

SUPPLEMENT TO “FEAR OF MISCOORDINATION  
AND THE ROBUSTNESS OF COOPERATION IN  
DYNAMIC GLOBAL GAMES WITH EXIT”  
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This appendix provides additional results on equilibria of exit game  $\Gamma_\sigma$  as the noise parameter  $\sigma$  becomes small. Section S.1 shows that all Markovian equilibria will be approximately symmetric for  $\sigma$  small. Section S.2 studies the structure of time-dependent Markovian equilibria. Finally, Section S.3 provides sufficient conditions under which game  $\Gamma_\sigma$  is asymptotically dominance solvable.

S.1. ASYMPTOTICALLY SYMMETRIC PLAY IN MARKOVIAN EQUILIBRIA

THEOREM 1 ESTABLISHES that for noise parameter  $\sigma$  small enough, the set of rationalizable strategies of  $\Gamma_\sigma$  is bounded by a most cooperative and a least cooperative equilibria, that are Markovian and take a threshold form.

The following proposition shows that for  $\sigma$  small enough, *all* Markovian equilibria take a threshold form and are asymptotically symmetric.

PROPOSITION S.1: *There exists  $\bar{\sigma} > 0$  such that for all  $\sigma \in (0, \bar{\sigma})$ , every Markovian equilibrium  $s$  of  $\Gamma_\sigma$  takes a threshold form with thresholds  $(x_{i,\sigma}, x_{-i,\sigma})$ . Furthermore, we have that  $|x_{i,\sigma} - x_{-i,\sigma}| < 2\sigma$ .*

PROOF: The fact that for  $\sigma$  small enough, all Markovian equilibria take a threshold form is a direct consequence of Lemma A.1, applied to the class of one-shot games augmented with the continuation values associated to Markovian equilibria.

We now show that thresholds  $(x_{i,\sigma}, x_{-i,\sigma})$  satisfy  $|x_{i,\sigma} - x_{-i,\sigma}| < 2\sigma$ . This follows from the fact that given an equilibrium threshold  $x_{-i,\sigma}$ , then player  $i$  knows that if  $x_{i,t} < x_{-i,\sigma} - \sigma$  player  $-i$  will play  $E$ , so that player  $i$ 's best reply is to play  $E$  as well. Similarly, if  $x_{i,t} > x_{-i,\sigma} + \sigma$ , then player  $i$  knows that player  $-i$  will play  $S$  and her best reply is to play  $S$  as well. *Q.E.D.*

This result highlights that, asymptotically, the likelihood of actual miscoordination is vanishing. Note that since the players' payoffs may be asymmetric, the continuation values associated with approximately symmetric Markovian equilibria may be quite different.

In games where payoffs are symmetric, and error terms  $\varepsilon_i$  and  $\varepsilon_{-i}$  have identical distributions, Proposition S.1 can be strengthened to show that all Markovian equilibria are symmetric.

PROPOSITION S.2: *Whenever payoff functions are symmetric, and error terms  $\varepsilon_{i,t}$  and  $\varepsilon_{-i,t}$  have identical distributions, there exists  $\bar{\sigma} > 0$  such that for all  $\sigma \in$*

$(0, \bar{\sigma})$ , every Markovian equilibrium takes a threshold form with thresholds  $x_{i,\sigma} = x_{-i,\sigma}$ .

PROOF: Consider  $\sigma$  small enough that all Markovian equilibria take a threshold form. Consider such a Markovian equilibrium with thresholds  $(x_{i,\sigma}, x_{-i,\sigma})$ , associated with values  $(V_{i,\sigma}, V_{-i,\sigma})$ . The proof proceeds by contradiction. Assume, for instance, that  $x_{i,\sigma} > x_{-i,\sigma}$ . Because of Assumption 5—that staying benefits one’s partner—it follows that  $V_{i,\sigma} > V_{-i,\sigma}$ . Given that player  $i$  is indifferent between staying and exiting at signal  $x_{i,t} = x_{i,\sigma}$  and that  $V_{-i,\sigma} < V_{i,\sigma}$ , player  $-i$  must strictly prefer exiting to staying when observing signal  $x_{-i,t} = x_{i,\sigma}$ . This contradicts  $x_{i,\sigma} > x_{-i,\sigma}$  and implies that  $x_{i,\sigma} = x_{-i,\sigma}$ . *Q.E.D.*

## S.2. TIME-DEPENDENT MARKOVIAN EQUILIBRIA

Section 4 used a dynamic programming approach à la Abreu, Pearce, and Stacchetti (1990) to characterize Markovian equilibria of  $\Gamma_\sigma$ . The same approach can be used to characterize time-dependent Markovian equilibria.

DEFINITION S.1: A strategy  $s_i$  is time-dependent Markovian if and only if  $s_i(h_{i,t})$  depends only on time  $t$  and player  $i$ ’s current signal  $x_{i,t}$ .

For  $\sigma$  small enough and for any pair of values  $\mathbf{V} \in \prod_{i \in \{1,2\}} [m_i, M_i]$ , we consider the mappings  $\mathbf{x}_\sigma^*(\mathbf{V})$ , and  $\phi_\sigma(\mathbf{V})$  defined in Appendix A.2. Recall that  $\mathbf{x}_\sigma^*(\mathbf{V})$  is the unique equilibrium threshold of the augmented one-shot global game  $\Psi_\sigma(\mathbf{V})$  and that  $\phi_\sigma(\mathbf{V})$  is the value of playing  $\Psi_\sigma(\mathbf{V})$  according to its unique equilibrium threshold.

A profile of time-dependent Markovian strategies  $\mathbf{s} = (s_i, s_{-i})$  is associated with the sequence of values  $(\mathbf{V}_t)_{t \in \mathbb{N}} = (V_{i,t}, V_{-i,t})_{t \in \mathbb{N}}$ , where  $\mathbf{V}_t$  is the pair of values associated with playing according to strategies  $\mathbf{s}$  in the subgame starting at date  $t$ . A sequence of values  $(\mathbf{V}_t)_{t \in \mathbb{N}}$  is supported by a time-dependent Markovian equilibrium of  $\Gamma_\sigma$  if and only if the sequence  $(\mathbf{V}_t)_{t \in \mathbb{N}}$  is bounded and satisfies the recurrence equation  $\mathbf{V}_t = \phi_\sigma(\mathbf{V}_{t+1})$  for all  $t \in \mathbb{N}$ . Furthermore, such a sequence of continuation values is sustained by a unique perfect Bayesian equilibrium such that players choose to stay in period  $t$  according to threshold  $\mathbf{x}_\sigma^*(\mathbf{V}_t)$ . The proof of these results is straightforward, and essentially identical to that of Theorem 2.

To say more about time-dependent Markovian equilibria, the rest of the section focuses on symmetric games and symmetric equilibria. Mapping  $\Phi$  can be reduced to a mapping from  $\mathbb{R}$  to  $\mathbb{R}$ . Denote by  $\mathcal{US}(\Phi)$  the set of unstable fixed points of mapping  $\Phi$  and denote by  $\mathcal{S}(\Phi)$  the set of stable fixed points of mapping  $\Phi$ . The analysis assumes that all the fixed points of  $\Phi$  are nonsingular.

Consider  $s$  a symmetric time-dependent Markovian equilibrium and  $(V_t)_{t \in \mathbb{N}}$  the associated sequence of values. Since  $V_t$  must be bounded, we can always extract a converging sequence converging to some value  $V_{\sigma, \infty}$ .

**PROPOSITION S.3:** *Pick any  $\eta > 0$ . There exists  $\bar{\sigma} > 0$  such that for all  $\sigma \in (0, \bar{\sigma})$ , the following conditions hold:*

(i) *If  $V_{\sigma, \infty} \in \mathbb{R} \setminus \bigcup_{V \in \mathcal{US}(\Phi)} [V - \eta, V + \eta]$ , then there exists  $V^* \in \mathcal{S}(\Phi)$  such that for all  $t \in \mathbb{N}$ ,  $V_t \in [V^* - \eta, V^* + \eta]$ .*

(ii) *If  $V_{\sigma, \infty} \in \mathbb{R} \setminus \bigcup_{V \in \mathcal{S}(\Phi)} [V - \eta, V + \eta]$ , then there exists  $V^* \in \mathcal{US}(\Phi)$  and  $T > 0$  such that for all  $t \geq T$ ,  $V_t \in [V^* - \eta, V^* + \eta]$ .*

**PROOF:** The fixed points of  $\Phi$  belong to some compact interval  $[m, M]$ . Since by assumption every fixed point of  $\Phi$  is nonsingular, this means that there are only finitely many of them.

Furthermore, since  $\Phi$  is increasing and all its fixed points are nonsingular, then for every  $\zeta > 0$ , there exist  $k \in \mathbb{N}$  and  $\nu \in (0, \zeta)$  such that the following statements hold:

- For all  $V \in [m, M] \setminus \bigcup_{V \in \mathcal{US}(\Phi)} [V - \zeta, V + \zeta]$ ,  $\Phi^k(V) \in \bigcup_{V \in \mathcal{S}(\Phi)} [V - \zeta + \nu, V + \zeta - \nu]$ .

- For all  $V^* \in \mathcal{S}(\Phi)$ ,  $\Phi([V^* - \zeta, V^* + \zeta]) \subset [V^* - \zeta + \nu, V^* + \zeta - \nu]$ .

Since  $\phi_\sigma$  converges uniformly to  $\Phi$  as  $\sigma$  goes to 0, there exists  $\bar{\sigma} > 0$  such that for all  $\sigma \in (0, \bar{\sigma})$ , the following conditions hold:

(a) For all  $V \in [m, M] \setminus \bigcup_{V \in \mathcal{US}(\Phi)} [V - \zeta, V + \zeta]$ ,  $\phi_\sigma^k(V) \in \bigcup_{V \in \mathcal{S}(\Phi)} [V - \zeta, V + \zeta]$ .

(b) For all  $V^* \in \mathcal{S}(\Phi)$ ,  $\phi_\sigma([V^* - \zeta, V^* + \zeta]) \subset [V^* - \zeta, V^* + \zeta]$ .

This implies Proposition S.3(i). Indeed, pick  $\eta > 0$  and apply (a) and (b) above with  $\zeta < \eta$ . Since  $V_{\sigma, \infty} \in [m, M] \setminus \bigcup_{V \in \mathcal{US}(\Phi)} [V - \eta, V + \eta]$ , there are infinitely many times  $t \in \mathbb{N}$  such that  $V_t \in [m, M] \setminus \bigcup_{V \in \mathcal{US}(\Phi)} [V - \zeta, V + \zeta]$ . By point (a) above, this implies that there exists  $V^* \in \mathcal{S}(\Phi)$  such that there are infinitely many times  $t$  at which  $V_t \in [V^* - \zeta, V^* + \zeta]$ . By point (b) it follows that in every earlier period, and hence in every period  $s$ ,  $V_s \in [V^* - \zeta, V^* + \zeta] \subset [V^* - \eta, V^* + \eta]$ . This proves point (i).

We now move to point (ii). Pick the same  $\bar{\sigma}$  as in the proof above. The fact that  $V_{\sigma, \infty} \in \mathbb{R} \setminus \bigcup_{V \in \mathcal{S}(\Phi)} [V - \eta, V + \eta]$  implies that there exists  $T_1 > 0$  large enough such that for all  $t > T_1$ ,  $V_t \in \bigcup_{V \in \mathcal{US}(\Phi)} [V - \eta, V + \eta]$ . Otherwise we would be in case (i), which implies that  $V_{\sigma, \infty}$  should be within a small neighborhood of  $\mathcal{S}(\Phi)$ . Furthermore, since  $\Phi$  is increasing, it is not possible to transition from a unstable fixed point of  $\Phi$  to another unstable fixed point of  $\Phi$  without being in the neighborhood of a stable fixed point of  $\Phi$ . Hence this means that there exist  $T_2$  and  $V^* \in \mathcal{US}(\Phi)$  such that for all  $t \geq T_2$ ,  $V_t \in [V^* - \eta, V^* + \eta]$ . Q.E.D.

Furthermore, note that if we are in case (ii) of Proposition S.3, then for  $\eta$  small, the continuation equilibrium after time  $T$  is in an arbitrarily small

neighborhood of a Markovian equilibrium that is asymptotically unstable in the sense developed in Section 4.4. This follows from the fact that unstable fixed points of  $\phi$  are associated to unstable fixed points of  $\xi$ .

Altogether this means that a time-dependent Markovian equilibrium is either very close to an asymptotically stable Markovian equilibrium or is arbitrarily close to an asymptotically unstable Markovian equilibrium sufficiently far away in the future.

### S.3. SUFFICIENT CONDITIONS FOR UNIQUENESS

Theorem 2 implies that whenever the mapping  $\Phi$  has a unique fixed point, then the set of rationalizable strategies of  $\Gamma_\sigma$  converges to a singleton as  $\sigma$  goes to 0. The following proposition provides sufficient conditions under which mapping  $\Phi$  has a unique fixed point.

**PROPOSITION S.4—Uniqueness:** *Pick  $K$  a compact of  $\mathbb{R}^2$ . There exists a constant  $\eta > 0$ , depending only on payoff functions and  $K$ , such that whenever (i) players have individually rational values for playing game  $\Gamma_0$  that belong to  $K$  and (ii) the distribution of states of the world  $f$  satisfies  $\max f < \eta$ , mapping  $\Phi$  admits a unique fixed point and the set of rationalizable strategies of  $\Gamma_\sigma$  converges to a singleton as  $\sigma$  goes to 0.*

**PROOF:** Let  $\|\cdot\|_1$  denote the norm on  $\mathbb{R}^2$  defined by  $\|\mathbf{V}\|_1 = |V_i| + |V_{-i}|$  and let  $\|\cdot\|_\infty$  denote the sup norm. It results from Theorem 2(iii) that

$$\begin{aligned} & \|\Phi(\mathbf{V}) - \Phi(\mathbf{V}')\|_1 \\ & \leq \beta \|\mathbf{V} - \mathbf{V}'\|_1 \\ & \quad + \|f\|_\infty \sum_{i \in \{1,2\}} \|g_{11}^i + \beta V_i - W_{22}^i\|_\infty \left\| \frac{\partial x^{\text{RD}}}{\partial V_i} + \frac{\partial x^{\text{RD}}}{\partial V_{-i}} \right\|_\infty \|\mathbf{V} - \mathbf{V}'\|_1. \end{aligned}$$

Since

$$\sum_{i \in \{1,2\}} \|g_{11}^i + \beta V_i - W_{22}^i\|_\infty \left\| \frac{\partial x^{\text{RD}}}{\partial V_i} + \frac{\partial x^{\text{RD}}}{\partial V_{-i}} \right\|_\infty$$

is finite, for any  $\delta \in (\beta, 1)$ , there exists  $\|f\|_\infty$  small enough such that  $\|\Phi(\mathbf{V}) - \Phi(\mathbf{V}')\|_1 \leq \delta \|\mathbf{V} - \mathbf{V}'\|_1$ . Hence  $\Phi$  is a contraction mapping, which concludes the proof. *Q.E.D.*

Intuitively, Proposition S.4 implies that when the state of the world  $w_i$  has sufficient variance, then game  $\Gamma_\sigma$  is asymptotically dominance solvable. Indeed, when the density of distribution  $f$  becomes arbitrarily small, a given change in cooperation levels induces an arbitrarily small change in continuation values,

which is not enough to make the original change in cooperation levels self-sustaining.

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