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SUPPLEMENT TO "WAITING FOR NEWS IN THE MARKET FOR LEMONS": DISCRETE-TIME APPENDIX

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The purpose of this supplement is to establish the strong connection between the continuous-time model in the main paper and a discrete-time analog. First, when time periods are short, there exists an equilibrium with nearly identical structure to $\Xi(\alpha^*, \beta^*)$ (the equilibrium of interest in the main paper). Second, this equilibrium is unique among stationary equilibria satisfying a similar refinement on off-path beliefs and nondecreasing value functions (a condition we include merely for convenience). Third, as we take the period length to zero, the sequence of discrete-time equilibria converges to $\Xi(\alpha^*, \beta^*)$.

S.1. INTRODUCTION

THE PURPOSE OF THIS SUPPLEMENT is to establish the strong connection between the continuous-time model in the main paper and a discrete-time analog. For the sake of brevity, we only consider the case where the static lemons condition (SLC) holds. We show three main results. First, when time periods are short, there exists an equilibrium with nearly identical structure to $\Xi(\alpha^*, \beta^*)$. Second, this equilibrium is unique among stationary equilibria satisfying a similar refinement on off-path beliefs and nondecreasing value functions (a condition we include merely for convenience). Third, as we take the period length to zero, the sequence of discrete-time equilibria converges to $\Xi(\alpha^*, \beta^*)$.

There are three reasons to carry out this exercise. First, some concepts and intuitions may be easier to capture in a discrete-time setting. The reader can appeal to whichever framework s/he finds most useful for understanding any aspect of the model. Second, we can regard this as a robustness check. It is reassuring that the limit of the equilibrium of the discrete-time model is the equilibrium of the model posed directly in continuous time. Finally, after seeing the "integer problems" encountered in discrete time, the reader should be convinced that working in continuous time has considerable advantages.

REMARK S.1: To maintain consistency with the continuous-time model, we continue to represent the buyer side of the market as an offer process and retain the solution concepts given by Definition 2.1, to establish existence, and Definition 5.2, for uniqueness. However, identical results obtain under the alternative modeling specification in which two short-lived strategic buyers make private offers in each period (as in Remark 2.2) and the solution concept is given by perfect Bayesian equilibrium (supplemented with stationarity and belief monotonicity for Theorem S.2).

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For expediency, some arguments given here are less formal than the proofs in the main Appendix. There is, however, enough content that the interested reader should understand the mechanics of the discrete-time equilibrium and be able to fill in desired details.

S.2. SETUP

Everything is as in Section 2 with the following exceptions.

(i) The SLC holds: $K_H > V_L$.
(ii) W_t is defined if and only if $t = \Delta, 2\Delta, \ldots$ for some $\Delta > 0$.
(iii) For each type, the news process follows a random walk.

Let $q^\Delta \equiv \frac{1}{2}(1 + \frac{\mu}{\sigma}\sqrt{\Delta})$ for some $(\mu, \sigma) \in \mathbb{R}^2_{++}$, and, given any $X_{n\Delta}$, the conditional distribution of $X_{(n+1)\Delta}$ is as follows:

If $\theta = H$,

$$X_{(n+1)\Delta} = \begin{cases} X_{n\Delta} + \sigma\sqrt{\Delta} & \text{with probability } q^\Delta, \\ X_{n\Delta} - \sigma\sqrt{\Delta} & \text{with probability } 1 - q^\Delta. \end{cases}$$

If $\theta = L$,

$$X_{(n+1)\Delta} = \begin{cases} X_{n\Delta} + \sigma\sqrt{\Delta} & \text{with probability } 1 - q^\Delta, \\ X_{n\Delta} - \sigma\sqrt{\Delta} & \text{with probability } q^\Delta. \end{cases}$$

It is well known that these processes converge to the continuous-time processes in the body (with $\mu_H = \mu$, $\mu_L = -\mu$) as $\Delta \to 0$ (Cox and Miller (1965)). The assumption that the high type will get good news (i.e., X will increase) with the same probability that the low type gets bad news (i.e., X will decrease) is convenient: starting from any prior, the belief of a Bayesian who observes one instance of good news followed by one instance of bad news will arrive back at his prior. Therefore, given parameters μ, σ, and Δ, for any z, there exists a unique grid G_z on which $\hat{Z}$ (beliefs based only on observation of X) will reside.[1] G_z is a countable sequence of points that span the real line with $z \in G_z$. Let $z_{+n} \in G_z$ denote the posterior a Bayesian would arrive at starting from a prior of z after observing $n \in \mathbb{I}_+$ more instances of good news than bad news, and define z_{-n} analogously.

S.3. MAIN RESULTS

THEOREM S.1: *When Δ is small, there exists a unique pair $\alpha^{\Delta*} < \beta^{\Delta*}$ and an equilibrium that is almost identical to $\Xi(\alpha^{\Delta*}, \beta^{\Delta*})$, adapted to the countable time grid. The only exception is that $\Psi(\beta^{\Delta*})$ is offered at $z = \beta^{\Delta*}$ with probability λ, which may or may not be uniquely determined, depending on parameters.*

[1]Recall that “beliefs are z” translates to $\Pr(\theta = H) = \frac{e^z}{1+e^z}$.

For analysis of the discrete-time model, modify the definition of Belief Monotonicity (condition (vi) of Definition 5.2) as follows: If an unexpected rejection occurs when beliefs are Z_t, then $Z_{t+\Delta} - Z_t = \hat{Z}_{t+\Delta} - \hat{Z}_t$. That is, beliefs are unchanged by the act of rejecting unexpectedly, and so the condition is slightly stronger than the original definition of Belief Monotonicity. This simplifies matters by not having to consider that an unexpected rejection leads to a higher belief level that resides on a different grid. Notice that in continuous time, the belief process in $\Xi(\alpha^*, \beta^*)$—the unique SBM equilibrium (Theorem 5.1)—satisfies the continuous-time analog of this condition.

THEOREM S.2: *When Δ is small, the equilibrium of Theorem S.1 is the unique SBM equilibrium satisfying nondecreasing value functions (NDVF).*

Restricting attention to equilibria satisfying NDVF (Section 5.2) is only for convenience (and in the interest of keeping this supplement a reasonable length). It is possible to demonstrate uniqueness, following many of the same arguments employed in the proof of Theorem 5.1, without this restriction.

Finally,

THEOREM S.3: *As $\Delta \to 0$, the equilibrium of Theorem S.1 converges to $\Xi(\alpha^*, \beta^*)$ from the continuous-time model in the following ways:*
(i) *Equilibrium payoffs converge pointwise.*
(ii) *The stationary offer function converges pointwise.*
(iii) *Equilibrium seller strategies converge in distribution.*

S.4. B^Δ FUNCTIONS

The arguments for the equilibrium construction—uniqueness and convergence—will rely on the discrete-time analogs of the B_θ functions analyzed in Appendix B.3.

DEFINITION S.1: Fix a pair (α, β), $\beta \geq \alpha$, $\beta \in G_\alpha$. Define $\widetilde{F}^\Delta_L(z|\alpha, \beta)$ as the continuation value to the low type of being at $z \in G_\alpha$, $\alpha_{-1} \leq z \leq \beta$ under the following conditions:
(i) Z evolves based on Bayesian updating on the observation of X.
(ii) $\widetilde{F}^\Delta_L(\alpha_{-1}|\alpha, \beta) = V_L$.
(iii) $\widetilde{F}^\Delta_L(\beta|\alpha, \beta) = \Psi(\beta)$.

DEFINITION S.2: Fix a pair (α, β), $\beta \geq \alpha$, $\beta \in G_\alpha$. Define $\widetilde{F}^\Delta_H(z|\alpha, \beta)$ as the continuation value to the high type of being at $z \in G_\alpha$, $\alpha_{-1} \leq z \leq \beta_{+1}$ under the following conditions:
(i) Z evolves based on Bayesian updating on the observation of X, except as stated in (iii).
(ii) $\widetilde{F}^\Delta_H(\beta_{+1}|\alpha, \beta) = \Psi(\beta_{+1})$.

(iii) If $Z_t = \alpha_{-1}$, then $Z_{t+\Delta} = \alpha_{+1}$ with probability q^{Δ}, and $Z_{t+\Delta} = \alpha_{-1}$ with probability $(1 - q^{\Delta})$.

DEFINITION S.3: $B_L^{\Delta}(\alpha)$ is the maximum $\beta \geq \alpha$, $\beta \in G_{\alpha}$, such that $\widetilde{F}_L^{\Delta}(\alpha|\alpha, \beta) \geq V_L$.

DEFINITION S.4: $B_H^{\Delta}(\alpha)$ is the minimum $\beta \geq \alpha$, $\beta \in G_{\alpha}$, such that $\widetilde{F}_H^{\Delta}(\beta|\alpha, \beta) \leq \Psi(\beta)$.

Recall from the analysis of the continuous-time model in Appendix B.3 that B_H and B_L are well defined and continuous. They intersect at a single point α^*, below which $B_L < B_H$ and above which $B_L > B_H$. The following convergence result is very useful.

LEMMA S.1: *For $\theta \in \{L, H\}$, B_{θ}^{Δ} converges to B_{θ} uniformly as $\Delta \to 0$.*

The proof of Lemma S.1 is found in Section S.7.

S.5. EQUILIBRIUM CONSTRUCTION AND VERIFICATION

To establish equilibrium existence and uniqueness, we need to delve a little deeper into the structures of B_L^{Δ} and B_H^{Δ}.

FACT S.1: *B_L^{Δ} is right-continuous and strictly increasing (and therefore continuous almost everywhere).*

That B_L^{Δ} is increasing when it is continuous is obvious. To understand why it increases at points of discontinuity, consider an α and its corresponding $B_L^{\Delta}(\alpha)$, such that $\widetilde{F}_L^{\Delta}(\alpha|\alpha, B_L^{\Delta}(\alpha)) = V_L$. Let $B_L^{\Delta}(\alpha) = \alpha_{+n}$. Now, for an $\alpha' = \alpha - \varepsilon$, its corresponding $B_L^{\Delta}(\alpha')$ must be $\alpha'_{+(n-1)}$. Why? Suppose it maintained the same distance in grid points as the original α and $B_L^{\Delta}(\alpha)$. Then every path of X results in the same behavior up and down a subgrid of equal size (n), endowing the same flow payoff and payoff at the lower terminal node. However, every path that ends at the upper terminal node receives a slightly lower terminal payoff. Because the original terminal payoff of $\Psi(B_L^{\Delta}(\alpha))$ was the exact amount to make $\widetilde{F}_L^{\Delta}(\alpha|\alpha, B_L^{\Delta}(\alpha)) = V_L$, the new lower terminal payoff of $\Psi(B_L^{\Delta}(\alpha'))$ will cause $\widetilde{F}_L^{\Delta}(\alpha'|\alpha', B_L^{\Delta}(\alpha'))$ to fall below V_L. The solution is to decrease $B_L^{\Delta}(\alpha')$ by one grid point to $\alpha'_{+(n-1)}$. This problem only occurs when $\widetilde{F}_L^{\Delta}(\alpha|\alpha, B_L^{\Delta}(\alpha)) = V_L$. If $\widetilde{F}_L^{\Delta}(\alpha|\alpha, B_L^{\Delta}(\alpha)) > V_L$ (and ε is small enough), then α' and $B_L^{\Delta}(\alpha')$ will also be separated by the same number of grid points as the original α and $B_L^{\Delta}(\alpha)$ were. This yields the continuous intervals in B_L^{Δ}. We can see the same logic from the opposite perspective, one of increasing α. Start with some $(\alpha, B_L^{\Delta}(\alpha))$ pair separated by n grid points. As we continuously increase α, if we keep $B_L^{\Delta}(\alpha)$ the same number of grid points away, $\widetilde{F}_L^{\Delta}(\alpha|\alpha, \alpha_{+n})$ increases as well. Eventually, $\widetilde{F}_L^{\Delta}(\alpha|\alpha, \alpha_{n+1}) = V_L$ and $B_L^{\Delta}(\alpha)$ therefore jumps up by one grid point.

FACT S.2: *B^{Δ}_H is right-continuous and continuous almost everywhere. B^{Δ}_H is strictly increasing on any interval on which it is continuous, but* decreases *at every point of discontinuity.*

The argument for this property is similar to the one above with the following difference. Consider an α^0 such that $B^{\Delta}_H(\alpha^0) = \alpha^0_{+n}$. Now, as we continuously increase α from α^0, we can check whether $B^{\Delta}_H(\alpha)$ can remain n grid points above. The answer is it cannot: as α increases, the payoff to waiting at $\alpha_{+(n-1)}$ falls until $\widetilde{F}^{\Delta}_H(\alpha_{+(n-1)}|\alpha, \alpha_{+n}) = \Psi(\alpha_{+(n-1)})$. Therefore, $B^{\Delta}_H(\alpha)$ must jump down by one grid point to $\alpha_{+(n-1)}$ when this occurs. The reason for this is that, just as in the continuous-time model, for any α, $B^{\Delta}_H(\alpha)$ is on the decreasing portion of (the discrete-time analog of) MB_H (Appendix A.2), meaning that the marginal gain from waiting at higher beliefs is getting lower.

Not surprisingly, the equilibrium results will rely on intersection of B^{Δ}_L and B^{Δ}_H. Generically, the two functions intersect. The convergence of these functions to their continuous-time analogs gives that, for small enough Δ, $B^{\Delta}_L < B^{\Delta}_H$ for small α and $B^{\Delta}_L > B^{\Delta}_H$ for large α. The fact that the discontinuities in both functions are always on the order of one grid point implies that the only way the two functions will fail to intersect is if they are both discontinuous at some α such that $B^{\Delta}_L < B^{\Delta}_H$ for all $\alpha' < \alpha$ and $B^{\Delta}_L > B^{\Delta}_H$ for all $\alpha' > \alpha$. Designate this occurrence as a "near intersection." In addition to being nongeneric, a near intersection does not pose a problem for equilibrium existence or uniqueness (see Case 2 below).

Identifying $(\alpha^{\Delta *}, \beta^{\Delta *})$ and λ

Analogous to the notation in the continuous-time model, F^{Δ}_θ refers to the equilibrium value function of type θ in a model with period length Δ. Notice that for the candidate equilibrium of Theorem S.1, $F^{\Delta}_\theta(z) \approx \widetilde{F}^{\Delta}_\theta(z|\alpha^{\Delta *}, \beta^{\Delta *})$ for all $z \in G_{\alpha^{\Delta *}}, \alpha^{\Delta *}_{-1} \le z \le \beta^{\Delta *}_{+1}$. There are two cases to consider: Case 1, B^{Δ}_L and B^{Δ}_H intersect or Case 2, they do not. Start with Case 1 and let α be a point of intersection. α is an element of two half-open intervals $[\underline{\alpha}_L, \overline{\alpha}_L)$ and $[\underline{\alpha}_H, \overline{\alpha}_H)$, where B^{Δ}_θ is continuous on $[\underline{\alpha}_\theta, \overline{\alpha}_\theta)$. Given that the B^{Δ}_θ curves intersect at α, $\max\{\underline{\alpha}_L, \underline{\alpha}_H\}$ is also a point of intersection. Let $\alpha^{\Delta *} = \max\{\underline{\alpha}_L, \underline{\alpha}_H\}$.

Case 1. (A) If $\underline{\alpha}_L \le \underline{\alpha}_H$, then $\beta^{\Delta *} = B^{\Delta}_\theta(\alpha^{\Delta *})$ for $\theta = L, H$. Define λ to be the probability that $w(\beta^{\Delta *}) = \Psi(\beta^{\Delta *})$. Recall that $\widetilde{F}^{\Delta}_L(\alpha^{\Delta *}|\alpha^{\Delta *}, B^{\Delta}_L(\alpha^{\Delta *})) \ge V_L$ and $\widetilde{F}^{\Delta}_L(\alpha^{\Delta *}|\alpha^{\Delta *}, (B^{\Delta}_L(\alpha^{\Delta *}))_{+1}) < V_L$. There is then a unique probability $\lambda \in (0, 1]$ such that if $w(B^{\Delta}_L(\alpha^{\Delta *})) = \Psi(B^{\Delta}_L(\alpha^{\Delta *}))$ with probability λ and $w((B^{\Delta}_L(\alpha^{\Delta *}))_{+1}) = \Psi((B^{\Delta}_L(\alpha^{\Delta *}))_{+1})$ with probability 1 that the low type's value at $\alpha^{\Delta *}$ will be exactly V_L. Let this be the equilibrium value of λ.

(B) If $\underline{\alpha}_L > \underline{\alpha}_H$, then $(B^{\Delta}_\theta(\alpha^{\Delta *}))_{-1} < \beta^{\Delta *} < B^{\Delta}_\theta(\alpha^{\Delta *})$ for $\theta = L, H$ and any $\lambda \in [0, 1]$ is consistent with equilibrium. We will describe how to precisely pin down $\beta^{\Delta *}$ in the verification stage of this argument.

Case 2. Finally, if B_L^Δ and B_H^Δ do not intersect, as discussed above, there is then a z where they nearly intersect. This z is $\alpha^{\Delta*}$. $\beta^{\Delta*} = B_H^\Delta(\alpha^{\Delta*}) = (B_L^\Delta(\alpha^{\Delta*}))_{-1}$ and $\lambda = 0$.

Equilibrium Verification

Let us take the cases in turn:

Case 1. (A) The Belief Consistency and Zero Profit conditions are immediate. For No Deals, as in the verification of the continuous-time model, it is sufficient to check that the condition holds for all $z \in [\alpha^{\Delta*}, \beta^{\Delta*}]$. First, consider only states z in $G_{\alpha^{\Delta*}}$. For $\theta = H$, suppose there exists a state $z \in [\alpha^{\Delta*}, \beta^{\Delta*}] \cap G_{\alpha^{\Delta*}}$ such that $F_H^\Delta(z) < \Psi(z)$, and let z' be the maximum such z. Then $\widetilde{F}_H^\Delta(z'|\alpha^{\Delta*}, z'_{+1}) \leq F_H^\Delta(z') < \Psi(z')$, contradicting the premise that $\beta^{\Delta*} = B_H^\Delta(\alpha^{\Delta*})$ (Definition S.4). For $\theta = L$, we prove by induction. Suppose that for some z, (i) $F_L^\Delta(z) \geq F_L^\Delta(z_{-1}) \geq V_L$ and (ii) the low type rejects with positive probability at z. Then

$$F_L^\Delta(z) = \int_0^\Delta e^{-rt} r K_L\, dt + e^{-r\Delta}[q^\Delta F_L^\Delta(z_{-1}) + (1-q^\Delta) F_L^\Delta(z_{+1})]. \tag{S.1}$$

Given premise (i) and $K_L < V_L$, (S.1) implies that $F_L^\Delta(z_{+1}) > F_L^\Delta(z)$. Now notice that premise (ii) is true of all $z \in [\alpha^{\Delta*}, \beta^{\Delta*}] \cap G_{\alpha^{\Delta*}}$ and that premise (i) is true of $z = \alpha^{\Delta*}$: $V_L = F_L^\Delta(\alpha^{\Delta*}) \geq F_L^\Delta(\alpha_{-1}^{\Delta*}) = V_L$. Hence, No Deals holds for all $z \in [\alpha^{\Delta*}, \beta^{\Delta*}] \cap G_{\alpha^{\Delta*}}$. To extend the result to states in the no-trade region but off the grid, consider now a $z \in [z_1, z_2)$, where $z_1, z_2 \in [\alpha^{\Delta*}, \beta^{\Delta*}] \cap G_{\alpha^{\Delta*}}$. Because the low type is never supposed to accept V_L with probability 1, her equilibrium payoff must be identical to the one she achieves by rejecting V_L always and waiting until $Z \geq \beta^{\Delta*}$ and Ψ is offered (i.e., playing the same strategy as the high type). Therefore, regardless of type, for any realization of $\{X_t\}_{t=0,1,\dots}$ the seller's payoff is weakly increasing (and differentiable) for $z \in [z_1, z_2)$ because it takes the same number of periods for Z_t to rise above $\beta^{\Delta*}$, but the payoff upon doing so is increasing in the first value of $Z_t > \beta^{\Delta*}$, which is in turn weakly increasing in the starting value z. This completes the argument for $\theta = L$. For $\theta = H$, because the high type is indifferent at β^{Δ^*}, her payoff is continuous at z_2 as well (to see this, note first that it is true for $z_1, z_2 = \beta_{-1}^{\Delta^*}, \beta^{\Delta^*}$ because of the high type's indifference at β^{Δ^*}; then iterate the argument down the grid). The result then follows from Taylor's approximation theorem. Finally, for Seller Optimality, the only consideration that is not immediate is ensuring that the high type does not want to reject for any $z > \beta^{\Delta*}$. However, this follows from an argument that the discrete-time analog of $\mathrm{MB}_H(z) < 0$ for all $z > \beta^{\Delta*}$, just as in Lemma B.3, since the high-type seller's problem is approximately the same when Δ is small.

(B) The only difference from the first case is that at $B_H^\Delta(\alpha^{\Delta*})$, the high type *strictly* prefers to accept, which will imply $\beta^{\Delta*} < B_H^\Delta(\alpha^{\Delta*})$. In this case,

$w(B_H^\Delta(\alpha^{\Delta*})) = \Psi(B_H^\Delta(\alpha^{\Delta*}))$ with probability 1. Because $\alpha^{\Delta*} = \underline{\alpha}_L$, it must be that $F_L^\Delta(\alpha^{\Delta*}) = V_L$, creating the necessary indifference between acceptance and continuation for $z < \alpha^{\Delta*}$. The high type strictly prefers to accept at $z = B_H^\Delta(\alpha^{\Delta*})$, but she strictly prefers to reject $\Psi((B_H^\Delta(\alpha^{\Delta*}))_{-1})$ at $z = (B_H^\Delta(\alpha^{\Delta*}))_{-1}$. It follows from the structure of the high-type seller's problem that there exists a unique $\beta^{\Delta*} \in ((B_\theta^\Delta(\alpha^{\Delta*}))_{-1}, B_\theta^\Delta(\alpha^{\Delta*}))$, where the high type will reject $\Psi(z)$ for $z < \beta^{\Delta*}$ and accept $\Psi(z)$ for $z > \beta^{\Delta*}$. At $\beta^{\Delta*}$, the high type is indifferent between accepting and rejecting $\Psi(\beta^{\Delta*})$. Because $\beta^{\Delta*} \notin G_{\alpha^{\Delta*}}$, the behavior at $\beta^{\Delta*}$ does not affect the low type's value at $\alpha^{\Delta*}$ and, therefore, λ can be any element of $[0, 1]$.

Case 2. The case of near intersection is very much like Case 1(B). The high type strictly prefers to accept $\Psi(z)$ at $z > B_H^\Delta(\alpha^{\Delta*}) = \beta^{\Delta*}$, but she is indifferent at $z = B_H^\Delta(\alpha^{\Delta*})$. $B_L^\Delta(\alpha^{\Delta*}) = (B_H^\Delta(\alpha^{\Delta*}))_{+1}$ implies that $B_H^\Delta(\alpha^{\Delta*})$ is an element of $G_{\alpha^{\Delta*}}$. Therefore, unlike Case 1(B), the probability with which $\Psi(\beta^{\Delta*})$ is offered at $\beta^{\Delta*}$ does affect $F_L^\Delta(\alpha^{\Delta*})$. To maintain $F_L^\Delta(\alpha^{\Delta*}) = V_L$, λ must be zero.

S.6. EQUILIBRIUM UNIQUENESS

Here we provide an argument for the uniqueness claim of Theorem S.2. It is immediate that by extending $\Xi(\alpha^{\Delta*}, \beta^{\Delta*})$ off the equilibrium path in the same manner as done in Section 5.2, we have an SBM equilibrium. The verification argument above demonstrated that F_H^Δ and F_L^Δ are nondecreasing.

Individual rationality implies that the high type will never trade when beliefs are $z < \underline{z}$. Therefore, just as in the continuous-time model, there exists a z_0 such that the low type trades at a price of V_L with positive probability (that is less than 1) when beliefs are z_0. Let $\mathcal{Z}$ be the set of all such z_0, and denote its supremum as z_0^s. Therefore, for all $z > z_0^s$, the evolution of the market belief is governed completely by the realization of news.

Starting from any $z_0 \in \mathcal{Z}$, rejection leads to a discontinuous increase (or jump) in beliefs to some $j(z_0) > z_0$. For the low type to be indifferent, it must be that continuation from $j(z_0)$ *when beliefs evolve only based on news* endows expected utility V_L; that is,

$$\text{(S.2)}\qquad \int_0^\Delta e^{-rt} r K_L\, dt + e^{-r\Delta}\big[q^\Delta F_L^\Delta(j(z_0)_{-1}) + (1-q^\Delta) F_L^\Delta(j(z_0)_{+1})\big] = V_L.$$

From (S.2), we can deduce the following. Given that $F_L^\Delta(j(z_0)_{-1}) \geq V_L$ (from No Deals), when Δ is small, $F_L^\Delta(j(z_0)_{+1}) \approx V_L$ to maintain equation (S.2) and F_L^Δ nondecreasing. So $F_L^\Delta(j(z_0)) \approx V_L$ as well. But then, it must be that $F_L^\Delta(j(z_0)) = V_L$. This follows because F_L^Δ is the maximum of the current offer and the continuation value. The continuation value is V_L (equation (S.2)), and it cannot be that the current offer is arbitrarily close to, but larger than, V_L because such an offer attracts only the low type and, therefore, violates No Deals. Hence $F_L^\Delta(j(z_0)) = V_L$, and $F_L^\Delta(j(z_0)_{-1}) = V_L$ by F_L^Δ nondecreasing and bounded by V_L. Finally, substituting V_L for $F_L^\Delta(j(z_0)_{-1})$ into equation (S.2) uniquely pins down $F_L^\Delta(j(z_0)_{+1}) > V_L$.

Let $\mathcal{J} = \{j : j = j(z_0), z_0 \in \mathcal{Z}\}$. F_L^Δ nondecreasing and No Deals imply that $F_L^\Delta(z) = V_L$ for all $z < \sup \mathcal{J}$. $F_L^\Delta(j_{+1}) > V_L$ for all j implies that $\inf \mathcal{J} \geq (\sup \mathcal{J})_{-1}$. Our next step is to demonstrate that $\mathcal{J}$ is a singleton.

Just as knowing $F_L^\Delta(j_{-1})$ and $F_L^\Delta(j)$ allowed us to pin down $F_L^\Delta(j_{+1}) > F_L^\Delta(j)$, knowing $F_L^\Delta(j)$ and $F_L^\Delta(j_{+1})$ allows us to pin down $F_L^\Delta(j_{+2}) > F_L^\Delta(j_{+1})$, and so on as long as F_L^Δ is governed by its continuation value. This will hold until $F_L^\Delta(j_{+n}) \leq \Psi(j_{+n})$ and (if driven solely by continuation value) $F_L^\Delta(j_{+n+1}) > \Psi(j_{+n+1})$ (i.e., $j_{+n} = B_L^\Delta(j)$).[2] Since the latter is not possible in equilibrium, it must be that $F_L^\Delta(B_L^\Delta(j))$ is not derived solely from continuation value. By similar argument used in Section S.5, there is a uniquely determined $\nu \in (0, 1]$ such that $\Psi(B_L^\Delta(j))$ must be offered with probability ν at $B_L^\Delta(j)$ and accepted with probability 1.[3] Therefore, for any $j \in \mathcal{J}$, equilibrium behavior is uniquely determined for all $z \leq \Psi(B_L^\Delta(j)_{+1})$ residing on the same grid as j. We will now show that this behavior is consistent with optimization by the high type for exactly one value of j.

To establish that $\mathcal{J}$ contains a single element, suppose it does not. For clarity, assume that B_L^Δ and B_H^Δ intersect.[4] Recall that B_L^Δ and B_H^Δ, therefore, coincide on a half-open interval $[\underline{\alpha}, \overline{\alpha})$.

Consider $\underline{j} \equiv \min(j \in \mathcal{J})$.[5] We claim that $\underline{j} \geq \underline{\alpha}$. Suppose that it were not so. The low type's value at $\underline{j}$ is V_L, where $\underline{j}$ is the lowest point of a no-trade region on grid $G_{\underline{j}}$. Hence, there is a unique profile of offers for beliefs on $G_{\underline{j}}$ consistent with $F_L^\Delta(\underline{j}) = V_L$:

- No trade at any z, $\underline{j} \leq z < B_L^\Delta(\underline{j})$.
- At $B_L^\Delta(\underline{j})$, $\Psi(B_L^\Delta(\underline{j}))$ is offered with a uniquely determined $\nu_{\underline{j}} \in (0, 1]$.
- At $B_L^\Delta(\underline{j}_{+1})$, $\Psi(B_L^\Delta(\underline{j}_{+1}))$ is offered with probability 1.

However, from our analysis of the B_θ^Δ curves, we know this cannot hold in equilibrium, because $\underline{j} < \underline{\alpha}$, $B_L^\Delta(\underline{j}) < B_H^\Delta(\underline{j})$. Therefore, if reflection occurred as in the definition of B_H^Δ, the high type prefers to reject $\Psi(B_L^\Delta(\underline{j}))$ in favor of continuation when $z = B_L^\Delta(\underline{j})$. The reflection, however, may not take place as in the definition of B_H^Δ. But because $\underline{j}$ is the minimum $j \in \mathcal{J}$, the reflection process must be at least as favorable to the high type as the one in the definition of B_H^Δ (if Z_t falls below $\underline{j}$, it will jump to some $j \geq \underline{j}$). She is then at least as willing to reject when $z = B_L^\Delta(\underline{j})$. If the high type will not accept $\Psi(B_L^\Delta(\underline{j}))$ if

[2]It is straightforward to show (by induction) that $F_L^\Delta(z_{+n}) - F_L^\Delta(z_{+n-1})$ (constructed in this way) is increasing in n, which ensures that such an n exists.

[3]Notice that ν can be distinct from λ if and only if $B_L^\Delta(j) \neq \beta^{\Delta *}$.

[4]The argument for the case when the curves only "nearly intersect" is analogous, though slightly more nuanced.

[5]This assumes the minimum exists. However, there is a straightforward extension of the argument considering an element of $\mathcal{J}$ arbitrarily close to $\inf \mathcal{J}$.

it is offered, it will not be offered. This contradiction implies that $\underline{j} \geq \underline{\alpha}$. The analogous argument demonstrates that every element of $\mathcal{J}$ is less than $\overline{\alpha}$.

We now know that every element of $\mathcal{J}$ is an element of $[\underline{\alpha}, \overline{\alpha})$. The final step is to establish that $\mathcal{J} = \{\underline{\alpha}\}$. Consider $\overline{j} \equiv \max(j \in \mathcal{J})$.[6] If $\overline{j} > \underline{\alpha}$, it implies two things: (i) that $\nu_{\overline{j}} < 1$ to ensure low-type indifference and (ii) that if reflection took place as in the definition of B_H^Δ, the high type would strictly prefer to accept at $B_L^\Delta(\overline{j})$. Again, $\overline{j}$ being the maximum element of $\mathcal{J}$ implies that the reflection process is no more favorable to the high type than the reflection process in the definition of B_H^Δ. Hence, the high type strictly prefers to accept at $B_L^\Delta(\overline{j})$. It is immediate that these scenarios are inconsistent—if the high type prefers to accept Ψ rather than continue, Ψ must be offered with probability 1 (No Deals). Hence, $\mathcal{J} = \{\underline{\alpha}\}$ and $\alpha^{\Delta*} = \underline{\alpha}$.

This uniquely determines equilibrium behavior for all $z \leq \beta^{\Delta*}$ (as identified by Section S.5). Given that beliefs evolve solely based on the realization of news when $z \geq \alpha^{\Delta*}$, trade at a price of $\Psi(z)$ occurs with probability 1 for all $z > \beta^{\Delta*}$. Again, this follows from an argument that the discrete-time analog of $\text{MB}_H(z) < 0$ for all $z > \beta^{\Delta*}$, just as in Lemma B.3, since the high-type seller's problem is approximately the same when Δ is small.

S.7. EQUILIBRIUM CONVERGENCE

All three statements in Theorem S.3 are corollaries to the equilibrium analysis we have conducted and Lemma S.1. Recall that in the continuous-time model, the equilibrium is characterized by the unique pair (α^*, β^*) such that $B_H(\alpha^*) = B_L(\alpha^*) = \beta^*$. The discrete-time model is similarly characterized by a pair $(\alpha^{\Delta*}, \beta^{\Delta*})$, where B_H^Δ and B_L^Δ intersect (or nearly intersect). It is immediate that $(\alpha^{\Delta*}, \beta^{\Delta*}) \to (\alpha^*, \beta^*)$ as $\Delta \to 0$. The theorem follows.

PROOF OF LEMMA S.1: We start by constructing the continuous-time analogs of $\widetilde{F}_L^\Delta$ and $\widetilde{F}_H^\Delta$. In the continuous-time model, for any triple $\alpha \leq z \leq \beta$, let $Z_t = \hat{Z}_t + Q_t^\alpha$ and let the following statements hold:

- Let $\widetilde{F}_L(z|\alpha, \beta)$ be the continuation value for the low type in state z when boundary conditions (16) and (17) from Section 3 hold.
- Let $\widetilde{F}_H(z|\alpha, \beta)$ be the continuation value for the high type in state z when boundary conditions (18) and (19) from Section 3 hold.

Definitions S.1 and S.2 only specify $\widetilde{F}_\theta^\Delta(\cdot|\alpha, \beta)$ on G_α. Extend them now as follows: If z, z_{+1} are two states such that $\widetilde{F}_\theta^\Delta(z|\alpha, \beta)$ and $\widetilde{F}_\theta^\Delta(z_{+1}|\alpha, \beta)$ are given by Definition S.1 or S.2, then for any $a \in (0, 1)$,

$$\widetilde{F}_\theta^\Delta(az + (1-a)z_{+1}|\alpha, \beta) = a\widetilde{F}_\theta^\Delta(z|\alpha, \beta) + (1-a)\widetilde{F}_\theta^\Delta(z_{+1}|\alpha, \beta).$$

[6] Again, if one does not wish to assume the maximum exists, the argument is easily extended by considering an element arbitrarily close to $\sup \mathcal{J}$.

The subsequent fact follows easily from the convergence of the discrete-time news process to its continuous-time counterpart.

FACT S.3: *Let* $\{\beta^\Delta\}$ *be a sequence that converges to a limit* $\widehat{\beta}$ *as* $\Delta \to 0$. *Then, for any* $\alpha < \widehat{\beta}$, *as* $\Delta \to 0$,

$$\widetilde{F}_L^\Delta(z|\alpha, \beta^\Delta) \to \widetilde{F}_L(z|\alpha, \widehat{\beta}) \quad \textit{uniformly},$$
$$\widetilde{F}_H^\Delta(z|\alpha, \beta^\Delta) \to \widetilde{F}_H(z|\alpha, \widehat{\beta}) \quad \textit{uniformly}.$$

In both discrete and continuous time, the value functions are determined by the exogenously given values of the end-point states, the likelihood of reaching each end-point state, and the distribution on first hitting times of the end-point states. The discrete-time versions of each of these components are converging to their continuous-time analogs, giving the result.

Now we show that $B_H^\Delta(\alpha) \to B_H(\alpha)$ for any fixed α. Recall that $B_H(\alpha)$ is the *unique* β such that, given α, three conditions hold:

$$\widetilde{F}_H(\beta|\alpha, \beta) = \Psi(\beta),$$
$$\widetilde{F}'_H(\alpha|\alpha, \beta) = 0,$$
$$\widetilde{F}'_H(\beta|\alpha, \beta) = \Psi'(\beta).$$

Fix an α. Let $\widehat{\beta}_H^\Delta$ be the limit of $B_H^\Delta(\alpha)$ as $\Delta \to 0$. It is sufficient to show that

$$\widetilde{F}_H(\widehat{\beta}_H^\Delta|\alpha, \widehat{\beta}_H^\Delta) = \Psi(\widehat{\beta}_H^\Delta),$$
$$\widetilde{F}'_H(\alpha|\alpha, \widehat{\beta}_H^\Delta) = 0,$$
$$\widetilde{F}'_H(\widehat{\beta}_H^\Delta|\alpha, \widehat{\beta}_H^\Delta) = \Psi'(\widehat{\beta}_H^\Delta).$$

The first point is immediate from the definition of B_H^Δ. To see the second point, from the definition of value function, the nature of the reflection of Z at α, and only writing terms that are (at least) first order in Δ,

$$\begin{aligned}\widetilde{F}_H^\Delta(\alpha|\alpha, B_H^\Delta(\alpha)) = rK_H\Delta + (1 - r\Delta)\big[q^\Delta \widetilde{F}_H^\Delta(\alpha_{+1}|\alpha, B_H^\Delta(\alpha)) \\ + (1 - q^\Delta)\widetilde{F}_H^\Delta(\alpha_{-1}|\alpha, B_H^\Delta(\alpha))\big].\end{aligned}$$

Suppressing the dependence on $(\alpha, B_H^\Delta(\alpha))$, notice that property (iii) of Definition S.2 implies that $\widetilde{F}_H^\Delta(\alpha_{-1}) = \widetilde{F}_H^\Delta(\alpha)$. Subtract $\widetilde{F}_H^\Delta(\alpha)$ from both sides to get

$$\begin{aligned}0 = rK_H\Delta + q^\Delta[\widetilde{F}_H^\Delta(\alpha_{+1}) - \widetilde{F}_H^\Delta(\alpha)] \\ - r\Delta[q^\Delta \widetilde{F}_H^\Delta(\alpha_{+1}) + (1 - q^\Delta)\widetilde{F}_H^\Delta(\alpha)].\end{aligned}$$

Divide by $(\alpha_{+1} - \alpha)$ and take limits as $(\alpha_{+1} - \alpha) \to 0$. It is routine to show that $(\alpha - \alpha_{-1})$ converges to zero at the same rate as $\sqrt{\Delta}$ does. We are left with $\widetilde{F}'_H(\alpha|\alpha, \widehat{\beta}^\Delta_H) = 0$.

To verify the third point, we start by bounding $\widetilde{F}^\Delta_H(B^\Delta_H(\alpha)|\alpha, B^\Delta_H(\alpha))$, which we shorten to $\widetilde{F}^\Delta_H(B^\Delta_H)$ hereafter. By definition, $\widetilde{F}^\Delta_H(B^\Delta_H) \leq \Psi(B^\Delta_H)$, $\widetilde{F}^\Delta_H((B^\Delta_H)_{+1}) = \Psi((B^\Delta_H)_{+1})$, and $\widetilde{F}^\Delta_H((B^\Delta_H)_{-1}) > \Psi((B^\Delta_H)_{-1})$, implying

$$\begin{aligned}\widetilde{F}^\Delta_H(B^\Delta_H) > rK_H\Delta + (1 - r\Delta)\big[&q^\Delta\Psi((B^\Delta_H)_{+1})\\ &+ (1 - q^\Delta)\Psi((B^\Delta_H)_{-1})\big].\end{aligned}$$

We will now evaluate

$$\lim_{\Delta\to 0}\left(\frac{\widetilde{F}^\Delta_H((B^\Delta_H)_{+1}) - \widetilde{F}^\Delta_H(B^\Delta_H)}{(B^\Delta_H)_{+1} - B^\Delta_H}\right) \tag{S.3}$$

with $\widetilde{F}^\Delta_H(B^\Delta_H)$ set first to its upper bound and then to its lower bound (for all Δ), and show that in both cases it produces $\Psi'(\widehat{\beta}^\Delta_H)$, giving the third point by the squeeze theorem. First, because $\widetilde{F}^\Delta_H((B^\Delta_H)_{+1}) = \Psi((B^\Delta_H)_{+1})$, setting $\widetilde{F}^\Delta_H(B^\Delta_H)$ to its upper bound of $\Psi(B^\Delta_H)$ makes (S.3) equivalent to the *definition* of $\Psi'(\widehat{\beta}^\Delta_H)$. Second, set $\widetilde{F}^\Delta_H(B^\Delta_H)$ to its lower bound for all Δ and evaluate

$$\begin{aligned}\frac{\widetilde{F}^\Delta_H((B^\Delta_H)_{+1}) - \widetilde{F}^\Delta_H(B^\Delta_H)}{(B^\Delta_H)_{+1} - B^\Delta_H} &= \big(\Psi((B^\Delta_H)_{+1}) - \big(rK_H\Delta + (1 - r\Delta)\\ &\quad\times\big[q^\Delta\Psi((B^\Delta_H)_{+1}) + (1 - q^\Delta)\Psi((B^\Delta_H)_{-1})\big]\big)\big)\\ &\quad/((B^\Delta_H)_{+1} - B^\Delta_H).\end{aligned}$$

Rearranging and eliminating additive terms that tend to zero as $\Delta \to 0$ gives

$$\begin{aligned}&\frac{\widetilde{F}^\Delta_H((B^\Delta_H)_{+1}) - \widetilde{F}^\Delta_H(B^\Delta_H)}{(B^\Delta_H)_{+1} - B^\Delta_H}\\ &\quad= (1 - q^\Delta)\frac{\Psi((B^\Delta_H)_{+1}) - \Psi((B^\Delta_H)_{-1})}{(B^\Delta_H)_{+1} - B^\Delta_H}\\ &\quad= (1 - q^\Delta)2\frac{\Psi((B^\Delta_H)_{+1}) - \Psi((B^\Delta_H)_{-1})}{2((B^\Delta_H)_{+1} - B^\Delta_H)}\\ &\quad= (1 - q^\Delta)2\frac{\Psi((B^\Delta_H)_{+1}) - \Psi((B^\Delta_H)_{-1})}{((B^\Delta_H)_{+1} - (B^\Delta_H)_{-1})}.\end{aligned}$$

Taking the limit as $\Delta \to 0$, we get that $\widetilde{F}'_H(\widehat{\beta}^\Delta_H|\alpha, \widehat{\beta}^\Delta_H) = \Psi'(\widehat{\beta}^\Delta_H)$. Therefore, $B^\Delta_H(\alpha) \to B_H(\alpha)$ for any fixed α.

Now we show that $B_L^{\Delta}(\alpha) \to B_L(\alpha)$ for any fixed α. Recall that $B_L(\alpha)$ is the *unique* β such that, given α, three conditions hold:

$$\widetilde{F}_L(\alpha|\alpha, \beta) = V_L,$$
$$\widetilde{F}_L(\beta|\alpha, \beta) = \Psi(\beta),$$
$$\widetilde{F}'_L(\alpha|\alpha, \beta) = 0.$$

Fix an α. Let $\widehat{\beta}_L^{\Delta}$ be the limit of $B_L^{\Delta}(\alpha)$ as $\Delta \to 0$. It is sufficient to show that

$$\widetilde{F}_L(\alpha|\alpha, \widehat{\beta}_L^{\Delta}) = V_L,$$
$$\widetilde{F}_L(\widehat{\beta}_L^{\Delta}|\alpha, \widehat{\beta}_L^{\Delta}) = \Psi(\widehat{\beta}_L^{\Delta}),$$
$$\widetilde{F}'_L(\alpha|\alpha, \widehat{\beta}_L^{\Delta}) = 0.$$

For the first point, the continuity of $\widetilde{F}_L(\cdot|\alpha, \beta)$ and Fact S.3 imply that

$$\left|\widetilde{F}_L^{\Delta}(\alpha|\alpha, B_L^{\Delta}(\alpha)) - \widetilde{F}_L^{\Delta}(\alpha_{-1}|\alpha, B_L^{\Delta}(\alpha))\right| \to 0.$$

That $\widetilde{F}_L^{\Delta}(\alpha_{-1}|\alpha, B_L^{\Delta}(\alpha)) = V_L$ for all Δ gives the result. Next, $\widetilde{F}_L^{\Delta}(B_L^{\Delta}(\alpha)|\alpha, B_L^{\Delta}(\alpha)) = \Psi(B_L^{\Delta}(\alpha))$ for all Δ, giving the second point.

We now verify the third point. Notice that as $\Delta \to 0$, $B_L^{\Delta}(\alpha)$ and $B_L^{\Delta}(\alpha)_{+1}$ both limit to $\widehat{\beta}_L^{\Delta}$. By definition of B_L^{Δ}, for every $\Delta > 0$,

$$\frac{\widetilde{F}_L^{\Delta}(\alpha|\alpha, B_L^{\Delta}(\alpha)) - \widetilde{F}_L^{\Delta}(\alpha_{-1}|\alpha, B_L^{\Delta}(\alpha))}{\alpha - \alpha_{-1}} \geq 0 \tag{S.4}$$

and

$$\frac{\widetilde{F}_L^{\Delta}(\alpha|\alpha, B_L^{\Delta}(\alpha)_{+1}) - \widetilde{F}_L^{\Delta}(\alpha_{-1}|\alpha, B_L^{\Delta}(\alpha)_{+1})}{\alpha - \alpha_{-1}} \leq 0. \tag{S.5}$$

It follows, that in the limit as $\Delta \to 0$, (S.4) and (S.5) both limit to zero, giving the desired result. This establishes that $B_L^{\Delta}(\alpha) \to B_L(\alpha)$ for any fixed α and completes the proof. *Q.E.D.*

ADDITIONAL REFERENCE

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