Econometrica Supplementary Material

SUPPLEMENT TO "PREFERENCE FOR FLEXIBILITY AND RANDOM CHOICE"

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In this supplement, we provide the axiomatic characterization of two representations used in the main paper (henceforth denoted AS). In Section S1, we show that with only slight modification of the axioms, the representation theorem from Dekel, Lipman, and Rustichini (2001) can be adapted to our setting of preferences over *finite* menus of lotteries. In Section S2, we show that the representation theorem for random choice rules from Gul and Pesendorfer (2006) can be adapted to obtain a random expected-utility representation defined on the *Borel σ-algebra* of the space of *twice-normalized* utility functions $\mathcal{U}$. We then introduce an additional finiteness axiom that allows this representation to be formulated using a probability measure over a finite state space, followed by a tie-breaking procedure. Proofs are contained in Section S3.

S1. DLR AXIOMS AND REPRESENTATION RESULT

The framework, notation, and definitions from the main text are all continued.

Dekel, Lipman, and Rustichini (2001) defined an additive expected-utility representation on the domain of all subsets of $\Delta(Z)$. To ensure compatibility with the choice domain of Gul and Pesendorfer (2006), we instead work with the space $\mathcal{A}$ of finite subsets of $\Delta(Z)$. In this section, we show that the restriction to finite menus requires only slight changes to the original DLR axioms.

For completeness, recall the definition of the DLR representation used in AS:

DEFINITION S1: A *DLR representation* of $\succsim$ is a triple (S, U, μ), where S is a finite state space, $U : S \times \Delta(Z) \to \mathbb{R}$ is a state-dependent expected-utility function, and μ is a probability distribution on S, such that the following statements hold:

(i) $A \succsim B$ if and only if $V(A) \geq V(B)$, where $V : \mathcal{A} \to \mathbb{R}$ is defined by

$$V(A) = \sum_{s\in S} \mu(s) \max_{p\in A} U_s(p). \tag{S1}$$

(ii) *Nonredundancy*. For any two distinct states $s, s' \in S$, U_s and U'_s do not represent the same vNM preference on $\Delta(Z)$.

(iii) *Minimality*. $\mu(s) > 0$ and U_s is nonconstant for all $s \in S$.

In what follows, for any $A, B \in \mathcal{A}$ and $\alpha \in [0, 1]$, the convex combination of these two menus is defined by $\alpha A + (1-\alpha)B \equiv \{\alpha p + (1-\alpha)q : p \in A \text{ and } q \in B\}$. Although our setting of finite subsets of $\Delta(Z)$ differs from that of Dekel,

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Lipman, and Rustichini (2001) and Dekel, Lipman, Rustichini, and Sarver (2007), Axioms DLR 1–5 below are exact restatements of their axioms.[1]

AXIOM DLR 1—Weak Order: *The relation* $\succsim$ *is complete and transitive*.

AXIOM DLR 2—Continuity: *The sets* $\{B \in \mathcal{A} : A \succ B\}$ *and* $\{B \in \mathcal{A} : B \succ A\}$ *are open for all* $A \in \mathcal{A}$.

AXIOM DLR 3—Independence: *If* $A \succ B$, *then for any* C *and* $\alpha \in (0, 1)$, $\alpha A + (1 - \alpha)C \succ \alpha B + (1 - \alpha)C$.

AXIOM DLR 4—Monotonicity: *If* $A \subset B$, *then* $B \succsim A$.

AXIOM DLR 5—Nontriviality: *There exists some* A *and* B *such that* $A \succ B$.

For analytical tractability, our DLR representation also imposes a finite state space, which requires additional restrictions on the preference. However, our different domain necessitates that we use a different finiteness axiom than that considered previously by Dekel, Lipman, and Rustichini (2009). In the case of monotone preferences, their axiom states that for every closed subset A of $\Delta(Z)$, there exists a finite $B \subset A$ such that $B \sim A$. Since our domain only includes finite subsets of $\Delta(Z)$, their axiom would be vacuous in our setting. Therefore, we adopt the following finiteness axiom.[2]

AXIOM DLR 6—Finiteness: *There exists* $K \in \mathbb{N}$ *such that for any* A, *there exists* $B \subset A$ *such that* $|B| \leq K$ *and* $B \sim A$.

Dekel, Lipman, and Rustichini (2001) and Dekel et al. (2007) constructed a representation for preferences over menus using a countably additive measure on the space of twice-normalized utility functions $\mathcal{U} = \{u \in \mathbb{R}^Z : \sum_{z \in Z} u_z = 0, \sum_{z \in Z} u_z^2 = 1\}$. Let $\Delta(\mathcal{U})$ denote the set of countably additive Borel probability measures on $\mathcal{U}$. Axiom DLR 6 will correspond to finiteness of the support of the representing measure, where support is defined as follows.

DEFINITION S2: The support of a measure $\mu \in \Delta^f(\mathcal{U})$ is

$$\operatorname{supp}(\mu) = \Big(\bigcup \{V \subset \mathcal{U} : V \text{ is open and } \mu(V) = 0\}\Big)^c.$$

[1]Dekel et al. (2007) established in the Supplemental Material that in the case of monotone preferences, continuity (Axiom DLR 2) could be weakened to von Neumann–Morgenstern continuity (see their Theorem S2). However, due to the difference in domain, we need to use the stronger form of continuity in our proof of Theorem S1.

[2]A related finiteness axiom was also considered by Kopylov (2009).

Since we will use finitely additive measures extensively in later results, Definition S2 defines the support on the larger space $\Delta^f(\mathcal{U})$ of all finitely additive Borel probability measures on $\mathcal{U}$. By definition, $\text{supp}(\mu)$ is a closed subset of $\mathcal{U}$, and $u \in \text{supp}(\mu)$ if and only if $\mu(V) > 0$ for every open set V containing u. Moreover, it is a standard result that if μ is a countably additive measure, then for any open set V, $\mu(V) > 0 \iff V \cap \text{supp}(\mu) \neq \emptyset$.[3]

THEOREM S1: *The relation $\succsim$ satisfies Axioms DLR 1–5 if and only if there exists $\mu \in \Delta(\mathcal{U})$ such that*

$$A \succsim B \iff \int_{\mathcal{U}} \max_{p \in A} u(p)\mu(du) \geq \int_{\mathcal{U}} \max_{p \in B} u(p)\mu(du), \tag{S2}$$

and this probability measure μ is unique. Moreover, $\succsim$ also satisfies Axiom DLR 6 if and only if the support of μ is finite.

Theorem S1 is a direct adaptation of the representation result from Dekel, Lipman, and Rustichini (2001) and Dekel et al. (2007) to our framework of finite menus of lotteries. While the result is not conceptually novel, there are some technical issues associated with this adaptation. Details and the complete proof can be found in Section S3.1. Briefly, the key steps are the following: First, show that by associating any finite menu with its convex hull, there is a well defined extension of any preference that satisfies Axioms DLR 1–5 to the space of convex polytopes in $\Delta(Z)$. Second, using continuity, extend this preference again to the space of all closed and convex subsets of $\Delta(Z)$. At this point, it is then possible to appeal to the construction in Dekel et al. (2007) to obtain the representation in Equation (S2).

The characterization of the DLR representation defined in Definition S1 now follows as a corollary to Theorem S1. To see this, suppose $\mu \in \Delta(\mathcal{U})$ is a measure with finite support satisfying Equation (S2). Construct a DLR representation (Definition S1) as follows: Let $S = \text{supp}(\mu)$, and for each $s \in S$ and $p \in \Delta(Z)$, let $U_s(p) = s(p)$. Then, abusing notation slightly to denote the restriction of μ to S also by μ, it follows that for any $A \in \mathcal{A}$,

$$\sum_{s \in S} \mu(s) \max_{p \in A} U_s(p) = \int_{\mathcal{U}} \max_{p \in A} u(p)\mu(du).$$

Therefore, (S, U, μ) is a DLR representation for $\succsim$. Since the necessity of Axioms DLR 1–6 for any DLR representation is immediate, we have established the following corollary.

COROLLARY S1: *The relation $\succsim$ satisfies Axioms DLR 1–6 if and only if it has a DLR representation (S, U, μ).*

[3]Lemma S6 establishes this property for countably additive measures and shows that we get a similar (but slightly weaker) result for the case of finitely additive measures.

S2. GP AXIOMS AND REPRESENTATION RESULT

Recall the definition of the GP representation used in AS:

DEFINITION S3: A *GP representation* of λ is a quadruple (S, U, μ, τ), where S is a finite state space, $U: S \times \Delta(Z) \to \mathbb{R}$ is a state-dependent utility function, μ is a probability distribution on S, and τ is a tie-breaking rule over S such that the following statements hold:

(i) For every $A \in \mathcal{A}$ and $p \in A$,

$$\lambda^A(p) = \sum_{s \in S} \mu(s)\tau_s\big(\{u \in \mathcal{U} : p \in M\big(M(A, U_s), u\big)\}\big).$$

(ii) *Nonredundancy*. For any two distinct states $s, s' \in S$, U_s and U'_s do not represent the same vNM preference on $\Delta(Z)$.

(iii) *Minimality*. $\mu(s) > 0$ and U_s is nonconstant for all $s \in S$.

The GP representation in Definition S3 corresponds to a special case of the tie-breaker representation from Gul and Pesendorfer (2006, Supplemental Material), where the state space is finite. Therefore, we show in this section that the GP representation is characterized by the axioms of Gul and Pesendorfer (2006) together with a technical axiom ensuring this finiteness.

In what follows, the space $\Delta(\Delta(Z))$ is endowed with the topology of weak convergence, and $\text{ext}(A)$ denotes the set of extreme points of A. Axioms GP 1–4 below are exact restatements of Gul and Pesendorfer's axioms.

AXIOM GP 1—Mixture Continuity: *$\lambda^{\alpha A+(1-\alpha)B}$ is continuous in α for all A, B.*

AXIOM GP 2—Monotonicity: *$p \in A \subset B$ implies $\lambda^B(p) \leq \lambda^A(p)$.*

AXIOM GP 3—Linearity: *If $p \in A$ and $\alpha \in (0, 1)$, then $\lambda^{\alpha A+(1-\alpha)\{q\}}(\alpha p + (1-\alpha)q) = \lambda^A(p)$.*

AXIOM GP 4—Extreme: *$\lambda^A(\text{ext}(A)) = 1$.*

For any menu $A \in \mathcal{A}$ and lottery $p \in A$, let $N(A, p)$ be the set of expected-utility functions in $\mathcal{U}$ for which p is a maximizer in A:

$$N(A, p) = \Big\{u \in \mathcal{U} : u(p) = \max_{q \in A} u(q)\Big\}.$$

Also, let $N^+(A, p)$ be the expected-utility functions for which p is the unique maximizer in A:

$$N^+(A, p) = \big\{u \in \mathcal{U} : u(p) > u(q), \forall q \in A \setminus \{p\}\big\}.$$

As above, let $\Delta^f(\mathcal{U})$ denote the set of all finitely additive Borel probability measures on the set of normalized utility functions $\mathcal{U}$. Gul and Pesendorfer (2006) defined a random utility function to be a finitely additive measure over utility functions.

DEFINITION S4: A *random utility function* (RUF) is a probability $\nu \in \Delta^f(\mathcal{U})$. A RUF is *regular* if $\nu(N^+(A, p)) = \nu(N(A, p))$ for all $A \in \mathcal{A}$ and $p \in A$.

As in the case of the tie-breaking rule (Definition 3 in AS), the regularity condition for a random utility function requires that ties occur with probability zero.

DEFINITION S5: A random choice rule λ maximizes a regular random utility function ν if $\lambda^A(p) = \nu(N(A, p))$ for all $A \in \mathcal{A}$ and $p \in A$.

THEOREM S2: *The RCR λ satisfies Axioms GP 1–4 if and only if there exists a regular RUF ν such that λ maximizes ν.*

There are two key differences between Definition S4 and the definition of an RUF in Gul and Pesendorfer (2006). The first is that we impose a different normalization on the set of utility functions. Instead of using $\mathcal{U}$, they used the space $\mathcal{U}^{\mathrm{GP}} = \{u \in \mathbb{R}^Z : u_{\bar{z}} = 0\}$ for some fixed $\bar{z}$ (they took this to be the last element in the enumeration of Z). The second difference is that instead of using the Borel σ-algebra, they endowed this space with the algebra $\mathcal{F}^{\mathrm{GP}}$ generated by the sets $N_{\mathrm{GP}}(A, p) \equiv \{u \in \mathcal{U}^{\mathrm{GP}} : u(p) = \max_{q \in A} u(q)\}$ for $A \in \mathcal{A}$ and $p \in A$.[4] In Section S3.2, we prove Theorem S2 by showing that neither of these differences is substantive and, hence, the representation theorem from Gul and Pesendorfer (2006) can be applied.

In additional supplementary material, Gul and Pesendorfer (2006) considered the case of a (possibly nonregular) RUF followed by a tie-breaker. The GP representation in Definition S3 is special case of such a representation, where the first measure puts all probability on a finite set of utility functions. Gul and Pesendorfer did not consider such a specialization explicitly; therefore, we must introduce the following new axiom.

AXIOM GP 5—Finiteness: *There exists $K \in \mathbb{N}$ such that for any A, there exists $B \subset A$ with $|B| \le K$ such that for every $p \in A \setminus B$, there are sequences $p_n \to p$ and $B_n \to B$ with $\lambda^{B_n \cup \{p_n\}}(p_n) = 0$ for all n.*

[4]In contrast with the Borel algebra, this algebra does not contain all singleton sets. In particular, this algebra does not separate a utility function $u \in \mathcal{U}^{\mathrm{GP}}$ from its scalar multiples αu for $\alpha > 0$.

Intuitively, if there are K possible utility functions that are assigned positive probability, then absent ties, at most K elements will be selected from any menu with positive probability. However, in the case of ties, more than K elements of a menu may be selected with positive probability, depending on the tie-breaking rule. The sequences in Axiom GP 5 allow the menus A and B to be perturbed so that ties occur with probability zero.[5]

The following theorem shows that Axiom GP 5 implies that the RUF ν representing λ has finite support (see Definition S2), which in turn implies there is a GP representation for λ.

THEOREM S3: *For any RCR* λ, *the following assertions are equivalent*:

(i) λ *satisfies Axioms GP* 1–5.

(ii) *There exists a regular RUF* ν *with finite support such that* λ *maximizes* ν.

(iii) λ *has a GP representation* (S, U, μ, τ).

Theorem S3 is proved in Appendix S3.3. There are some subtleties to the arguments. If ν is finitely additive, we need not have $\nu(\text{supp}(\nu)) = 1$. In fact, if ν is a regular RUF and $\text{supp}(\nu)$ is finite, it must be that $\nu(\text{supp}(\nu)) = 0$. Nonetheless, any neighborhood of $\text{supp}(\nu)$ must have probability 1 (see Lemma S6). As a result, we are able to show that the behavior corresponding to this RUF is equivalent to maximizing a random utility function that assigns positive probability only to $\text{supp}(\nu)$, followed by a tie-breaker. This allows us to construct a GP representation for λ.

S3. PROOFS

S3.1. *Proof of Theorem S1*

To prove this representation theorem, we show in the following lemmas that a preference $\succsim$ satisfying the DLR axioms on $\mathcal{A}$ induces a unique DLR preference on the space of closed and convex menus. Then we can appeal directly to the construction in Dekel, Lipman, and Rustichini (2001) and Dekel et al. (2007).

Let $\text{co}(A)$ denote the convex hull of the menu A. Dekel, Lipman, and Rustichini (2001) and Dekel et al. (2007) showed that weak order and independence imply that for any menu A, $A \sim \text{co}(A)$. Since the convex hull of any nonsingleton menu A is infinite, $\text{co}(A)$ is not a part of our domain. However, the next two lemmas establish a similar conclusion, namely, that the individual is indifferent between any two menus that have the same convex hull.

LEMMA S1: *If* $\succsim$ *satisfies weak order and independence* (*Axioms DLR* 1 *and* 3), *then for all* $A \in \mathcal{A}$, $A \sim \frac{1}{2}A + \frac{1}{2}A$.

[5]Note that the interpretation of the sequences in Axiom GP 5 is similar to the interpretation of Axiom 2 of AS. There is also a formal connection: It can be shown that if the pair $(\succsim, \lambda)$ satisfies Axiom 2 of AS and $\succsim$ satisfies Axioms DLR 1, 4, and 6, then λ satisfies Axiom GP 5.

PROOF: Fix any $A \in \mathcal{A}$. If $A \nsim \frac{1}{2}A + \frac{1}{2}A$, then independence implies that for any $B \in \mathcal{A}$ and $\alpha \in (0,1)$, $\alpha A + (1-\alpha)B \nsim \alpha(\frac{1}{2}A + \frac{1}{2}A) + (1-\alpha)B$. By weak order, this is contradicted if we find a menu B and scalar α such that

$$\alpha A + (1-\alpha)B = \alpha\left(\frac{1}{2}A + \frac{1}{2}A\right) + (1-\alpha)B.$$

Let $k = |A|$ and let

$$B = \underbrace{\frac{1}{k-1}A + \cdots + \frac{1}{k-1}A}_{k-1}.$$

Then, for $\alpha = \frac{2}{k+1}$, we have

$$\alpha A + (1-\alpha)B = \frac{2}{k+1}A + \underbrace{\frac{1}{k+1}A + \cdots + \frac{1}{k+1}A}_{k-1},$$

$$\alpha\left(\frac{1}{2}A + \frac{1}{2}A\right) + (1-\alpha)B = \underbrace{\frac{1}{k+1}A + \cdots + \frac{1}{k+1}A}_{k+1}.$$

Clearly, $\alpha A + (1-\alpha)B \subset \alpha(\frac{1}{2}A + \frac{1}{2}A) + (1-\alpha)B$. To see the converse, fix any $p \in \alpha(\frac{1}{2}A + \frac{1}{2}A) + (1-\alpha)B$. Then $p = \sum_{i=1}^{k+1} \frac{1}{k+1} p^i$ for some $p^1, \dots, p^{k+1} \in A$. However, since $|A| = k$, we must have $p^i = p^j$ for some $i \neq j$. Without loss of generality, assume the p^i's are ordered so that $p^1 = p^2$. Then $p = \frac{2}{k+1}p^1 + \sum_{i=3}^{k+1} \frac{1}{k+1} p^i \in \alpha A + (1-\alpha)B$. This establishes the desired contradiction. *Q.E.D.*

LEMMA S2: *If $\succsim$ satisfies weak order, continuity, independence, and monotonicity (Axioms DLR 1–4), then* $\text{co}(B) \subset \text{co}(A)$ *implies* $A \succsim B$. *In particular, if* $\text{co}(A) = \text{co}(B)$, *then* $A \sim B$.

PROOF: Suppose $A, B \in \mathcal{A}$ satisfy $\text{co}(B) \subset \text{co}(A)$. Define a sequence of sets inductively by $A_0 = A$ and $A_k = \frac{1}{2}A_{k-1} + \frac{1}{2}A_{k-1}$ for $k \geq 1$. By Lemma S1 and transitivity, we have $A \sim A_k$ for all k. Suppose for a contradiction that $B \succ A$. Then, by continuity, there exists $\varepsilon > 0$ such that $C \succ A$ whenever $d_h(B, C) < \varepsilon$. We will contradict this by finding a menu $C \in \mathcal{A}$ such that $C \subset A_k$ for some k and $d_h(B, C) < \varepsilon$. Then monotonicity requires that $A \sim A_k \succsim C$ and $d_h(B, C) < \varepsilon$ requires that $C \succ A$, a contradiction.

To construct the desired menu C, first note that $d_h(A_k, \text{co}(A)) \to 0$. Now fix k such that $d_h(A_k, \text{co}(A)) < \varepsilon$ and let $C = \{p \in A_k : d(p, q) < \varepsilon \text{ for some } q \in B\}$. Since $B \subset \text{co}(B) \subset \text{co}(A)$ for every $q \in B$, there exists $p \in A_k$ such that

$d(p,q) < \varepsilon$. Thus, for every $q \in B$, there exists $p \in C$ with $d(p,q) < \varepsilon$. Conversely, $p \in C$ implies by definition that $d(p,q) < \varepsilon$ for some $q \in B$. Therefore, conclude that $d_h(B,C) < \varepsilon$. *Q.E.D.*

One difficulty in working with the space $\mathcal{A}$ is that it is not a mixture space. For example, we may have $\alpha A + (1-\alpha)A \neq A$. However, making use of the previous lemma, we can define an induced preference on the space $\mathcal{P}$ of convex polytopes in $\Delta(Z)$. Note that $\mathcal{P} = \{\text{co}(A) : A \in \mathcal{A}\}$. Moreover, $\mathcal{P}$ is a mixture space. Define a binary relation $\succsim^*$ on $\mathcal{P}$ by $P \succsim^* Q$ if there exists $A, B \in \mathcal{A}$ such that $P = \text{co}(A)$, $Q = \text{co}(B)$, and $A \succsim B$. Then, by definition, $A \succsim B$ implies $\text{co}(A) \succsim^* \text{co}(B)$. Since Lemma S2 ensures that $A \sim A'$ whenever $\text{co}(A) = \text{co}(A')$, we also obtain the converse: $\text{co}(A) \succsim^* \text{co}(B)$ implies $A \succsim B$.

LEMMA S3: *If $\succsim$ satisfies Axioms DLR 1–4, then there exists an affine, monotone, and Lipschitz continuous utility representation $V : \mathcal{P} \to \mathbb{R}$ of $\succsim^*$. Moreover, V is unique up to a positive affine transformation.*

PROOF: As noted above, $\mathcal{P}$ is a mixture space, and Lemma S2 implies that $\succsim^*$ is well defined. We now verify that $\succsim^*$ satisfies the axioms of the mixture-space theorem. The weak order axiom on $\succsim$ (Axiom DLR 1) implies $\succsim^*$ is also a weak order. It is a standard result that the continuity axiom on $\succsim$ (Axiom DLR 2) implies that $\succsim$ also satisfies von Neumann–Morgenstern continuity. Suppose $P, Q, R \in \mathcal{P}$ satisfy $P = \text{co}(A) \succ^* Q = \text{co}(B) \succ^* R = \text{co}(C)$. Then $A \succ B \succ C$ and, hence, there exist $\alpha, \beta \in (0,1)$ such that $\alpha A + (1-\alpha)C \succ B \succ \beta A + (1-\beta)C$. Together with the identity $\text{co}(\alpha A + (1-\alpha)C) = \alpha\,\text{co}(A) + (1-\alpha)\,\text{co}(C)$, this implies $\succsim^*$ satisfies von Neumann–Morgenstern continuity:

$$\begin{aligned} \alpha P + (1-\alpha)R &= \text{co}\big(\alpha A + (1-\alpha)C\big) \\ &\succ^* \quad \text{co}(B) = Q \\ &\succ^* \quad \text{co}\big(\beta A + (1-\beta)C\big) = \beta P + (1-\beta)R. \end{aligned}$$

A similar argument shows that the independence axiom on $\succsim$ induces the independence axiom on $\succsim^*$. Therefore, we can apply the mixture-space theorem to conclude that there exists an affine function $V : \mathcal{P} \to \mathbb{R}$ that represents $\succsim^*$ and that this representation is unique up to a positive affine transformation. Since $\succsim^*$ is monotone by Lemma S2, so is this representation V.

It remains only to show that V is Lipschitz continuous. However, by Lemma S1 in the Supplemental Material of Dekel et al. (2007), monotonicity of $\succsim^*$ implies that it also satisfies their L-continuity axiom.[6] Therefore, their Lemma S8 can be applied to conclude that V is Lipschitz continuous.[7] *Q.E.D.*

[6]Although Dekel et al. (2007) worked with the space of all closed subsets of $\Delta(Z)$, it is easy to see that the arguments in their Lemma S1 carry through even when restricted to convex polytopes.

[7]Again, the restriction to polytopes does not affect this result.

The following result shows that V has a unique continuous extension to the set of all convex menus $\mathcal{A}^c \equiv \{A \subset \Delta(Z) : A \text{ is closed and convex}\}$, which was the domain used in the proofs in Dekel, Lipman, and Rustichini (2001) and Dekel et al. (2007).

LEMMA S4: *If $V : \mathcal{P} \to \mathbb{R}$ is Lipschitz continuous, affine, and monotone, then there exists an extension $\hat{V} : \mathcal{A}^c \to \mathbb{R}$ of V that is also Lipschitz continuous, affine, and monotone. Moreover, any continuous extension of V from $\mathcal{P}$ to $\mathcal{A}^c$ is unique.*

PROOF: By Theorem 1.8.13 in Schneider (1993), $\mathcal{P}$ is dense in $\mathcal{A}^c$. Therefore, since V is Lipschitz (and hence uniformly) continuous, Lemma 3.11 in Aliprantis and Border (2006) implies that V has a unique continuous extension $\hat{V}$ to $\mathcal{A}^c$.

To see that $\hat{V}$ satisfies the desired properties, fix any $A, B \in \mathcal{A}^c$, and take sequences $\{P_n\}, \{Q_n\} \subset \mathcal{P}$ such that $P_n \to A$ and $Q_n \to B$. Let $K > 0$ be a Lipschitz constant for V on $\mathcal{P}$. To see that $\hat{V}$ is Lipschitz continuous with the same constant K, note that $V(P_n) = \hat{V}(P_n) \to \hat{V}(A)$ and $V(Q_n) = \hat{V}(Q_n) \to \hat{V}(B)$, and hence

$$\begin{aligned}
\left|\hat{V}(A) - \hat{V}(B)\right| &= \lim_{n\to\infty} \left|V(P_n) - V(Q_n)\right| \\
&\leq \lim_{n\to\infty} K d_h(P_n, Q_n) = K d_h(A, B).
\end{aligned}$$

To see that $\hat{V}$ is affine, fix any $\alpha \in (0, 1)$. Since $\alpha P_n + (1-\alpha)Q_n \to \alpha A + (1-\alpha)B$, we have

$$\begin{aligned}
\hat{V}\big(\alpha A + (1-\alpha)B\big) &= \lim_{n\to\infty} V\big(\alpha P_n + (1-\alpha)Q_n\big) \\
&= \lim_{n\to\infty} \big[\alpha V(P_n) + (1-\alpha)V(Q_n)\big] \\
&= \alpha\hat{V}(A) + (1-\alpha)\hat{V}(B).
\end{aligned}$$

Finally, to see that $\hat{V}$ is monotone, suppose $A \subset B$. For each n, let $R_n = \text{co}(P_n \cup Q_n) \in \mathcal{P}$. Then $P_n \subset R_n$, which implies $V(P_n) \leq V(R_n)$. Moreover, it is a standard result that $R_n \to \text{co}(A \cup B) = B$. Therefore, $\hat{V}(A) = \lim_{n\to\infty} V(P_n) \leq \lim_{n\to\infty} V(R_n) = \hat{V}(B)$. *Q.E.D.*

We now proceed to proving Theorem S1.

Representation. The necessity of the axioms for the representation in Equation (S2) is straightforward. For the other direction, suppose $\succsim$ satisfies Axioms DLR 1–5. By the preceding lemmas, there exists a Lipschitz continuous, affine, and monotone function $\hat{V} : \mathcal{A}^c \to \mathbb{R}$ such that for any $A, B \in \mathcal{A}$, $A \succsim B \iff \text{co}(A) \succsim^* \text{co}(B) \iff \hat{V}(\text{co}(A)) \geq \hat{V}(\text{co}(B))$. Moreover, by the uniqueness properties in Lemma S3, we can normalize $\hat{V}$ so that the uniform distribution gets utility 0. Then, from the construction in the Supplemental Material of Dekel et al. (2007) (in particular, see Lemmas S9–S12 and the surrounding discussion), there exists a finite Borel measure μ on $\mathcal{U}$ such that for every $A \in \mathcal{A}^c$,

$$\hat{V}(A) = \int_{\mathcal{U}} \max_{p\in A} u(p)\mu(du).$$

Therefore, for any $A, B \in \mathcal{A}$,

$$\begin{aligned} A \succsim B \quad &\Longleftrightarrow \quad \int_{\mathcal{U}} \max_{p\in \mathrm{co}(A)} u(p)\mu(du) \geq \int_{\mathcal{U}} \max_{p\in \mathrm{co}(B)} u(p)\mu(du) \\ &\Longleftrightarrow \quad \int_{\mathcal{U}} \max_{p\in A} u(p)\mu(du) \geq \int_{\mathcal{U}} \max_{p\in B} u(p)\mu(du), \end{aligned}$$

where the last equivalence follows from the linearity of each $u \in \mathcal{U}$. Finally, by nontriviality (Axiom DLR 5), $\mu \neq 0$ and, therefore, μ can be normalized to be a probability measure.

Uniqueness. Suppose $\mu_1, \mu_2 \in \Delta(\mathcal{U})$ both satisfy Equation (S2). Define $V_1, V_2 : \mathcal{A}^c \to \mathbb{R}$ for $A \in \mathcal{A}^c$ by

$$V_i(A) = \int_{\mathcal{U}} \max_{p\in A} u(p)\mu_i(du).$$

By the linearity of each $u \in \mathcal{U}$, for any $A \in \mathcal{A}$,

$$V_i\big(\mathrm{co}(A)\big) = \int_{\mathcal{U}} \max_{p\in \mathrm{co}(A)} u(p)\mu_i(du) = \int_{\mathcal{U}} \max_{p\in A} u(p)\mu_i(du).$$

Since both μ_1 and μ_2 satisfy Equation (S2), this implies that for all $A, B \in \mathcal{A}$,

$$\begin{aligned} V_1\big(\mathrm{co}(A)\big) \geq V_1\big(\mathrm{co}(B)\big) \quad &\Longleftrightarrow \quad A \succsim B \\ &\Longleftrightarrow \quad V_2\big(\mathrm{co}(A)\big) \geq V_2\big(\mathrm{co}(B)\big). \end{aligned}$$

Thus, V_1 and V_2 are ordinally equivalent on $\mathcal{P} = \{\mathrm{co}(A) : A \in \mathcal{A}\}$. Therefore, by the uniqueness part of Lemma S3, there exists $\alpha > 0$ and $\beta \in \mathbb{R}$ such that $V_1(P) = \alpha V_2(P) + \beta$ for all $P \in \mathcal{P}$. By the uniqueness part of Lemma S4, it follows that $V_1(A) = \alpha V_2(A) + \beta$ for all $A \in \mathcal{A}^c$. Hence,

$$\text{(S3)} \qquad \int_{\mathcal{U}} \max_{p\in A} u(p)\mu_1(du) = \alpha \int_{\mathcal{U}} \max_{p\in A} u(p)\mu_2(du) + \beta \quad \forall A \in \mathcal{A}^c.$$

Let $p^* = (1/|Z|, \dots, 1/|Z|)$ be the uniform distribution. Then, for $A = \{p^*\}$, we have $\max_{p\in A} u(p) = 0$ for all $u \in \mathcal{U}$ by the normalization of $\mathcal{U}$.

Applying Equation (S3) to this menu A yields $\beta = 0$. Now, for $B = \{p \in \Delta(Z) : d(p, p^*) \leq 1/|Z|\}$, we have $\max_{p\in B} u(p) = 1/|Z|$ for all $u \in \mathcal{U}$ (see, e.g., Lemma 6 in Sarver (2008)). Thus, applying Equation (S3) to this menu B yields $\mu_1(\mathcal{U})/|Z| = \alpha\mu_2(\mathcal{U})/|Z|$. Since μ_1 and μ_2 are probabilities, this implies $\alpha = 1$. Therefore, for every $A \in \mathcal{A}^c$,

$$\int_{\mathcal{U}} \max_{p\in A} u(p)\mu_1(du) = \int_{\mathcal{U}} \max_{p\in A} u(p)\mu_2(du).$$

By Lemma 18 in Sarver (2008), this implies $\mu_1 = \mu_2$, as desired.

Finiteness. If $\operatorname{supp}(\mu)$ is finite, then $\succsim$ clearly satisfies Axiom DLR 6 with $K = |\operatorname{supp}(\mu)|$: For any A, simply let B contain a maximizer of each $u \in \operatorname{supp}(\mu)$ on A.

Conversely, suppose $\succsim$ is represented by μ and $\succsim$ satisfies the finiteness axiom. Fix K from this axiom. We will show that $|\operatorname{supp}(\mu)| > K$ (in particular, the support being infinite) leads to a contradiction. If $|\operatorname{supp}(\mu)| > K$, then choose $F \subset \operatorname{supp}(\mu)$ such that $|F| = K + 1$. Take $A \in \mathcal{A}$ as described in Lemma 1 of AS for this set F. Then, by part (i) of the lemma, for each $u \in F$, there exists $p \in A$ such that $u(p) > u(q)$ for all $q \in A \setminus \{p\}$. Denote this lottery by p^u. Moreover, since $F \subset \mathcal{U}$ and no two distinct $u, v \in \mathcal{U}$ represent the same expected-utility preference, part (ii) of Lemma 1 in AS implies that $p^u \neq p^v$ for any $u, v \in F$, $u \neq v$. In particular, $|\{p^u : u \in F\}| = K + 1$.

To see that these assumptions contradict Axiom DLR 6, take any $B \subset A$ with $|B| \leq K$. Then there exists $\bar{u} \in F$ such that $p^{\bar{u}} \notin B$. Thus, $\bar{u}(p^{\bar{u}}) > \bar{u}(q)$ for all $q \in B$. Consider the set

$$E \equiv \left\{u \in \mathcal{U} : \max_{p\in A} u(p) > \max_{p\in B} u(p)\right\}.$$

This set is open by the continuity of the mappings $u \mapsto \max_{p\in A} u(p)$ and $u \mapsto \max_{p\in B} u(p)$. Moreover, since $\bar{u} \in E$, we have $E \cap \operatorname{supp}(\mu) \neq \emptyset$. Therefore, $\mu(E) > 0$ by part (i) of Lemma S6 below. This implies

$$\int_{\mathcal{U}} \max_{p\in A} u(p)\mu(du) > \int_{\mathcal{U}} \max_{p\in B} u(p)\mu(du),$$

contradicting Axiom DLR 6. Therefore, conclude that $|\operatorname{supp}(\mu)| \leq K$. *Q.E.D.*

S3.2. *Proof of Theorem S2*

Gul and Pesendorfer (2006) used the following set of utility functions, where $\bar{z}$ is some fixed element of Z (they took $\bar{z}$ to be the last element in the enumeration of Z)[8]:

$$\mathcal{U}^{\text{GP}} = \{u \in \mathbb{R}^Z : u_{\bar{z}} = 0\}.$$

The sets $N_{\text{GP}}(A, p)$ and $N^+_{\text{GP}}(A, p)$ are defined as subsets of $\mathcal{U}^{\text{GP}}$ analogously to the definitions of $N(A, p)$ and $N^+(A, p)$ in $\mathcal{U}$:

$$N_{\text{GP}}(A, p) = \Big\{u \in \mathcal{U}^{\text{GP}} : u(p) = \max_{q \in A} u(q)\Big\},$$

$$N^+_{\text{GP}}(A, p) = \{u \in \mathcal{U}^{\text{GP}} : u(p) > u(q), \forall q \in A \setminus \{p\}\}.$$

Let $\mathcal{F}^{\text{GP}}$ denote the algebra generated by the sets $N_{\text{GP}}(A, p)$ for $A \in \mathcal{A}$ and $p \in A$. Gul and Pesendorfer (2006) defined random utility functions and regularity just as in Definition S4 in Section S2, but on the space $(\mathcal{U}^{\text{GP}}, \mathcal{F}^{\text{GP}})$. The following theorem is a restatement of their Theorem 2.

THEOREM S4—Gul and Pesendorfer (2006): *The RCR λ satisfies Axioms GP 1–4 if and only if there exists a finitely additive probability measure ν^{GP} on $(\mathcal{U}^{\text{GP}}, \mathcal{F}^{\text{GP}})$ such that $\lambda^A(p) = \nu^{\text{GP}}(N_{\text{GP}}(A, p)) = \nu^{\text{GP}}(N^+_{\text{GP}}(A, p))$ for all $A \in \mathcal{A}$ and $p \in A$.*[9]

To prove Theorem S2, we will show that any finitely additive probability measure ν^{GP} on $(\mathcal{U}^{\text{GP}}, \mathcal{F}^{\text{GP}})$ can be transformed into a RUF $\nu \in \Delta^f(\mathcal{U})$ that is maximized by the same random choice rule. The following lemma provides a key step in this argument.

LEMMA S5: *There exists a (unique) function $f : \mathcal{U}^{GP} \to \mathcal{U} \cup \{0\}$ such that u and $f(u)$ represent the same preference over lotteries, and for any $A \in \mathcal{A}$ and $p \in A$,*

$$f^{-1}(N(A, p)) = N_{\text{GP}}(A, p) \setminus \{0\}, \tag{S4}$$

$$f^{-1}(N^+(A, p)) = N^+_{\text{GP}}(A, p) \setminus \{0\}.$$

Moreover, letting $\mathcal{F}$ denote the algebra on $\mathcal{U}$ generated by the sets $N(A, p)$, $f^{-1}(E) \in \mathcal{F}^{\text{GP}}$ for every $E \in \mathcal{F}$.

[8]Aside from simply being normalized differently, the set $\mathcal{U}^{\text{GP}}$ also differs from $\mathcal{U}$ in a substantive way: Unlike with $\mathcal{U}$, multiple utility functions in $\mathcal{U}^{\text{GP}}$ may represent the same expected utility preference. Specifically, for any $u \in \mathcal{U}^{\text{GP}}$, we also have $\alpha u \in \mathcal{U}^{\text{GP}}$ for any $\alpha > 0$.

[9]The first equality $\lambda^A(p) = \nu^{\text{GP}}(N_{\text{GP}}(A, p))$ implies that ν^{GP} is maximized by λ, and the second equality $\nu^{\text{GP}}(N_{\text{GP}}(A, p)) = \nu^{\text{GP}}(N^+_{\text{GP}}(A, p))$ is the regularity condition applied to ν^{GP}. However, the regularity condition does not need to be included explicitly in this result since it can be shown that regularity is implied whenever ν^{GP} is maximized by a RCR λ. In fact, Gul and Pesendorfer (2006) showed that regularity is both a necessary and sufficient condition on ν^{GP} for there to exist a RCR λ that maximizes ν^{GP}.

PROOF: Representing expected-utility functions as vectors in $\mathbb{R}^Z$ and letting $\mathbf{1} = (1, \dots, 1)$ denote the unit vector, let $f(u) = 0$ if $u \in \mathbb{R}^Z$ is constant (i.e., $u = (\alpha, \dots, \alpha)$) and otherwise let

$$f(u) = \frac{u - \left(\frac{1}{|Z|}\sum_{z\in Z} u_z\right)\mathbf{1}}{\left\| u - \left(\frac{1}{|Z|}\sum_{z\in Z} u_z\right)\mathbf{1}\right\|}.$$

By construction, $f(u) \in \mathcal{U} \cup \{0\}$ for all $u \in \mathcal{U}^{\text{GP}}$ (in fact, for any $u \in \mathbb{R}^Z$), and since $f(u)$ is simply an affine transformation of u, it represents the same expected-utility preference. Therefore, it is immediate that p is optimal (strictly optimal) in A with respect to the utility function $u \in \mathcal{U}^{\text{GP}}$ if and only if p is optimal (strictly optimal) in A with respect to $f(u) \in \mathcal{U} \cup \{0\}$. Since $f(u) \in \mathcal{U}$ whenever $u \neq 0$, the equalities in Equation (S4) follow directly.

To verify the last claim, first note that $\{0\} \in \mathcal{F}^{\text{GP}}$ and hence $f^{-1}(N(A, p)) = N_{\text{GP}}(A, p) \setminus \{0\} \in \mathcal{F}^{\text{GP}}$ for all $A \in \mathcal{A}$ and $p \in A$. Since $\mathcal{F}$ is the algebra generated by the sets $N(A, p)$, standard arguments can be used to show this implies $f^{-1}(E) \in \mathcal{F}^{\text{GP}}$ for every $E \in \mathcal{F}$. *Q.E.D.*

We now proceed to proving Theorem S2. The necessity of the axioms is straightforward and directly replicates the arguments used in Gul and Pesendorfer (2006).

To show sufficiency of the axioms, suppose the RCR λ satisfies Axioms GP 1–4. By Theorem S4, there exists a finitely additive probability measure ν^{GP} on $(\mathcal{U}^{\text{GP}}, \mathcal{F}^{\text{GP}})$ such that $\lambda^A(p) = \nu^{\text{GP}}(N_{\text{GP}}(A, p)) = \nu^{\text{GP}}(N^+_{\text{GP}}(A, p))$ for all $A \in \mathcal{A}$ and $p \in A$. Note that for any $p, q \in \Delta(Z)$, $p \neq q$, we have $0 \in N_{\text{GP}}(\{p, q\}, p)$ and $0 \notin N^+_{\text{GP}}(\{p, q\}, p)$. Therefore, since $\nu^{\text{GP}}(N^+_{\text{GP}}(\{p, q\}, p)) = \nu^{\text{GP}}(N_{\text{GP}}(\{p, q\}, p))$ by regularity, it must be that $\nu^{\text{GP}}(\{0\}) = 0$.

Take f and $\mathcal{F}$ as in Lemma S5. Define a measure ν on $(\mathcal{U}, \mathcal{F})$ by $\nu(E) = \nu^{\text{GP}}(f^{-1}(E))$ for $E \in \mathcal{F}$. Finite additivity of ν follows from the finite additivity of ν^{GP}. Also, since $\nu^{\text{GP}}(\{0\}) = 0$, we have $\nu(\mathcal{U}) = \nu^{\text{GP}}(f^{-1}(\mathcal{U})) = \nu^{\text{GP}}(\mathcal{U}^{\text{GP}} \setminus \{0\}) = 1$. Hence, ν is a probability. By Equation (S4),

$$\nu\big(N(A, p)\big) = \nu^{\text{GP}}\big(N_{\text{GP}}(A, p) \setminus \{0\}\big) = \nu^{\text{GP}}\big(N_{\text{GP}}(A, p)\big),$$
$$\nu\big(N^+(A, p)\big) = \nu^{\text{GP}}\big(N^+_{\text{GP}}(A, p) \setminus \{0\}\big) = \nu^{\text{GP}}\big(N^+_{\text{GP}}(A, p)\big),$$

and therefore $\lambda^A(p) = \nu(N(A, p)) = \nu(N^+(A, p))$ for all $A \in \mathcal{A}$ and $p \in A$. Finally, since each $N(A, p)$ is a closed subset of $\mathcal{U}$, the algebra $\mathcal{F}$ is contained in the Borel σ-algebra $\mathbf{B}_{\mathcal{U}}$. By Theorem 3.4.4 in Rao and Rao (1983), there exists a finitely additive measure $\bar{\nu}$ on $(\mathcal{U}, \mathbf{B}_{\mathcal{U}})$ which is an extension of ν from $\mathcal{F}$ to $\mathbf{B}_{\mathcal{U}}$. *Q.E.D.*

S3.3. *Proof of Theorem S3*

S3.3.1. *A Preliminary Result*

The following lemma gives some useful properties of the support of a finitely additive measure (see Definition S2). It was used implicitly in the construction of the DLR representation in Corollary S1 and will be used in several parts of the proof of Theorem S3.

LEMMA S6: *Fix any finitely additive measure $\mu \in \Delta^f(\mathcal{U})$. The following statements hold*:

(i) *For any open set V, $V \cap \text{supp}(\mu) \neq \emptyset \Longrightarrow \mu(V) > 0$.*

(ii) *For any compact set C, $C \cap \text{supp}(\mu) = \emptyset \Longrightarrow \mu(C) = 0$.*

(iii) *If μ is countably additive, then $\mu(\text{supp}(\mu)) = 1$. In particular, for any measurable set $V \in \mathbf{B}_{\mathcal{U}}$, $V \cap \text{supp}(\mu) = \emptyset \Longrightarrow \mu(V) = 0$.*

PROOF: Part (i). If V is open and $\mu(V) = 0$, then $V \in \{V' \subset \mathcal{U} : V' \text{ is open and } \mu(V') = 0\}$, so $V \subset \text{supp}(\mu)^c$. By contrapositive, $V \cap \text{supp}(\mu) \neq \emptyset \Longrightarrow \mu(V) > 0$.

Part (ii). Suppose C is compact and $C \cap \text{supp}(\mu) = \emptyset$. Then $C \subset \text{supp}(\mu)^c = \bigcup\{V \subset \mathcal{U} : V \text{ is open and } \mu(V) = 0\}$. Therefore, there exists a finite subcover $\{V_1, \dots, V_n\}$ of C, and by finite subadditivity, $\mu(C) \leq \mu(\bigcup_{i=1}^n V_i) \leq \sum_{i=1}^n \mu(V_i) = 0$.

Part (iii). Since $\mathcal{U}$ is a separable metric space, it is second countable and hence $\text{supp}(\mu)^c = \bigcup\{V \subset \mathcal{U} : V \text{ is open and } \mu(V) = 0\}$ can be expressed as a countable union of sets of measure zero. If μ is countably additive, this implies $\mu(\text{supp}(\mu)^c) = 0$. Now suppose $V \in \mathbf{B}_{\mathcal{U}}$ and $V \cap \text{supp}(\mu) = \emptyset$. Then $V \subset \text{supp}(\mu)^c$, and hence $\mu(V) \leq \mu(\text{supp}(\mu)^c) = 0$. *Q.E.D.*

S3.3.2. *Proof of (i) ⇒ (ii)*

Suppose λ satisfies Axioms GP 1–5. By Theorem S2, since λ satisfies Axioms GP 1–4, there exists a regular RUF ν such that λ maximizes ν. Now take K as in Axiom GP 5. We will show that $|\text{supp}(\nu)| > K$ (in particular, the support being infinite) leads to a contradiction. If $|\text{supp}(\nu)| > K$, then choose $F \subset \text{supp}(\nu)$ such that $|F| = K + 1$. Take $A \in \mathcal{A}$ as described in Lemma 1 of AS for this set F. Then, by part (i) of the lemma, for each $u \in F$, there exists $p \in A$ such that $u(p) > u(q)$ for all $q \in A \setminus \{p\}$. Denote this lottery by p^u. Moreover, since $F \subset \mathcal{U}$ and no two distinct $u, v \in \mathcal{U}$ represent the same expected-utility preference, part (ii) of Lemma 1 of AS implies that $p^u \neq p^v$ for any $u, v \in F$, $u \neq v$. In particular, $|\{p^u : u \in F\}| = K + 1$.

To see that these assumptions contradict Axiom GP 5, take any $B \subset A$ with $|B| \leq K$. Then there exists $u \in F$ such that $p^u \notin B$. Thus, $u(p^u) > u(q)$ for all $q \in B$. Fix any sequences $p_n \to p^u$ and $B_n \to B$. By continuity, there exists $N \in \mathbb{N}$ such that for $n \geq N$, $u(p_n) > u(q)$ for all $q \in B_n$. Thus, $u \in N^+(B_n \cup \{p_n\}, p_n)$

for all $n \geq N$. Since $u \in \text{supp}(\nu)$ and $N^+(B_n \cup \{p_n\}, p_n)$ is an open set, part (i) of Lemma S6 implies $\nu(N^+(B_n \cup \{p_n\}, p_n)) > 0$ for $n \geq N$. Hence, for all $n \geq N$,

$$\lambda^{B_n \cup \{p_n\}}(p_n) = \nu\big(N\big(B_n \cup \{p_n\}, p_n\big)\big) = \nu\big(N^+\big(B_n \cup \{p_n\}, p_n\big)\big) > 0,$$

contradicting Axiom GP 5. Therefore, conclude that $|\text{supp}(\nu)| \leq K$.

S3.3.3. *Proof of (ii) $\Rightarrow$ (iii)*

We begin by considering the case of a RUF ν with singleton support, $\text{supp}(\nu) = \{u\}$. In this case, the following lemma shows that the only lotteries that are selected with positive probability by this RUF are those that maximize u.

LEMMA S7: *Suppose $\nu \in \Delta^f(\mathcal{U})$ and* $\text{supp}(\nu) = \{u\}$ *for some $u \in \mathcal{U}$. Then, for any $A \in \mathcal{A}$ and $p \in A$, $\nu(N(A, p)) = \nu(N(M(A, u), p))$.*[10]

PROOF: Fix any $A \in \mathcal{A}$ and $p \in A$. First, note that for any $q \in A$,

$$q \notin M(A, u) \implies u \notin N(A, q) \implies \nu\big(N(A, q)\big) = 0, \tag{S5}$$

where the last implication follows from part (ii) of Lemma S6. Therefore, if $p \notin M(A, u)$, then $\nu(N(A, p)) = 0 = \nu(N(M(A, u), p))$. Consider the remaining case of $p \in M(A, u)$. Then

$$N(A, p) \subset N\big(M(A, u), p\big) \subset \left(N(A, p) \cup \bigcup_{q \in A \setminus M(A,u)} N(A, q)\right)$$

implies that

$$\begin{aligned} \nu\big(N(A, p)\big) &\leq \nu\big(N\big(M(A, u), p\big)\big) \\ &\leq \nu\big(N(A, p)\big) + \sum_{q \in A \setminus M(A,u)} \nu\big(N(A, q)\big) = \nu\big(N(A, p)\big), \end{aligned}$$

where the last equality follows from Equation (S5). *Q.E.D.*

The following lemma uses the preceding result to construct a GP representation for any regular RUF ν with finite support. Recall that $\mathbf{B}_{\mathcal{U}}$ denotes the Borel σ-algebra on the set $\mathcal{U}$. For any $u \in \mathcal{U}$ and $\varepsilon > 0$, denote the open ball of radius ε around u by $B_\varepsilon(u) \equiv \{v \in \mathcal{U} : \|u - v\| < \varepsilon\}$.

[10]It may be that $p \notin M(A, u)$ even if $p \in A$, in which case $N(M(A, u), p)$ is not defined. Therefore, we adopt the convention that $N(A, p) = \emptyset$ if $p \notin A$. As a result, we have $N(M(A, u), p) = \{v \in \mathcal{U} : p \in M(M(A, u), v)\}$ for all A and p.

LEMMA S8: *Suppose* $\nu \in \Delta^f(\mathcal{U})$ *is a regular RUF with finite support. Fix any* $\varepsilon > 0$ *such that* $B_\varepsilon(u) \cap B_\varepsilon(v) = \emptyset$ *for all* $u, v \in \operatorname{supp}(\nu)$, $u \neq v$. *Let* $S = \operatorname{supp}(\nu)$.

(i) *Taking* $\mu = \sum_{s \in S} \nu(B_\varepsilon(s))\delta_s$ *defines a probability measure on* S, *and* $\mu(s) > 0$ *for all* $s \in S$.

(ii) *Taking* $\tau_s(V) = \frac{\nu(V \cap B_\varepsilon(s))}{\nu(B_\varepsilon(s))}$ *for* $V \in \mathbf{B}_\mathcal{U}$ *defines a tie-breaking rule* $\tau_s \in \Delta^f(\mathcal{U})$. *Moreover, for these measures, for any* $A \in \mathcal{A}$ *and* $p \in A$,

$$\nu\big(N(A, p)\big) = \sum_{s \in S} \mu(s)\tau_s\big(\{u \in \mathcal{U} : p \in M\big(M(A, s), u\big)\}\big). \tag{S6}$$

PROOF: Part (i). Since $C = \mathcal{U} \setminus (\bigcup_{s \in S} B_\varepsilon(s))$ is compact and $C \cap \operatorname{supp}(\nu) = \emptyset$, part (ii) of Lemma S6 implies $\nu(C) = 0$. Therefore,

$$\sum_{s \in S} \nu\big(B_\varepsilon(s)\big) = \nu\left(\bigcup_{s \in S} B_\varepsilon(s)\right) = 1, \tag{S7}$$

implying that μ is a probability measure. Also, for any $s \in S$, $\mu(s) = \nu(B_\varepsilon(s)) > 0$ by part (i) of Lemma S6.

Part (ii). First note that for any $s \in S$, τ_s is well defined since $\nu(B_\varepsilon(s)) > 0$. Therefore, by construction, $\tau_s \in \Delta^f(\mathcal{U})$ since $\nu \in \Delta^f(\mathcal{U})$. To see that τ_s satisfies the regularity condition, fix any $A \in \mathcal{A}$ and $p \in A$. Since ν is regular, $\nu(N^+(A, p)) = \nu(N(A, p))$. Together with the fact that $N^+(A, p) \subset N(A, p)$, this implies that $\tau_s(N^+(A, p)) = \tau_s(N(A, p))$.

Equation (S6). Note that $\operatorname{supp}(\tau_s) = \{s\}$ for each $s \in S$. Therefore, by Lemma S7, $\tau_s(N(A, p)) = \tau_s(N(M(A, s), p))$ for any $A \in \mathcal{A}$ and $p \in A$. As a result,

$$\begin{aligned}
&\sum_{s \in S} \mu(s)\tau_s\big(\{u \in \mathcal{U} : p \in M\big(M(A, s), u\big)\}\big) \\
&\quad = \sum_{s \in S} \mu(s)\tau_s\big(N\big(M(A, s), p\big)\big) \\
&\quad = \sum_{s \in S} \mu(s)\tau_s\big(N(A, p)\big) = \sum_{s \in S} \nu\big(N(A, p) \cap B_\varepsilon(s)\big) \\
&\quad = \nu\left(N(A, p) \cap \left(\bigcup_{s \in S} B_\varepsilon(s)\right)\right) = \nu\big(N(A, p)\big),
\end{aligned}$$

where the last equality follows from Equation (S7). *Q.E.D.*

To complete the proof of (ii) $\Rightarrow$ (iii), suppose there exists a regular RUF ν with finite support such that λ maximizes ν. Take S, μ, and τ as in Lemma S8. For each $s \in S$, define $U_s : \Delta(Z) \to \mathbb{R}$ by $U_s(p) = s(p)$ for $p \in \Delta(Z)$. We claim that (S, U, μ, τ) is a GP representation for λ. By Equation (S6), for any $A \in \mathcal{A}$ and $p \in A$,

$$\lambda^A(p) = \nu\big(N(A, p)\big) = \sum_{s\in S} \mu(s)\tau_s\big(\{u \in \mathcal{U} : p \in M\big(M(A, U_s), u\big)\}\big).$$

The other conditions in the definition of the GP representation are readily verified.

S3.3.4. *Proof of (iii) ⇒ (i)*

Suppose the λ has a GP representation (S, U, μ, τ). The necessity of Axioms GP 1–4 is straightforward and replicates the arguments used in the proof of Theorem S1 in the Supplemental Material of Gul and Pesendorfer (2006). We now show that λ satisfies Axiom GP 5. Let $K = |S|$. Fix any $A \in \mathcal{A}$. For each $s \in S$, choose $q^s \in A$ such that $U_s(q^s) = \max_{q\in A} U_s(q)$ and let $B = \{q^s : s \in S\}$. Then $|B| \leq K$ and for any $p \in A \setminus B$, $U_s(p) \leq \max_{q\in B} U_s(q)$. By the same arguments used in the proof of Proposition 2 in AS, for every $\varepsilon > 0$, there exist r and C with $d(p, r) < \varepsilon$ and $d_h(B, C) < \varepsilon$ such that $\lambda^{C\cup\{r\}}(r) = 0$. Intuitively, if $U_s(p) \leq \max_{q\in B} U_s(q)$ for all $s \in S$, then p and B can be perturbed slightly to get nearby r and C for which the inequality is strict for all s, so that r will never be chosen from C. Consequently, for every $n \in \mathbb{N}$, there exists p_n and B_n with $d(p, p_n) < 1/n$ and $d_h(B, B_n) < 1/n$ such that $\lambda^{B_n\cup\{p_n\}}(p_n) = 0$. Thus, $B_n \to B$ and $p_n \to p$, and hence λ satisfies Axiom GP 5.

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