{1/\sigma})^\sigma / A^\sigma$,

$$a_i = \frac{\bar{\varepsilon}_i^{1/\sigma}}{\pi_1^* \bar{\varepsilon}_1^{1/\sigma} + \pi_2^* \bar{\varepsilon}_2^{1/\sigma}} A.$$

Differentiate this expression with respect to κ to find that $\partial a_2 / \partial \kappa$ has the sign of $(\varepsilon_2 - \varepsilon_1)$ and $\partial a_1 / \partial \kappa$ has the opposite sign. Since $\varepsilon_1 < \varepsilon_2$, $da_1/d\kappa < 0 < da_2/d\kappa$. To find $\frac{d\mathcal{V}}{d\kappa}$, we consider two cases. (a) An increase in κ caused by a decrease in η (keeping α constant). For this case,

$$\frac{d\mathcal{V}}{d\kappa} = \frac{da_2}{d\kappa} - \frac{da_1}{d\kappa} > 0.$$

(b) An increase in κ caused by an increase in α , which implies

$$\begin{aligned} \frac{d\mathcal{V}}{d\kappa} &= \left[\frac{\delta}{\alpha + \delta(\pi_{12} + \pi_{21})} \right]^2 \pi_{12}\pi_{21}(a_2 - a_1) \\ &\quad + \frac{\alpha\delta\pi_{12}\pi_{21}}{[\alpha + \delta(\pi_{12} + \pi_{21})](\pi_{12} + \pi_{21})} \left(\frac{da_2}{d\kappa} - \frac{da_1}{d\kappa} \right) > 0. \end{aligned}$$

(ii) Let $\varkappa = (r + \kappa)^{-1}$. Under $u_i(a) = \varepsilon_i \ln a$, (47) implies $a_i = \bar{\varepsilon}_i / (rp)$, where $\bar{\varepsilon}_i = \sum_{k=0}^{\infty} \mu_k \bar{\varepsilon}_i^{(k)}$ with $\bar{\varepsilon}_i^{(k)} = \sum_{j=1}^I \pi_{ij}^{(k)} \varepsilon_j$ and $\mu_k = \left(\frac{1}{1+\delta\varkappa}\right) \left(\frac{\delta\varkappa}{1+\delta\varkappa}\right)^k$. Differentiate with respect to \varkappa to find

$$(55) \quad \frac{da_i}{d\varkappa} = \frac{1}{(1 + \delta\varkappa)\varkappa} \sum_{k=0}^{\infty} (k - \delta\varkappa) \mu_k \bar{\varepsilon}_i^{(k)}.$$

We know from Proposition 7 that under this preference specification the equilibrium price is independent of κ , so (55) captures the general equilibrium effect of κ on a_i . Let $\varkappa \rightarrow 0$ as in part (ii) of the proof of Proposition 6 to find

$$\lim_{\varkappa \rightarrow 0} \frac{da_i}{d\varkappa} = \frac{\delta[\bar{\varepsilon}_i^{(1)} - \bar{\varepsilon}_i^{(0)}]}{rp}.$$

Therefore,

$$\frac{d(a_j - a_i)}{d\varkappa} = \frac{\delta}{rp} \{ \bar{\varepsilon}_j^{(1)} - \varepsilon_j - [\bar{\varepsilon}_i^{(1)} - \varepsilon_i] \}.$$

The assumption that $\varepsilon_j - \bar{\varepsilon}_j^{(1)} > \varepsilon_i - \bar{\varepsilon}_i^{(1)}$ if $\bar{\varepsilon}_j > \bar{\varepsilon}_i$ implies $d(a_j - a_i)/d\varkappa < 0$ for $a_j > a_i$ and $d(a_j - a_i)/d\varkappa > 0$ for $a_j < a_i$, so an increase in \varkappa decreases the size of every trade. If the increase in \varkappa is due to an increase in η (i.e., keeping α constant), then the weights n_{ij} in (24) remain constant and \mathcal{V} decreases.

(iii) From (40), $\bar{u}_i(a) = \sum_{j=1}^I \omega_{ij}(\bar{r}) u_j(a)$, where $\omega_{ij}(\bar{r}) = \sum_{k=0}^{\infty} \left(\frac{\bar{r}}{\bar{r}+\delta}\right) \left(\frac{\delta}{\bar{r}+\delta}\right)^k \times \pi_{ij}^{(k)}$ and $\bar{r} = r + \kappa$. We first show that for any $\varepsilon > 0$, $|\omega_{ij}(\bar{r}) - \pi_j^*| < \varepsilon$ obtains for all \bar{r} close enough to 0. For any $\bar{r} > 0$ and any $N \in \mathbb{Z}_+$,

$$\begin{aligned} |\omega_{ij}(\bar{r}) - \pi_j^*| &\leq \left| \sum_{k=0}^N \left(\frac{\bar{r}}{\bar{r}+\delta}\right) \left(\frac{\delta}{\bar{r}+\delta}\right)^k [\pi_{ij}^{(k)} - \pi_j^*] \right| \\ &\quad + \left| \sum_{k=N+1}^{\infty} \left(\frac{\bar{r}}{\bar{r}+\delta}\right) \left(\frac{\delta}{\bar{r}+\delta}\right)^k [\pi_{ij}^{(k)} - \pi_j^*] \right|. \end{aligned}$$

Since $\lim_{k \rightarrow \infty} \pi_{ij}^{(k)} = \pi_j^*$, choose N large enough so that the second term is strictly smaller than $\varepsilon/2$ for any $\bar{r} > 0$. The first term is bounded above by

$|1 - (\frac{\delta}{\bar{r} + \delta})^{N+1}|$, so it is strictly less than $\varepsilon/2$ for all \bar{r} close enough to 0. Therefore, $\lim_{\bar{r} \rightarrow 0} \omega_{ij}(\bar{r}) = \pi_j^*$ and $\lim_{\bar{r} \rightarrow 0} \bar{u}_i(a) = \sum_{j=1}^I \pi_j^* u_j(a)$ for every i . In turn, (39) approaches $\max_{a' \geq 0} [\sum_{j=1}^I \pi_j^* u_j(a') - q(t)a']$, so $a_i \rightarrow A$ for all i . With this, $\mathcal{V} \rightarrow 0$ as $r + \kappa \rightarrow 0$ is immediate from (24). *Q.E.D.*

Part (i) of Proposition 8 is a generalization of part (i) of Proposition 3. Part (ii) of Proposition 8 is analogous to part (ii) of Proposition 3. The focus of the former on the frictionless limit simplifies the analysis of the effects of trading frictions on individual asset demands (see, e.g., part (ii) of Proposition 6). The additional assumption is a condition on the speed with which preference shocks revert to their unconditional mean. For example, suppose $\bar{\varepsilon}_j > \bar{\varepsilon}_i$, which means that the expected marginal valuation over the holding period for an investor who currently has preference type j is larger than for an investor with preference type i . Then the assumption requires that the expected change in the marginal valuation after a single preference shock (e.g., $\varepsilon_j - \sum_{k=1}^I \pi_{jk} \varepsilon_k$ for the agent with preference type j) must be larger for the investor with the higher current expected valuation over the holding period. Part (iii) of Proposition 8 generalizes part (iii) of Proposition 3 as well as the notion—which for the i.i.d. case was proved in part (ii) of Proposition 2 and used in the proof of Proposition 4—that if the investor is patient, the influence of his current valuation at the time of the trade on his choice of asset holdings vanishes as the market becomes very illiquid. In other words, as $r + \kappa \rightarrow 0$, the distribution of asset holdings converges to a mass point at A and trade volume approaches zero. This has important implications for intermediation fees and dealer revenue: both approach zero as trade sizes vanish, just as in the i.i.d. case. Note that intermediation fees and revenue also go to zero as κ becomes large, so they are nonmonotonic functions of κ . Therefore, the nonmonotonicity results we established for i.i.d. preference shocks (Proposition 4) generalize. Finally, these nonmonotonicities can generate multiple steady-state equilibria, so the multiplicity that we find for the i.i.d. case (Proposition 8 in Lagos and Rocheteau (2008)) can also be generalized.

APPENDIX C: STRATEGIC BARGAINING

In the body of the paper we assumed that when an investor and a dealer trade, the new asset position of the investor, a' , and the fee, ϕ , are the solution to a Nash bargaining problem where the dealer has bargaining power $\eta \in [0, 1]$ and disagreement point $W(t)$, and the investor has disagreement point $V_i(a, t)$. In this appendix we describe a strategic bargaining game with a unique subgame perfect equilibrium outcome that coincides with the solution of the axiomatic Nash bargaining problem we have adopted.

Our theory is meant to model a fast-moving market where investors and dealers do not form long-lasting relationships, but rather contact each other

at relatively high frequencies and must trade on the spot, instantaneously, before they part ways. With this in mind, consider the following natural and simple strategic bargaining game. Upon contact, with probability η , Nature selects the dealer to make an instantaneous take-it-or-leave-it offer, which the investor must either accept or reject on the spot. With complementary probability, Nature selects the investor to make an instantaneous take-it-or-leave-it offer, which the dealer must either accept or reject on the spot. The whole process is instantaneous, and the dealer and the investor part ways regardless of the outcome.²⁴

Let $\langle a_i^1(t), \phi_i^1(a, t) \rangle$ denote the proposal that the dealer makes to an investor of type i who is holding a at time t and let $\langle a_i^2(t), \phi_i^2(a, t) \rangle$ denote the offer that the latter makes to the former. The set of offers that an investor of type i who is holding asset position a finds acceptable at time t is

$$\mathcal{A}_i^2(a, t) = \{(a', \phi) : V_i(a', t) - p(t)(a' - a) - \phi \geq V_i(a, t)\}.$$

Similarly, the set of offers that a dealer finds acceptable at time t is $\mathcal{A}^1 = \{(a', \phi) : \phi \geq 0\}$. If the dealer is selected as the proposer, he will offer

$$\langle a_i^1(t), \phi_i^1(a, t) \rangle = \arg \max_{(a', \phi)} \phi \mathbb{I}_{\mathcal{A}_i^2(a, t)}(a', \phi),$$

where the maximization is subject to $a' \geq 0$ and $\mathbb{I}_{\mathcal{A}_i^2(a, t)}(a', \phi)$ is an indicator function that is equal to 1 if $(a', \phi) \in \mathcal{A}_i^2(a, t)$. It is easy to see that $a_i^1(t) = a_i(t)$, where $a_i(t)$ is as in (3), and $\eta \phi_i^1(a, t) = \phi_i(a, t)$, where $\phi_i(a, t)$ is as in (4). If the investor makes the offer, he chooses

$$\begin{aligned} \langle a_i^2(t), \phi_i^2(a, t) \rangle = \arg \max_{(a', \phi)} \{ & [V_i(a', t) - p(t)(a' - a) - \phi] \mathbb{I}_{\mathcal{A}^1}(a', \phi) \\ & + [1 - \mathbb{I}_{\mathcal{A}^1}(a', \phi)] V_i(a, t) \}, \end{aligned}$$

where the maximization is subject to $a' \geq 0$ and $\mathbb{I}_{\mathcal{A}^1}(a', \phi)$ is an indicator function that is equal to 1 if $(a', \phi) \in \mathcal{A}^1$. Hence, $a_i^2(t) = a_i(t)$ and $\phi_i^2(a, t) = 0$. Note that regardless of who gets selected to make the offer, the outcome of the negotiation is that the investor exits the meeting with asset position $a_i(t)$. The transaction fee equals $\phi_i(a, t)/\eta$ if the dealer makes the offer and equals 0 if the investor makes the offer, so the expected fee (before Nature decides who will make the offer) equals $\phi_i(a, t)$. It is easy to check that with these equilibrium outcomes, the investors' and dealers' value functions are just as in the body of the paper and all our results go through (subject to the obvious reinterpretation of $\phi_i(a, t)$ as an *expected* intermediation fee, which is inconsequential).

²⁴This type of bargaining procedure has been used extensively in search models of money, for example, Burdett, Trejos, and Wright (2001), as well as in search models of the labor market, for example, Kiyotaki and Lagos (2007).

APPENDIX D: PRINCIPLE OF OPTIMALITY

Consider an investor who effectively contacts the market with Poisson intensity κ and who is subject to preference shocks with Poisson intensity δ . Let $\{T_n\}_{n=1}^\infty$ denote the sequence of contact times and let N_t denote the number of contacts over the time interval $[0, t)$. Similarly, let $\{T'_n\}_{n=1}^\infty$ denote the sequence of times at which he receives preference shocks. We adopt the convention that $T_0 = T'_0 = 0$. Define the function $k: \mathbb{R}_+ \rightarrow \mathbb{X}$, and interpret $k(t)$ as the investor's preference type at time t . The process for preference shocks implies $k(t) = k(T'_n)$ for $t \in [T'_n, T'_{n+1})$ for any integer $n \geq 0$. The realization $\omega = (N_t, k(t))_{t \in [0, \infty)}$ summarizes an investor's individual history of shocks. Let Ω be the set of all such histories. Similarly, let $\omega^t = (N_s, k(s))_{s \in [0, t]}$ denote a history of shocks up to time t and let Ω^t be the collection of all such histories. We work with the probability space $(\Omega, \mathcal{H}, \mathbb{P})$, where \mathcal{H} is an appropriate σ -field of subsets of Ω (e.g., the σ -field generated by Ω^t for all finite t), and \mathbb{P} is the probability measure on \mathcal{H} induced by the independent Poisson processes for preference shocks and effective contacts with the market. Let $\mathcal{H}^t \subseteq \mathcal{H}$ be a partition of Ω such that $H^t_\varpi \in \mathcal{H}^t$ is a set of histories that coincide over $[0, t]$, that is, $H^t_\varpi = \{\omega \in \Omega: \omega^t = \varpi \text{ for some } \varpi \in \Omega^t\}$. The σ -field generated by \mathcal{H}^t , denoted \mathcal{F}^t , captures the information available to the investor at time t , and the filtration $\{\mathcal{F}^t, t \in \mathbb{R}_+\}$ represents how information is revealed over time.

An *asset plan*, $\mathbf{a} = (\mathbf{a}_t)_{t \in [0, \infty)}$, for the investor is a set of functions $\mathbf{a}_t: \Omega \rightarrow [0, \bar{a}]$ for all $t \geq 0$, such that \mathbf{a}_t is \mathcal{F}^t -measurable.²⁵ An asset plan $(\mathbf{a}_t)_{t \in [0, \infty)}$ is feasible if for every ω , $\mathbf{a}_0(\omega)$ equals the given initial asset holding of the investor and $\mathbf{a}_t(\omega) = \mathbf{a}_{T_n}(\omega)$ for all $t \in [T_n, T_{n+1})$. Let \mathbb{A} denote the set of all feasible asset plans. Let $\mathcal{U}_{k(t)}^M(\cdot, t)$ be the utility functional over the time interval $[t, T_M]$ of an investor with preference type $k(t)$ at time t . His utility over the period $[t, T_M]$ from following asset plan $\mathbf{a} = (\mathbf{a}_s)_{s \in [0, \infty)}$ is

$$\begin{aligned}
(56) \quad & \mathcal{U}_{k(t)}^M(\mathbf{a}, t) \\
&= \mathbb{E}_t \left[\int_t^{T_{N_t+1}} e^{-r(s-t)} u_{k(s)}(\mathbf{a}_t(\omega)) ds \right. \\
&\quad + \sum_{n=1}^{M-1} \int_{T_{N_t+n}}^{T_{N_t+n+1}} e^{-r(s-t)} u_{k(s)}(\mathbf{a}_{T_{N_t+n}}(\omega)) ds \\
&\quad - e^{-r(T_{N_t+1}-t)} p(T_{N_t+1}) [\mathbf{a}_{T_{N_t+1}}(\omega) - \mathbf{a}_t(\omega)] \\
&\quad \left. - \sum_{n=1}^{M-1} e^{-r(T_{N_t+n+1}-t)} p(T_{N_t+n+1}) [\mathbf{a}_{T_{N_t+n+1}}(\omega) - \mathbf{a}_{T_{N_t+n}}(\omega)] \right],
\end{aligned}$$

²⁵The upper bound \bar{a} is imposed for technical reasons (to ensure that the investor's utility is bounded above) and is chosen to be sufficiently large so that it does not affect the investor's decision.

where \mathbb{E}_t is shorthand for the conditional expectation $\mathbb{E}[\cdot|\mathcal{F}^t]$.²⁶ The first M terms on the right side of (56) represent the expected discounted sum of utility flows from holding the asset position prescribed by the asset plan \mathbf{a} over the time interval $[t, T_{N_t+M})$. The first term, for instance, is the expected utility from holding the asset position $\mathbf{a}_t(\omega)$ from the initial time t until the next time the investor gains effective access to the market, T_{N_t+1} . Similarly, each term in the summation represents the utility from holding the asset over the period $[T_{N_t+n}, T_{N_t+n+1})$, that is, between the effective contact number $N_t + n$ and the next one. The second M terms represent the expected net utility cost to the investor from readjusting his asset holdings at the times he contacts the market. The term on the second line of (56), for instance, is the (expected, discounted to time t) disutility the investor incurs to buy $\mathbf{a}_{T_{N_t+1}}(\omega)$ on his $(N_t + 1)$ th effective contact with the market, net of the utility he gets from selling the assets he is holding at this time, $\mathbf{a}_t(\omega)$. In what follows, we will leave the dependence of the function \mathbf{a}_t on ω implicit to simplify the notation. By the law of iterated expectations, the utility functional in (56) can be rewritten as

$$(57) \quad \mathcal{U}_{k(t)}^M(\mathbf{a}, t) = \frac{\bar{u}_{k(t)}(\mathbf{a}_t)}{r + \kappa} + \left[p(t) - \frac{q(t)}{r + \kappa} \right] \mathbf{a}_t - \mathbb{E}_t \left[e^{-r(T_{N_t+M}-t)} p(T_{N_t+M}) \mathbf{a}_{T_{N_t+M}} \right] + \frac{1}{r + \kappa} \mathbb{E}_t \left\{ \sum_{n=1}^{M-1} e^{-r(T_{N_t+n}-t)} [\bar{u}_{k(T_{N_t+n})}(\mathbf{a}_{T_{N_t+n}}) - q(T_{N_t+n}) \mathbf{a}_{T_{N_t+n}}] \right\},$$

where

$$(58) \quad \bar{u}_{k(T_{N_t+n})}(\mathbf{a}_{T_{N_t+n}}) \equiv (r + \kappa) \mathbb{E}_{T_{N_t+n}} \int_{T_{N_t+n}}^{T_{N_t+n+1}} e^{-r(s-T_{N_t+n})} u_{k(s)}(\mathbf{a}_{T_{N_t+n}}) ds, \\ q(T_{N_t+n}) \equiv (r + \kappa) [p(T_{N_t+n}) - \mathbb{E}_{T_{N_t+n}} e^{-r(T_{N_t+n+1}-T_{N_t+n})} p(T_{N_t+n+1})].$$

Notice that the function $\bar{u}_i(a)$ is as in (7), and since

$$\mathbb{E}_{T_{N_t+n}} e^{-r(T_{N_t+n+1}-T_{N_t+n})} p(T_{N_t+n+1}) = \kappa \int_0^\infty e^{-(r+\kappa)s} p(T_{N_t+n} + s) ds,$$

the function $q(t)$ is the one defined in (8). For any finite M and any t , the utility functional $\mathcal{U}_{k(t)}^M(\mathbf{a}, t)$ is well defined for any feasible asset plan \mathbf{a} .²⁷

²⁶Notice that the stochastic process $\{T_n\}_{n=1}^\infty$ can be thought of as being a function of the process ω (since $(N_t)_{t \in [0, \infty)}$ is a right-continuous step function with jumps at $\{T_n\}_{n=1}^\infty$), so for any \mathcal{F}^t -measurable function $f: \Omega \rightarrow \mathbb{R} \cup \{\pm\infty\}$, the expectation $\mathbb{E}[f(\omega)|\mathcal{F}^t]$ is also integrating over $\{T_n\}_{n=1}^\infty$.

²⁷From (7), it is clear that the first term on the right side of (57) is a well behaved function of \mathbf{a}_t , which is itself a bounded and \mathcal{F}^t -measurable function. Since throughout the paper we have

Next, for any given nonnegative measurable price function $p(t)$, we define the infinite-horizon utility for the investor from following a feasible asset plan \mathbf{a} by

$$\mathcal{U}_{k(t)}(\mathbf{a}, t) = \limsup_{M \rightarrow \infty} \mathcal{U}_{k(t)}^M(\mathbf{a}, t).$$

For any feasible asset plan, the sequence

$$\left\{ \sum_{n=1}^{M-1} e^{-r(T_{N_t+n}-t)} \bar{u}_{k(T_{N_t+n})}(\mathbf{a}_{T_{N_t+n}}) \right\}_{M=1}^{\infty}$$

has a limit. This limit may be a finite number or $-\infty$.²⁸ The sequence

$$\left\{ \sum_{n=1}^{M-1} e^{-r(T_{N_t+n}-t)} q(T_{N_t+n}) \mathbf{a}_{T_{N_t+n}} \right\}_{M=1}^{\infty}$$

is nondecreasing, so it has a limit, which may be $+\infty$. Let

$$(59) \quad f_M(\omega) \equiv \sum_{n=1}^{M-1} e^{-r(T_{N_t+n}-t)} [\bar{u}_{k(T_{N_t+n})}(\mathbf{a}_{T_{N_t+n}}) - q(T_{N_t+n}) \mathbf{a}_{T_{N_t+n}}].$$

Then we have shown that $\lim_{M \rightarrow \infty} f_M$ exists (it may be finite or $-\infty$). If we rescale u_i for each i so that $u_i(\bar{a}) \leq 0$ for all i , we see that the sequence $\{-f_M\}_{M=1}^{\infty}$ is a monotone increasing sequence of measurable functions that converge pointwise to $-\lim_{M \rightarrow \infty} f_M$, so by the monotone convergence theorem (e.g., Theorem 7.8 in Stokey and Lucas (1989)), we have $\lim_{M \rightarrow \infty} \mathbb{E}_t[f_M] = \mathbb{E}_t[\lim_{M \rightarrow \infty} f_M]$. All this implies that, given a price path $p(t)$, an investor's expected lifetime utility from following a feasible asset plan $\mathbf{a} = (\mathbf{a}_s)_{s \in [0, \infty)}$

specialized the analysis to price paths with the property that $p(t)$ is measurable, $q(t)$ is well defined for any t and the second term on the right side of (57) is well defined. Since $e^{-rt} p(t) \mathbf{a}_t$ is a nonnegative measurable function, the integral in the third term is well defined (although it need not be finite). As for the last term, notice that

$$\bar{u}_{k(T_{N_t+n})}(\mathbf{a}) = \sum_{i=1}^I \bar{u}_i(\mathbf{a}) \mathbb{1}_{\{k(T_{N_t+M})=i\}},$$

where $\bar{u}_i(\mathbf{a})$ is a continuous function for each i , so the integral of $e^{-rt} \bar{u}_{k(t)}(\mathbf{a}_t)$ is well defined. Finally, the integral of $q(t) \mathbf{a}_t$ is well defined since $p(t)$ and \mathbf{a}_t are nonnegative and measurable.

²⁸This limit is finite if u_i is bounded below for all i , since in that case we can rescale each utility function so that $u_i(0) \geq 0$ for all i , and the sequence of partial sums is nondecreasing and bounded above (because $a_t \leq \bar{a}$ for all t and u_i is continuous for each i). Conversely, if some u_i is unbounded below, we can rescale u_i and every other u_j so that $u_k(\bar{a}) \leq 0$ for all k . Then since the sequence of partial sums is nonincreasing, it has a limit, which could be $-\infty$.

is

$$\begin{aligned} \mathcal{U}_{k(t)}(\mathbf{a}, t) &= \frac{\bar{u}_{k(t)}(\mathbf{a}_t)}{r + \kappa} + \left[p(t) - \frac{q(t)}{r + \kappa} \right] \mathbf{a}_t \\ &\quad - \limsup_{M \rightarrow \infty} \mathbb{E}_t \left[e^{-r(T_{N_t+M}-t)} p(T_{N_t+M}) \mathbf{a}_{T_{N_t+M}} \right] \\ &\quad + \frac{1}{r + \kappa} \mathbb{E}_t \left\{ \sum_{n=1}^{\infty} e^{-r(T_{N_t+n}-t)} \left[\bar{u}_{k(T_{N_t+n})}(\mathbf{a}_{T_{N_t+n}}) - q(T_{N_t+n}) \mathbf{a}_{T_{N_t+n}} \right] \right\}, \end{aligned}$$

which is well defined for any feasible path.²⁹ The investor's problem at t is

$$(60) \quad \max_{\mathbf{a} \in \mathbb{A}} \mathcal{U}_{k(t)}(\mathbf{a}, t), \quad \text{s.t.} \quad \mathbf{a}_t = \mathbf{a} \geq 0 \quad \text{and} \quad k(t) \in \mathbb{X} \text{ given.}$$

The investor's maximum attainable utility is then

$$V_{k(t)}^*(a, t) = \max_{\mathbf{a} \in \mathbb{A}} \mathcal{U}_{k(t)}(\mathbf{a}, t).$$

PROPOSITION 9: *A feasible plan $\mathbf{a}^* = (\mathbf{a}_s^*(\omega))_{s \in [t, \infty), \omega \in \Omega}$ is optimal from a given initial date $t \geq 0$ if and only if it satisfies*

$$(61) \quad \mathbf{a}_{T_n}^*(\omega) = \arg \max_{\mathbf{a} \in (0, \bar{a}]} \left[\bar{u}_{k(T_n)}(\mathbf{a}) - q(T_n) \mathbf{a} \right] \quad \forall \omega \in \Omega, \forall \{T_n\}_{n=T_{N_t+1}}^{\infty}$$

and

$$(62) \quad \lim_{n \rightarrow \infty} \mathbb{E}_i \left\{ e^{-r(T_{N_t+n}-t)} p(T_{N_t+n}) \mathbf{a}_{T_{N_t+n}}^* \right\} = 0.$$

Moreover, if there exists a number $B > \max_j \bar{u}'_j(\infty)$ such that $q(s) \geq B$ for all s , then an optimal plan exists and is unique.

PROOF: The proof proceeds in three steps.

(i) We first show that (61) and (62) are sufficient for an optimum. Let \mathbf{a}^* be the asset plan that satisfies (61) and (62), and let \mathbf{a} be any other feasible plan.

²⁹We have chosen to define the lifetime utility as $\limsup_{M \rightarrow \infty} \mathcal{U}^M(\mathbf{a}, t)$ rather than $\lim_{M \rightarrow \infty} \mathcal{U}^M(\mathbf{a}, t)$, because $\lim_{M \rightarrow \infty} \mathbb{E}_t[\exp(-r(T_{N_t+M}-t)) p(T_{N_t+M}) \mathbf{a}_{T_{N_t+M}}]$ need not exist for every feasible asset plan. The definition we have adopted guarantees that the payoff from every feasible asset plan can be evaluated using the investor's utility function. As we show below, the optimal asset plan, \mathbf{a}^* , has the property that $\lim_{M \rightarrow \infty} \mathbb{E}_t[\exp(-r(T_{N_t+M}-t)) p(T_{N_t+M}) \mathbf{a}_{T_{N_t+M}}^*] = 0$, which means that, equivalently, we could define the utility function as $\lim_{M \rightarrow \infty} \mathcal{U}^M(\mathbf{a}, t)$ and simply restrict the investor's choices to the set of feasible paths for which $\lim_{M \rightarrow \infty} \mathbb{E}_t[\exp(-r \times (T_{N_t+M}-t)) p(T_{N_t+M}) \mathbf{a}_{T_{N_t+M}}]$ exists.

For any t , let $\Delta \equiv \mathcal{U}_{k(t)}(\mathbf{a}^*, t) - \mathcal{U}_{k(t)}(\mathbf{a}, t)$. Then

$$\begin{aligned} \Delta &\geq \frac{1}{r + \kappa} \mathbb{E}_t \left\{ \sum_{n=1}^{\infty} e^{-r(T_{N_t+n-t})} [\bar{u}_{k(T_{N_t+n})}(\mathbf{a}_{T_{N_t+n}}^*) - q(T_{N_t+n})\mathbf{a}_{T_{N_t+n}}^*] \right\} \\ &\quad - \frac{1}{r + \kappa} \mathbb{E}_t \left\{ \sum_{n=1}^{\infty} e^{-r(T_{N_t+n-t})} [\bar{u}_{k(T_{N_t+n})}(\mathbf{a}_{T_{N_t+n}}) - q(T_{N_t+n})\mathbf{a}_{T_{N_t+n}}] \right\} \\ &\quad - \limsup_{M \rightarrow \infty} \mathbb{E}_t [e^{-r(T_{N_t+M-t})} p(T_{N_t+M})\mathbf{a}_{T_{N_t+M}}^*]. \end{aligned}$$

From (61) and (62), it follows that $\Delta \geq 0$.

(ii) Next we show that an optimal plan must satisfy (61) and (62). The first step is to notice that the objective function on the right side of (61) is strictly concave and differentiable, so $u'_i[\mathbf{a}_s^*(\omega)] - q(s) \leq 0$ (“=” if $\mathbf{a}_i^*(s) > 0$) is necessary and sufficient for an optimum. Since $q(s) > \bar{u}'_i(\infty)$ for all i , we can choose \bar{a} large enough so that $q(s) > \bar{u}'_i(\bar{a})$ for all i and, therefore, (61) is the *unique* solution to the investor’s problem at time s , for history ω , when his preference type is $k(s)$. Suppose that the asset plan $\tilde{\mathbf{a}}$ is optimal, with $\tilde{\mathbf{a}}_s(\omega) \neq \mathbf{a}_s^*(\omega)$ for some history ω at some date $s > t$. Since both $\tilde{\mathbf{a}}$ and \mathbf{a}^* are feasible, $\tilde{\mathbf{a}}_{T_{N_s}}(\omega) \neq \mathbf{a}_{T_{N_s}}^*(\omega)$. Then the investor could maintain his asset plan $\tilde{\mathbf{a}}$ unchanged except at date T_{N_s} for history ω , where he could choose $\mathbf{a}_{T_{N_s}}^*(\omega)$. By (61), this deviation is feasible. Since the maximization in (61) has a unique solution, the proposed deviation strictly increases the investor’s expected utility, so $\tilde{\mathbf{a}}$ could not have been optimal—a contradiction. Next, we show that any optimal policy must satisfy (62). Let \mathbf{a}^* be an optimal plan and consider the feasible plan $(1 - \varepsilon)\mathbf{a}^*$ for some small $\varepsilon > 0$. Let $\Delta_\varepsilon \equiv \mathcal{U}_{k(t)}(\mathbf{a}^*, t) - \mathcal{U}_{k(t)}[(1 - \varepsilon)\mathbf{a}^*, t]$. Then

$$\begin{aligned} \Delta_\varepsilon &= \mathbb{E}_t \left\{ \sum_{n=1}^{\infty} \frac{e^{-r(T_{N_t+n-t})}}{r + \kappa} [\bar{u}_{k(T_{N_t+n})}(\mathbf{a}_{T_{N_t+n}}^*) - \bar{u}_{k(T_{N_t+n})}[(1 - \varepsilon)\mathbf{a}_{T_{N_t+n}}^*] \right. \\ &\quad \left. - \varepsilon q(T_{N_t+n})\mathbf{a}_{T_{N_t+n}}^*] \right\} \\ &\quad - \varepsilon \limsup_{M \rightarrow \infty} \mathbb{E}_t [e^{-r(T_{N_t+M-t})} p(T_{N_t+M})\mathbf{a}_{T_{N_t+M}}^*]. \end{aligned}$$

Divide the previous expression by ε and take the limit as $\varepsilon \rightarrow 0$ (applying l’Hôpital’s rule) to arrive at

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \frac{\Delta_\varepsilon}{\varepsilon} &= \frac{1}{r + \kappa} \mathbb{E}_t \left\{ \sum_{n=1}^{\infty} e^{-r(T_{N_t+n-t})} [\bar{u}'_{k(T_{N_t+n})}(\mathbf{a}_{T_{N_t+n}}^*) - q(T_{N_t+n})]\mathbf{a}_{T_{N_t+n}}^* \right\} \\ &\quad - \limsup_{M \rightarrow \infty} \mathbb{E}_t [e^{-r(T_{N_t+M-t})} p(T_{N_t+M})\mathbf{a}_{T_{N_t+M}}^*]. \end{aligned}$$

Since the asset plan \mathbf{a}^* is optimal, the first-order condition for the investor's problem (61), that is, $[\bar{u}'_{k(T_n)}(\mathbf{a}_{T_n}^*) - q(T_n)]\mathbf{a}_{T_n}^* = 0$ for all $\{T_n\}_{n=T_{N_i+1}}^\infty$, implies

$$\lim_{\varepsilon \rightarrow 0} \frac{\Delta_\varepsilon}{\varepsilon} = - \limsup_{M \rightarrow \infty} \mathbb{E}_t \left[e^{-r(T_{N_i+M}-t)} p(T_{N_i+M}) \mathbf{a}_{T_{N_i+M}}^* \right]$$

and the optimality of \mathbf{a}^* requires

$$0 \leq - \limsup_{M \rightarrow \infty} \mathbb{E}_t \left[e^{-r(T_{N_i+M}-t)} p(T_{N_i+M}) \mathbf{a}_{T_{N_i+M}}^* \right].$$

Then, since $e^{-rT} p(T) \mathbf{a}_T^* \geq 0$ for all T , we have

$$\begin{aligned} 0 &\leq \liminf_{M \rightarrow \infty} \mathbb{E}_t \left[e^{-r(T_{N_i+M}-t)} p(T_{N_i+M}) \mathbf{a}_{T_{N_i+M}}^* \right] \\ &\leq \limsup_{M \rightarrow \infty} \mathbb{E}_t \left[e^{-r(T_{N_i+M}-t)} p(T_{N_i+M}) \mathbf{a}_{T_{N_i+M}}^* \right] \leq 0, \end{aligned}$$

so the optimality of \mathbf{a}^* requires

$$\lim_{M \rightarrow \infty} \mathbb{E}_t \left[e^{-r(T_{N_i+M}-t)} p(T_{N_i+M}) \mathbf{a}_{T_{N_i+M}}^* \right] = 0.$$

(iii) Finally, since the necessary conditions (61) and (62) determine a unique $\mathbf{a}^* = (\mathbf{a}_t^*(\omega))_{t \in [0, \infty), \omega \in \Omega}$, the optimal plan exists and is unique. *Q.E.D.*

The formulation we have laid out in this appendix is quite general in that it allows the investor to choose among feasible asset plans $\mathbf{a} = (\mathbf{a}_t(\omega))_{t \in [0, \infty), \omega \in \Omega}$, where \mathbf{a}_t can be any \mathcal{F}^t -measurable function of the whole history of shocks, ω , as well as time, t . From (61), however, notice that the optimal asset plan $\mathbf{a}^* = (\mathbf{a}_t^*(\omega))_{t \in [0, \infty), \omega \in \Omega}$ is not history dependent: when the investor gains effective access to the market at time T_n , his optimal decision depends only on T_n and his preference type at that time, $k(T_n)$. For this reason, we can simplify the notation as we did in the body of the paper, by letting $a_{k(T_n)}(T_n) \equiv \mathbf{a}_{T_n}^*(\omega)$. With this notation, we can denote the optimal plan \mathbf{a}^* simply by a sequence of functions $\{(a_i(t), t \in [0, \infty))\}_{i=1}^I$, with $a_i(t) = a_i(T_n)$ for all $t \in [T_n, T_{n+1})$ and every i . Also as in the body of the paper, we can use $\mathbb{E}_{k(t)}$ to denote \mathbb{E}_t , which stresses the fact that $k(t)$ summarizes all the relevant information available to the investor at time t that enables him to form the conditional expectation over ω . With this notation, consider an investor at time t , with asset holdings $a_t = a \geq 0$ and preference type $k(t) = i \in \mathbb{X}$ both given. His maximum attainable utility is $V_i^*(a, t) = \mathcal{U}_i(\mathbf{a}^*, t)$, that is,

$$(63) \quad V_i^*(a, t) = \frac{\bar{u}_i(a)}{r + \kappa} + \left[p(t) - \frac{q(t)}{r + \kappa} \right] a + K_i(t),$$

where

$$K_i(t) = \mathbb{E}_i \left\{ \sum_{n=1}^{\infty} e^{-r(T_{N_t+n}-t)} \left[\frac{\bar{u}_{k(T_{N_t+n})}[a_{k(T_{N_t+n})}(T_{N_t+n})]}{r + \kappa} - \frac{q(T_{N_t+n})}{r + \kappa} a_{k(T_{N_t+n})}(T_{N_t+n}) \right] \right\}.$$

From Proposition 9 we know that if there exists a number $B > \max_j \bar{u}'_j(\infty)$ such that $q(s) \geq B$ for all s , then an optimal plan $\{(a_i(t), t \in [0, \infty))\}_{i=1}^I$ exists and \bar{B} is unique, so $K_i(t)$ is well defined. If, in addition, there exists a real number \bar{B} such that $q(t) \leq \bar{B}$ for all t , then $K_i(t) \in \mathbb{R}$ for all t and every i .

Instead of considering (60), in the body of the paper we described the investor's problem using a recursive functional equation (i.e., (1)) with asset holdings and fees given by (2), which we showed to be equivalent to (25). Lemma 8 formalizes the relationship between both formulations of the investor's problem, (25) and (60). Before we prove this result, it is convenient to establish a preliminary result.

LEMMA 7: For any $t \geq 0$,

$$\begin{aligned} K_{k(t)}(t) &= \frac{1}{r + \kappa} \mathbb{E}_{k(t)} \left\{ e^{-r(T_{N_t+1}-t)} [\bar{u}_{k(T_{N_t+1})}[a_{k(T_{N_t+1})}(T_{N_t+1})] \right. \\ &\quad \left. - q(T_{N_t+1})a_{k(T_{N_t+1})}(T_{N_t+1})] \right\} \\ &\quad + \mathbb{E}_{k(t)} [e^{-r(T_{N_t+1}-t)} K_{k(T_{N_t+1})}(T_{N_t+1})]. \end{aligned}$$

PROOF: First, notice that for all integers $n \geq 0$, we have $N_s = N_t + n$ if $s = T_{N_t+n}$, so the definition of $K_{k(t)}(t)$ implies

$$\begin{aligned} (64) \quad &K_{k(T_{N_t+1})}(T_{N_t+1}) \\ &= \frac{1}{r + \kappa} \mathbb{E}_{k(T_{N_t+1})} \left\{ \sum_{n=2}^{\infty} e^{-r(T_{N_t+n}-T_{N_t+1})} \bar{u}_{k(T_{N_t+n})}[a_{k(T_{N_t+n})}(T_{N_t+n})] \right\} \\ &\quad - \frac{1}{r + \kappa} \mathbb{E}_{k(T_{N_t+1})} \left\{ \sum_{n=2}^{\infty} e^{-r(T_{N_t+n}-T_{N_t+1})} q(T_{N_t+n}) a_{k(T_{N_t+n})}(T_{N_t+n}) \right\}. \end{aligned}$$

Also from the definition of $K_{k(t)}(t)$,

$$\begin{aligned} K_{k(t)}(t) &= \frac{1}{r + \kappa} \mathbb{E}_{k(t)} \left\{ e^{-r(T_{N_t+1}-t)} [\bar{u}_{k(T_{N_t+1})}[a_{k(T_{N_t+1})}(T_{N_t+1})] \right. \\ &\quad \left. - q(T_{N_t+1})a_{k(T_{N_t+1})}(T_{N_t+1})] \right\} \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{r + \kappa} \mathbb{E}_{k(t)} \left\{ \sum_{n=2}^{\infty} e^{-r(T_{N_t+n}-t)} [\bar{u}_{k(T_{N_t+n})} [a_{k(T_{N_t+n})}(T_{N_t+n})] \right. \\
& \qquad \qquad \qquad \left. - q(T_{N_t+n}) a_{k(T_{N_t+n})}(T_{N_t+n})] \right\} \\
& = \frac{1}{r + \kappa} \mathbb{E}_{k(t)} \left\{ e^{-r(T_{N_t+1}-t)} [\bar{u}_{k(T_{N_t+1})} [a_{k(T_{N_t+1})}(T_{N_t+1})] \right. \\
& \qquad \qquad \qquad \left. - q(T_{N_t+1}) a_{k(T_{N_t+1})}(T_{N_t+1})] \right\} \\
& \quad + \mathbb{E}_{k(t)} e^{-r(T_{N_t+1}-t)} \left[\frac{1}{r + \kappa} \right. \\
& \quad \times \mathbb{E}_{k(T_{N_t+1})} \left\{ \sum_{n=2}^{\infty} e^{-r(T_{N_t+n}-T_{N_t+1})} \bar{u}_{k(T_{N_t+n})} [a_{k(T_{N_t+n})}(T_{N_t+n})] \right\} \\
& \quad \left. - \mathbb{E}_{k(t)} e^{-r(T_{N_t+1}-t)} \left[\frac{1}{r + \kappa} \right. \right. \\
& \quad \times \mathbb{E}_{k(T_{N_t+1})} \left\{ \sum_{n=2}^{\infty} e^{-r(T_{N_t+n}-T_{N_t+1})} q(T_{N_t+n}) a_{k(T_{N_t+n})}(T_{N_t+n}) \right\} \left. \right. \\
& = \frac{1}{r + \kappa} \mathbb{E}_{k(t)} \left\{ e^{-r(T_{N_t+1}-t)} [\bar{u}_{k(T_{N_t+1})} [a_{k(T_{N_t+1})}(T_{N_t+1})] \right. \\
& \qquad \qquad \qquad \left. - q(T_{N_t+1}) a_{k(T_{N_t+1})}(T_{N_t+1})] \right\} \\
& \quad + \mathbb{E}_{k(t)} [e^{-r(T_{N_t+1}-t)} K_{k(T_{N_t+1})}(T_{N_t+1})].
\end{aligned}$$

The last equality follows from (64).

Q.E.D.

LEMMA 8: Consider an investor who, at some initial time $t \geq 0$, starts with asset position a and preference type $k(t) \in \mathbb{X}$, and suppose that there exists a number $B > \max_j \bar{u}'_j(\infty)$ such that $q(s) \geq B$ for all $s \geq t$.

(i) The maximum value of (60), that is, $V_{k(t)}^*(a, t)$, satisfies the functional equation (25).

(ii) The asset plan that solves (60), that is, $(a_{k(T_{N_s})}(s), s \in [t, \infty))$, satisfies

$$\begin{aligned}
& V_{k(t)}^* [a_{k(T_{N_t})}(T_{N_t}), t] \\
& = \frac{\bar{u}_{k(t)} [a_{k(T_{N_t})}(T_{N_t})]}{r + \kappa} + \mathbb{E}_{k(t)} [e^{-r(T_{N_t+1}-t)} p(T_{N_t+1}) a_{k(T_{N_t})}(T_{N_t})] \\
& \quad + \mathbb{E}_{k(t)} [e^{-r(T_{N_t+1}-t)} \{ V_{k(T_{N_t+1})}^* [a_{k(T_{N_t+1})}(T_{N_t+1}), T_{N_t+1}] \\
& \qquad \qquad \qquad - p(T_{N_t+1}) a_{k(T_{N_t+1})}(T_{N_t+1}) \}].
\end{aligned}$$

(iii) Let $(a_{k(T_{N_s})}(s), s \in [t, \infty))$ be the asset plan induced by (25), that is, the asset plan in (6), with

$$(65) \quad \lim_{n \rightarrow \infty} \mathbb{E}_i \left[e^{-r(T_{N_t+n}-t)} p(T_{N_t+n}) a_{k(T_{N_t+n})}(T_{N_t+n}) \right] = 0$$

for each $i \in \mathbb{X}$. Then this asset plan achieves the maximum in (60).

(iv) Let $(a_{k(T_{N_s})}(s), s \in [t, \infty))$ be the asset plan induced by (25) and assume it satisfies (65). If $V_i(a, t)$ solves (25) and satisfies

$$(66) \quad \lim_{n \rightarrow \infty} \mathbb{E}_i \left[e^{-r(T_{N_t+n}-t)} V_{k(T_{N_t+n})} \left[a_{k(T_{N_t+n})}(T_{N_t+n}), T_{N_t+n} \right] \right] = 0$$

for each $i \in \mathbb{X}$, then $V_i(a, t) = V_i^*(a, t)$.

PROOF: (i) If we let $V^*(a, t) \equiv \{V_i^*(a, t)\}_{i=1}^I$ and regard the right side of (25) as a map F , we need to show $FV^* = V^*$. Substitute $V^*(a, t)$ as given by (63), into (25):

$$\begin{aligned} & (FV^*)(a, t, i) \\ &= \frac{\bar{u}_i(a)}{r + \kappa} + \mathbb{E}_i \left[e^{-r(T_{N_t+1}-t)} \left\{ p(T_{N_t+1}) a \right. \right. \\ & \qquad \qquad \qquad \left. \left. + \max_{a' \geq 0} [V_{k(T_{N_t+1})}^*(a', T_{N_t+1}) - p(T_{N_t+1}) a'] \right\} \right] \\ &= \frac{\bar{u}_i(a)}{r + \kappa} + \left[p(t) - \frac{q(t)}{r + \kappa} \right] a + \mathbb{E}_i \left[e^{-r(T_{N_t+1}-t)} K_{k(T_{N_t+1})}(T_{N_t+1}) \right] \\ & \quad + \frac{1}{r + \kappa} \mathbb{E}_i \left\{ e^{-r(T_{N_t+1}-t)} \left[\bar{u}_{k(T_{N_t+1})} \left[a_{k(T_{N_t+1})}(T_{N_t+1}) \right] \right. \right. \\ & \qquad \qquad \qquad \left. \left. - q(T_{N_t+1}) a_{k(T_{N_t+1})}(T_{N_t+1}) \right] \right\} \\ &= \frac{\bar{u}_i(a)}{r + \kappa} + \left[p(t) - \frac{q(t)}{r + \kappa} \right] a + K_i(t) \\ &= V_i^*(a, t), \end{aligned}$$

where the third equality follows from Lemma 7.

(ii) From (63),

$$\begin{aligned} & V_{k(t)}^* \left[a_{k(T_{N_t})}(T_{N_t}), t \right] \\ &= \frac{\bar{u}_{k(t)} \left[a_{k(T_{N_t})}(T_{N_t}) \right]}{r + \kappa} + \left[p(t) - \frac{q(t)}{r + \kappa} \right] a_{k(T_{N_t})}(T_{N_t}) + K_{k(t)}(t) \\ &= \frac{\bar{u}_{k(t)} \left[a_{k(T_{N_t})}(T_{N_t}) \right]}{r + \kappa} + \mathbb{E}_{k(t)} \left[e^{-r(T_{N_t+1}-t)} p(T_{N_t+1}) a_{k(T_{N_t})}(T_{N_t}) \right] \end{aligned}$$

$$\begin{aligned}
& + K_{k(t)}(t) \\
& = \frac{\bar{u}_{k(t)}[a_{k(T_{N_t})}(T_{N_t})]}{r + \kappa} + \mathbb{E}_{k(t)}[e^{-r(T_{N_{t+1}}-t)} p(T_{N_{t+1}}) a_{k(T_{N_t})}(T_{N_t})] \\
& \quad + \frac{1}{r + \kappa} \mathbb{E}_{k(t)} \{ e^{-r(T_{N_{t+1}}-t)} [\bar{u}_{k(T_{N_{t+1}})}[a_{k(T_{N_{t+1}})}(T_{N_{t+1}})] \\
& \quad \quad \quad - q(T_{N_{t+1}}) a_{k(T_{N_{t+1}})}(T_{N_{t+1}})] \} \\
& \quad + \mathbb{E}_{k(t)}[e^{-r(T_{N_{t+1}}-t)} K_{k(T_{N_{t+1}})}(T_{N_{t+1}})] \\
& = \frac{\bar{u}_{k(t)}[a_{k(T_{N_t})}(T_{N_t})]}{r + \kappa} + \mathbb{E}_{k(t)}[e^{-r(T_{N_{t+1}}-t)} p(T_{N_{t+1}}) a_{k(T_{N_t})}(T_{N_t})] \\
& \quad + \mathbb{E}_{k(t)}[e^{-r(T_{N_{t+1}}-t)} \{ V_{k(T_{N_{t+1}})}^* [a_{k(T_{N_{t+1}})}(T_{N_{t+1}}), T_{N_{t+1}}] \\
& \quad \quad \quad - p(T_{N_{t+1}}) a_{k(T_{N_{t+1}})}(T_{N_{t+1}}) \} \}.
\end{aligned}$$

The second equality follows from the definition of $q(t)$, the third equality follows from Lemma 7, and the fourth equality follows from the fact that

$$\begin{aligned}
& V_{k(T_{N_{t+1}})}^* [a_{k(T_{N_{t+1}})}(T_{N_{t+1}}), T_{N_{t+1}}] \\
& = \frac{\bar{u}_{k(T_{N_{t+1}})}[a_{k(T_{N_{t+1}})}(T_{N_{t+1}})]}{r + \kappa} \\
& \quad + \left[p(T_{N_{t+1}}) - \frac{q(T_{N_{t+1}})}{r + \kappa} \right] a_{k(T_{N_{t+1}})}(T_{N_{t+1}}) + K_{k(T_{N_{t+1}})}(T_{N_{t+1}}).
\end{aligned}$$

(iii) Part (iii) is immediate from Proposition 9.

(iv) By (3) and (6), we can write (25) as

$$\begin{aligned}
V_{k(t)}(a, t) & = \frac{\bar{u}_{k(t)}(a)}{r + \kappa} + \mathbb{E}_{k(t)} \{ e^{-r(T_{N_{t+1}}-t)} p(T_{N_{t+1}}) [a - a_{k(T_{N_{t+1}})}(T_{N_{t+1}})] \} \\
& \quad + \mathbb{E}_{k(t)} \{ e^{-r(T_{N_{t+1}}-t)} V_{k(T_{N_{t+1}})} [a_{k(T_{N_{t+1}})}(T_{N_{t+1}}), T_{N_{t+1}}] \}.
\end{aligned}$$

Iterate this expression forward $M - 1$ times (using the law of iterated expectations and (58)) to arrive at

$$\begin{aligned}
(67) \quad & V_{k(t)}^M(a, t) \\
& = \frac{\bar{u}_{k(t)}(a)}{r + \kappa} + \left[p(t) - \frac{q(t)}{r + \kappa} \right] a \\
& \quad - \mathbb{E}_{k(t)} [e^{-r(T_{N_{t+M}}-t)} p(T_{N_{t+M}}) a_{k(T_{N_{t+M}})}(T_{N_{t+M}})]
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{r + \kappa} \mathbb{E}_{k(t)} \left\{ \sum_{n=1}^{M-1} e^{-r(T_{N_t+n}-t)} [\bar{u}_{k(T_{N_t+n})} [a_{k(T_{N_t+n})}(T_{N_t+n})] \right. \\
& \qquad \qquad \qquad \left. - q(T_{N_t+n}) a_{k(T_{N_t+n})}(T_{N_t+n})] \right\} \\
& + \mathbb{E}_{k(t)} [e^{-r(T_{N_t+M}-t)} V_{k(T_{N_t+M})} [a_{k(T_{N_t+M})}(T_{N_t+M}), T_{N_t+M}]].
\end{aligned}$$

A function $V_{k(t)}(a, t)$ that solves (25) must satisfy (67) for all M , so the solution is $V_{k(t)}(a, t) = \lim_{M \rightarrow \infty} V_{k(t)}^M(a, t)$, provided this limit exists. From (67),

$$\begin{aligned}
& \lim_{M \rightarrow \infty} V_{k(t)}^M(a, t) \\
& = \frac{\bar{u}_{k(t)}(a)}{r + \kappa} + \left[p(t) - \frac{q(t)}{r + \kappa} \right] a \\
& \quad - \lim_{M \rightarrow \infty} \mathbb{E}_{k(t)} [e^{-r(T_{N_t+M}-t)} p(T_{N_t+M}) a_{k(T_{N_t+M})}(T_{N_t+M})] \\
& \quad + \lim_{M \rightarrow \infty} \mathbb{E}_{k(t)} \left\{ \sum_{n=1}^{M-1} e^{-r(T_{N_t+n}-t)} \left[\frac{\bar{u}_{k(T_{N_t+n})} [a_{k(T_{N_t+n})}(T_{N_t+n})]}{r + \kappa} \right. \right. \\
& \qquad \qquad \qquad \left. \left. - \frac{q(T_{N_t+n})}{r + \kappa} a_{k(T_{N_t+n})}(T_{N_t+n}) \right] \right\} \\
& \quad + \lim_{M \rightarrow \infty} \mathbb{E}_{k(t)} [e^{-r(T_{N_t+M}-t)} V_{k(T_{N_t+M})} [a_{k(T_{N_t+M})}(T_{N_t+M}), T_{N_t+M}]] \\
& = \frac{\bar{u}_{k(t)}(a)}{r + \kappa} + \left[p(t) - \frac{q(t)}{r + \kappa} \right] a \\
& \quad + \lim_{M \rightarrow \infty} \mathbb{E}_{k(t)} \left\{ \sum_{n=1}^{M-1} e^{-r(T_{N_t+n}-t)} \left[\frac{\bar{u}_{k(T_{N_t+n})} [a_{k(T_{N_t+n})}(T_{N_t+n})]}{r + \kappa} \right. \right. \\
& \qquad \qquad \qquad \left. \left. - \frac{q(T_{N_t+n})}{r + \kappa} a_{k(T_{N_t+n})}(T_{N_t+n}) \right] \right\}.
\end{aligned}$$

The second equality follows from (65) and (66). Since $\{a_{k(T_{N_t+n})}(T_{N_t+n})\}_{n=1}^{\infty}$ is bounded and \bar{u}_i is continuous for every i , $\{\bar{u}_{k(T_{N_t+n})} [a_{k(T_{N_t+n})}(T_{N_t+n})]\}_{n=1}^{\infty}$ is bounded above. Hence, without loss of generality, we can rescale u_i for each i so that $\{-\bar{u}_{k(T_{N_t+n})} [a_{k(T_{N_t+n})}(T_{N_t+n})]\}_{n=1}^{\infty}$ is a nonnegative sequence. Then the sequence $\{\bar{f}_M\}_{M=1}^{\infty}$, where

$$\begin{aligned}
\bar{f}_M(\omega) \equiv & \sum_{n=1}^{M-1} e^{-r(T_{N_t+n}-t)} [\bar{u}_{k(T_{N_t+n})} [a_{k(T_{N_t+n})}(T_{N_t+n})] \\
& \qquad \qquad \qquad - q(T_{N_t+n}) a_{k(T_{N_t+n})}(T_{N_t+n})],
\end{aligned}$$

is a nonincreasing sequence and hence it has a limit, $\lim_{M \rightarrow \infty} \bar{f}_M$, which could be $-\infty$. Since $\{-\bar{f}_M\}_{M=1}^\infty$ is a monotone increasing sequence of measurable functions that converge pointwise to $-\lim_{M \rightarrow \infty} \bar{f}_M$, by the monotone convergence theorem (e.g., Theorem 7.8 in Stokey and Lucas (1989)), we have $\lim_{M \rightarrow \infty} \mathbb{E}_{k(t)}[\bar{f}_M] = \mathbb{E}_{k(t)}[\lim_{M \rightarrow \infty} \bar{f}_M] = (r + \kappa)K_{k(t)}(t)$ and, therefore, for every $k(t) \in \mathbb{X}$,

$$\lim_{M \rightarrow \infty} V_{k(t)}^M(a, t) = \frac{\bar{u}_{k(t)}(a)}{r + \kappa} + \left[p(t) - \frac{q(t)}{r + \kappa} \right] a + K_{k(t)}(t) = V_{k(t)}^*(a, t).$$

This concludes the proof. Q.E.D.

Lemma 8 establishes a version of Bellman's principle of optimality for the economy we analyze: Part (i) shows that $V_{k(t)}^*(a, t)$, the maximum value of the investor's problem given in (60), satisfies the functional equation (1) with asset holdings and fees given by (2) (which is equivalent to the functional equation (25)). Part (ii) establishes that the asset plan that solves (60) is an optimal plan implied by the functional equation (1) when this functional equation is evaluated at $V_{k(t)}^*(a, t)$. Part (iii) is a partial converse of part (ii): it proves that the asset plan that is optimal according to the functional equation (25) and that satisfies the boundedness condition (65) is the same asset plan that achieves the maximum of (60). Part (iv) is a partial converse of part (i): it shows that $V_{k(t)}^*(a, t)$ is the only solution of the functional equation (25) that satisfies the boundedness condition (66).

APPENDIX E: RELATED LITERATURE

In this appendix we draw connections to some related literature.

E.1. Search Models of Over-the-Counter Markets

Traders who operate in markets with OTC-style frictions will seek to mitigate these trading frictions by adjusting their asset positions so as to reduce their trading needs. Our analysis has shown that this is a critical aspect of investor behavior in illiquid markets. To illustrate this point, in this section we derive the main predictions of a version of DGP's model and contrast them with those of a special case of our formulation. This comparison will underscore the fact that the type of "liquidity hedging" that we have identified—and that only becomes possible with unrestricted asset holdings—generates new insights on how trading frictions shape the various dimensions of market liquidity, alters the empirical predictions of the theory, and leads to a different assessment of their normative implications.

We will contrast the empirical predictions of DGP's model with those of a special case of our model with $\mathbb{X} = \{1, 2\}$ and $u_i(a) = \varepsilon_i a^{1-\sigma}/(1-\sigma)$ for $i \in \mathbb{X}$

and $\sigma > 0$. We focus on the version of DGP's model with no interinvestor meetings (e.g., the version that DGP use in their Theorem 4 and part (i) of Theorem 6). DGP restricted $a \in \{0, 1\}$, and let u_{ij} denote the flow utility of an investor with asset position $i \in \{0, 1\}$ and preference type $j \in \{0, 1\}$.³⁰ DGP assumed $u_{00} = u_{01} = 0$, so for comparison purposes, we do the same hereafter. To simplify the notation, in both models we let π denote the steady-state fraction of investors with high valuation.³¹

Price

Since asset holdings are indivisible in DGP, equilibrium in the interdealer market requires investors who are on the long side of the market to be indifferent between trading and not trading. It is easy to show that in steady state, investors who want to sell are on the short side if and only if $A < \pi$. The equilibrium price in the interdealer market is

$$(68) \quad p = \begin{cases} \frac{1}{r} \frac{(r + \kappa)u_{11} + \delta\bar{u}}{r + \kappa + \delta} & \text{if } A < \pi, \\ \frac{1}{r} \frac{(r + \kappa)u_{10} + \delta\bar{u}}{r + \kappa + \delta} & \text{if } \pi < A, \end{cases}$$

where $\bar{u} \equiv \pi_1 u_{11} + \pi_0 u_{10}$.³²

The asset holding restrictions in DGP are also the reason why the asset price in their theory is independent of the stock of assets, A , for any $A < \pi$ and for any $A > \pi$, with a discontinuity at $A = \pi$. In contrast, the asset price in our model is smooth and decreasing in A . For example, in the special case of our model that we are considering in this section, $p = (\sum_i \pi_i \bar{\varepsilon}_i^{1/\sigma})^\sigma / r A^\sigma$.³³ The behavior of the asset price in response to changes in the trading frictions in DGP depends critically on the level of A . From (68), p is increasing in α (decreasing in η) if $A < \pi$, but decreasing in α (increasing in η) if $A > \pi$. In contrast, with unrestricted asset holdings these extensive-margin considerations are irrelevant to assess the impact of trading frictions on the asset price (recall Proposition 5).

³⁰DGP stated their restriction on asset holdings as $a \in [0, 1]$ but only studied equilibria in which agents hold either 0 or 1 unit of the asset, which is effectively equivalent to imposing the restriction $a \in \{0, 1\}$.

³¹“High valuation” corresponds to the index 2 in our formulation and 1 in DGP.

³²If $A = \pi$, $p \in [\frac{(r+\kappa)u_{10} + \delta\bar{u}}{r(r+\kappa+\delta)}, \frac{(r+\kappa)u_{11} + \delta\bar{u}}{r(r+\kappa+\delta)}]$ and the equilibrium price in the interdealer market is indeterminate.

³³Notice that we obtain DGP's formulation with $A < \pi$ as a special case of ours when $\sigma \rightarrow 0$.

Trade Volume

Trade volume is

$$\mathcal{V} = \alpha \frac{\delta \pi (1 - \pi)}{\alpha + \delta} \frac{(\bar{\varepsilon}_2)^{1/\sigma} - (\bar{\varepsilon}_1)^{1/\sigma}}{\pi (\bar{\varepsilon}_2)^{1/\sigma} + (1 - \pi) (\bar{\varepsilon}_1)^{1/\sigma}} A$$

in our model and

$$\mathcal{V}_{\text{DGP}} = \alpha \frac{\delta \pi (1 - \pi)}{\alpha + \delta} \min \left\{ \frac{A}{\pi}, \frac{1 - A}{1 - \pi} \right\}$$

in DGP. The latter is independent of the dealers' bargaining power, η , and of all preference parameters and holding payoffs. In contrast, these parameters are critical determinants of trade volume in our theory, as they influence the investors' choices of asset holdings (the second factor in \mathcal{V}). Our model predicts that markets in which dealers have less market power will tend to exhibit larger trade volume.³⁴

Transaction Costs

DGP's transaction costs can be expressed in terms of the intermediation fees ϕ_{01} and ϕ_{10} that dealers charge investors who want to buy and sell, respectively. The equilibrium spread is $s = \eta(u_{11} - u_{10}) / (r + \kappa + \delta)$.³⁵ Conditional on having contacted an investor, the expected intermediation fee that accrues to a dealer in DGP is $\Phi_{\text{DGP}} = \frac{\delta \pi (1 - \pi)}{\alpha + \delta} \min \left\{ \frac{A}{\pi}, \frac{1 - A}{1 - \pi} \right\} s$. This key determinant of dealers' incentives to make markets is decreasing in the investors' contact rate with dealers, α , and increasing in the dealers' bargaining power, η . In contrast, as we have shown analytically in Proposition 4, in our model with no restrictions on asset holdings it is natural for the average fee to be nonmonotonic in α and η . Our theory suggests that this nonmonotonicity can be important. From an applied standpoint, it can help explain how OTC markets have reacted to recent changes in their market structure (see Lagos and Rocheteau (2006)).

³⁴Apart from these qualitative differences, the theory with unrestricted portfolios also has different quantitative implications for the relationship between trade volume and trading frictions. For example, DGP's model has a sharp empirical implication: the elasticity of trade volume with respect to trading frictions equals $\frac{\delta}{\alpha + \delta} \in (0, 1)$. In contrast, in the model with unrestricted asset holdings, the corresponding elasticity is larger by an amount that equals the elasticity of $(a_2 - a_1)$ with respect to α —which is positive, capturing the notion that each investor wishes to conduct a larger trade when frictions are reduced.

³⁵Since asset holdings in DGP are restricted to lie in $\{0, 1\}$, every trade is of size 1 and hence $\phi_{01} + \phi_{10} = s$. In addition, the indivisibility assumption implies that dealers either charge a fee on asset sales or on asset purchases, but not both. Specifically, if $A < \pi$, then $\phi_{01} = 0$ and investors only pay a fee $\phi_{10} = s$ when they sell. Conversely, if $\pi < A$, $\phi_{10} = 0$ and investors only pay a fee $\phi_{01} = s$ when they buy.

From a theoretical standpoint, it can be shown to generate self-fulfilling liquidity shortages in markets with free entry of dealers (see Proposition 8 in Lagos and Rocheteau (2008)).³⁶

Another key difference with DGP is the fact that since the equilibrium in the model with unrestricted portfolios implies a nondegenerate distribution of trade sizes, our theory has predictions for the relationship between transaction costs and transaction sizes. As we showed in Lemma 4, transaction costs are increasing in the size of the transaction. Thus, if $a_i - a_j > a_i - a_k > 0$, then the effective price at which the investor buys is $\hat{p}_{ji} > \hat{p}_{ki}$, that is, he effectively pays higher prices when he conducts larger purchases. Conversely, $\hat{p}_{ji} < \hat{p}_{ki}$ if $a_i - a_j < a_i - a_k < 0$, that is, he effectively receives lower prices when he conducts larger sales. In other words, the theory with unrestricted asset holdings naturally generates instances of *price concession*, which are commonplace in OTC markets.³⁷

Trading Delays

DGP endogenized trading delays by allowing a single monopolist dealer to choose search intensity once and for all at the beginning of time. Free entry of competing dealers or market-makers is a feature of most OTC markets; however, the implications of this microstructure have not yet been explored in the literature. We find that allowing for free entry of dealers is a natural way to endogenize trading delays and the amount of liquidity supplied by dealers, and that it provides an important channel through which changes in market conditions affect transaction costs and trade volume. In addition, the interaction between free entry and unrestricted asset holdings leads to a natural kind of strategic complementarity that can help rationalize self-fulfilling liquidity shortages in markets with OTC-style frictions (see Proposition 8 in Lagos and Rocheteau (2008)).

Welfare

The equilibrium allocation is always constrained to be efficient in the baseline model of DGP—regardless of the value of η —which stands in contrast to the finding we report in Proposition 2 in Lagos and Rocheteau (2008). The reason is that in our model investors choose asset holdings, while this intensive

³⁶The spread, s , is decreasing in α and increasing in η in this version of DGP with no inter-investor meetings. One can also verify that the average effective spread weighted by the sizes of each trade and expressed as a proportion of the price is also decreasing in α and increasing in η . The behavior of this measure of the marketwide spread (i.e., (38) in Lagos and Rocheteau (2006)) is much more complicated in our model, where the investors' expected holding payoffs, their individual asset demands, the asset price, and the whole distribution of asset holdings change in response to a change in α . Our numerical work, some of which we have reported in Lagos and Rocheteau (2006), is in accordance with the predictions of DGP.

³⁷See Section 4.3 in Harris (2003).

margin is absent in DGP. For the same reason, the inefficiency result we find in the context of the model with free entry also has no counterpart in DGP.

A paper that is closely related to ours is an independent contribution by Gârleanu (2008), which studied the asset pricing and volume implications of infrequent (Poisson) trading opportunities. Some of our findings are similar: he also finds that under certain conditions (e.g., a mean-reversion property of preference shocks), investors take more extreme positions when trading delays are short. Also, Gârleanu stressed that the asset price is not affected by the trading frictions—which is true in our model for a particular specification of the utility function (Proposition 5). In terms of differences, trades in Gârleanu (2008) are not intermediated by dealers, so he could not consider the implications of trading delays for transaction costs and dealers' incentives to provide liquidity, which are at the center of our analysis. Also, Gârleanu (2008) formalized the investors' motive for holding the asset by developing the “hedging needs” motive we mentioned in footnote 4. Despite the differences in the formulations, some of our results on the effects of α on trade volume are remarkably similar.³⁸

E.2. Search Models of Money

Here we discuss the relationship between our theory and the search-theoretic literature on monetary exchange. In contrast to the monetary literature, our model does not have fiat money as an asset and it does not aim to explain the use or emergence of a medium of exchange. However, it shares a common objective with modern monetary theory, which is to endogenize some relevant dimensions of “liquidity.” We organize the comparison around four types of results.

Endogenous Distribution of Asset Holdings

Because of idiosyncratic (trading) shocks, under incomplete markets, our model generates a nondegenerate distribution of wealth as in Green and Zhou (2002) and Molico (2006), but also Aiyagari (1994). The trading mechanism in our model is closer to the one in Molico: the asset is traded in bilateral matches and the transaction price is determined through bargaining. In terms of the methodology, both Aiyagari (1994) and Molico (2006) solved their models numerically. The model of Green and Zhou (2002) is closer to our analysis in that they can characterize the equilibrium and its distribution of money holdings analytically. Moreover, like us, they do not restrict their analysis to stationary equilibria. The pricing mechanism is different (Green and Zhou considered a double auction).

³⁸See the discussion around Proposition 6 in online Appendix B for details.

Bargaining and the Distribution of Prices

A key insight of our model is that the intermediation fee depends on the (endogenous) asset position of the investor. Similarly, in monetary search models with bargaining, the transaction price depends on the traders' money balances. This dependence occurs through (at least) two channels. First, the buyer can be constrained by his money balances. This mechanism is present even in models with a degenerate distribution of money balances, such as Shi (1997) and Lagos and Wright (2005). Second, the money holdings of an agent affect his marginal utility of wealth and, hence, the terms of trade. These two effects are absent from our model, since our investors never face binding borrowing constraints and the marginal utility of wealth is normalized to one due to the quasi-linear preferences. An investor's asset holdings influence the outcome of the bargaining in our model because this asset position determines the size of the gains from trade that will be generated for readjusting the investor's asset holdings.

Uniqueness of the Equilibrium

The equilibrium (not just the steady state) is unique in our model. In contrast, the model of fiat money of Green and Zhou can display multiple equilibria. This indeterminacy is a general feature of models of fiat money. Even in models with a degenerate distribution of money balances (e.g., Lagos and Wright (2005)), the equilibrium is typically not unique, unless one restricts attention to steady-state monetary equilibria. Models of monetary exchange consider environments where the asset being traded is fiat money, whose value emerges endogenously when it is valued as a medium of exchange that mitigates a double coincidence of wants problem. In contrast, in our model and the rest of the literature that deals with the trading process in OTC markets, the asset being traded is not used to facilitate trades; it is valued for its intrinsic characteristics (e.g., dividend flow).

Endogenous Trading Delays and Multiple Equilibria

In our model, the multiplicity of steady-state equilibria with dealer entry arises from complementarities between investors' asset demands and dealers' entry decisions. If more dealers participate in the market, it is easier for investors to readjust their asset holdings, which induces them to take more extreme positions, and this in turn makes it profitable for dealers to enter. Rocheteau and Wright (2005) considered a monetary search model with free entry of sellers and found that the strategic complementarities between the sellers' entry decision and the buyers' demand for real balances generate multiple steady-state equilibria. If buyers accumulate more real balances, the buyer and the seller are able to exploit larger gains from trade, which gives more incentives for sellers to participate in the market. In both models, the multiplicity does not require increasing returns to scale in the matching function

as in Diamond (1982) or as in most recent search models of financial markets (e.g., Vayanos and Weill (2008)). A key difference between our model and Rocheteau and Wright (2005) is the opportunity cost from holding real balances in the latter, which has no counterpart in our formulation. If the opportunity cost from holding cash balances to make a purchase is zero (e.g., if the nominal interest rate is zero), then the multiplicity of (active) steady-state equilibria in that model disappears. In contrast, the multiplicity in our model obtains even though investors do not bear any opportunity cost (e.g., forgone interest) while searching for an asset to purchase (since they have access to a technology to produce the numéraire good). Also notice that the gains from trade in Rocheteau and Wright (2005) depend on the mean of the distribution of real balances (since the distribution of real balances is degenerate as in Lagos and Wright (2005)), which is independent of trading frictions when the nominal interest rate is zero. In our model it is the second moment, which is endogenous and depends on the trading frictions, that gives rise to multiple steady-state equilibria.

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