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SUPPLEMENT TO "UNCONDITIONAL QUANTILE REGRESSIONS": ESTIMATION AND TESTING

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This supplement provides the detailed derivations of the asymptotic properties of the estimators proposed to compute the parameter defined as unconditional quantile partial effects (UQPE). This parameter is the ratio between an average derivative and the marginal density of the dependent variable at a given point. Because of the nonparametric nature of the density estimator, the convergence rate of final estimators will be slower than the usual root N. However, we provide a higher order asymptotic expansion that considers the parametric and faster components. Finally, we show that to test a zero null hypothesis, we can employ a test statistic that converges at the parametric rate.

KEYWORDS: Influence functions, unconditional quantile, RIF regressions, quantile regressions.

S1. INTRODUCTION

IN OUR PAPER "Unconditional Quantile Regressions" (Firpo, Fortin, and Lemieux (2009)),[1] we proposed a simple method to estimate the effect of changes in the distribution of explanatory variables on unconditional quantiles of the distribution of the outcome variable. In its simplest version, the method consists of running a regression of the recentered influence function of quantiles on the explanatory variables, the so-called RIF regression. The RIF regression enables us to easily recover the marginal effect of changes in the covariates distribution on the unconditional quantiles of a dependent variable.

Consider a specific change in the distribution of a covariate: a location shift holding everything else constant. In FFL, the marginal effect of such a change on the quantile of the unconditional distribution of the dependent variable was called the unconditional τ quantile partial effect (UQPE(τ)),

$$\mathrm{UQPE}(\tau) = \frac{1}{f_Y(q_\tau)} \cdot E\left[d\frac{\Pr[Y > q_\tau|X]}{dX}\right],$$

where q_τ is the population unconditional τ quantile of Y, the dependent variable, $f_Y(\cdot)$ is the density function of the marginal distribution of Y, and $E[d\Pr[Y > q_\tau|X]/dX]$ is the average derivative of the probability that Y is greater than q_τ given X, the vector of covariates.

Three estimators were proposed in FFL. The first one is the RIF-OLS, which consists of the regression coefficients of the recentered influence function of

[1] Henceforth FFL.

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the unconditional quantile of Y on X. The influence function of the unconditional quantile is defined as

$$\text{IF}(Y; q_\tau) = \frac{\mathbb{1}\{Y > q_\tau\} - (1 - \tau)}{f_Y(q_\tau)}$$

and its recentered version adds back the quantile: $\text{RIF}(Y; q_\tau) = \text{IF}(Y; q_\tau) + q_\tau$. Thus, the RIF-OLS is the vector of ordinary least squares (OLS) regression coefficients of a feasible version of $\text{RIF}(Y; q_\tau)$ on X. The expression for feasible RIF is

$$\widehat{\text{RIF}}(Y; \widehat{q}_\tau) = \widehat{q}_\tau + \frac{\mathbb{1}\{Y > \widehat{q}_\tau\} - (1 - \tau)}{\widehat{f}_Y(\widehat{q}_\tau)},$$

where is $\widehat{q}_\tau$ is the sample τ quantile of unconditional distribution of Y and $\widehat{f}_Y(\cdot)$ is Rosenblatt's kernel density estimator.

The second estimator directly computes the ratio between the sample average marginal effect

$$\frac{1}{N} \sum_{i=1}^{N} d \frac{\widehat{\Pr}[Y > \widehat{q}_\tau | X_i]}{dX}$$

and $\widehat{f}_Y(\widehat{q}_\tau)$, the kernel density estimator evaluated at $\widehat{q}_\tau$. This estimator was called RIF-Logit because $\widehat{\Pr}[Y > \widehat{q}_\tau | X_i] = \Lambda(X_i^\intercal \widehat{\theta}_\tau)$, where Λ is the logistic cumulative distribution function and $\widehat{\theta}_\tau$ is the maximum likelihood estimate of a logistic regression of $\mathbb{1}\{Y > \widehat{q}_\tau\}$ on X.

The last estimator proposed by FFL was called RIF-NP (where NP stands for nonparametric), which is also a ratio between a sample average marginal effect and $\widehat{f}_Y(\widehat{q}_\tau)$. The key difference, however, is that $E[d \Pr[Y > q_\tau | X]/dX]$ is nonparametrically estimated. More specifically, FFL used polynomial series estimation for $\Pr[Y > \widehat{q}_\tau | X]$ and its derivative with respect to X.

We provide detailed derivations of the asymptotic properties of these three estimators proposed in FFL. Because of the nonparametric nature of the density estimator, the convergence rate of the final estimators will be slower than the usual root N. However, we provide a higher order asymptotic expansion that includes such components that converge at the faster parametric rate. We also show that to test a zero null hypothesis, we can employ a test statistic that converges at the parametric rate. We derive the asymptotic distribution for all three estimators and establish conditions under which they will be asymptotically unbiased. Finally, we show how to consistently estimate their asymptotic variances.

S2. BASIC SETUP

S2.1. *Sampling*

We assume throughout that we observe a random sample of Y in the presence of covariates X, and that Y and X have a joint distribution.

ASSUMPTION 1: *We observe a random sample of size N from the joint distribution of Y and X, which is denoted $F_{Y,X}(\cdot,\cdot): \mathbb{R} \times \mathcal{X} \to [0,1]$, where $\mathcal{X} \subset \mathbb{R}^k$ is the support of X.*

S2.2. *Density*

The three estimators for UQPE(τ) use $\widehat{f}_Y(q)$, Rosenblatt's kernel density estimator for the density of Y at q, $f_Y(q_\tau)$:

$$\widehat{f}_Y(q) = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{h}\cdot\mathcal{K}\left(\frac{Y_i - q}{h}\right).$$

We invoke a general assumption regarding the kernel function $\mathcal{K}(\cdot)$.

ASSUMPTION 2—Kernel Function (I): $\mathcal{K}(\cdot)$ *is symmetric real-valued function around zero satisfying* (a) $\int \mathcal{K}(v)\cdot dv = 1$, (b) $\int v\cdot\mathcal{K}(v)\cdot dv = S_1^{\mathcal{K}} < \infty$, (c) $\int v^2\cdot\mathcal{K}(v)\cdot dv = S_2^{\mathcal{K}} < \infty$, (d) $\int \mathcal{K}^2(v)\cdot dv = M_2^{\mathcal{K}} < \infty$, (e) $\int |\mathcal{K}(v)|\cdot dv < \infty$, (f) $\lim_{v\to\infty} |\mathcal{K}(v)|\cdot|v| = 0$, *and* (g) $\sup_v |\mathcal{K}(v)| < \infty$.

We also impose some restrictions on the marginal density of Y, $f_Y(\cdot)$, and its second-order derivative:

ASSUMPTION 3—Density Function (I): (a) $f_Y(\cdot)$ *is positive and has second-order derivative that is bounded and continuous in a neighborhood of a grid of selected points* $q_\tau \in \mathbb{R}$. (b) $\int |f(y)|\cdot dy < \infty$.

Finally we impose a rate restriction on the bandwidth $h = h(N)$.

ASSUMPTION 4—Bandwidth (I): $\lim_{N\to\infty} h(N) = 0$ *and* $\lim_{N\to\infty} N\cdot h(N) = \infty$.

Below we get expressions for the bias and the variance of $\widehat{f}_Y(q)$. Before that, let us define the first and second derivatives of $f_Y(q)$ by $f'_Y(q)$ and $f''_Y(q)$, respectively, and denote

$$\psi_{f,i}(q,\mathcal{K},h) = \frac{1}{h}\cdot\mathcal{K}\left(\frac{Y_i - q}{h}\right) - E[\widehat{f}_Y(q_\tau)].$$

THEOREM 1: *Under Assumptions* 1–4, *the bias of* $\widehat{f}_Y(q)$ *will be*

$$\mathcal{B}_f(q,\mathcal{K},h)=E[\widehat{f}_Y(q)]-f_Y(q)=\frac{1}{2}\cdot h^2\cdot f_Y''(q)\cdot S_2^{\mathcal{K}}+O(h^4),$$

whereas the variance of $\widehat{f}_Y(q)$ *will be*

$$V[\widehat{f}_Y(q)]=\frac{1}{Nh}f_Y(q_\tau)\cdot M_2^{\mathcal{K}}+\frac{1}{N}\cdot(f_Y'(q_\tau)\cdot S_1^{\mathcal{K}}-f_Y(q_\tau))+o(N^{-1}).$$

Therefore,

$$\begin{aligned}\widehat{f}_Y(q)-f_Y(q)&=\frac{1}{N}\sum_{i=1}^{N}\psi_{f,i}(q,\mathcal{K},h)+\mathcal{B}_f(q,\mathcal{K},h)\\&=O_p\big((Nh)^{-1/2}\big)+O(h^2)+R_f(q,h),\end{aligned}$$

where

$$R_f(q,h)=O_p((Nh)^{-1})+O(h^4)+O_p\big(N^{-1/2}h^{3/2}\big).$$

S2.3. *Sample Quantile*

As q_τ, the population quantile of the marginal distribution of Y, is an unknown parameter, we have to estimate it. The sample quantile $\widehat{q}_\tau$ is obtained by minimizing the sum of check functions (Koenker and Bassett (1978)),

$$\widehat{q}_\tau=\arg\min_q\sum_{i=1}^{N}\rho_\tau(Y_i-q),$$

where the check function $\rho_\tau(\cdot)$ evaluated at v is $\rho_\tau(v)=v\cdot(\tau-\mathbb{1}\{v\le 0\})$.

We define

$$\psi_{Q,i}(q)=\frac{\mathbb{1}\{Y_i>q\}-(1-\tau)}{f_Y(q)}$$

and obtain an asymptotic linear representation of the sample quantile.

THEOREM 2—Sample Quantile: *Under Assumptions* 1 *and* 3,

$$\widehat{q}_\tau-q_\tau=\frac{1}{N}\sum_{i=1}^{N}\psi_{Q,i}(q_\tau)+O_p(N^{-1}).$$

S3. RIF-OLS

The first estimator corresponds to

$$\widehat{\mathrm{UQPE}}(\tau)_{\mathrm{OLS}} = \frac{\widehat{\gamma}(\widehat{q}_\tau)}{\widehat{f}_Y(\widehat{q}_\tau)},$$

where for any q in the support of Y, we have[2]

$$\widehat{\gamma}(q) = \widehat{\Omega}_X^{-1} \cdot \frac{1}{N} \sum_{i=1}^{N} \big\{ X_i \cdot (\mathbb{1}\{Y_i > q\} - (1-\tau)) \big\}$$

whose population counterpart is

$$\gamma(q) = \Omega_X^{-1} \cdot E\big[X \cdot (\mathbb{1}\{Y > q\} - (1-\tau))\big]$$

and where

$$\widehat{\Omega}_X = \frac{1}{N} \sum_{i=1}^{N} X_i \cdot X_i^{\intercal} \quad \text{and} \quad \Omega_X = E[X \cdot X^{\intercal}].$$

We also define

$$\widehat{\beta}(q) = \frac{\widehat{\gamma}(q)}{\widehat{f}_Y(q)} \quad \text{and} \quad \beta(q) = \frac{\gamma(q)}{f_Y(q)}.$$

Now, we consider each term of the following sum separately:

$$\widehat{\mathrm{UQPE}}(\tau)_{\mathrm{OLS}} - \mathrm{UQPE}(\tau)$$

$$= \widehat{\beta}(\widehat{q}_\tau) - \widehat{\beta}(q_\tau) \tag{S.1}$$

$$+ \widehat{\beta}(q_\tau) - \beta(q_\tau) \tag{S.2}$$

$$+ \beta(q_\tau) - \mathrm{UQPE}. \tag{S.3}$$

Consider the term (S.1), $\widehat{\beta}(\widehat{q}_\tau) - \widehat{\beta}(q_\tau)$. We now define two quantities. The first one, $\widehat{\gamma}'^{*}(\cdot)$, is an approximate derivative of $\widehat{\gamma}$ with respect to q:

$$\widehat{\gamma}'^{*}(q) = -\widehat{\Omega}_X^{-1} \frac{1}{N} \sum_{i}^{N} X_i \cdot f_{Y|X}(q|X_i).$$

[2]Note that we have used the influence function and not its recentered version as the dependent variable in the regression. In fact, all regression coefficients are the same in both cases, the only exception being the intercept.

The second term, $\widehat{f'}_Y(\cdot)$, is the derivative of $\widehat{f}_Y$ with respect to q, or the Bhattacharya density derivative estimator,

$$\widehat{f'}_Y(q) = -\frac{1}{Nh^2}\sum_i^N \mathcal{K}'\left(\frac{Y_i - q}{h}\right),$$

where $\mathcal{K}'(v) = \mathrm{d}\mathcal{K}(v)/\mathrm{d}v$. We also define

$$\widehat{\beta}'^*(q) = \frac{\widehat{\gamma}'^*(q)}{\widehat{f}_Y(q)} - \frac{\widehat{\gamma}(q)}{\widehat{f}_Y(q)} \cdot \frac{\widehat{f}'_Y(q)}{\widehat{f}_Y(q)}.$$

An expression for the first term $\widehat{\beta}(\widehat{q}_\tau) - \widehat{\beta}(q_\tau)$ is derived in the following lemma.

LEMMA 1: *Suppose the assumptions of Theorems* 1 *and* 2 *hold*. *Then*

$$\widehat{\beta}(\widehat{q}_\tau) - \widehat{\beta}(q_\tau) = \widehat{\beta}'^*(q_\tau) \cdot (\widehat{q}_\tau - q_\tau) + O_p(N^{-1}).$$

It is now useful to define the following quantities. The population derivative of γ is

$$\gamma'(q) = -\Omega_X^{-1} \cdot E\left[X \cdot f_{Y|X}(q|X)\right]$$

and the population derivative of β is

$$\beta'(q) = \frac{\gamma'(q)}{f_Y(q)} - \frac{\gamma(q)}{f_Y(q)} \cdot \frac{f'_Y(q)}{f_Y(q)}.$$

Thus

$$\widehat{\beta}(\widehat{q}_\tau) - \widehat{\beta}(q_\tau) = \beta'(q_\tau) \cdot (\widehat{q}_\tau - q_\tau) + O_p(N^{-1}) + \mathrm{rem}_{\mathrm{OLS},1}$$

where $\mathrm{rem}_{\mathrm{OLS},1} = (\widehat{\beta}'^*(q_\tau) - \beta'(q_\tau)) \cdot (\widehat{q}_\tau - q_\tau)$. We will investigate its rate of convergence later.

First we turn our attention to term (S.2), $\widehat{\beta}(q_\tau) - \beta(q_\tau)$. For a fixed value q, we have

$$\widehat{\beta}(q) - \beta(q) = \frac{\widehat{\gamma}(q) - \gamma(q)}{f_Y(q)} - \frac{\gamma(q)}{f_Y(q)} \cdot \left(\frac{\widehat{f}_Y(q) - f_Y(q)}{f_Y(q)}\right) + \mathrm{rem}_{\mathrm{OLS},2}(q),$$

where

$$O_p(\mathrm{rem}_{\mathrm{OLS},2}(q)) = O_p\big((\widehat{f}_Y(q) - f_Y(q))^2\big) + O_p\big((\widehat{f}_Y(q) - f_Y(q)) \cdot \|\widehat{\gamma}(q) - \gamma(q)\|\big).$$

We define for all $i = 1, \dots, N$,

$$u_i(q) = \mathbb{1}\{Y_i > q\} - (1-\tau) - X_i^\intercal \gamma(q)$$

such that $E[X \cdot u(q)] = 0$ and

$$\psi_{\gamma,i}(q) = \Omega_X^{-1} \cdot X_i \cdot u_i(q).$$

ASSUMPTION 5—RIF-OLS: (i) $\frac{1}{N}\sum_{i=1}^N X_i X_i^\intercal \xrightarrow{P} E[XX^\intercal]$, *which is invertible*. (ii) *For each* $q \in \mathbb{R}$, $\frac{1}{\sqrt{N}}\sum_{i=1}^N X_i u_i(q) \xrightarrow{D} N(0, V_{Xu}(q))$, *where* $V_{Xu}(q) = E[u^2(q)XX^\intercal]$ *has bounded matrix norm*.

We now state a convergence order result regarding $\widehat{\beta}(q_\tau) - \beta(q_\tau)$.

LEMMA 2: *Suppose the assumptions of Lemma* 1 *and Assumption* 5 *hold. Then*

$$\begin{aligned}
&\widehat{\beta}(q_\tau) - \beta(q_\tau) \\
&\quad = \frac{1}{N}\sum_{i=1}^N \frac{\psi_{\gamma,i}(q_\tau)}{f_Y(q_\tau)} - \frac{\beta(q_\tau)}{f_Y(q_\tau)} \cdot \psi_{f,i}(q_\tau, \mathcal{K}, h) \\
&\quad\quad - \frac{\beta(q_\tau)}{f_Y(q_\tau)} \cdot \mathcal{B}_f(q, \mathcal{K}, h) + O_p\big((Nh)^{-1}\big) + O(h^4) + O_p\big(N^{-1/2}h^{3/2}\big).
\end{aligned}$$

Note that $\psi_{\gamma,i}(q_\tau)/f_Y(q_\tau)$ is the contribution to estimation of $\beta(q_\tau)$ given by estimation of the projection coefficient $\gamma(q_\tau)$ by $\widehat{\gamma}(q_\tau)$, and that $(\beta(q_\tau)/f_Y(q_\tau)) \cdot \psi_{f,i}(q_\tau, \mathcal{K}, h)$ is the contribution to estimation of $\beta(q_\tau)$ given by estimation of the density at q_τ by $\widehat{f}_Y(q_\tau)$. Finally, we go back to $\text{rem}_{\text{OLS},1}$.

ASSUMPTION 6—Kernel Function (II): *The kernel function* $\mathcal{K}(\cdot)$ *has first derivative satisfying* (a) $\int \mathcal{K}'(v) \cdot dv = 0$, (b) $\int v \cdot \mathcal{K}'(v) \cdot dv = -1$, (c) $\int v^2 \cdot \mathcal{K}'(v) \cdot dv = 0$, *and* (d) $\int v^3 \mathcal{K}'(v) \cdot dv = D_3^{\mathcal{K}} < \infty$.

ASSUMPTION 7—Density Function (II): $f_Y(\cdot)$ *has second-order derivative that is bounded and continuous in a neighborhood of a grid of selected points* $q_\tau \in \mathbb{R}$.

LEMMA 3: *Suppose the assumptions of Lemma* 2 *and Assumptions* 6 *and* 7 *hold. Then*

$$\begin{aligned}
O_p(\text{rem}_{\text{OLS},1}) &= O_p\big(\|\widehat{\beta}'^{*}(q_\tau) - \beta'(q_\tau)\| \cdot |\widehat{q}_\tau - q_\tau|\big) \\
&= O_p\big(N^{-1/2}h^2\big) + O_p\big(N^{-1}h^{-3/2}\big) + O_p\big(N^{-3/2}h^{-3}\big).
\end{aligned}$$

A combination of previous results yields

$$
\begin{aligned}
&\widehat{\beta}(\widehat{q}_\tau) - \beta(q_\tau) \\
&\quad = \frac{1}{N}\sum_{i=1}^{N} \frac{\psi_{\gamma,i}(q_\tau)}{f_Y(q_\tau)} + \beta'(q_\tau)\cdot\psi_{Q,i}(q_\tau) - \frac{\beta(q_\tau)}{f_Y(q_\tau)}\cdot\psi_{f,i}(q_\tau,\mathcal{K},h) \\
&\qquad - \frac{\beta(q_\tau)}{f_Y(q_\tau)}\cdot\mathcal{B}_f(q,\mathcal{K},h) + O(h^4) + O_p\bigl(N^{-1/2}h^{3/2}\bigr) \\
&\qquad + O_p\bigl(N^{-1}h^{-3/2}\bigr) + O_p\bigl(N^{-3/2}h^{-3}\bigr).
\end{aligned}
$$

COROLLARY 1: *Suppose the assumptions of Lemma* 3 *hold*. *Then*

$$
\begin{aligned}
&\sqrt{Nh}\cdot\bigl(\widehat{\mathrm{UQPE}}(\tau)_{\mathrm{OLS}} - \beta(q_\tau)\bigr) \\
&\quad = \frac{1}{\sqrt{Nh}}\sum_{i}^{N}\psi_{\mathrm{OLS},i}(q_\tau,h,\mathcal{K}) \\
&\qquad - \sqrt{Nh}\cdot\frac{\beta(q_\tau)}{f_Y(q_\tau)}\cdot\mathcal{B}_f(q,\mathcal{K},h) + O\bigl(N^{1/2}h^{9/2}\bigr) \\
&\qquad + O_p(h^2) + O_p\bigl(N^{-1/2}h^{-1}\bigr) + O_p\bigl(N^{-1}h^{-5/2}\bigr),
\end{aligned}
$$

where

$$
\begin{aligned}
&\psi_{\mathrm{OLS},i}(q_\tau,h,\mathcal{K}) \\
&\quad = h\cdot\left(\frac{\psi_{\gamma,i}(q_\tau)}{f_Y(q_\tau)} + \beta'(q_\tau)\cdot\psi_{Q,i}(q_\tau) - \frac{\beta(q_\tau)}{f_Y(q_\tau)}\cdot\psi_{f,i}(q_\tau,\mathcal{K},h)\right).
\end{aligned}
$$

S3.1. *Estimators' Contributions*

Now, we rewrite $\psi_{\mathrm{OLS},i}(q_\tau,h,\mathcal{K})$ as

$$
\psi_{\mathrm{OLS},i}(q_\tau,h,\mathcal{K}) = \frac{h}{f_Y(q_\tau)}\cdot\psi_{\gamma,i}(q_\tau) \tag{S.4}
$$

$$
+ h\cdot\beta'(q)\cdot\psi_{Q,i}(q_\tau) \tag{S.5}
$$

$$
- h\cdot\frac{\beta(q_\tau)}{f_Y(q_\tau)}\cdot\psi_{f,i}(q_\tau,h,\mathcal{K}). \tag{S.6}
$$

The first term in the previous equation (S.4) is the asymptotic contribution of the uncertainty about the projection coefficient $\gamma(\tau)$. If the projection coefficient were known, that term would be zero. The second term (S.5) is the contribution when estimation of the τ quantile q_τ is taken into account. Finally, the third term (S.6) is the contribution of the density. Again, if there was no density estimation involved, such term would be null.

S3.2. *Asymptotic Normality*

We now get a limiting distribution for a suitable normalized version of the RIF-OLS estimator. For that, we establish the usual condition that makes the kernel density estimator have an asymptotic squared bias that goes faster to zero than the variance.

ASSUMPTION 8—Bandwidth (II): $\lim_{N\to\infty}(Nh^5)^{1/2} = 0$.

THEOREM 3: *Suppose the assumptions of Corollary* 1 *and Assumption* 8 *hold. Then*

$$\sqrt{Nh}\cdot(\widehat{\text{UQPE}}(\tau)_{\text{OLS}} - \beta(q_\tau)) \xrightarrow{D} N(0, V_{\text{OLS}}(q_\tau, \mathcal{K})),$$

where

$$V_{\text{OLS}}(q_\tau, \mathcal{K}) = \lim_{h\downarrow 0} h^{-1}\cdot E[\psi_{\text{OLS}}(q_\tau, h, \mathcal{K})\cdot \psi_{\text{OLS}}(q_\tau, h, \mathcal{K})^\intercal].$$

S3.3. *The Bias* $\beta(q_\tau) - \text{UQPE}(\tau)$

Finally, we consider the bias term given by

$$\beta(q_\tau) - \text{UQPE}(\tau) = \frac{\Omega_X^{-1}\cdot E[X\cdot(\tau - F_{Y|X}(q_\tau|X))] - E\left[d\dfrac{\Pr[Y > q_\tau|X]}{dX}\right]}{f_Y(q_\tau)}.$$

The condition for that asymptotic bias term to be zero is given below.

ASSUMPTION 9: $E[u(q_\tau)|X] = 0$.

LEMMA 4: *If Assumption* 9 *holds, then* $\beta(q_\tau) = \text{UQPE}(\tau)$.

Finally, under the conditions that make $\beta(q_\tau)$ equal to $\text{UQPE}(\tau)$ we can extend the result of Theorem 3.

THEOREM 4: *Suppose the assumptions of Theorem* 3 *and Assumption* 9 *hold. Then*

$$\sqrt{Nh}\cdot(\widehat{\text{UQPE}}(\tau)_{\text{OLS}} - \text{UQPE}(\tau)) \xrightarrow{D} N(0, V_{\text{OLS}}(q_\tau, \mathcal{K})).$$

S3.4. *Consistent Estimation of the Asymptotic Variance*

Now, we consider a simpler expression for the asymptotic $V_{\mathrm{OLS}}(q_\tau, \mathcal{K})$. Note first that

$$
\begin{aligned}
&E\big[(\widehat{f}_Y(q_\tau))^2\big] \\
&\quad = \frac{1}{f_Y^2(q_\tau)} \beta(q_\tau)\beta^\intercal(q_\tau) \cdot f_Y(q_\tau) \cdot \int \mathcal{K}^2(v) \cdot dv \\
&\qquad + h \cdot \frac{1}{f_Y^2(q_\tau)} \beta(q_\tau)\beta^\intercal(q_\tau) \cdot \left(f_Y'(q_\tau) \int v\mathcal{K}^2(v)\, dv - f_Y(q_\tau) \right).
\end{aligned}
$$

PROPOSITION 1: *The asymptotic variance can be expressed as*

$$
\begin{aligned}
V_{\mathrm{OLS}}(q_\tau, \mathcal{K}) = \lim_{h \downarrow 0} \bigg\{ & \frac{1}{f_Y^2(q_\tau)} \beta(q_\tau)\beta^\intercal(q_\tau) E\big[(\widehat{f}_Y(q_\tau))^2\big] \\
& + \frac{1}{f_Y^2(q_\tau)} \operatorname{Var}\big[h^{1/2} \cdot \big(\Omega_X^{-1} X u(q_\tau) \\
& + \beta'(q_\tau) \cdot (\mathbb{1}\{Y > q_\tau\} - (1-\tau))\big)\big] \bigg\}.
\end{aligned}
$$

A simple estimator for the asymptotic variance of $V_{\mathrm{OLS}}(q_\tau, \mathcal{K})$ is $\widehat{V}_{\mathrm{OLS}}(\widehat{q}_\tau, h, \mathcal{K})$:

$$
\begin{aligned}
&\widehat{V}_{\mathrm{OLS}}(\widehat{q}_\tau, h, \mathcal{K}) \\
&\quad = \frac{1}{\widehat{f}_Y^2(\widehat{q}_\tau)} \widehat{\beta}(\widehat{q}_\tau)\widehat{\beta}^\intercal(\widehat{q}_\tau) \cdot \widehat{f}_Y(\widehat{q}_\tau) \cdot \int \mathcal{K}^2(v) \cdot dv \\
&\qquad + h \cdot \frac{1}{\widehat{f}_Y^2(\widehat{q}_\tau)} \widehat{\beta}(\widehat{q}_\tau)\widehat{\beta}^\intercal(\widehat{q}_\tau) \cdot \left(\widehat{f'}_Y(\widehat{q}_\tau) \int v\mathcal{K}^2(v)\, dv - \widehat{f}_Y(\widehat{q}_\tau) \right) \\
&\qquad + h \cdot \frac{1}{\widehat{f}_Y^2(\widehat{q}_\tau)} \frac{1}{N} \sum_{i=1}^{N} \big(\widehat{\Omega}_X^{-1} X_i \widehat{u}_i(\widehat{q}_\tau) \\
&\qquad + \widehat{\beta}'(\widehat{q}_\tau) \cdot (\mathbb{1}\{Y_i > \widehat{q}_\tau\} - (1-\tau)) \big) \\
&\qquad \cdot \big(\widehat{\Omega}_X^{-1} X_i \widehat{u}_i(\widehat{q}_\tau) + \widehat{\beta}'(\widehat{q}_\tau) \cdot (\mathbb{1}\{Y_i > \widehat{q}_\tau\} - (1-\tau)) \big)^\intercal,
\end{aligned}
$$

where

$$
\widehat{\beta}'(q) = -\frac{\widehat{\Omega}_X^{-1}}{\widehat{f}_Y(q)} \cdot \big(\widehat{E}\big[X \cdot f_{Y|X}(q|X)\big] - \widehat{\beta}(q) \cdot \widehat{f'}_Y(q) \big),
$$

$$\widehat{E}\big[X \cdot f_{Y|X}(q|X)\big] = \frac{1}{Nh}\sum_{i}^{N} X_i \cdot \mathcal{K}\left(\frac{Y_i - q}{h}\right),$$

and

$$\widehat{u}_i(q) = \mathbb{1}\{Y_i > q\} - (1-\tau) - X_i^{\intercal}\widehat{\gamma}(q).$$

PROPOSITION 2: *Under the assumptions of Theorem* 4, $\text{plim}_{h\downarrow 0}\widehat{V}_{\text{OLS}}(\widehat{q}_\tau, h, \mathcal{K}) = V_{\text{OLS}}(q_\tau, \mathcal{K})$.

S3.5. *Testing the Zero Null Hypothesis*

Along this section, we assume that the requirements for Theorem 4 to hold are met. To test the null hypotheses of

$$H_0 : R \cdot \text{UQPE}(\tau) = \alpha$$

for $\alpha \neq 0$, one can proceed trivially by constructing a test statistic

$$Nh \cdot (\widehat{\text{UQPE}}(\tau) - \alpha)^{\intercal} R^{\intercal}(R \cdot \widehat{V}_{\text{OLS}}(q_\tau, \mathcal{K}) \cdot R^{\intercal})^{-} \times R(\widehat{\text{UQPE}}(\tau) - \alpha) \xrightarrow{D} \chi^2_r,$$

where $r = \text{rank}(R) = \#\,\text{row}(R)$.

Things are different if we are testing $\alpha = 0$. In this case, we can come up with a test statistic that converges at the usual parametric rate. Note first that $R \cdot \text{UQPE}(\tau) = 0 \Leftrightarrow R \cdot E[d((\Pr[Y > q_\tau|X])/dX)] = 0$. Remember that under Assumption 9, $E[d((\Pr[Y > q_\tau|X])/dX)] = \gamma(q_\tau)$, which is estimated by $\widehat{\gamma}(\widehat{q}_\tau)$. Therefore, we have that

$$\begin{aligned}
&\widehat{\gamma}(\widehat{q}_\tau) - E\left[d\frac{\Pr[Y > q_\tau|X]}{dX}\right] \\
&\quad = \frac{1}{N}\sum_{i=1}^{N} \psi_{\gamma,i}(q_\tau) + \gamma'(q_\tau) \cdot \psi_{Q,i}(q_\tau) + O_p(N^{-1}) \\
&\quad = \Omega_X^{-1} \cdot \Bigg(\frac{1}{N}\sum_{i=1}^{N}\big(X_i \cdot u_i(q_\tau) \\
&\qquad - E[X|Y = q_\tau] \cdot (\mathbb{1}\{Y_i > q_\tau\} - (1-\tau))\big)\Bigg) + O_p(N^{-1})
\end{aligned}$$

as

$$E[X|Y = q_\tau] = \frac{E[X \cdot f_{Y|X}(q_\tau|X)]}{f_Y(q_\tau)}.$$

We now invoke a central limit theorem that yields

$$\sqrt{N} \cdot (\widehat{\gamma}(\widehat{q}_\tau) - \gamma(q_\tau)) \xrightarrow{D} N(0, V_\gamma(q_\tau)),$$

where

$$\begin{aligned} V_\gamma(q_\tau) &= \Omega_X^{-1} \cdot \Xi(q_\tau) \cdot \Omega_X^{-1}, \\ \Xi(q_\tau) &= E[u^2(q_\tau)XX^\intercal] + \tau \cdot (1-\tau) \cdot E[X|Y=q_\tau]E[X|Y=q_\tau]^\intercal \\ &\quad - E[u^2(q_\tau)X] \cdot E[X|Y=q_\tau]^\intercal - E[X|Y=q_\tau]E[X^\intercal u^2(q_\tau)]. \end{aligned}$$

A consistent estimator for the variance is

$$\widehat{V}_\gamma = \widehat{\Omega}_X^{-1}\widehat{\Xi}(\widehat{q}_\tau)\widehat{\Omega}_X^{-1},$$

where

$$\begin{aligned} \widehat{\Xi}(\widehat{q}_\tau) &= \frac{1}{N}\sum_{i=1}^{N} \widehat{u}_i^2(\widehat{q}_\tau)X_iX_i^\intercal + \tau \cdot (1-\tau) \\ &\quad \cdot \left(\frac{\widehat{E}[X \cdot f_{Y|X}(\widehat{q}_\tau|X)]}{\widehat{f}_Y(\widehat{q}_\tau)}\right) \cdot \left(\frac{\widehat{E}[X^\intercal \cdot f_{Y|X}(\widehat{q}_\tau|X)]}{\widehat{f}_Y(\widehat{q}_\tau)}\right) \\ &\quad - \left(\frac{1}{N}\sum_{i=1}^{N} \widehat{u}_i^2(\widehat{q}_\tau)X_i\right) \cdot \left(\frac{\widehat{E}[X^\intercal \cdot f_{Y|X}(\widehat{q}_\tau|X)]}{\widehat{f}_Y(\widehat{q}_\tau)}\right) \\ &\quad - \left(\frac{\widehat{E}[X \cdot f_{Y|X}(\widehat{q}_\tau|X)]}{\widehat{f}_Y(\widehat{q}_\tau)}\right) \cdot \left(\frac{1}{N}\sum_{i=1}^{N} \widehat{u}_i^2(\widehat{q}_\tau)X_i^\intercal\right). \end{aligned}$$

Therefore, if we want to construct a test, for example, for $\mathrm{UQPE}(\tau)_j = 0$, all we need is a t-statistic for γ_j. In general, we can construct a test statistic which will converge in law to a chi-square under H_0:

$$N \cdot \widehat{\gamma}(\widehat{q}_\tau)^\intercal R^\intercal (R\widehat{V}_\gamma R^\intercal)^- R\widehat{\gamma}(\widehat{q}_\tau) \xrightarrow{D} \chi_r^2.$$

The t-statistic for γ_j will be the particular square root of the left-hand side with $R = [0, \dots, 0, 1, 0, \dots, 0]$, where 1 is at the jth entry.

S4. RIF-LOGIT

We estimate UQPE(τ) as[3]

$$\widehat{\text{UQPE}}_{\text{logit}}(\tau) = \widehat{\theta}(\widehat{q}_\tau) \cdot \frac{1}{N} \cdot \sum_{i=1}^{N} \frac{\Lambda'(X_i^\intercal \widehat{\theta}(\widehat{q}_\tau))}{\widehat{f}_Y(\widehat{q}_\tau)} = \frac{\widehat{\delta}(\widehat{q}_\tau)}{\widehat{f}_Y(\widehat{q}_\tau)},$$

where

$$\widehat{\delta}(q) = \widehat{\theta}(q) \cdot \frac{1}{N} \cdot \sum_{i=1}^{N} \Lambda'(X_i^\intercal \widehat{\theta}(q)),$$

$$\widehat{\theta}(q) = \arg\max_{\theta} \sum_{i}^{N} \mathbb{1}\{Y_i > q\} \cdot X_i^\intercal \theta + \log(1 - \Lambda(X_i^\intercal \theta)),$$

and

$$\delta(q) = \theta(q) \cdot E\big[\Lambda'(X^\intercal \theta(q))\big],$$

$$\theta(q) = \arg\max_{\theta} E\big[\mathbb{1}\{Y > q\} \cdot X^\intercal \theta + \log(1 - \Lambda(X^\intercal \theta))\big].$$

Now we define

$$\widehat{\varsigma}(q) = \frac{\widehat{\delta}(q)}{\widehat{f}_Y(q)} \quad \text{and} \quad \varsigma(q) = \frac{\delta(q)}{f_Y(q)}.$$

S4.1. *Decomposition*

We decompose the difference $\widehat{\varsigma}(\widehat{q}_\tau) - \varsigma(q_\tau)$ into three terms, exactly as we did for RIF-OLS. The assumptions required for the linearization are similar to the ones previously used. The key difference is basically that now the average derivative $\delta(q)$ is not a regression coefficient. We omit derivation details as there are no new insights from this case relative to RIF-OLS.

We proceed by analogy with the previous estimator (RIF-OLS) and obtain

$$\text{(S.7)} \qquad \widehat{\varsigma}(\widehat{q}_\tau) - \varsigma(q_\tau) = \frac{1}{Nh} \sum_{i=1}^{N} \psi_{\text{logit},i}(q_\tau, h, \mathcal{K}) + r_{\text{logit}}(q_\tau, h, \mathcal{K}),$$

[3]It will be useful to remember that by the properties of the logistic distribution we have that $\Lambda(z) = (1 + \exp(-z))^{-1}$, $\Lambda'(z) = \Lambda(z) \cdot (1 - \Lambda(z))$, $\Lambda''(z) = \Lambda'(z) \cdot (1 - 2 \cdot \Lambda(z))$, and $\Lambda'''(z) = \Lambda''(z) \cdot (1 - 2 \cdot \Lambda(z)) - 2 \cdot (\Lambda'(z))^2$.

where

$$\begin{aligned}
\text{(S.8)}\qquad & \psi_{\mathrm{logit},i}(q_\tau, h, \mathcal{K}) \\
&\quad = h \cdot \left(\frac{\psi_{\delta,i}(q_\tau)}{f_Y(q_\tau)} + \varsigma'(q_\tau) \cdot \psi_{Q,i}(q_\tau) - \frac{\varsigma(q_\tau)}{f_Y(q_\tau)} \cdot \psi_{f,i}(q_\tau, h, \mathcal{K}) \right), \\
& \varsigma'(q) = \frac{\delta'(q)}{f_Y(q)} - \frac{\delta(q)}{f_Y(q)} \cdot \frac{f'_Y(q)}{f_Y(q)}, \\
& \delta'(q) = E\big[\Lambda'(X^\intercal \theta(q))\big] \cdot \theta'(q) + \theta(q) \cdot E\big[X^\intercal \theta'(q) \Lambda''(X^\intercal \theta(q))\big] \\
&\qquad = G(q) \cdot \theta'(q),
\end{aligned}$$

where $G(q)$ is the implicit derivative of δ with respect to θ,

$$G(q) = 2E\big[\Lambda(X^\intercal \theta(q)) \cdot \big(1 - \Lambda(X^\intercal \theta(q))\big)^2 \cdot \theta(q) \cdot X^\intercal\big],$$

and, finally, by the implicit function theorem,

$$\theta'(q) = -\big(E\big[XX^\intercal \Lambda'(X^\intercal \theta(q))\big]\big)^{-1} \cdot E\big[X \cdot f_{Y|X}(q)\big].$$

The first term of the sum in equation (S.8) is proportional to $\psi_{\delta,i}(q_\tau)$ and refers to the contribution of uncertainty regarding $\delta(q_\tau)$, the average derivative. The influence function $\psi_{\delta,i}(q_\tau)$ is decomposed into two terms, because one can note that

$$\begin{aligned}
\frac{1}{N}\sum_{i=1}^{N} \psi_{\delta,i}(q_\tau) &= G(q_\tau) \cdot \big(\widehat{\theta}(q_\tau) - \theta(q_\tau)\big) \\
&\qquad + \theta(q_\tau) \cdot \frac{1}{N}\sum_{i=1}^{N} \psi_{\Lambda',i}(q_\tau) + O_p(N^{-1}),
\end{aligned}$$

where

$$\begin{aligned}
& \widehat{\theta}(q_\tau) - \theta(q_\tau) = \frac{1}{N}\sum_{i=1}^{N} \psi_{\theta,i}(q_\tau) + O_p(N^{-1}), \\
& \psi_{\theta,i}(q) = \big(E\big[XX^\intercal \Lambda'(X^\intercal \theta(q))\big]\big)^{-1} X_i \cdot v(q), \\
& v(q) = \mathbb{1}\{Y > q\} - \Lambda(X^\intercal \theta(q)),
\end{aligned}$$

and

$$\psi_{\Lambda',i}(q) = \Lambda'(X_i^\intercal \theta(q)) - E\big[\Lambda'(X^\intercal \theta(q))\big].$$

That is, $\psi_\delta(q_\tau)$ is further decomposed into the part regarding θ and the average of $\Lambda'(X^\intercal \theta(q_\tau))$.

The second term of equation (S.8) is proportional to $\psi_{Q,i}(q_\tau)$ and has to do with the fact that we are estimating the unconditional quantile q_τ. The last term refers to the density estimation. Finally, the remainder term $r_{\text{logit}}(q_\tau, h, \mathcal{K})$ converges at the order established by the following theorem.

ASSUMPTION 10—RIF-Logit: *For each* $q \in \mathbb{R}$, (i) $\frac{1}{N}\sum_{i=1}^{N} X_i X_i^\intercal \Lambda'(X_i^\intercal \times \theta(q)) \xrightarrow{P} E[XX^\intercal \Lambda'(X^\intercal\theta(q))]$ *which is invertible at* $q = q_\tau$; (ii) $\frac{1}{\sqrt{N}} \times \sum_{i=1}^{N} X_i v_i(q) \xrightarrow{D} N(0, V_{Xv}(q))$, *where* $V_{Xv}(q) = E[v^2(q)XX^\intercal]$ *has bounded matrix norm*; (iii) $\frac{1}{\sqrt{N}}\sum_{i=1}^{N} \Lambda'(X_i^\intercal\theta(q)) - E[\Lambda'(X^\intercal\theta(q))] \xrightarrow{D} N(0, V_{\Lambda'}(q))$, *where* $V_{\Lambda'}(q) < \infty$.

THEOREM 5: *Suppose the assumptions of Lemma* 1 *and Assumptions* 6, 7, *and* 10 *hold*. *Then*

$$\sqrt{Nh} \cdot (\widehat{\text{UQPE}}(\tau)_{\text{logit}} - \varsigma(q_\tau)) = \frac{1}{\sqrt{Nh}} \sum_{i}^{N} \psi_{\text{logit},i}(q_\tau, h, \mathcal{K}) + r_{\text{logit}}(q_\tau, h, \mathcal{K}),$$

where

$$r_{\text{logit}}(q_\tau, h, \mathcal{K}) = -\sqrt{Nh} \cdot \frac{\delta(q_\tau)}{f_Y(q_\tau)} \cdot \mathcal{B}_f(q, \mathcal{K}, h) + O\big(N^{1/2}h^{9/2}\big) + O_p(h^2) + O_p\big(N^{-1/2}h^{-1}\big) + O_p\big(N^{-1}h^{-5/2}\big).$$

THEOREM 6: *Suppose the assumptions of Theorem* 5 *and Assumption* 8 *hold*. *Then*

$$\sqrt{Nh} \cdot (\widehat{\text{UQPE}}(\tau)_{\text{logit}} - \varsigma(q_\tau)) \xrightarrow{D} N(0, V_{\text{logit}}(q_\tau, \mathcal{K})),$$

where

$$V_{\text{logit}}(q_\tau, \mathcal{K}) = \lim_{h \downarrow 0} h^{-1} \cdot E[\psi_{\text{logit}}(q_\tau, h, \mathcal{K}) \cdot \psi_{\text{logit}}(q_\tau, h, \mathcal{K})^\intercal].$$

S4.2. *The Bias* $\varsigma(q_\tau) - \text{UQPE}(\tau)$

Finally, we consider the bias term given by

$$\varsigma(q_\tau) - \text{UQPE}(\tau) = \frac{\theta(q) \cdot E[\Lambda'(X^\intercal\theta(q))] - E\left[d\dfrac{\Pr[Y > q_\tau|X]}{dX}\right]}{f_Y(q_\tau)}.$$

The condition for that asymptotic bias term to be zero is given below.

ASSUMPTION 11: $E[v(q_\tau)|X] = 0$.

LEMMA 5: *If Assumption* 11 *holds*, *then* $\varsigma(q_\tau) = \text{UQPE}(\tau)$.

THEOREM 7: *Suppose the assumptions of Theorem* 6 *and Assumption* 11 *hold*. *Then*

$$\sqrt{Nh} \cdot (\widehat{\text{UQPE}}(\tau)_{\text{logit}} - \text{UQPE}(\tau)) \xrightarrow{D} N(0, V_{\text{logit}}(q_\tau, \mathcal{K})).$$

S4.3. *Consistent Estimation of the Asymptotic Variance*

PROPOSITION 3: *The asymptotic variance can be expressed as*

$$V_{\text{logit}}(q_\tau, \mathcal{K}) = \lim_{h\downarrow 0}\left\{ \frac{1}{f_Y^2(q_\tau)} \varsigma(q_\tau)\varsigma^\intercal(q_\tau) E\big[(\widehat{f}_Y(q_\tau))^2\big] + \frac{1}{f_Y^2(q_\tau)} \text{Var}\big[h^{1/2} \cdot (\psi_\delta(q_\tau) + \varsigma'(q_\tau) \cdot \mathbb{1}\{Y > q_\tau\})\big] \right\}.$$

A simple estimator for the asymptotic variance of $V_{\text{logit}}(q_\tau, \mathcal{K})$ is

$$\begin{aligned}
&\widehat{V}_{\text{logit}}(\widehat{q}_\tau, h, \mathcal{K}) \\
&\quad = \frac{1}{\widehat{f}_Y^2(\widehat{q}_\tau)} \widehat{\varsigma}(\widehat{q}_\tau)\widehat{\varsigma}^\intercal(\widehat{q}_\tau) \cdot \widehat{f}_Y(\widehat{q}_\tau) \cdot \int \mathcal{K}^2(v) \cdot dv \\
&\qquad + h \cdot \frac{1}{\widehat{f}_Y^2(\widehat{q}_\tau)} \widehat{\varsigma}(\widehat{q}_\tau)\widehat{\varsigma}^\intercal(\widehat{q}_\tau) \cdot \left(\widehat{f'}_Y(\widehat{q}_\tau) \int v\mathcal{K}^2(v)\, dv - \widehat{f}_Y(\widehat{q}_\tau) \right) \\
&\qquad + h \cdot \frac{1}{\widehat{f}_Y^2(\widehat{q}_\tau)} \frac{1}{N} \sum_{i=1}^{N} \big(\widehat{\psi}_{\delta,i}(\widehat{q}_\tau) + \widehat{\varsigma}'(\widehat{q}_\tau)(\mathbb{1}\{Y_i > \widehat{q}_\tau\} - (1-\tau))\big) \\
&\qquad \cdot \big(\widehat{\psi}_{\delta,i}(\widehat{q}_\tau) + \widehat{\varsigma}'(\widehat{q}_\tau)(\mathbb{1}\{Y_i > \widehat{q}_\tau\} - (1-\tau))\big)^\intercal,
\end{aligned}$$

where $\widehat{f'}_Y(q)$ is the density derivative estimator previously discussed,

$$\begin{aligned}
&\widehat{\varsigma}'(q) = -\widehat{f}_Y^{-1}(q)(\widehat{\varsigma}(q) \cdot \widehat{f'}_Y(q) - \widehat{\delta}'(q)), \\
&\widehat{\delta}'(q) = \widehat{G}(q) \cdot \widehat{\theta}'(q), \\
&\widehat{G}(q) = \frac{2}{N} \sum_{i}^{N} \big\{ \Lambda(X_i^\intercal \widehat{\theta}(q)) \cdot \big(1 - \Lambda(X_i^\intercal \widehat{\theta}(q))\big)^2 \cdot \widehat{\theta}(q) \cdot X_i^\intercal \big\},
\end{aligned}$$

where

$$\begin{aligned}
\widehat{\theta}'(q) = &-\left(\frac{1}{N} \sum_{i}^{N} X_i X_i^\intercal \cdot \Lambda(X_i^\intercal \widehat{\theta}(q)) \cdot \big(1 - \Lambda(X_i^\intercal \widehat{\theta}(q))\big) \right)^{-1} \\
&\cdot \widehat{E}\big[X \cdot f_{Y|X}(q|X)\big],
\end{aligned}$$

$$\widehat{\psi}_{\delta,i}(q) = \widehat{G}(q) \cdot \left(\frac{1}{N} \sum_{i}^{N} X_i X_i^{\intercal} \cdot \Lambda(X_i^{\intercal} \widehat{\theta}(q)) \cdot \bigl(1 - \Lambda(X_i^{\intercal} \widehat{\theta}(q))\bigr) \right)^{-1}$$
$$\times X_i \cdot \widehat{v}_i(q) + \widehat{\theta}(q) \cdot \Bigl(\Lambda(X_i^{\intercal} \widehat{\theta}(q)) \cdot \bigl(1 - \Lambda(X_i^{\intercal} \widehat{\theta}(q))\bigr)$$
$$- \overline{\Lambda(X^{\intercal} \widehat{\theta}(q)) \cdot \bigl(1 - \Lambda(X^{\intercal} \widehat{\theta}(q))\bigr)} \Bigr),$$

where

$$\widehat{v}_i(q) = \mathbb{1}\{Y_i > q\} - \Lambda(X_i^{\intercal} \widehat{\theta}(q)),$$
$$\overline{\Lambda(X^{\intercal} \widehat{\theta}(q)) \cdot \bigl(1 - \Lambda(X^{\intercal} \widehat{\theta}(q))\bigr)}$$
$$= \frac{1}{N} \sum_{i}^{N} \Lambda(X_i^{\intercal} \widehat{\theta}(q)) \cdot \bigl(1 - \Lambda(X_i^{\intercal} \widehat{\theta}(q))\bigr).$$

PROPOSITION 4: *Under the assumptions of Theorem* 7, $\operatorname{plim}_{h \downarrow 0} \widehat{V}_{\text{logit}}(\widehat{q}_\tau, h, \mathcal{K}) = V_{\text{logit}}(q_\tau, \mathcal{K})$.

S4.4. *Testing the Zero-Null Hypothesis*

Everything follows by analogy with RIF-OLS in the $H_0 : R \cdot \text{UQPE}(\tau) = \alpha$ case, so we do not discuss this. The interesting case, in which we have a test statistic that converges at the usual parametric rate, occurs if we have a zero null hypothesis of the type $R \cdot \text{UQPE}(\tau) = 0$. Remember that this is equivalent to $R \cdot E[d((\Pr[Y > q_\tau|X])/dX)] = 0$. Under Assumption 11, $E[d((\Pr[Y > q_\tau|X])/dX)] = \delta(q_\tau)$, we have that

$$\widehat{\delta}(\widehat{q}_\tau) - E\left[d \frac{\Pr[Y > q_\tau|X]}{dX} \right]$$
$$= \frac{1}{N} \sum_{i=1}^{N} \psi_{t,i}(q_\tau) + O_p(N^{-1}),$$

where

$$\psi_{t,i}(q) = \psi_{\delta,i}(q) + \delta'(q) \cdot \psi_{Q,i}(q)$$
$$= H(q) \cdot \bigl(X_i \cdot v_i(q) - E[X|Y = q] \cdot (\mathbb{1}\{Y_i > q\} - (1 - \tau)) \bigr),$$

where

$$H(q) = G(q) \cdot \bigl(E\bigl[X X^{\intercal} \Lambda'(X^{\intercal} \theta(q)) \bigr] \bigr)^{-1}.$$

Invoking a central limit theorem, we have

$$\sqrt{N} \cdot (\widehat{\delta}(\widehat{q}_\tau) - \delta(q_\tau)) \xrightarrow{D} N(0, V_\delta(q_\tau)),$$

where

$$V_\delta(q_\tau) = E[\psi_t(q_\tau)\cdot\psi_t^\intercal(q_\tau)] = H(q_\tau)\cdot\Theta(q_\tau)\cdot H(q_\tau)^\intercal$$

and

$$\begin{aligned}\Theta(q_\tau) &= E[v^2(q_\tau)XX^\intercal] + \tau\cdot(1-\tau)\cdot E[X|Y=q_\tau]E[X|Y=q_\tau]^\intercal \\ &\quad - E[v^2(q_\tau)X]\cdot E[X|Y=q_\tau]^\intercal - E[X|Y=q_\tau]E[X^\intercal v^2(q_\tau)].\end{aligned}$$

A consistent estimator for the variance is

$$\widehat{V}_\delta(\widehat{q}_\tau) = \widehat{H}(\widehat{q}_\tau)\cdot\widehat{\Theta}(\widehat{q}_\tau)\cdot\widehat{H}(\widehat{q}_\tau),$$

where

$$\widehat{H}(\widehat{q}_\tau) = \widehat{G}(\widehat{q}_\tau)\cdot\left(\frac{1}{N}\sum_i^N X_iX_i^\intercal\cdot\Lambda(X_i^\intercal\widehat{\theta}(\widehat{q}_\tau))\cdot\left(1-\Lambda(X_i^\intercal\widehat{\theta}(\widehat{q}_\tau))\right)\right)^{-1}$$

and

$$\begin{aligned}\widehat{\Theta}(\widehat{q}_\tau) &= \frac{1}{N}\sum_{i=1}^N \widehat{v}_i^2(\widehat{q}_\tau)X_iX_i^\intercal + \tau\cdot(1-\tau) \\ &\quad \cdot\left(\frac{\widehat{E}[X\cdot f_{Y|X}(\widehat{q}_\tau|X)]}{\widehat{f}_Y(\widehat{q}_\tau)}\right)\cdot\left(\frac{\widehat{E}[X^\intercal\cdot f_{Y|X}(\widehat{q}_\tau|X)]}{\widehat{f}_Y(\widehat{q}_\tau)}\right) \\ &\quad - \left(\frac{1}{N}\sum_{i=1}^N \widehat{v}_i^2(\widehat{q}_\tau)X_i\right)\cdot\left(\frac{\widehat{E}[X^\intercal\cdot f_{Y|X}(\widehat{q}_\tau|X)]}{\widehat{f}_Y(\widehat{q}_\tau)}\right) \\ &\quad - \left(\frac{\widehat{E}[X\cdot f_{Y|X}(\widehat{q}_\tau|X)]}{\widehat{f}_Y(\widehat{q}_\tau)}\right)\cdot\left(\frac{1}{N}\sum_{i=1}^N \widehat{v}_i^2(\widehat{q}_\tau)X_i^\intercal\right).\end{aligned}$$

Therefore, if we want to construct a test, for example, for $\mathrm{UQPE}(\tau)_j = 0$, all we need is a t-statistic for δ_j. In general, we can construct a test statistic which will converge in law to a chi-square under H_0:

$$N\cdot\widehat{\delta}(\widehat{q}_\tau)^\intercal R^\intercal\cdot(R\cdot\widehat{V}_\delta(\widehat{q}_\tau)\cdot R^\intercal)^-\cdot R\cdot\widehat{\delta}(\widehat{q}_\tau)\xrightarrow{D}\chi_r^2.$$

The t-statistic for δ_j will be the particular square root of the left-hand side with $R = [0,\dots,0,1,0,\dots,0]$, where 1 is at the jth entry.

S5. RIF-NP

The RIF-NP estimator is the fully nonparametric estimator of $\mathrm{UQPE}(\tau)$. We estimate $E[d\Pr[Y > q_\tau|X]/dX]$ by a series of polynomials, exactly as proposed by Newey (1994). The density estimation part remains the same as the previous methods. To simplify notation, we define $m(x;q) = \Pr[Y > q|X = x]$ and $m'(x;q) = \partial m(x;q)/\partial x = d\Pr[Y > q|X = x]/dx$. We call $\pi(q)$ the expectation of $m'(X;q)$: $\pi(q) = E[m'(X;q)]$. We also define $w_i(q) = \mathbb{1}\{Y_i > q\} - m(X_i;q)$.

We estimate the average derivative $\pi(q)$ as

$$\widehat{\pi}_K(q) = \frac{1}{N}\sum_{i=1}^{N}\frac{dR_K(X_i)}{dX}\cdot\widehat{\lambda}_K(q),$$

$$\widehat{\lambda}_K(q) = \left(\sum_{i=1}^{N}R_K(X_i)R_K(X_i)^{\intercal}\right)^{-1}\cdot\sum_{i=1}^{N}\{R_K(X_i)\cdot\mathbb{1}\{Y_i > q\}\},$$

where $R_K(x)$ is a vector of length K of polynomials of X and $dR_K(x)/dx$ is a $k\times K$ matrix of derivatives of $R_K(x)$ with respect to x.

Following Newey (1994), we expect that under some regularity conditions, $\widehat{\pi}_K(q_\tau) - \pi(q_\tau)$ is asymptotic linear and converges in distribution at the $O_p(N^{-1/2})$ parametric rate. We define first $\partial^a m(X;q) = \partial^{\overline{a}}m(X;q)/\partial X_1^{a_1}\cdots\partial X_k^{a_k}$, where $\overline{a} = \sum_{j=1}^{k}a_j$, and invoke the following assumption.

ASSUMPTION 12: (i) *X is continuously distributed*; (ii) *the support of X is $\mathcal{X} = [x_{l1}, x_{u1}]\times\cdots\times[x_{lk}, x_{uk}]$*; (iii) *for all $x\in\mathcal{X}$, the density $f_X(x)$* (a) *is continuously differentiable, and* (b) *is bounded below by $C\prod_{j=1}^{k}(x - x_{lj})^d(x_{uj} - x)^d$*; (iv) *for all $x\in d\mathcal{X}$, the boundary $d\mathcal{X}$ of $\mathcal{X}$, $f_X(x) = 0$*; (v) *$E[\|l_X(X)\|^2] < \infty$, where $l_X(\cdot) = -f_X'(\cdot)/f_X(\cdot)$*, (vi) *$m(X;q)$ is continuously differentiable in X for all orders and all q in the support of Y*; (vii) *there is a constant C such that $\|\partial^a m(X;q)\| \le C^{\overline{a}}$*; (viii) *$K = K(N) = O(N^\varepsilon)$ for some $\varepsilon > 0$*; (ix) *$K^{7+2d} = o(N)$*.

According to Theorem 7.2 in Newey (1994), we have the following theorem.

THEOREM 8: *If Assumption 12 holds, then*

$$\sqrt{N}(\widehat{\pi}_K(q_\tau) - \pi(q_\tau)) = \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\psi_{\pi,i}(q_\tau) + o_p(1),$$

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\psi_{\pi,i}(q_\tau)\xrightarrow{D}N(0, V_\pi(q_\tau)),$$

where

$$\psi_{\pi,i}(q_\tau) = m'(X_i;q_\tau) - \pi(q_\tau) + l_X(X_i)\cdot w_i(q_\tau)$$

and

$$V_\pi(q_\tau) = E[\psi_\pi(q_\tau)\psi_\pi(q_\tau)^\intercal].$$

S5.1. *Convergence Order*

The nonparametric estimator of $\mathrm{UQPE}(\tau) = \kappa(q_\tau)$ is

$$\widehat{\mathrm{UQPE}}(\tau)_{\mathrm{NP}} = \widehat{\kappa}_K(\widehat{q}_\tau).$$

Letting

$$\widehat{\kappa}_K(q) = \frac{\widehat{\pi}_K(q)}{\widehat{f}_Y(q)} \quad \text{and} \quad \kappa(q) = \frac{\pi(q)}{f_Y(q)},$$

we now decompose the difference into two parts:

$$\begin{aligned}\widehat{\mathrm{UQPE}}(\tau)_{\mathrm{NP}} - \mathrm{UQPE}(\tau) &= \widehat{\kappa}_K(\widehat{q}_\tau) - \kappa(q_\tau) \\ &= \widehat{\kappa}_K(\widehat{q}_\tau) - \widehat{\kappa}_K(q_\tau) + \widehat{\kappa}_K(q_\tau) - \kappa(q_\tau).\end{aligned}$$

We write the difference as

$$\begin{aligned}&\widehat{\kappa}_K(\widehat{q}_\tau) - \widehat{\kappa}_K(q_\tau) \\ &\quad = \left(\frac{\widehat{\pi}_K'^*(q_\tau)}{\widehat{f}_Y(q_\tau)} - \frac{\widehat{\pi}_K(q_\tau)}{\widehat{f}_Y(q_\tau)} \cdot \frac{\widehat{f}_Y'(q_\tau)}{\widehat{f}_Y(q_\tau)}\right) \cdot (\widehat{q}_\tau - q_\tau) + \mathrm{rem}_{\mathrm{NP},2} \\ &\quad = \left(\frac{\pi'(q_\tau)}{f_Y(q_\tau)} - \frac{\pi(q_\tau)}{f_Y(q_\tau)} \cdot \frac{f_Y'(q_\tau)}{f_Y(q_\tau)}\right) \cdot (\widehat{q}_\tau - q_\tau) + \mathrm{rem}_{\mathrm{NP},3},\end{aligned}$$

where

$$\widehat{\pi}_K'^*(q) = \frac{1}{N}\sum_{i=1}^N \frac{dR_K(X_i)}{dX} \cdot \widehat{\lambda}_K'^*(q),$$

$$\pi'(q) = E\left[d\frac{f_{Y|X}(q|X)}{dX}\right],$$

and

$$\widehat{\lambda}_K'^*(q) = -\left(\sum_{i=1}^N R_K(X_i)R_K(X_i)^\intercal\right)^{-1} \cdot \sum_{i=1}^N \{R_K(X_i) \cdot f_{Y|X}(q|X_i)\}.$$

We write the second difference as

$$\widehat{\kappa}_K(q_\tau) - \kappa(q_\tau) = \frac{\widehat{\pi}_K(q_\tau) - \pi(q_\tau)}{f_Y(q_\tau)} - \frac{\pi(q_\tau)}{f_Y(q_\tau)} \cdot \left(\frac{\widehat{f}_Y(q_\tau) - f_Y(q_\tau)}{f_Y(q_\tau)} \right) + \mathrm{rem}_{\mathrm{NP},4}\,.$$

We now provide bounds on the remainders.

Bounds on $\| \mathrm{rem}_{\mathrm{NP},2} \|$

We invoke the following assumption:

ASSUMPTION 13: $f^2_{Y|X}(q|x)$ *is bounded and* $f_{Y|X}(q|x)$ *is continuously differentiable in* X *for all orders and all* q *in the support of* Y.

According to Theorem 7.2 in Newey (1994), we have that the parametric rate of convergence of average derivative estimators would prevail.

LEMMA 6: *Under Assumptions* 12 *and* 13, $\sqrt{N}(\widehat{\pi}'^*_K(q_\tau) - \pi'(q_\tau)) = O_p(1)$.

LEMMA 7: *Suppose the assumptions of Theorems* 1 *and* 2 *and Assumptions* 12 *and* 13 *hold*. *Then*

$$\mathrm{rem}_{\mathrm{NP},2} = O_p(N^{-1}).$$

Bounds on $\| \mathrm{rem}_{NP,4} \|$

We have

$$\begin{aligned}
\mathrm{rem}_{\mathrm{NP},4} &= \widehat{\kappa}_K(q_\tau) - \kappa(q_\tau) \\
&\quad - \left(\frac{\widehat{\pi}_K(q_\tau) - \pi(q_\tau)}{f_Y(q_\tau)} - \frac{\pi(q_\tau)}{f_Y(q_\tau)} \cdot \left(\frac{\widehat{f}_Y(q_\tau) - f_Y(q_\tau)}{f_Y(q_\tau)} \right) \right) \\
&= -\left(\frac{\widehat{f}_Y(q_\tau) - f_Y(q_\tau)}{\widehat{f}_Y(q_\tau) \cdot f_Y(q_\tau)} \right) \\
&\quad \cdot \left(\frac{\widehat{\pi}_K(q_\tau) - \pi(q_\tau)}{f_Y(q_\tau)} - \frac{\pi(q_\tau)}{f_Y(q_\tau)} \cdot \left(\frac{\widehat{f}_Y(q_\tau) - f_Y(q_\tau)}{f_Y(q_\tau)} \right) \right) \\
&= O_p((Nh)^{-1}) + O(h^4) + O_p\big(N^{-1/2} h^{3/2}\big),
\end{aligned}$$

which follows by Theorems 1, 2, and 8.

Bounds on $\| \mathrm{rem}_{\mathrm{NP},3} \|$

LEMMA 8: *Suppose the assumptions of Theorems* 1 *and* 2 *and Assumptions* 12 *and* 13 *hold*. *Then*

$$\mathrm{rem}_{\mathrm{NP},3} = O_p\big(N^{-1/2}h^2\big) + O_p\big(N^{-1}h^{-3/2}\big) + O_p\big(N^{-3/2}h^{-3}\big).$$

S5.2. *Combining Terms*

THEOREM 9: *Suppose the assumptions of Lemmas* 1 *and* 6 *hold*. *Then*

$$\begin{aligned}&\sqrt{Nh}\cdot(\widehat{\mathrm{UQPE}}(\tau)_{\mathrm{NP}} - \mathrm{UQPE}(\tau))\\ &\quad = \frac{1}{\sqrt{Nh}}\sum_{i}^{N}\psi_{\mathrm{NP},i}(q_\tau, h, \mathcal{K}) + r_{\mathrm{NP}}(q_\tau, h, \mathcal{K}),\end{aligned}$$

where

$$\begin{aligned}&\psi_{\mathrm{NP},i}(q_\tau, h, \mathcal{K})\\ &\quad = h\cdot\left(\frac{\psi_{\pi,i}(q_\tau)}{f_Y(q_\tau)} + \kappa'(q_\tau)\cdot\psi_{Q,i}(q_\tau) - \frac{\kappa(q_\tau)}{f_Y(q_\tau)}\cdot\psi_{f,i}(q_\tau, h, \mathcal{K})\right)\end{aligned}$$

and

$$\begin{aligned}r_{\mathrm{NP}}(q_\tau, h, \mathcal{K}) &= -\sqrt{Nh}\cdot\frac{\delta(q_\tau)}{f_Y(q_\tau)}\cdot\mathcal{B}_f(q, \mathcal{K}, h) + O\big(N^{1/2}h^{9/2}\big)\\ &\quad + O_p(h^2) + O_p\big(N^{-1/2}h^{-1}\big) + O_p\big(N^{-1}h^{-5/2}\big).\end{aligned}$$

THEOREM 10: *Suppose the assumptions of Theorem* 9 *and Assumption* 8 *hold*. *Then*

$$\sqrt{Nh}\cdot(\widehat{\mathrm{UQPE}}(\tau)_{\mathrm{NP}} - \mathrm{UQPE}(\tau)) \xrightarrow{D} N(0, V_{\mathrm{NP}}(q_\tau, \mathcal{K})),$$

where

$$V_{\mathrm{NP}}(q_\tau, \mathcal{K}) = \lim_{h\downarrow 0} h^{-1}\cdot E[\psi_{\mathrm{NP}}(q_\tau, h, \mathcal{K})\cdot\psi_{\mathrm{NP}}(q_\tau, h, \mathcal{K})^{\intercal}].$$

S5.3. *Consistent Estimation of the Asymptotic Variance*

PROPOSITION 5: *The asymptotic variance can be expressed as*

$$\begin{aligned}V_{\mathrm{NP}}(q_\tau, \mathcal{K}) = \lim_{h\downarrow 0}\bigg\{&\frac{1}{f_Y^2(q_\tau)}\kappa(q_\tau)\kappa^{\intercal}(q_\tau)E\big[(\widehat{f}_Y(q_\tau))^2\big]\\ &+ \frac{1}{f_Y^2(q_\tau)}\mathrm{Var}\big[h^{1/2}\cdot(\psi_\pi(q_\tau) + \kappa'(q_\tau)\cdot\mathbb{1}\{Y > q_\tau\})\big]\bigg\}.\end{aligned}$$

A simple estimator for the asymptotic variance of $V_{\mathrm{NP}}(q_\tau, \mathcal{K})$ is

$$
\begin{aligned}
&\widehat{V}_{\mathrm{NP}}(\widehat{q}_\tau, h, \mathcal{K}) \\
&= \frac{1}{\widehat{f}_Y^2(\widehat{q}_\tau)} \widehat{\kappa}(\widehat{q}_\tau)\widehat{\kappa}_K^\intercal(\widehat{q}_\tau) \cdot \widehat{f}_Y(\widehat{q}_\tau) \cdot \int \mathcal{K}^2(v) \cdot dv \\
&\quad + h \cdot \frac{1}{\widehat{f}_Y^2(\widehat{q}_\tau)} \widehat{\kappa}_K(\widehat{q}_\tau)\widehat{\kappa}_K^\intercal(\widehat{q}_\tau) \cdot \left(\widehat{f'}_Y(\widehat{q}_\tau) \int v\mathcal{K}^2(v)\, dv - \widehat{f}_Y(\widehat{q}_\tau) \right) \\
&\quad + h \cdot \frac{1}{\widehat{f}_Y^2(\widehat{q}_\tau)} \frac{1}{N} \sum_{i=1}^{N} \big(\widehat{\psi}_{\pi,i}(\widehat{q}_\tau) + \widehat{\kappa'}(\widehat{q}_\tau)(\mathbb{1}\{Y_i > \widehat{q}_\tau\} - (1-\tau)) \big) \\
&\quad \cdot \big(\widehat{\psi}_{\pi,i}(\widehat{q}_\tau) + \widehat{\kappa'}(\widehat{q}_\tau)(\mathbb{1}\{Y_i > \widehat{q}_\tau\} - (1-\tau)) \big)^\intercal,
\end{aligned}
$$

where $\widehat{f'}_Y(q)$ is the density derivative estimator previously discussed,

$$
\widehat{\kappa'}_K(q) = -\widehat{f}_Y^{-1}(q) \cdot (\widehat{\kappa}_K(q) \cdot \widehat{f'}_Y(q) - \widehat{\pi'}_K(q)),
$$

and

$$
\begin{aligned}
\widehat{\pi'}_K(q) = -\frac{1}{Nh} \sum_{i=1}^{N} \frac{dR_K(X_i)}{dX} \cdot \left(\sum_{i=1}^{N} R_K(X_i) R_K(X_i)^\intercal \right)^{-1} \\
\cdot \sum_{i=1}^{N} R_K(X_i) \cdot \mathcal{K}\left(\frac{Y_i - q}{h} \right).
\end{aligned}
$$

PROPOSITION 6: *Under the assumptions of Theorem* 4, $\operatorname{plim}_{h\downarrow 0} \widehat{V}_{\mathrm{NP}}(\widehat{q}_\tau, h, \mathcal{K}) = V_{\mathrm{NP}}(q_\tau, \mathcal{K})$.

S5.4. *Testing the Zero Null Hypothesis*

We proceed exactly as in the case of RIF-OLS and RIF-Logit. We are interested in a test statistic for the zero null hypothesis that converges at the parametric rate. We note that

$$
\begin{aligned}
&\widehat{\pi}_K(\widehat{q}_\tau) - E\left[d\frac{\Pr[Y > q_\tau | X]}{dX} \right] \\
&= \frac{1}{N} \sum_{i=1}^{N} \psi_{\pi,i}(q_\tau) + \pi'(q_\tau) \cdot \psi_{Q,i}(q_\tau) + O_p(N^{-1}).
\end{aligned}
$$

We now invoke a central limit theorem that yields

$$
\sqrt{N} \cdot (\widehat{\pi}_K(\widehat{q}_\tau) - \pi_K(q_\tau)) \xrightarrow{D} N(0, V_{\pi,q}(q_\tau)),
$$

where

$$V_{\pi,q}(q_\tau) = V_\pi(q_\tau) + \frac{\tau \cdot (1-\tau)}{f_Y^2(q_\tau)} \pi'(q_\tau)\pi'(q_\tau)^\intercal + E[\psi_{Q,i}(q_\tau) \cdot \psi_{\pi,i}(q_\tau)] \cdot \pi'(q_\tau)^\intercal + \pi'(q_\tau) \cdot E[\psi_{Q,i}(q_\tau) \cdot \psi_{\pi,i}(q_\tau)^\intercal].$$

S6. PROOFS

PROOF OF THEOREM 1: The bias and variance parts of the proof can be found in the proof of Theorem 2.2 of Pagan and Ullah (1999). The convergence order (third equation) follows from the bias and variance expression and an application of Chebyshev's inequality. The rate of convergence for the remainder term (fourth equation) follows from an Edgeworth type expansion of $(V[\widehat{f}_Y(q)])^{-1/2} \cdot (\widehat{f}_Y(q) - E[\widehat{f}_Y(q)])$ and from the bias expression. *Q.E.D.*

A proof of Theorem 2 is provided by Ferguson (2002) in his corollary after Theorem 13.

PROOF OF LEMMA 1: We start with

$$\begin{aligned}
&\widehat{\beta}(\widehat{q}_\tau) - \widehat{\beta}(q_\tau) \\
&\quad = \frac{\widehat{\gamma}(\widehat{q}_\tau)}{\widehat{f}_Y(\widehat{q}_\tau)} - \frac{\widehat{\gamma}(q_\tau)}{\widehat{f}_Y(q_\tau)} \\
&\quad = \frac{\widehat{f}_Y(q_\tau) \cdot (\widehat{\gamma}(\widehat{q}_\tau) - \widehat{\gamma}(q_\tau)) - \widehat{\gamma}(q_\tau) \cdot (\widehat{f}_Y(\widehat{q}_\tau) - \widehat{f}_Y(q_\tau))}{\widehat{f}_Y^2(q_\tau)} \\
&\qquad + \mathrm{rem}_{A,1},
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{rem}_{A,1} &= -\left(\frac{\widehat{f}_Y(q_\tau) \cdot (\widehat{\gamma}(\widehat{q}_\tau) - \widehat{\gamma}(q_\tau)) - \widehat{\gamma}(q_\tau) \cdot (\widehat{f}_Y(\widehat{q}_\tau) - \widehat{f}_Y(q_\tau))}{\widehat{f}_Y^2(q_\tau)}\right) \\
&\quad \cdot \left(\frac{\widehat{f}_Y(\widehat{q}_\tau) - \widehat{f}_Y(q_\tau)}{\widehat{f}_Y(\widehat{q}_\tau)}\right) \\
&= O_p(|\widehat{q}_\tau - q_\tau|^2).
\end{aligned}$$

Now we turn to $\widehat{\gamma}(\widehat{q}_\tau) - \widehat{\gamma}(q_\tau)$ and $\widehat{f}_Y(\widehat{q}_\tau) - \widehat{f}_Y(q_\tau)$. The first term is not differentiable in q as it is inside an indicator function. Therefore, we have to approximate the derivative of $\widehat{\gamma}$. The second term is differentiable as long we choose a smooth enough kernel function. We start with $\widehat{\gamma}(\widehat{q}_\tau) - \widehat{\gamma}(q_\tau)$:

$$\begin{aligned}
\widehat{\gamma}(\widehat{q}_\tau) &= \widehat{\Omega}_X^{-1}\frac{1}{N}\sum_{i=1}^{N} X_i \cdot (\mathbb{1}\{Y_i > \widehat{q}_\tau\} - (1-\tau)) \\
&= \widehat{\Omega}_X^{-1}\frac{1}{N}\sum_{i=1}^{N} X_i \cdot (\mathbb{1}\{Y_i > q_\tau\} - (1-\tau)) \\
&\quad + \widehat{\Omega}_X^{-1}\frac{1}{N}\sum_{i=1}^{N} X_i \cdot (\mathbb{1}\{Y_i > \widehat{q}_\tau\} - \mathbb{1}\{Y_i > q_\tau\}) \\
&= \widehat{\gamma}(q_\tau) - \widehat{\Omega}_X^{-1}\frac{1}{N}\sum_{i=1}^{N} X_i \cdot f_{Y|X}(\widetilde{q}_\tau|X_i) \cdot (\widehat{q}_\tau - q_\tau) + \mathrm{rem}_{A,2} \\
&\Rightarrow \widehat{\gamma}(\widehat{q}_\tau) - \widehat{\gamma}(q_\tau) = \widehat{\gamma}'^{*}(q_\tau) \cdot (\widehat{q}_\tau - q_\tau) + \mathrm{rem}_{A,2},
\end{aligned}$$

where $\widetilde{q}_\tau$ is a number between $\widehat{q}_\tau$ and q_τ. Now, we look at the difference:

$$\begin{aligned}
&\frac{1}{N}\sum_{i=1}^{N} X_i \cdot (\mathbb{1}\{Y_i > \widehat{q}_\tau\} - \mathbb{1}\{Y_i > q_\tau\}) \\
&= -\frac{1}{N}\sum_{i=1}^{N} X_i \cdot (\mathbb{1}\{Y_i \le \widehat{q}_\tau\} - \mathbb{1}\{Y_i \le q_\tau\}) \\
&= -\frac{1}{N}\sum_{i=1}^{N} X_i \cdot \big(\mathbb{1}\{Y_i - (\widehat{q}_\tau - q_\tau) \le q_\tau\} - \mathbb{1}\{Y_i \le q_\tau\}\big) \\
&= -\frac{1}{N}\sum_{i=1}^{N} X_i \cdot \big(E\big[\mathbb{1}\{Y - (\widehat{q}_\tau - q_\tau) \le q_\tau\}|X_i\big] - E[\mathbb{1}\{Y \le q_\tau\}|X_i]\big) \\
&\quad - \frac{1}{N}\sum_{i=1}^{N}\big\{X_i \cdot \mathbb{1}\{Y_i - (\widehat{q}_\tau - q_\tau) \le q_\tau\} \\
&\quad - E\big[\mathbb{1}\{Y - (\widehat{q}_\tau - q_\tau) \le q_\tau\}|X_i\big]\big\} \\
&\quad + \frac{1}{N}\sum_{i=1}^{N}\big\{X_i \cdot \mathbb{1}\{Y_i \le q_\tau\} - E[\mathbb{1}\{Y \le q_\tau\}|X_i]\big\} \\
&= -\frac{1}{N}\sum_{i=1}^{N} X_i \cdot \big(E\big[\mathbb{1}\{Y - (\widehat{q}_\tau - q_\tau) \le q_\tau\}|X_i\big] - E[\mathbb{1}\{Y \le q_\tau\}|X_i]\big) \\
&\quad + O_p\big(N^{-1/2}\big) \cdot (\widehat{q}_\tau - q_\tau) + O_p(N^{-1})
\end{aligned}$$

by Chebyschev inequality. Now, we define for convenience $U_N/\sqrt{N} = \widehat{q}_\tau - q_\tau = O_p(N^{-1/2})$. Note that for any $\widetilde{q}_\tau$ between $\widehat{q}_\tau$ and q_τ we can write $\widetilde{q}_\tau - q_\tau = C \cdot U_N/\sqrt{N} = O_p(N^{-1/2})$ for a constant C. We have then

$$\begin{aligned}
&\big|E\big[\mathbb{1}\{Y - (\widehat{q}_\tau - q_\tau) \le q_\tau\}|X\big] - E[\mathbb{1}\{Y \le q_\tau\}|X_i] \\
&\quad - f_{Y|X}(q_\tau|X)\cdot(\widehat{q}_\tau - q_\tau)\big| \\
&= \big|\Pr[Y \le \widehat{q}_\tau|X] - \Pr[Y \le q_\tau|X] - f_{Y|X}(q_\tau|X)\cdot(\widehat{q}_\tau - q_\tau)\big| \\
&= \big|E\big[\Pr[Y \le q_\tau + U_N/\sqrt{N}|X, U_N] - \Pr[Y \le q_\tau|X, U_N] \\
&\quad - f_{Y|X,U_N}(q_\tau|X, U_N)\cdot U_N/\sqrt{N}|X\big]\big| \\
&= \big|E\big[\big(f_{Y|X,U_N}(\widetilde{q}_\tau|X, U_N) - f_{Y|X,U_N}(q_\tau|X, U_N)\big)\cdot U_N/\sqrt{N}|X\big]\big| \\
&\le \frac{C}{N}\cdot\big|E\big[\big(f'_{Y|X,U_N}(\widetilde{q}^*_\tau|X, U_N)\big)\cdot U_N^2|X\big]\big| \\
&= O_p(N^{-1}).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\mathrm{rem}_{A,2} &= \widehat{\gamma}(\widehat{q}_\tau) - \widehat{\gamma}(q_\tau) + \widehat{\Omega}_X^{-1}\frac{1}{N}\sum_{i=1}^N X_i\cdot f_{Y|X}(\widetilde{q}_\tau|X_i)\cdot(\widehat{q}_\tau - q_\tau) \\
&= -\widehat{\Omega}_X^{-1}\frac{1}{N}\sum_{i=1}^N X_i\cdot(\mathbb{1}\{Y_i \le \widehat{q}_\tau\} - \mathbb{1}\{Y_i \le q_\tau\}) \\
&\quad + \widehat{\Omega}_X^{-1}\frac{1}{N}\sum_{i=1}^N X_i\cdot f_{Y|X}(\widetilde{q}_\tau|X_i)\cdot(\widehat{q}_\tau - q_\tau) \\
&= -\widehat{\Omega}_X^{-1}\frac{1}{N}\sum_{i=1}^N X_i \\
&\quad \cdot\big(\mathbb{1}\{Y_i \le \widehat{q}_\tau\} - \mathbb{1}\{Y_i \le q_\tau\} - f_{Y|X}(\widetilde{q}_\tau|X_i)\cdot(\widehat{q}_\tau - q_\tau)\big) \\
&= O_p(N^{-1}) = O_p(|\widehat{q}_\tau - q_\tau|^2).
\end{aligned}$$

Thus

$$\widehat{\gamma}(\widehat{q}_\tau) - \widehat{\gamma}(q_\tau) = \widehat{\gamma}'^*(q_\tau)\cdot(\widehat{q}_\tau - q_\tau) + O_p(|\widehat{q}_\tau - q_\tau|^2).$$

Now look at

$$\widehat{f}_Y(\widehat{q}_\tau) - \widehat{f}_Y(q_\tau) = \frac{1}{N}\sum_{i=1}^N \frac{1}{h}\cdot\left(\mathcal{K}\left(\frac{Y_i - \widehat{q}_\tau}{h}\right) - \mathcal{K}\left(\frac{Y_i - q_\tau}{h}\right)\right)$$

$$= \widehat{f'}_Y(\widetilde{q}_\tau) \cdot (\widehat{q}_\tau - q_\tau)$$
$$= \widehat{f'}_Y(q_\tau) \cdot (\widehat{q}_\tau - q_\tau) + O_p((\widehat{q}_\tau - q_\tau)^2).$$

We can write, therefore,

$$\widehat{\beta}(\widehat{q}_\tau) - \widehat{\beta}(q_\tau) = \left(\frac{\widehat{\gamma}'^*(q_\tau)}{\widehat{f}_Y(q_\tau)} - \frac{\widehat{\gamma}(q_\tau)}{\widehat{f}_Y(q_\tau)} \cdot \frac{\widehat{f}'_Y(q_\tau)}{\widehat{f}_Y(q_\tau)}\right) \cdot (\widehat{q}_\tau - q_\tau) + \mathrm{rem}_{A,3}$$
$$= \widehat{\beta}'^*(q_\tau) \cdot (\widehat{q}_\tau - q_\tau) + \mathrm{rem}_{A,3},$$

where $O_p(\mathrm{rem}_{A,3}) = O_p(\mathrm{rem}_{A,1}) + O_p(\mathrm{rem}_{A,2}) = O_p(|\widehat{q}_\tau - q_\tau|^2) = O_p(N^{-1})$.
Q.E.D.

PROOF OF LEMMA 2: We have that

$$\widehat{\gamma}(q) - \gamma(q) = \frac{1}{N}\sum_{i=1}^{N} \psi_{\gamma,i}(q) + O_p(N^{-1}),$$

where

$$\psi_{\gamma,i}(q) = \Omega_X^{-1} \cdot X_i \cdot u_i(q)$$

and

$$O_p\left(\widehat{\gamma}(q) - \gamma(q) - \frac{1}{N}\sum_{i=1}^{N} \psi_{\gamma,i}(q)\right) = O_p(N^{-1})$$

by Chebyschev inequality. Therefore,

$$O_p(r_{A,5}(q)) = O_p((Nh)^{-1}) + O(h^4) + O_p\big(N^{-1/2}h^{3/2}\big).$$

We then write

$$\widehat{\beta}(q_\tau) - \beta(q_\tau)$$
$$= \frac{\widehat{\gamma}(q_\tau) - \gamma(q_\tau)}{f_Y(q_\tau)} - \frac{\gamma(q_\tau)}{f_Y(q_\tau)} \cdot \left(\frac{\widehat{f}_Y(q_\tau) - f_Y(q_\tau)}{f_Y(q_\tau)}\right) + r_{A,5}(q_\tau)$$
$$= \frac{1}{f_Y(q_\tau)}\left(\frac{1}{N}\sum_{i=1}^{N} \psi_{\gamma,i}(q_\tau) + r_{A,6}(q_\tau)\right)$$
$$- \frac{\beta(q_\tau)}{f_Y(q_\tau)} \cdot \left(\frac{1}{N}\sum_{i=1}^{N} \psi_{f,i}(q_\tau, \mathcal{K}, h) + \mathcal{B}_f(q_\tau, \mathcal{K}, h)\right) + r_{A,5}(q_\tau)$$

$$= \frac{1}{N}\sum_{i=1}^{N}\frac{\psi_{\gamma,i}(q_\tau)}{f_Y(q_\tau)} - \frac{\beta(q_\tau)}{f_Y(q_\tau)}\cdot\psi_{f,i}(q_\tau,\mathcal{K},h) - \frac{\beta(q_\tau)}{f_Y(q_\tau)}\mathcal{B}_f(q_\tau,\mathcal{K},h)$$
$$+ O_p((Nh)^{-1}) + O(h^4) + O_p\big(N^{-1/2}h^{3/2}\big). \qquad \textit{Q.E.D.}$$

PROOF OF LEMMA 3: We have that

$$\begin{aligned}
\text{rem}_{\text{OLS},1} &= \Bigg(\bigg(\frac{\widehat{\gamma}'^*(q_\tau)}{\widehat{f}_Y(q_\tau)} - \frac{\widehat{\gamma}(q_\tau)}{\widehat{f}_Y(q_\tau)}\cdot\frac{\widehat{f}'_Y(q_\tau)}{\widehat{f}_Y(q_\tau)}\bigg) \\
&\quad - \bigg(\frac{\gamma'(q_\tau)}{f_Y(q_\tau)} - \frac{\gamma(q_\tau)}{f_Y(q_\tau)}\cdot\frac{f'_Y(q_\tau)}{f_Y(q_\tau)}\bigg)\Bigg)\cdot(\widehat{q}_\tau - q_\tau) \\
&= \Bigg(\frac{\widehat{\gamma}'^*(q_\tau)}{\widehat{f}_Y(q_\tau)} - \frac{\gamma'(q_\tau)}{f_Y(q_\tau)} \\
&\quad - \bigg[\bigg(\frac{\widehat{\gamma}(q_\tau)}{\widehat{f}_Y(q_\tau)}\cdot\frac{\widehat{f}'_Y(q_\tau)}{\widehat{f}_Y(q_\tau)}\bigg) - \bigg(\frac{\gamma(q_\tau)}{f_Y(q_\tau)}\cdot\frac{f'_Y(q_\tau)}{f_Y(q_\tau)}\bigg)\bigg]\Bigg) \\
&\quad \cdot(\widehat{q}_\tau - q_\tau).
\end{aligned}$$

Now, we look at the terms multiplying $\widehat{q}_\tau - q_\tau$:

$$\begin{aligned}
\frac{\widehat{\gamma}'^*(q_\tau)}{\widehat{f}_Y(q_\tau)} - \frac{\gamma'(q_\tau)}{f_Y(q_\tau)} &= \frac{1}{f_Y(q_\tau)}\cdot(\widehat{\gamma}'^*(q_\tau) - \gamma'(q_\tau)) \\
&\quad - \frac{\gamma'(q_\tau)}{f_Y^2(q_\tau)}\cdot(\widehat{f}_Y(q_\tau) - f_Y(q_\tau)) + \text{rem}_{\text{OLS},1a},
\end{aligned}$$

where

$$\begin{aligned}
\text{rem}_{\text{OLS},1a} &= -\bigg(\frac{1}{f_Y(q_\tau)}\cdot(\widehat{\gamma}'^*(q_\tau) - \gamma'(q_\tau)) \\
&\quad - \frac{\gamma'(q_\tau)}{f_Y^2(q_\tau)}\cdot(\widehat{f}_Y(q_\tau) - f_Y(q_\tau))\bigg)\cdot\bigg(\frac{\widehat{f}_Y(q_\tau) - f_Y(q_\tau)}{\widehat{f}_Y(q_\tau)}\bigg) \\
&= O_p((Nh)^{-1}) + O(h^4) + O_p\big(N^{-1/2}h^{3/2}\big)
\end{aligned}$$

because

$$\begin{aligned}
&\widehat{\gamma}'^*(q_\tau) - \gamma'(q_\tau) \\
&\quad = -\bigg(\widehat{\Omega}_X^{-1}\frac{1}{N}\sum_{i=1}^{N}X_i\cdot f_{Y|X}(q_\tau|X_i) - \Omega_X^{-1}E\big[X\cdot f_{Y|X}(q_\tau|X)\big]\bigg) \\
&\quad = O_p(N^{-1/2}).
\end{aligned}$$

The second term is

$$\begin{aligned}
&\left(\frac{\widehat{\gamma}(q_\tau)}{\widehat{f}_Y(q_\tau)}\cdot\frac{\widehat{f}'_Y(q_\tau)}{\widehat{f}_Y(q_\tau)}\right)-\left(\frac{\gamma(q_\tau)}{f_Y(q_\tau)}\cdot\frac{f'_Y(q_\tau)}{f_Y(q_\tau)}\right)\\
&=\frac{f_Y^2(q_\tau)f'_Y(q_\tau)}{f_Y^4(q_\tau)}\cdot(\widehat{\gamma}(q_\tau)-\gamma(q_\tau))\\
&\quad-2\frac{\gamma(q_\tau)f_Y(q_\tau)f'_Y(q_\tau)}{f_Y^4(q_\tau)}\cdot(\widehat{f}_Y(q_\tau)-f_Y(q_\tau))\\
&\quad+\frac{\gamma'(q_\tau)f_Y^2(q_\tau)}{f_Y^4(q_\tau)}\cdot(\widehat{f}'_Y(q_\tau)-f'_Y(q_\tau))+\mathrm{rem}_{\mathrm{OLS},1b},
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{rem}_{\mathrm{OLS},1b}&=f_Y(q_\tau)^{-4}\cdot\big(f_Y(q_\tau)^2(\widehat{\gamma}(q_\tau)-\gamma(q_\tau))(\widehat{f}'_Y(q_\tau)-f'_Y(q_\tau))\\
&\quad-\gamma(q_\tau)f'_Y(q_\tau)(\widehat{f}_Y(q_\tau)-f_Y(q_\tau))(\widehat{f}_Y(q_\tau)-f_Y(q_\tau))\big)\\
&\quad\cdot\left(\frac{\widehat{f}_Y(q_\tau)^2+f_Y(q_\tau)^2}{\widehat{f}_Y(q_\tau)^2}\right)\\
&\quad-f_Y(q_\tau)^{-4}\cdot\big(f_Y(q_\tau)^2\big(\gamma(q_\tau)(\widehat{f}'_Y(q_\tau)-f'_Y(q_\tau))\\
&\quad+f'_Y(q_\tau)(\widehat{\gamma}(q_\tau)-\gamma(q_\tau))\big)\\
&\quad-2\gamma(q_\tau)f_Y(q_\tau)f'_Y(q_\tau)(\widehat{f}_Y(q_\tau)-f_Y(q_\tau))\big)\\
&\quad\cdot\left(\frac{\widehat{f}_Y(q_\tau)^2-f_Y(q_\tau)^2}{\widehat{f}_Y(q_\tau)^2}\right)\\
&=O_p\big(|\widehat{\gamma}(q_\tau)-\gamma(q_\tau)|\cdot|\widehat{f}'_Y(q_\tau)-f'_Y(q_\tau)|\big)\\
&\quad+O_p\big(|\widehat{f}_Y(q_\tau)-f_Y(q_\tau)|^2\big)\\
&\quad+O_p\big(|\widehat{f}_Y(q_\tau)-f_Y(q_\tau)|\cdot|\widehat{f}'_Y(q_\tau)-f'_Y(q_\tau)|\big)\\
&=O(h^4)+O_p((Nh^3)^{-1})+O_p\big(N^{-1/2}h^{1/2}\big)
\end{aligned}$$

because $\widehat{f}'_Y(q_\tau)-f'_Y(q_\tau)=O_p((Nh^3)^{-1/2})+O(h^2)$. Putting together these results, we have

$$\begin{aligned}
\mathrm{rem}_{\mathrm{OLS},1}&=(\widehat{\beta}'^*(q_\tau)-\beta(q_\tau))\cdot(\widehat{q}_\tau-q_\tau)\\
&=O_p\big(N^{-1/2}h^2\big)+O_p\big(N^{-1}h^{-3/2}\big)+O_p\big(N^{-3/2}h^{-3}\big). \qquad \textit{Q.E.D.}
\end{aligned}$$

The proof of Corollary 1 follows from straightforward multiplication of $\widehat{\beta}(\widehat{q}_\tau)-\beta(q_\tau)$ by $(Nh)^{1/2}$.

PROOF OF THEOREM 3: Making $h = o(N^{-1/5})$, we have

$$\begin{aligned}
&\sqrt{Nh} \cdot \frac{\beta(q_\tau)}{f_Y(q_\tau)} \cdot \mathcal{B}_f(q, \mathcal{K}, h) + O\big(N^{1/2}h^{9/2}\big) \\
&\quad + O_p(h^2) + O_p\big(N^{-1/2}h^{-1}\big) + O_p\big(N^{-1}h^{-5/2}\big) \\
&= O_p\big(N^{1/2}h^{1/2}\big) \cdot O(h^2) + O\big(N^{1/2}h^{9/2}\big) + O_p(h^2) \\
&\quad + O_p\big(N^{-1/2}h^{-1}\big) + O_p\big(N^{-1}h^{-5/2}\big) \\
&= O_p\big(N^{1/2}h^{5/2}\big) + O_p(h^2) + O_p\big(N^{-1/2}h^{-1}\big) + O_p\big(N^{-1}h^{-5/2}\big) \\
&= o_p\big(N^{1/2-1/2}\big) + o_p\big(N^{-2/5}\big) + o_p\big(N^{-1/2+1/5}\big) + o_p\big(N^{-1+1/2}\big) \\
&= o_p(1) + o_p\big(N^{-4/10}\big) + o_p\big(N^{-3/10}\big) + o_p\big(N^{-5/10}\big) = o_p(1).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\sqrt{Nh} \cdot \big(\widehat{\mathrm{UQPE}}(\tau)_{\mathrm{OLS}} - \beta(q_\tau)\big) \\
&= \frac{1}{\sqrt{Nh}} \sum_i^N \psi_{\mathrm{OLS},i}(q_\tau, h, \mathcal{K}) + o_p(1)
\end{aligned}$$

and, by a central limit theorem,

$$\begin{aligned}
&\frac{1}{\sqrt{Nh}} \sum_i^N \psi_{\mathrm{OLS},i}(q_\tau, h, \mathcal{K}) \\
&= \frac{1}{\sqrt{N}} \sum_i^N h^{-1/2} \psi_{\mathrm{OLS},i}(q_\tau, h, \mathcal{K}) \\
&\xrightarrow{D} N\left(0, \lim_{h \downarrow 0} \frac{E[\psi_{\mathrm{OLS}}(q_\tau, h, \mathcal{K}) \cdot \psi_{\mathrm{OLS}}(q_\tau, h, \mathcal{K})^\intercal]}{h}\right).
\end{aligned}$$

Q.E.D.

PROOF OF LEMMA 4: If $E[u(q_\tau)|X] = 0$, then

$$\begin{aligned}
0 &= E[u(q_\tau)|X] = E[\mathbb{1}\{Y_i > q_\tau\} - (1-\tau) - X_i^\intercal \gamma(q_\tau)|X] \\
&\Rightarrow \Pr[Y > q_\tau|X] = 1 - \tau + X^\intercal \gamma(q)
\end{aligned}$$

and because $\Pr[Y > q_\tau|X] = 1 - \tau + X^\intercal \gamma(q)$, we have that $d\Pr[Y > q_\tau|X]/dX = \gamma(q_\tau)$. Thus,

$$\beta(q_\tau) = \frac{\gamma(q_\tau)}{f_Y(q_\tau)} = \frac{E\left[d\dfrac{\Pr[Y > q_\tau|X]}{dX}\right]}{f_Y(q_\tau)} = \mathrm{UQPE}(\tau).$$

Q.E.D.

Theorem 4 follows trivially and so no proof is provided.

PROOF OF PROPOSITION 1: We have

$$
\begin{aligned}
V_{\mathrm{OLS}}(q_\tau, \mathcal{K}) = \lim_{h \downarrow 0} \Bigg\{ & \frac{1}{f_Y^2(q_\tau)} \beta(q_\tau)\beta^\intercal(q_\tau) \cdot h^{-1} V\left(\mathcal{K}\left(\frac{Y - q_\tau}{h}\right)\right) \\
& + h \frac{1}{f_Y^2(q_\tau)} \big(\Omega_X^{-1} E[u^2(q_\tau) \cdot X X^\intercal] \Omega_X^{-1} \\
& + \tau(1-\tau)\beta'(q_\tau)\beta'^\intercal(q_\tau)\big) \\
& + h \frac{1}{f_Y^2(q_\tau)} \big(\Omega_X^{-1} E[u(q_\tau) \cdot X \cdot \mathbb{1}\{Y > q_\tau\}]\beta'^\intercal(q_\tau) \\
& + \beta'(q_\tau) E[X^\intercal \cdot u(q_\tau) \cdot \mathbb{1}\{Y > q_\tau\}]\Omega_X^{-1}\big) \\
& - h \frac{1}{f_Y^2(q_\tau)} \Omega_X^{-1} E\left[X \cdot u(q_\tau) \cdot \mathcal{K}\left(\frac{Y - q_\tau}{h}\right)\right] \beta^\intercal(q_\tau) \\
& - h \frac{1}{f_Y^2(q_\tau)} \beta(q_\tau) E\left[X^\intercal \cdot u(q_\tau) \cdot \mathcal{K}\left(\frac{Y - q_\tau}{h}\right)\right] \Omega_X^{-1} \\
& - h \frac{1}{f_Y^2(q_\tau)} E\left[(\mathbb{1}\{Y > q_\tau\} - (1-\tau)) \cdot \mathcal{K}\left(\frac{Y - q_\tau}{h}\right)\right] \\
& \cdot (\beta'(q_\tau)\beta^\intercal(q_\tau) + \beta(q_\tau)\beta'^\intercal(q_\tau)) \Bigg\},
\end{aligned}
$$

where

$$
\begin{aligned}
& h^{-1} V\left(\mathcal{K}\left(\frac{Y - q_\tau}{h}\right)\right) \\
& \quad = h^{-1} E\left(\mathcal{K}^2\left(\frac{Y - q_\tau}{h}\right)\right) - E^2\left(\mathcal{K}\left(\frac{Y - q_\tau}{h}\right)\right) \\
& \quad = \int \mathcal{K}^2(v) \cdot f_Y(vh + q_\tau) \cdot dv - h \cdot \left(\int \mathcal{K}(v) \cdot f_Y(vh + q_\tau) \cdot dv\right)^2 \\
& \quad = f_Y(q_\tau) \cdot \int \mathcal{K}^2(v) \cdot dv + h \cdot f_Y'(q_\tau) \cdot \int v \cdot \mathcal{K}^2(v) \cdot dv \\
& \qquad + h^2 \cdot \frac{f_Y''(q_\tau)}{2} \cdot \int v^2 \cdot \mathcal{K}^2(v) \cdot dv \\
& \qquad - h \cdot \left(f_Y(q_\tau) + h^2 \cdot \frac{f_Y''(q_\tau)}{2} \cdot \int v^2 \cdot \mathcal{K}(v) \cdot dv\right)^2
\end{aligned}
$$

$$= f_Y(q_\tau) \cdot \int \mathcal{K}^2(v) \cdot dv$$
$$+ h \cdot \left(f'_Y(q_\tau) \int v\mathcal{K}^2(v)\, dv - f_Y(q_\tau) \right) + o(h)$$

and

$$E\left[(\mathbb{1}\{Y > q_\tau\} - (1-\tau)) \cdot \mathcal{K}\left(\frac{Y - q_\tau}{h} \right) \right]$$
$$= E\left[(\mathbb{1}\{Y > q_\tau\}) \cdot \mathcal{K}\left(\frac{Y - q_\tau}{h} \right) \right] - (1-\tau) \cdot E\left[\mathcal{K}\left(\frac{Y - q_\tau}{h} \right) \right]$$
$$= h f_Y(q_\tau) \left(\frac{1}{2} - (1-\tau) \right) + O(h^2)$$

because

$$E\left[\mathcal{K}\left(\frac{Y - q_\tau}{h} \right) \right] = \int \mathcal{K}\left(\frac{y - q_\tau}{h} \right) f_Y(y) \cdot dy$$
$$= h \int \mathcal{K}(v) f_Y(hv + q_\tau) \cdot dv$$
$$= h \int \mathcal{K}(v) \left(f_Y(q_\tau) + hv f'_Y(q_\tau) \right.$$
$$\left. + (hv)^2 \frac{f''_Y(q_\tau)}{2} + o(h^2) \right) \cdot dv$$
$$= h \cdot f_Y(q_\tau) + h^3 \int \mathcal{K}(v) v^2 \cdot dv + o(h^3)$$

and

$$E\left[(\mathbb{1}\{Y_i > q_\tau\}) \cdot \mathcal{K}\left(\frac{Y - q_\tau}{h} \right) \right]$$
$$= \int_{q_\tau}^{\infty} \mathcal{K}\left(\frac{y - q_\tau}{h} \right) f_Y(y) \cdot dy$$
$$= h \int_0^{\infty} \mathcal{K}(v) \left(f_Y(q_\tau) + hv f'_Y(q_\tau) + (hv)^2 \frac{f''_Y(q_\tau)}{2} + o(h^2) \right) \cdot dv$$
$$= h \frac{f_Y(q_\tau)}{2} + O(h^2)$$

and because

$$E\left[X \cdot u(q_\tau) \cdot \mathcal{K}\left(\frac{Y-q_\tau}{h}\right)\right] = E\left[X \cdot E\left[u(q_\tau) \cdot \mathcal{K}\left(\frac{Y-q_\tau}{h}\right)\Big| X\right]\right],$$

where

$$\begin{aligned}
&E\left[u(q_\tau) \cdot \mathcal{K}\left(\frac{Y-q_\tau}{h}\right)\Big| X = x\right] \\
&\quad = E\left[(\mathbb{1}\{Y > q_\tau\} - (1-\tau) - x^\intercal \gamma(q)) \cdot \mathcal{K}\left(\frac{Y-q_\tau}{h}\right)\Big| X = x\right] \\
&\quad = E\left[(\mathbb{1}\{Y > q_\tau\}) \cdot \mathcal{K}\left(\frac{Y-q_\tau}{h}\right)\Big| X = x\right] \\
&\qquad - ((1-\tau) + x^\intercal \gamma(q_\tau)) \cdot E\left[\mathcal{K}\left(\frac{Y-q_\tau}{h}\right)\Big| X = x\right] \\
&\quad = h f_{Y|X}(q_\tau|x)\left(\frac{1}{2} - (1-\tau) - x^\intercal \gamma(q_\tau)\right) + O(h^2).
\end{aligned}$$

Therefore,

$$\begin{aligned}
V_{\mathrm{OLS}}(q_\tau, \mathcal{K}) = \lim_{h\downarrow 0}\Bigg\{ &\frac{1}{f_Y^2(q_\tau)} \beta(q_\tau)\beta^\intercal(q_\tau) \cdot f_Y(q_\tau) \cdot \int \mathcal{K}^2(v) \cdot dv \\
&+ h \cdot \frac{1}{f_Y^2(q_\tau)} \beta(q_\tau)\beta^\intercal(q_\tau) \\
&\cdot \left(f_Y'(q_\tau) \int v\mathcal{K}^2(v)\, dv - f_Y(q_\tau)\right) + h \frac{1}{f_Y^2(q_\tau)} \\
&\times \left(\Omega_X^{-1} E[u^2(q_\tau) \cdot XX^\intercal]\Omega_X^{-1} + \tau(1-\tau)\beta'(q_\tau)\beta'^\intercal(q_\tau)\right) \\
&+ h \frac{1}{f_Y^2(q_\tau)} \big(\Omega_X^{-1} E[u(q_\tau) \cdot X \cdot \mathbb{1}\{Y > q_\tau\}]\beta'^\intercal(q_\tau) \\
&+ \beta'(q_\tau) E[X^\intercal \cdot u(q_\tau) \cdot \mathbb{1}\{Y > q_\tau\}]\Omega_X^{-1}\big) + O(h^2)\Bigg\}
\end{aligned}$$

or

$$\begin{aligned}
V_{\mathrm{OLS}}(q_\tau, \mathcal{K}) = \lim_{h\downarrow 0}\Bigg\{ &\frac{1}{f_Y^2(q_\tau)} \beta(q_\tau)\beta^\intercal(q_\tau) E\big[(\widehat{f}_Y(q_\tau))^2\big] \\
&+ \frac{1}{f_Y^2(q_\tau)} \operatorname{Var}\big[h^{1/2} \cdot \big(\Omega_X^{-1} X u(q_\tau) \\
&+ \beta'(q_\tau) \cdot (\mathbb{1}\{Y > q_\tau\} - (1-\tau))\big)\big]\Bigg\}. \qquad \textit{Q.E.D.}
\end{aligned}$$

PROOF OF PROPOSITION 2: All estimators involved in $\widehat{V}_{\mathrm{OLS}}(q_\tau, h, \mathcal{K})$ can be trivially shown to be consistent for their population counterparts. The only one that is missing is $\widehat{E}[X \cdot f_{Y|X}(q|X)] = (1/Nh)\sum_i^N X_i \cdot \mathcal{K}((Y_i - q)/h)$:

$$\begin{aligned}
&\lim_{h\downarrow 0} E\big[\widehat{E}\big[X \cdot f_{Y|X}(q|X)\big]\big] \\
&\quad = \lim_{h\downarrow 0} E\left[\frac{1}{Nh}\sum_i^N X_i \cdot \mathcal{K}\left(\frac{Y_i - q}{h}\right)\right] \\
&\quad = \lim_{h\downarrow 0} \frac{1}{h} E\left[X \cdot \mathcal{K}\left(\frac{Y - q}{h}\right)\right] \\
&\quad = \lim_{h\downarrow 0} \frac{1}{h} \int\int x \cdot \mathcal{K}\left(\frac{y - q}{h}\right) f_{Y,X}(y, x)\, dy\, dx \\
&\quad = \lim_{h\downarrow 0} \int\int x \cdot \mathcal{K}(u) f_{Y,X}(uh + q, x)\, du\, dx \\
&\quad = \lim_{h\downarrow 0} \int\int x \cdot \mathcal{K}(u)(f_{Y,X}(q, x) + h \cdot O(1))\, du\, dx \\
&\quad = \int x \cdot f_{Y,X}(q, x)\, dx \int \mathcal{K}(u)\, du = \int x \cdot f_{Y,X}(q, x)\, dx \\
&\quad = \int x \cdot f_{Y|X}(q|x) \cdot f_X(x)\, dx = E\big[X \cdot f_{Y|X}(q|X)\big].
\end{aligned}$$

Q.E.D.

All proofs of Results in Section S4 are similar to the ones provided for RIF-OLS and are therefore omitted.

For the proofs of Theorem 8 and Lemma 6, see Newey (1994, Theorem 7.2).

PROOF OF LEMMA 7: We write

$$\begin{aligned}
&\widehat{\kappa}_K(\widehat{q}_\tau) - \widehat{\kappa}_K(q_\tau) \\
&\quad = \frac{\widehat{\pi}_K(\widehat{q}_\tau)}{\widehat{f}_Y(\widehat{q}_\tau)} - \frac{\widehat{\pi}_K(q_\tau)}{\widehat{f}_Y(q_\tau)} \\
&\quad = \frac{\widehat{f}_Y(q_\tau) \cdot (\widehat{\pi}_K(\widehat{q}_\tau) - \widehat{\pi}_K(q_\tau)) - \widehat{\pi}_K(q_\tau) \cdot (\widehat{f}_Y(\widehat{q}_\tau) - \widehat{f}_Y(q_\tau))}{\widehat{f}_Y(\widehat{q}_\tau) \cdot \widehat{f}_Y(q_\tau)} \\
&\quad = \frac{\widehat{\pi}_K(\widehat{q}_\tau) - \widehat{\pi}_K(q_\tau)}{\widehat{f}_Y(q_\tau)} - \frac{\widehat{\pi}_K(q_\tau)}{\widehat{f}_Y(q_\tau)} \cdot \left(\frac{\widehat{f}_Y(\widehat{q}_\tau) - \widehat{f}_Y(q_\tau)}{\widehat{f}_Y(q_\tau)}\right) \\
&\qquad + \operatorname{rem} A,
\end{aligned}$$

where

$$\begin{aligned}
\operatorname{rem} A &= -\big(\widehat{f}_Y(q_\tau)\cdot(\widehat{\pi}_K(\widehat{q}_\tau)-\widehat{\pi}_K(q_\tau))-\widehat{\pi}_K(q_\tau)\cdot(\widehat{f}_Y(\widehat{q}_\tau)-\widehat{f}_Y(q_\tau))\big)\\
&\qquad\cdot\left(\frac{\widehat{f}_Y(\widehat{q}_\tau)-\widehat{f}_Y(q_\tau)}{\widehat{f}_Y(\widehat{q}_\tau)\cdot\widehat{f}_Y^2(q_\tau)}\right)\\
&= O_p\big(\|\widehat{\pi}_K(\widehat{q}_\tau)-\widehat{\pi}_K(q_\tau)\|\cdot|\widehat{f}_Y(\widehat{q}_\tau)-\widehat{f}_Y(q_\tau)|\big)\\
&\qquad+O_p\big(|\widehat{f}_Y(\widehat{q}_\tau)-\widehat{f}_Y(q_\tau)|^2\big)\\
&= O_p(N^{-1}).
\end{aligned}$$

Now, we look at the difference:

$$\begin{aligned}
\widehat{\lambda}_K(\widehat{q}_\tau)-\widehat{\lambda}_K(q_\tau) &= \left(\sum_{i=1}^N R_K(X_i)R_K(X_i)^\intercal\right)^{-1}\cdot\sum_{i=1}^N R_K(X_i)\\
&\qquad\cdot(\mathbb{1}\{Y_i\le\widehat{q}_\tau\}-\mathbb{1}\{Y_i\le q_\tau\})\\
&= \left(\sum_{i=1}^N R_K(X_i)R_K(X_i)^\intercal\right)^{-1}\cdot\sum_{i=1}^N R_K(X_i)\\
&\qquad\cdot\big(\mathbb{1}\{Y_i-(\widehat{q}_\tau-q_\tau)\le q_\tau\}-\mathbb{1}\{Y_i\le q_\tau\}\big).
\end{aligned}$$

Thus

$$\begin{aligned}
&\widehat{\pi}_K(\widehat{q}_\tau)-\widehat{\pi}_K(q_\tau)\\
&\quad=\frac{1}{N}\sum_{i=1}^N\frac{dR_K(X_i)}{dX}\cdot(\widehat{\lambda}_K(\widehat{q}_\tau)-\widehat{\lambda}_K(q_\tau))\\
&\quad=\frac{1}{N}\sum_{i=1}^N\frac{dR_K(X_i)}{dX}\cdot\left(\sum_{i=1}^N R_K(X_i)R_K(X_i)^\intercal\right)^{-1}\\
&\qquad\cdot\sum_{i=1}^N R_K(X_i)\cdot\big(\mathbb{1}\{Y_i-(\widehat{q}_\tau-q_\tau)\le q_\tau\}-\mathbb{1}\{Y_i\le q_\tau\}\big)\\
&\quad=\frac{1}{N}\sum_{i=1}^N\frac{dR_K(X_i)}{dX}\cdot\left(\sum_{i=1}^N R_K(X_i)R_K(X_i)^\intercal\right)^{-1}\\
&\qquad\cdot\sum_{i=1}^N R_K(X_i)\cdot E\big[\big(\mathbb{1}\{Y-(\widehat{q}_\tau-q_\tau)\le q_\tau\}-\mathbb{1}\{Y\le q_\tau\}\big)|X_i\big]\\
&\qquad+\operatorname{rem} B,
\end{aligned}$$

where

$$
\begin{aligned}
\operatorname{rem} B = {} & \frac{1}{N}\sum_{i=1}^{N}\frac{dR_K(X_i)}{dX}\cdot\left(\sum_{i=1}^{N}R_K(X_i)R_K(X_i)^{\intercal}\right)^{-1} \\
& \cdot\sum_{i=1}^{N}\Big\{R_K(X_i)\cdot\big(\mathbb{1}\{Y_i-(\widehat{q}_\tau-q_\tau)\le q_\tau\} \\
& - E\big[\mathbb{1}\{Y-(\widehat{q}_\tau-q_\tau)\le q_\tau\}|X_i\big] \\
& - \mathbb{1}\{Y_i\le q_\tau\}+E[\mathbb{1}\{Y\le q_\tau\}|X_i]\big)\Big\} \\
= {} & O_p\big(N^{-1/2}\big)\cdot(\widehat{q}_\tau-q_\tau)=O_p(N^{-1}).
\end{aligned}
$$

We then write

$$
\begin{aligned}
& \widehat{\pi}_K(\widehat{q}_\tau)-\widehat{\pi}_K(q_\tau) \\
& \quad = E\left[\frac{d}{dX}\big\{E\big[\big(\mathbb{1}\{Y-(\widehat{q}_\tau-q_\tau)\le q_\tau\}-\mathbb{1}\{Y\le q_\tau\}\big)|X\big]\big\}\right]+\operatorname{rem} C,
\end{aligned}
$$

where

$$
\begin{aligned}
\operatorname{rem} C = {} & \frac{1}{N}\sum_{i=1}^{N}\frac{dR_K(X_i)}{dX}\cdot\left(\sum_{i=1}^{N}R_K(X_i)R_K(X_i)^{\intercal}\right)^{-1} \\
& \cdot\sum_{i=1}^{N}R_K(X_i)\cdot E\big[\big(\mathbb{1}\{Y-(\widehat{q}_\tau-q_\tau)\le q_\tau\}-\mathbb{1}\{Y\le q_\tau\}\big)|X_i\big] \\
& - E\left[\frac{d}{dX}\big\{E\big[\big(\mathbb{1}\{Y-(\widehat{q}_\tau-q_\tau)\le q_\tau\}-\mathbb{1}\{Y\le q_\tau\}\big)|X\big]\big\}\right] \\
= {} & \frac{1}{N}\sum_{i=1}^{N}\frac{dR_K(X_i)}{dX}\cdot\left(\sum_{i=1}^{N}R_K(X_i)R_K(X_i)^{\intercal}\right)^{-1} \\
& \cdot\sum_{i=1}^{N}R_K(X_i)\cdot E\big[\mathbb{1}\{Y-(\widehat{q}_\tau-q_\tau)\le q_\tau\}|X_i\big] \\
& - E\left[\frac{d}{dX}\big\{E[\mathbb{1}\{Y-(\widehat{q}_\tau-q_\tau)\le q_\tau\}|X]\big\}\right] \\
& - \frac{1}{N}\sum_{i=1}^{N}\frac{dR_K(X_i)}{dX}\cdot\left(\sum_{i=1}^{N}R_K(X_i)R_K(X_i)^{\intercal}\right)^{-1}
\end{aligned}
$$

$$\cdot \sum_{i=1}^{N} R_K(X_i) \cdot E[\mathbb{1}\{Y \leq q_\tau\}|X_i] + E\left[\frac{d}{dX}\{E[\mathbb{1}\{Y \leq q_\tau\}|X]\}\right]$$

$$= O_p(N^{-1/2}) \cdot (\widehat{q}_\tau - q_\tau) = O_p(N^{-1}),$$

since the average derivative estimator should converge at the parametric rate according to Theorem 8. Thus

$$\widehat{\pi}_K(\widehat{q}_\tau) - \widehat{\pi}_K(q_\tau) = E\left[\frac{d}{dX}\{\Pr[Y - (\widehat{q}_\tau - q_\tau) \leq q_\tau|X] - \Pr[Y \leq q_\tau|X]\}\right] + O_p(N^{-1}).$$

Now, we define for convenience $U_N/\sqrt{N} = \widehat{q}_\tau - q_\tau = O_p(N^{-1/2})$. Note that for any $\widetilde{q}_\tau$ between $\widehat{q}_\tau$ and q_τ we can write $\widetilde{q}_\tau - q_\tau = C \cdot U_N/\sqrt{N} = O_p(N^{-1/2})$ for a constant C. We have, therefore,

$$\begin{aligned}
&\left|\Pr[Y \leq \widehat{q}_\tau|X] - \Pr[Y \leq q_\tau|X] - f_{Y|X}(q_\tau|X) \cdot (\widehat{q}_\tau - q_\tau)\right| \\
&= \big|E\big[\Pr[Y \leq q_\tau + U_N/\sqrt{N}|X, U_N] - \Pr[Y \leq q_\tau|X, U_N] \\
&\quad - f_{Y|X,U_N}(q_\tau|X, U_N) \cdot U_N/\sqrt{N}|X\big]\big| \\
&= \big|E\big[\big(f_{Y|X,U_N}(\widetilde{q}_\tau|X, U_N) - f_{Y|X,U_N}(q_\tau|X, U_N)\big) \cdot U_N/\sqrt{N}|X\big]\big| \\
&\leq \frac{C}{N} \cdot \big|E\big[\big(f'_{Y|X,U_N}(\widetilde{q}^*_\tau|X, U_N)\big) \cdot U_N^2|X\big]\big| \\
&= O_p(N^{-1}).
\end{aligned}$$

Therefore, we have

$$\widehat{\pi}_K(\widehat{q}_\tau) - \widehat{\pi}_K(q_\tau) = E\left[\frac{d}{dX} f_{Y|X}(q_\tau|X)\right] \cdot (\widehat{q}_\tau - q_\tau) + O_p(N^{-1})$$

and, moving backward, we can write

$$\begin{aligned}
\widehat{\pi}_K(\widehat{q}_\tau) - \widehat{\pi}_K(q_\tau) &= \frac{1}{N}\sum_{i=1}^{N} \frac{dR_K(X_i)}{dX}\left(\sum_{i=1}^{N} R_K(X_i)R_K(X_i)^\intercal\right)^{-1} \\
&\quad \cdot \sum_{i=1}^{N}\{R_K(X_i) \cdot f_{Y|X}(q_\tau|X_i)\} \cdot (\widehat{q}_\tau - q_\tau) + \operatorname{rem} D \\
&= \widehat{\pi}'^*_K(q_\tau) \cdot (\widehat{q}_\tau - q_\tau) + \operatorname{rem} D,
\end{aligned}$$

where

$$\operatorname{rem} D = -(\widehat{\pi}_K'^{*}(q_\tau) - \pi'(q_\tau)) \cdot (\widehat{q}_\tau - q_\tau) + O_p(N^{-1}).$$

According to Lemma 6,

$$\operatorname{rem} D = O_p(N^{-1}).$$

Now, by the same line of reasoning used before,

$$\begin{aligned}
&|\widehat{f}_Y(\widehat{q}_\tau) - \widehat{f}_Y(q_\tau) - \widehat{f}_Y'(q_\tau) \cdot (\widehat{q}_\tau - q_\tau)| \\
&= \left| \frac{1}{Nh} \sum_{i=1}^{N} K\left(\frac{Y_i - \widehat{q}_\tau}{h}\right) - K\left(\frac{Y_i - q_\tau}{h}\right) \right. \\
&\qquad \left. + \frac{1}{Nh^2} \sum_{i=1}^{N} K'\left(\frac{Y_i - q_\tau}{h}\right) \cdot (\widehat{q}_\tau - q_\tau) \right| \\
&= \left| -\frac{1}{Nh^2} \sum_{i=1}^{N} \left(K'\left(\frac{Y_i - \widetilde{q}_\tau}{h}\right) - K'\left(\frac{Y_i - q_\tau}{h}\right)\right) \cdot (\widehat{q}_\tau - q_\tau) \right| \\
&\leq C \cdot (\widehat{q}_\tau - q_\tau)^2 = O_p(N^{-1}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\widehat{\kappa}_K(\widehat{q}_\tau) - \widehat{\kappa}_K(q_\tau) \\
&= \frac{\widehat{\pi}_K(\widehat{q}_\tau) - \widehat{\pi}_K(q_\tau)}{\widehat{f}_Y(q_\tau)} - \frac{\widehat{\pi}_K(q_\tau)}{\widehat{f}_Y(q_\tau)} \cdot \left(\frac{\widehat{f}_Y(\widehat{q}_\tau) - \widehat{f}_Y(q_\tau)}{\widehat{f}_Y(q_\tau)} \right) + \operatorname{rem} A \\
&= \left(\frac{\widehat{\pi}_K'^{*}(q_\tau)}{\widehat{f}_Y(q_\tau)} - \frac{\widehat{\pi}_K(q_\tau)}{\widehat{f}_Y(q_\tau)} \cdot \frac{\widehat{f}_Y'(q_\tau)}{\widehat{f}_Y(q_\tau)} \right) \cdot (\widehat{q}_\tau - q_\tau) + \operatorname{rem}_{\mathrm{NP},2},
\end{aligned}$$

so we have obtained that

$$\begin{aligned}
\operatorname{rem}_{\mathrm{NP},2} &= \operatorname{rem} A + \frac{1}{\widehat{f}_Y(q_\tau)} \cdot \operatorname{rem} D \\
&\quad - \frac{\widehat{\pi}_K(q_\tau)}{\widehat{f}_Y^2(q_\tau)} \cdot (\widehat{f}_Y(\widehat{q}_\tau) - \widehat{f}_Y(q_\tau) - \widehat{f}_Y'(q_\tau) \cdot (\widehat{q}_\tau - q_\tau)) \\
&= O_p(N^{-1}). \qquad \textit{Q.E.D.}
\end{aligned}$$

PROOF OF LEMMA 8: We have

$$\operatorname{rem}_{\mathrm{NP},3} = \left(\left(\frac{\widehat{\pi}_K'^{*}(q_\tau)}{\widehat{f}_Y(q_\tau)} - \frac{\widehat{\pi}_K(q_\tau)}{\widehat{f}_Y(q_\tau)} \cdot \frac{\widehat{f}_Y'(q_\tau)}{\widehat{f}_Y(q_\tau)} \right) \right.$$

$$
\begin{aligned}
&\quad - \left(\frac{\pi'(q_\tau)}{f_Y(q_\tau)} - \frac{\pi(q_\tau)}{f_Y(q_\tau)} \cdot \frac{f'_Y(q_\tau)}{f_Y(q_\tau)} \right) \Bigg) \cdot (\widehat{q}_\tau - q_\tau) + O_p(N^{-1}) \\
&= \Bigg(\frac{\widehat{\pi}'^{*}_K(q_\tau)}{\widehat{f}_Y(q_\tau)} - \frac{\pi'(q_\tau)}{f_Y(q_\tau)} \\
&\quad - \left[\left(\frac{\widehat{\pi}_K(q_\tau)}{\widehat{f}_Y(q_\tau)} \cdot \frac{\widehat{f}'_Y(q_\tau)}{\widehat{f}_Y(q_\tau)} \right) - \left(\frac{\pi(q_\tau)}{f_Y(q_\tau)} \cdot \frac{f'_Y(q_\tau)}{f_Y(q_\tau)} \right) \right] \Bigg) \\
&\quad \cdot (\widehat{q}_\tau - q_\tau) + O_p(N^{-1}).
\end{aligned}
$$

Now, we look at the terms multiplying $\widehat{q}_\tau - q_\tau$:

$$
\begin{aligned}
\frac{\widehat{\pi}'^{*}_K(q_\tau)}{\widehat{f}_Y(q_\tau)} - \frac{\pi'(q_\tau)}{f_Y(q_\tau)} &= \frac{1}{f_Y(q_\tau)} \cdot (\widehat{\pi}'^{*}_K(q_\tau) - \pi'(q_\tau)) \\
&\quad - \frac{\pi'(q_\tau)}{f^2_Y(q_\tau)} \cdot (\widehat{f}_Y(q_\tau) - f_Y(q_\tau)) + \operatorname{rem} E,
\end{aligned}
$$

where

$$
\begin{aligned}
\operatorname{rem} E &= - \left(\frac{(\widehat{\pi}'^{*}_K(q_\tau) - \pi'(q_\tau))}{f_Y(q_\tau)} - \frac{\pi'(q_\tau)}{f_Y(q_\tau)} \cdot \frac{(\widehat{f}_Y(q_\tau) - f_Y(q_\tau))}{f_Y(q_\tau)} \right) \\
&\quad \cdot \left(\frac{\widehat{f}_Y(q_\tau) - f_Y(q_\tau)}{\widehat{f}_Y(q_\tau)} \right) \\
&= O_p((Nh)^{-1}) + O(h^4) + O_p\big(N^{-1/2} h^{3/2}\big)
\end{aligned}
$$

because $\widehat{\pi}'^{*}_K(q_\tau) - \pi'(q_\tau) = O_p(N^{-1/2})$. The second term is

$$
\begin{aligned}
&\left(\frac{\widehat{\pi}_K(q_\tau)}{\widehat{f}_Y(q_\tau)} \cdot \frac{\widehat{f}'_Y(q_\tau)}{\widehat{f}_Y(q_\tau)} \right) - \left(\frac{\pi(q_\tau)}{f_Y(q_\tau)} \cdot \frac{f'_Y(q_\tau)}{f_Y(q_\tau)} \right) \\
&\quad = \frac{f^2_Y(q_\tau) f'_Y(q_\tau)}{f^4_Y(q_\tau)} \cdot (\widehat{\pi}_K(q_\tau) - \pi(q_\tau)) \\
&\qquad - 2 \frac{\pi(q_\tau) f_Y(q_\tau) f'_Y(q_\tau)}{f^4_Y(q_\tau)} \cdot (\widehat{f}_Y(q_\tau) - f_Y(q_\tau)) \\
&\qquad + \frac{\pi'(q_\tau) f^2_Y(q_\tau)}{f^4_Y(q_\tau)} \cdot (\widehat{f}'_Y(q_\tau) - f'_Y(q_\tau)) + \operatorname{rem} F,
\end{aligned}
$$

where

$$
\begin{aligned}
\operatorname{rem} F &= f_Y(q_\tau)^{-4} \big(f_Y(q_\tau)^2 (\widehat{\pi}_K(q_\tau) - \pi(q_\tau))(\widehat{f}'_Y(q_\tau) - f'_Y(q_\tau)) \\
&\qquad - \pi(q_\tau) f'_Y(q_\tau)(\widehat{f}_Y(q_\tau) - f_Y(q_\tau))(\widehat{f}_Y(q_\tau) - f_Y(q_\tau)) \big)
\end{aligned}
$$

$$
\begin{aligned}
&\quad \cdot \left(\frac{\widehat{f}_Y(q_\tau)^2 + f_Y(q_\tau)^2}{\widehat{f}_Y(q_\tau)^2}\right) \\
&\quad - f_Y(q_\tau)^{-4}\big(f_Y(q_\tau)^2\big(\pi(q_\tau)(\widehat{f}'_Y(q_\tau) - f'_Y(q_\tau)) \\
&\quad + f'_Y(q_\tau)(\widehat{\pi}_K(q_\tau) - \pi(q_\tau))\big) \\
&\quad - 2\pi(q_\tau) f_Y(q_\tau) f'_Y(q_\tau)(\widehat{f}_Y(q_\tau) - f_Y(q_\tau))\big) \\
&\quad \cdot \left(\frac{\widehat{f}_Y(q_\tau)^2 - f_Y(q_\tau)^2}{\widehat{f}_Y(q_\tau)^2}\right) \\
&= O_p\big(|\widehat{\pi}_K(q_\tau) - \pi(q_\tau)| \cdot |\widehat{f}'_Y(q_\tau) - f'_Y(q_\tau)|\big) \\
&\quad + O_p\big(|\widehat{f}_Y(q_\tau) - f_Y(q_\tau)|^2\big) \\
&\quad + O_p\big(|\widehat{f}_Y(q_\tau) - f_Y(q_\tau)| \cdot |\widehat{f}'_Y(q_\tau) - f'_Y(q_\tau)|\big) \\
&= O(h^4) + O_p((Nh^3)^{-1}) + O_p\big(N^{-1/2}h^{1/2}\big)
\end{aligned}
$$

because $\widehat{f}'_Y(q_\tau) - f'_Y(q_\tau) = O_p((Nh^3)^{-1/2}) + O(h^2)$. Putting together these results, we have

$$
\begin{aligned}
&\mathrm{rem}_{\mathrm{NP},3} \\
&= \left(\left(\frac{\widehat{\pi}'^{*}_K(q_\tau)}{\widehat{f}_Y(q_\tau)} - \frac{\widehat{\pi}_K(q_\tau)}{\widehat{f}_Y(q_\tau)} \cdot \frac{\widehat{f}'_Y(q_\tau)}{\widehat{f}_Y(q_\tau)}\right) - \left(\frac{\pi'(q_\tau)}{f_Y(q_\tau)} - \frac{\pi(q_\tau)}{f_Y(q_\tau)} \cdot \frac{f'_Y(q_\tau)}{f_Y(q_\tau)}\right)\right) \\
&\quad \cdot (\widehat{q}_\tau - q_\tau) + O_p(N^{-1}) \\
&= \left(\frac{\widehat{\pi}'^{*}_K(q_\tau) - \pi'(q_\tau)}{f_Y(q_\tau)} - \frac{\pi'(q_\tau)}{f_Y(q_\tau)} \cdot \left(\frac{\widehat{f}_Y(q_\tau) - f_Y(q_\tau)}{f_Y(q_\tau)}\right) + \mathrm{rem}\, E\right. \\
&\quad - \left[\frac{f'_Y(q_\tau)}{f_Y(q_\tau)} \cdot \frac{(\widehat{\pi}_K(q_\tau) - \pi(q_\tau))}{f_Y(q_\tau)}\right. \\
&\quad - 2\frac{\pi(q_\tau)}{f_Y(q_\tau)} \cdot \frac{f'_Y(q_\tau)}{f_Y(q_\tau)} \cdot \left(\frac{\widehat{f}_Y(q_\tau) - f_Y(q_\tau)}{f_Y(q_\tau)}\right) \\
&\quad \left.\left. + \frac{\pi'(q_\tau)}{f_Y(q_\tau)} \cdot \left(\frac{\widehat{f}'_Y(q_\tau) - f'_Y(q_\tau)}{f_Y(q_\tau)}\right)\right] + \mathrm{rem}\, F\right) \\
&\quad \cdot (\widehat{q}_\tau - q_\tau) + O_p(N^{-1}) \\
&= O_p(N^{-1}) + O_p\big(N^{-1/2}\big(O_p\big((Nh)^{-1/2}\big) + O(h^2)\big)\big) \\
&\quad + O_p\big(N^{-1/2}\big(O_p((Nh)^{-1}) + O(h^4) + O_p\big(N^{-1/2}h^{3/2}\big)\big)\big) \\
&\quad + O_p(N^{-1}) + O_p\big(N^{-1/2}\big(O_p\big((Nh)^{-1/2}\big) + O(h^2)\big)\big)
\end{aligned}
$$

$$+ O_p\big(N^{-1/2}\big(O_p\big((Nh^3)^{-1/2}\big) + O(h^2)\big)\big)$$
$$+ O_p\big(N^{-1/2}\big(O(h^4) + O_p((Nh^3)^{-1}) + O_p\big(N^{-1/2}h^{1/2}\big)\big)\big)$$
$$= O_p\big(N^{-1/2}h^2\big) + O_p\big(N^{-1}h^{-3/2}\big) + O_p\big(N^{-3/2}h^{-3}\big). \qquad \textit{Q.E.D.}$$

The proof of Theorem 9 follows trivially from a combination of previous results. The proof of Theorem 10 follows from a central limit theorem and follows the same reasoning as Theorem 3. The proof of Proposition 5 follows the same lines as Proposition 1.

PROOF OF PROPOSITION 6: All estimators involved are consistent estimators to population parameters, which guarantees consistency of the variance estimator. *Q.E.D.*

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