SUPPLEMENT TO "TESTING MODELS WITH MULTIPLE EQUILIBRIA BY QUANTILE METHODS"

(*Econometrica*, Vol. 77, No. 4, July 2009, 1281–1297)

BY FEDERICO ECHENIQUE AND IVANA KOMUNJER

This supplement contains the proofs of Proposition 1 and Lemma 1 that were stated in the paper.

PROOF OF PROPOSITION 1: For any $(y, x) \in \mathbb{R} \times \mathcal{X}$, $F_{Y|X=x}(y) = \int_{-\infty}^{+\infty} \mathcal{P}_{xu}(y) f_{U|X=x}(u)\, du$ with $\mathcal{P}_{xu}(y) = \sum_{i=1}^{n_x} \pi_{ix} \mathbb{1}(\xi_{ixu} \leq y)$, where $\mathbb{1}$ denotes the standard indicator function: For any event A in $\mathcal{B}$, where $\mathcal{B}$ is the Borel σ-algebra on $\mathbb{R}$, $\mathbb{1}(A) = 1$ if A is true and $=0$ otherwise. Combining all of the above, we get

$$F_{Y|X=x}(y) = \sum_{i=1}^{n_x} \pi_{ix} \int_{-\infty}^{+\infty} \mathbb{1}(\xi_{ixu} \leq y) f_{U|X=x}(u)\, du.$$

For any $x \in \mathcal{X}$ and any $1 \leq i \leq n_x$, let $F_{iY|X=x}(y) = \int_{-\infty}^{+\infty} \mathbb{1}(\xi_{ixu} \leq y) f_{U|X=x}(u)\, du$ for all $y \in \mathbb{R}$. Then $F_{iY|X=x}(y) : \mathbb{R} \to \mathbb{R}$ is right-continuous, $\lim_{y\to-\infty} F_{iY|X=x}(y) = 0$, $\lim_{y\to+\infty} F_{iY|X=x}(y) = 1$, and $F_{iY|X=x}$ is nondecreasing in y. Hence, $F_{iY|X=x}$'s are distribution functions and the conditional distribution of the dependent variable can be written as in Proposition 1. Moreover, for any $(y, x) \in \mathbb{R} \times \mathcal{X}$, we have $F_{iY|X=x}(y) - F_{jY|X=x}(y) = \int_{-\infty}^{+\infty} \mathbb{1}(\xi_{ixu} \leq y < \xi_{jxu}) f_{U|X=x}(u)\, du \geq 0$ whenever $\xi_{jxu} \geq \xi_{ixu}$, that is, $F_{jY|X=x}(y) \leq F_{iY|X=x}(y)$ whenever $j \geq i$. So, $F_{jY|X=x}$ first-order stochastically dominates $F_{iY|X=x}$ for any $j \geq i$. *Q.E.D.*

PROOF OF LEMMA 1: Fix $(y_0, x) \in \mathbb{R} \times \mathcal{X}$: continuity and limit conditions on $r(y, x)$ in S1 then ensure that the envelope $r^{\mathrm{e}}(y, x)$ is well defined on $[y_0, +\infty)$. Now consider $y \geq y_0$. That $\mathbb{1}(\xi_{n_x xu} \leq y) = \mathbb{1}(u \leq r^{\mathrm{e}}(y, x))$ follows from showing that $r^{\mathrm{e}}(\xi_{n_x xu}, x) = r(\xi_{n_x xu}, x)$, as r^{e} is nonincreasing and $\xi_{n_x xu}$ is the largest equilibrium. We proceed in two steps. First, we show that for all $y > \xi_{n_x xu}$, we have $r(\xi_{n_x xu}, x) > r(y, x)$. If that were not the case, then there would exist a $y' > \xi_{n_x xu}$ such that $r(\xi_{n_x xu}, x) \leq r(y', x)$. But this is incompatible with $\xi_{n_x xu}$ being the largest equilibrium: we would have $u \leq r(y', x)$, so given the limit condition S1(ii) on r at $+\infty$, there would be an equilibrium larger than $\xi_{n_x xu}$. Second, we show that $r^{\mathrm{e}}(\xi_{n_x xu}, x) = r(\xi_{n_x xu}, x)$. By definition of r^{e}, we have $r^{\mathrm{e}}(\xi_{n_x xu}, x) \geq r(\xi_{n_x xu}, x)$, so we need to rule out that strict inequality holds. We again reason by contradiction: assume that $r^{\mathrm{e}}(\xi_{n_x xu}, x) > r(\xi_{n_x xu}, x)$. From the first step, we know that $r(\xi_{n_x xu}, x) > r(y, x)$ for all $y > \xi_{n_x xu}$. Then consider the function which coincides with $r^{\mathrm{e}}(y, x)$ for $y < \xi_{n_x xu}$ and with $\min\{r^{\mathrm{e}}(y, x), r(y, x)\}$ for $y \geq \xi_{n_x xu}$. This function is nonincreasing, larger than r, and smaller than r^{e} at $\xi_{n_x xu}$, which is impossible by the definition of r^{e}. *Q.E.D.*

 DOI: 10.3982/ECTA6223

Division of Humanities and Social Sciences, California Institute of Technology, 311 Baxter Hall, Pasadena, CA 91125, U.S.A.; fede@caltech.edu

and

Dept. of Economics, University of California at San Diego, 9500 Gilman Drive, La Jolla, CA 92093-0508, U.S.A.; komunjer@ucsd.edu.

Manuscript received December, 2005; final revision received November, 2008.