

PUBLIC VS. PRIVATE OFFERS: THE TWO-TYPE CASE
TO SUPPLEMENT “PUBLIC VS. PRIVATE OFFERS
IN THE MARKET FOR LEMONS”
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PUBLIC OFFERS, WITHOUT SHOCKS

SUPPOSE THAT THERE ARE only two seller types, v and 1 , which are also the buyer’s values for the unit. The seller’s cost is αv or α , depending on whether his type is low or high. The prior probability assigned by the buyers to the high type is $\mu = \mu_0$.

Suppose further that the seller is patient enough and that adverse selection is severe enough in the sense that

$$\alpha > (1 - \mu)v + \mu,$$
$$\delta\alpha + (1 - \delta)\alpha v > v.$$

The first equation states that the cost of the high type seller exceeds the average value of the unit to the buyer; the second equation requires that δ be high enough. Let μ_n be the prior on the high type after a given history h_n . We shall prove that the equilibrium outcome is similar to the case with a continuum of types. The first buyer makes an offer that is accepted with positive probability by the low type and rejected by the high type. If this offer is rejected, all future buyers submit losing offers.

We proceed with a series of claims.

CLAIM S1: If μ_n is close enough to 1, then independently of the history, buyer n offers α (which the seller accepts for sure).

Indeed, the only alternative is to make an offer that only (some or all) seller’s low types accept, yielding a payoff that is at most $(1 - \mu_n)v$. By offering α instead, the buyer guarantees $\mu_n + (1 - \mu_n)v - \alpha$. So the claim follows whenever μ_n is large enough that

$$\mu_n + (1 - \mu_n)v - \alpha > (1 - \mu_n)v \quad \text{or} \quad \mu_n > \alpha.$$

Let μ^* be the infimum over those values whose existence is established in Claim S1. That is, whenever $\mu_n > \mu^*$, then, independently of the history, the buyer offers α and the seller accepts for sure. Clearly, μ^* is bounded below, since by assumption $\alpha > (1 - \mu)v + \mu$, so that offering α with belief μ yields negative profits.

¹Hörner’s work on this project began when he was affiliated with Northwestern University.

CLAIM S2: *If $\mu_n \leq \mu^*$, buyer n 's equilibrium offer cannot lead to a posterior $\mu_{n+1} \in (\mu^*, 1)$.*

Suppose instead that this was the case in equilibrium. So, in particular, the seller's low type must accept with positive probability, but not the high type. The next offer will then be α , by Claim S1, which means that the offer must be $p = \delta\alpha + (1 - \delta)\alpha v$, since by accepting, the seller's low type gets $p - \alpha v$ and by rejecting, he gets $\delta(\alpha - \alpha v)$. However, by assumption, $\delta\alpha + (1 - \delta)\alpha v > v$, which means that this offer would be unprofitable for the buyer.

CLAIM S3: *The value μ^* solves $\mu^* + (1 - \mu^*)v - \alpha = 0$.*

To see this, observe that, by definition of μ^* , for every $\varepsilon > 0$, there exists a history such that, given belief $\mu^* - \varepsilon$, the buyer makes an offer that is not winning. Given Claim S2, this means that this offer can only lead to a posterior in $[\mu^* - \varepsilon, \mu^*]$, which implies that the probability of sale is at most ε/μ^* . By making the winning offer instead, he can secure $\mu^* - \varepsilon + (1 - (\mu^* - \varepsilon))v - \alpha$. Since this is true for every $\varepsilon > 0$, it follows that $\mu^* + (1 - \mu^*)v - \alpha \leq 0$. Since making a winning offer is profitable for a belief $\mu^* + \varepsilon$, for every $\varepsilon > 0$, the reverse inequality follows.

CLAIM S4: *Suppose that $\mu_n < \mu^*$ and that the equilibrium calls for an offer by buyer n leading to a posterior $\mu_{n+1} = \mu^*$. Then all future offers must be losing.*

Suppose they were not. Then buyer n must offer a price p_n strictly larger than αv (indeed, αv is the seller's low type reservation value, and by rejecting, he gets an offer α in the future with positive probability). Let \bar{p} be the supremum over all such offers p_n . Pick $\varepsilon > 0$ such that $\bar{p} - 2\varepsilon > \alpha v$ and pick a history such that $p_n > \bar{p} - \varepsilon$. If buyer n deviates by offering $\bar{p} - 2\varepsilon$ instead, then either the posterior μ_{n+1} would weakly exceed μ^* —in which case this would be a profitable deviation, since the price would be lower and the probability of acceptance higher—or $\mu_{n+1} < \mu^*$. In that case, by accepting the offer, the seller's low type gets $\bar{p} - 2\varepsilon - \alpha v$, while by rejecting, he cannot hope for more than $\delta(\bar{p} - \alpha v)$. Since he is supposed to be willing to reject, this is a contradiction for small enough ε .

It follows that starting from a belief strictly below μ^* , beliefs will never exceed μ^* and offers will not exceed αv .

CLAIM S5: *If $\mu_n < \mu^*$, then buyer n offers αv and the posterior is $\mu_{n+1} = \mu^*$.*

Clearly, the posterior cannot be larger than μ^* , since the next offer would be α and the seller's low type should have rejected. If $\mu_{n+1} < \mu^*$, consider a deviation by the seller to an offer $\alpha v + \varepsilon$. Again, if such an offer led to a posterior $\mu_{n+1} \geq \mu^*$, then it would be profitable. But if not, the seller's low type should accept it for sure, since he cannot hope for more than αv in the future—a contradiction. This claim completes the derivation of the unique equilibrium.

PUBLIC OFFERS WITH SHOCKS

Consider now the case in which the seller is *myopic* with probability $\varepsilon > 0$ in every period (“liquidity shock”), that is, in that event, he accepts the offer if and only if it exceeds its cost. Buyers do not observe those shocks. For convenience, we assume that the nonmyopic (or *strategic*) seller does not take into account the possibility that he might later become myopic: for ε small enough, this does not change the equilibrium structure, but significantly simplifies the formulas. We provide a detailed sketch of the derivation of the unique equilibrium outcome in that case.

By making a winning offer, buyer n gets $\mu_n + (1 - \mu_n)v - \alpha$. By making the offer αv instead, he gets $\varepsilon(1 - \mu_n)(1 - \alpha)v$. By the same reasoning as before, we can then define $\mu(0)$ as the solution of

$$\varepsilon(1 - \mu_n)(1 - \alpha)v = \mu_n + (1 - \mu_n)v - \alpha.$$

By following the same reasoning as before, we can show that if $\mu_n > \mu(0)$, buyer n submits a winning offer (independently of the specific history). Let $\lambda(0) := \mu(0)/(1 - \mu(0))$ and define $\lambda(k) := \lambda(0)(1 - \varepsilon)^k$. The sequence of odds ratio $\{\lambda(k)\}$ has the property that if a losing offer αv is rejected and the belief was such that the odds ratio was $\lambda(k)$, then the posterior odds ratio is $\lambda(k - 1)$. Let $\lambda_0 := \mu_0/(1 - \mu_0)$ be the initial odds ratio and let $I(k) := (\lambda(k + 1), \lambda(k))$. If $\lambda_0 \in I(0)$, then the outcome is clear. The first buyer makes a losing offer (αv) which the seller accepts if and only if there is a shock, and the second buyer makes a winning offer.

For which values of k does this structure extend? What makes the low cost seller indifferent (in the absence of a liquidity shock)? If the posterior μ_{n+1} is in $I(k)$ and thereafter buyers make losing offers until $\mu_{n'} > \mu(0)$, then by rejecting the offer, the seller can expect to get $\delta^{k+1}\alpha(1 - v)$, so indifference requires $p = \alpha v + \delta^{k+1}\alpha(1 - v)$. This price function has obviously a downward jump when we go from k to $k + 1$, but the trade-off is that this requires a lower probability of sale. So a buyer with odds ratio λ_n could make an offer that maximizes

$$J(\lambda_n) := \max_{\lambda \in I(k)} \frac{1 - \lambda_n/\lambda}{1 + \lambda_n} (v - (\alpha v + \delta^{k+1}\alpha(1 - v))),$$

where the first term is the probability of acceptance given that the prior odds ratio is λ_n and the posterior odds ratio is λ . Clearly, the only candidate for the maximizer in a given interval is $\lambda(k)$, and so we might as well redefine the intervals $I(k)$ as half closed to the right. As a function of k , this is concave, and as a function of λ_n , it is increasing.

We can thus define $\lambda'(0)$ as the infimum over all values of λ above which, for all k , this payoff falls short of the payoff from a losing offer. Starting from

$\lambda_n < \lambda'(0)$, any offer that is submitted must necessarily be serious. Given the continuity of this function, $\lambda'(0)$ is the unique value λ_n that solves

$$\max_k \left(1 - \frac{\lambda_n}{\lambda(0)(1-\varepsilon)^k} \right) ((1-\alpha)v - \delta^{k+1}\alpha(1-v)) = \varepsilon(1-\alpha)v.$$

Let K be the value for which $\lambda'(0) \in I(k)$. This should not be confused with k_0 , the value which we define as the maximizer of the left-hand side for $\lambda_n = \lambda'(0)$ (k_0 need not be 0). It is important to observe that $\lambda(0) - \lambda'(0)$ is of the order ε . It is also easily verified that the value of k that maximizes the left-hand side is nondecreasing in k . That is, we can now construct a sequence $\lambda'(k)$, $k = k_0, \dots, K$, such that for $I'(k) := (\lambda'(k+1), \lambda'(k)]$, the maximizing value in the left-hand side is precisely k ; that is, if $\lambda_n \in I'(k)$, the serious offer that would be submitted would lead to a posterior odds ratio equal to $\lambda(k)$. See Figure S1.

The next step is to determine $\lambda'(K+1)$. Observe that no serious offer would ever be submitted that led to a posterior in $I'(k)$, $k = k_0, \dots, K-1$: this is because the price that has to be submitted for such an offer to be accepted if the posterior is in $I'(k)$ is the same as if the posterior is in $I(k+1)$, so clearly, since the probability of acceptance is higher in the latter case, the latter dominates the former. However, the price that has to be submitted if the posterior is $\lambda'(K)$ is strictly lower than for all larger posterior odds (by an amount proportional to K and $1-\delta$, that is, much larger than ε). Since again, the difference $\lambda'(K) - \lambda(K)$ is of the order ε , this implies that we can find a largest value $\lambda''(0)$ such that, given $\lambda_n = \lambda''(0)$, it is more profitable to make a serious offer that leads

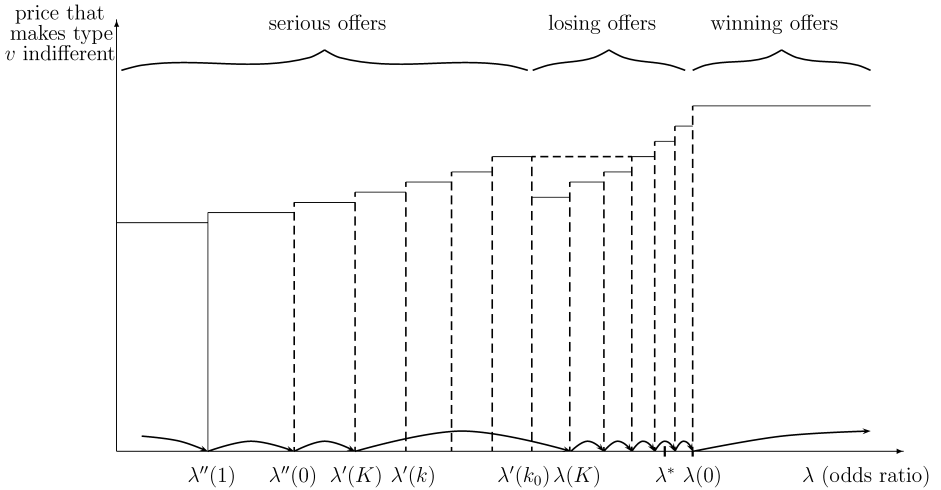


FIGURE S1.

to a posterior $\lambda'(K)$ than any other offer, and again $\lambda''(0) - \lambda'(K)$ is of the order ε . The value $\lambda'(K + 1)$ is set at $\lambda''(0)$.

We can define a third and final sequence $\lambda''(k)$: $\lambda''(1)$ is the largest value λ_n for which it is preferable to make a serious offer with posterior $\lambda''(0)$ than with posterior $\lambda'(K)$, and, more generally, $\lambda''(k + 1)$ is the largest value λ_n for which it is preferable to make a serious offer with posterior $\lambda''(k)$ than with posterior $\lambda''(k - 1)$. As $k \rightarrow \infty$, the distance $\lambda''(k + 1) - \lambda''(k)$ grows (because the corresponding price difference vanishes), so that eventually we “hit” the initial value λ_0 .

Figure S1 summarizes this description, and shows the serious offer that would lead to a posterior odds ratio in this interval. The arrows describe the actual dynamics of this odds ratio. That is, to summarize, starting from λ_0 , for small enough ε , on the equilibrium path there will be a finite number of consecutive serious offers (with posterior odds ratios $\lambda''(k)$) until $\lambda''(0)$, followed by one serious offer leading to posterior odds of $\lambda'(K)$, and a last serious offer leading to $\lambda(K)$. After that, there will be $K + 1$ consecutive losing offers and then a winning offer.

It is not hard to see that, as $\varepsilon \rightarrow 0$, $K \rightarrow \infty$, and the outcome approaches the outcome in the unperturbed game, conditional on a history in which the shock never occurs, a first serious offer arbitrarily close to (but above) αv leads to a posterior arbitrarily close to (but below) μ^* , followed by arbitrarily many serious (but “almost surely” rejected) offers, and then arbitrarily many losing offers, after which a final winning offer is submitted.

PRIVATE OFFERS: EQUILIBRIUM CHARACTERIZATION

Here is an equilibrium outcome, which we will later show to be part of an essentially unique equilibrium. Let

$$\mu^* = \frac{\alpha - v}{1 - v},$$

that is, μ^* solves $\mu + (1 - \mu)v = \alpha$.

- The first buyer offers $p = v$. The seller’s high type rejects and the low type accepts with probability q such that the posterior equals μ^* .

- All buyers after the first randomize between the offers $p^L = v$ and $p^H = \alpha$, with probability λ on the high offer α . The probability λ solves

$$p^L - \alpha v = \delta(\lambda\alpha + (1 - \lambda)p^L - \alpha v) \quad \text{or} \quad \lambda = \frac{(1 - \delta)(1 - \alpha)v}{\delta(\alpha - v)}.$$

The probability λ is in $[0, 1]$ because it is equivalent to the condition

$$v \leq \alpha\delta + (1 - \delta)\alpha v.$$

- In all periods but the first, on the equilibrium path, the seller rejects the low offer and accepts the high offer.

To see why this is part of an equilibrium, observe that offers in the range (v, α) are unprofitable deviations for the buyer anyway, since only the low type may accept them and the high type will reject them. Given λ , offers below p^L are rejected (since p^L is precisely the offer that makes the low type indifferent between accepting and rejecting the offer). So the buyers have no profitable deviation. Observe now that, given μ^* and $p^L = v$, both the low and high offer yield zero profits to the buyers (after the first one). So they are willing to randomize. The first buyer makes zero profits as well, but his offer is accepted with positive probability.

PRIVATE OFFERS: UNIQUENESS

Let us proceed with a series of claims. Proofs are more different from the case with a continuum of types than for the public case.

Denote μ_n the posterior probability assigned by buyer n to the high type.²

- Trade eventually takes place. Hence there is a stage n with $\mu_n \geq \mu^*$.

Trade of the low type must eventually take place with probability 1. (The argument goes as in the paper.) But then, if trade of the high type does not take place with probability 1, we would have both $\mu_n \rightarrow 1$ and equilibrium payoffs converging to 0. This is impossible.

- There is no stage n with $\mu_n > \mu^*$.

Denote $\tilde{\mu} \geq \mu^*$ the limit of μ_n . Assume to the contrary that $\tilde{\mu} > \mu^*$ and let n be such that μ_n is close to $\tilde{\mu}$. We claim that buyer n makes a winning offer with probability 1. The proof is a bit tedious.

Since $\mu_n > \mu^*$, buyer n 's equilibrium payoff is positive and bounded away from zero; hence (i) buyer n submits no losing offer and (ii) any equilibrium offer targeted at the low type must be accepted with probability bounded away from 0. Since μ_n is close to $\tilde{\mu}$, this implies that the probability that buyer n submits a nonwinning offer is very close to 0. The same is true of buyer $n + 1$, so that, in stage n , the low type of the seller expects to receive a winning offer in the next stage, with high probability. This implies that buyer n must offer a lot so that the low type accepts and in fact more than αv . Since such an offer would yield a negative payoff, this shows that buyer n actually makes a winning offer with probability 1.

Next, consider the last buyer, say N , who makes a nonwinning offer with positive probability (hence, $\mu_{N+1} = \tilde{\mu}$). As in the previous paragraph, an offer that attracts only the low type yields negative payoff. Hence buyer N submits only winning or losing offers; hence $\mu_{N+1} = \mu_N$. Hence $\mu_N > \mu^*$. Hence buyer N 's payoff is positive: he makes a winning offer with probability 1—this contradicts the definition of N .

²It is defined only for those buyers n who are called to play, with positive probability.

Denote $N_0 = \inf\{n : \mu_n = \mu^*\}$. Note that $N_0 > 1$.

- All buyers $n \geq N_0$ submit either losing or winning offers.

This is clear, since $\mu_n = \mu_{N_0}$ for all $n \geq N_0$.

We now consider buyers before N_0 . Note that $N_0 \geq 2$.

- Assume $N_0 > 2$. Then all buyers $n < N_0 - 1$ have a positive equilibrium payoff. Moreover, they all play a pure strategy.

Consider first buyer $N_0 - 1$. This buyer submits a serious offer with positive probability, which is acceptable to the low type (and only to the low type). This offer does not exceed v . Hence, in all earlier stages, the low type would settle for less than v . This proves the first part of the claim.

Note that in any given stage, there is only one offer that the low type is indifferent to accept. Since $\mu_n < \mu^*$ and since equilibrium payoffs are positive, buyer $n < N_0 - 1$ submits with probability 1 the offer that makes the low type indifferent.

- One has $N_0 = 2$.

Assume $N_0 > 2$. By the previous bulleted point, buyer 1 submits an offer that makes the low type indifferent. This offer is rejected with positive probability by the low type (for otherwise, $\mu_2 > \mu_*$). Hence, buyer 1 would increase his payoff when offering slightly more, since the low type would then accept with probability 1.

- The first buyer randomizes between a losing offer and v . The latter offer makes the low type indifferent.

We know that $\mu_2 = \mu^*$, so the first buyer submits a serious offer with positive probability and the low type rejects with positive probability. Hence, as above, the first buyer's payoff must be 0, for otherwise, a slightly higher offer would be accepted with probability 1 by the low type and would yield a higher payoff. Hence, this serious offer is v .

PUBLIC VS. PRIVATE OFFERS: COMPARISON

In the case of unobservable offers, the welfare is independent of the equilibrium and equal to

$$(1 - \alpha) \left(\frac{\mu^* - \mu}{\mu^*} v + \delta \frac{\lambda(\mu^* + (1 - \mu^*)v)}{1 - \delta(1 - \lambda)} \right) = (1 - \alpha)(1 - \mu)v.$$

On the other hand, with observable offers, the payoff is

$$(1 - \alpha) \frac{\mu^* - \mu}{\mu^*} v = (1 - \alpha) \left(1 - \mu \frac{1 - v}{\alpha - v} \right),$$

which is strictly lower whenever $\alpha < 1$. Remarkably, welfare is independent of discounting in both cases. In the public case, it is equal to the gains from trade from the low type only. Comparisons for the buyers is immediate, since they

all make zero profit in the unobservable case, while this is not the case for the first buyer in the observable case.

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