

Supplement to “Seeding a Simple Contagion”

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A Larger Seed Sets

The most prominent applications of targeted seeding in the economics literature involve small sets of seeds, but the theoretical framework can readily adapt to offer guidance for larger seed sets—this seems particularly relevant for mass marketing, in which susceptibility is relatively low, and advertisements can reach a large fraction of the relevant population. We can explicitly solve for the extent of contagion, given an arbitrary vector \mathbf{s} of seeded fractions. For convenience, I reproduce the key equations here:

$$y_t(\mathbf{s}) = \alpha_t \left(1 - (1 - s_t) \frac{g'_t(1 - \mathbf{y}(\mathbf{s}) \cdot \mathbf{q}^t)}{\mu_t} \right), \quad \pi_t(\mathbf{s}) = \alpha_t (1 - (1 - s_t) g_t(1 - \mathbf{y}(\mathbf{s}) \cdot \mathbf{q}^t)), \quad (1)$$

in which $g_t(x)$ is the probability generating function for the degree distribution D_t . Writing $\pi(\mathbf{s}) = \boldsymbol{\pi}(\mathbf{s}) \cdot \mathbf{p}$ for the infected fraction of the population, a natural way to formulate a

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planner's seeding problem is then

$$\begin{aligned} \max_{\mathbf{s} \geq \mathbf{0}} \quad & \pi(\mathbf{s}) \\ \text{s.t.} \quad & \mathbf{p} \cdot \mathbf{s} \leq c \end{aligned}$$

for a suitable seeding budget c . We can solve this optimization using standard methods—the optimal \mathbf{s}^* satisfies $\frac{\partial \pi(\mathbf{s}^*)}{\partial s_t} \leq \lambda p_t$, for some $\lambda \geq 0$, with equality for any type t such that $s_t^* > 0$.

To gain some insight into the properties of optimal seeding policies with larger budgets, I specialize the model, assuming each type t has a Poisson degree distribution with mean μ_t , and neighbor types follow a simple homophily model with parameter h —recall in the simple homophily model we have $\mathbf{q}^t = h\mathbf{e}_t + (1-h)\mathbf{q}^*$ in which $q_t^* = \frac{p_t \mu_t}{\sum_{s \in \Theta} p_s \mu_s}$. This allows us to obtain an explicit expression for the partial derivatives $\frac{\partial \pi(\mathbf{s})}{\partial s_t}$.

Proposition 1. *If each type has a Poisson degree distribution, and neighbor type distributions follow a simple homophily model with parameter h , then*

$$\frac{1}{p_t} \frac{\partial \pi(\mathbf{s})}{\partial s_t} = \frac{e^{-\mu_t(1-h)\pi(\mathbf{s}) \cdot \mathbf{q}^*}}{\sum_{r \in \Theta} p_r \mu_r} \frac{\mu_t e^{-\mu_t h \pi_t(\mathbf{s})}}{1 - h(\alpha_t - \pi_t(\mathbf{s}))} (p_t + X), \quad (2)$$

in which

$$X = \frac{(1-h) \sum_{r \in \Theta} p_r \frac{q_r^*(\alpha_r - \pi_r(\mathbf{s}))}{1 - h(\alpha_r - \pi_r(\mathbf{s}))}}{1 - (1-h) \sum_{r \in \Theta} \frac{q_r^*(\alpha_r - \pi_r(\mathbf{s}))}{1 - h(\alpha_r - \pi_r(\mathbf{s}))}}.$$

Proof. See Appendix C. □

At an optimal solution \mathbf{s}^* , the expression in (2) must be the same for each type t such that $s_t > 0$, and it must be lower for all other types. A higher value means seeding type t makes a larger impact on the margin. We can read off a number of intuitive results. All else equal, type t is a more attractive target if α_t is bigger and $\pi_t(\mathbf{s})$ is smaller—greater

susceptibility and less redundancy makes for a better target. Increasing the average degree μ_t has an ambiguous effect just like it did when we were calculating seed multipliers: high-degree types can spread the infection to more neighbors, but they are also more redundant.

B Alternative Objectives

At first glance, it appears as though we are limited to planners whose payoff is linear in the fraction $\pi(\mathbf{s})$ of the population that gets infected. In fact, this paper's focus on marginal effects means that the seed multipliers provide valid comparisons for a large class of objectives. Suppose f is any strictly increasing differentiable function, and the planner seeks to maximize the objective

$$f(\pi(\mathbf{s})).$$

The marginal impact of an additional type t seed, assuming \mathbf{s} is small, is then approximately

$$\lim_{\substack{\mathbf{s} \rightarrow \mathbf{0} \\ \mathbf{s} \gg \mathbf{0}}} \frac{1}{p_t} \frac{\partial}{\partial s_t} f(\pi(\mathbf{s})) = \lim_{\substack{\mathbf{s} \rightarrow \mathbf{0} \\ \mathbf{s} \gg \mathbf{0}}} \frac{1}{p_t} f'(\pi(\mathbf{s})) \frac{\partial \pi(\mathbf{s})}{\partial s_t} = \beta_t \lim_{\substack{\mathbf{s} \rightarrow \mathbf{0} \\ \mathbf{s} \gg \mathbf{0}}} f'(\pi(\mathbf{s})).$$

Since the last term is common to all types, the relative impact of seeding each type on the margin is still proportional to the seed multipliers.

In a more significant departure, we can study the complementary immunization problem. Suppose a planner wishes to *prevent* an infection from spreading beyond a small group and can immunize individuals based on observed types. Write m_t for the fraction of type t individuals that get immunized, so the effective susceptibility of type t is $\alpha_t(1 - m_t)$, and the effective entries of the matrix M' are $\alpha_s q_s^t \mu'_t (1 - m_t)$. If a fraction $\epsilon \approx 0$ of the population is initially exposed, then the total fraction of the population that gets infected is approximately zero if and only if there is no giant component, i.e. $\rho(M') \leq 1$. If immunization costs are

linear, then the planner can achieve this at lowest cost by solving

$$\begin{aligned} \min_{\mathbf{m} \in [0,1]^\Theta} \quad & \sum_{r \in \Theta} p_r m_r \\ \text{s.t.} \quad & \rho(M') \leq 1. \end{aligned} \tag{3}$$

Notice that the generating functions have no role at all in solving the immunization problem: the constraint depends only on the unconditional M' , whose entries are functions of the degree distributions' first two moments. In this sense, the immunization problem is simpler than the optimal choice of a larger seed set, or determining seed multipliers in the viral case. With some structure on the neighbor type distributions, we can obtain a closed form expression for the solutions.

Proposition 2. *Suppose neighbor type distributions follow a simple homophily model. If $h < 1$, an optimal solution \mathbf{m}^* to (3) satisfies $h\alpha_t\mu'_t(1 - m_t^*) < 1$ for each t , and*

$$1 = \frac{1 - h}{\sum p_r \mu_r} \sum_{t \in \Theta} \frac{\alpha_t p_t \mu_t \mu'_t (1 - m_t^*)}{1 - h\alpha_t \mu'_t (1 - m_t^*)}.$$

Moreover, there exists a constant C such that

$$m_t^* = \begin{cases} 0 & \text{if } \frac{\alpha_t \mu_t \mu'_t}{(1 - \alpha_t \mu'_t h)^2} < C \\ 1 & \text{if } \alpha_t \mu_t \mu'_t > C, \end{cases}$$

and otherwise $\frac{\alpha_t \mu_t \mu'_t}{(1 - \alpha_t \mu'_t h (1 - m_t^*))^2} = C$. If $h = 1$, the optimal solution \mathbf{m}^* sets

$$m_t^* = \max \left\{ 0, \frac{\alpha_t \mu'_t - 1}{\alpha_t \mu'_t} \right\}.$$

for each type t .

Proof. See Appendix C. □

The solution in Proposition 4 highlights a few features that should be obvious: types with high susceptibility, high degrees, and high forward degrees make the best targets for immunization, and if the network is more dense, we need to immunize more individuals to prevent large cascades of infection. The role of homophily is less obvious, but as a starting point, comparing solutions at the extremes with $h = 0$ and $h = 1$ is instructive.

If $h = 1$, meaning all individuals only interact with others of the same type, then we effectively have $|\Theta|$ separate networks, and we must immunize an appropriate fraction within *each* to get under the viral threshold. On the other hand, in the absence of any homophily ($h = 0$), the optimal policy lexicographically targets types with the highest values of $\alpha_t \mu_t \mu'_t$ —there is a constant C such that $m_t^* = 0$ if $\alpha_t \mu_t \mu'_t < C$ and $m_t^* = 1$ if $\alpha_t \mu_t \mu'_t > C$. We can focus exclusively on the highest degree types because they both have more neighbors they can infect and comprise a larger share of other types' neighbors—we do not need to immunize lower degree types because their neighbors are likely immunized.

These extremes underscore a general rule: with less homophily we can focus on the highest degree types, while with greater homophily optimal immunization becomes more uniform across the population. From the characterization in Proposition 4, we can deduce that whenever

$$\frac{\alpha_t \mu_t \mu'_t}{(1 - \alpha_t \mu'_t h (1 - m_t))^2} > \frac{\alpha_s \mu_s \mu'_s}{(1 - \alpha_s \mu'_s h (1 - m_s))^2},$$

it is more cost effective on the margin to immunize type t rather than type s . We can rearrange this inequality to obtain

$$\frac{1 - \alpha_s \mu'_s h + \alpha_s \mu'_s h m_s}{1 - \alpha_t \mu'_t h + \alpha_t \mu'_t h m_t} > \sqrt{\frac{\alpha_s \mu_s \mu'_s}{\alpha_t \mu_t \mu'_t}}.$$

From this we can see that whenever h is small, whether type t or type s is a better target

depends almost entirely on whether $\alpha_s \mu_s \mu'_s$ is greater than $\alpha_t \mu_t \mu'_t$. When h is large, the ratio on the left depends more on who is already immunized, with higher m_t making type t less attractive relative to type s —there is a force pushing towards more even immunization across types.

C Omitted Proofs

Proof of Theorem 1

Differentiating the first equation in (1) yields

$$\frac{\partial y_s(\mathbf{s})}{\partial s_t} = \alpha_s (1 - s_s) \frac{g''_s(1 - \mathbf{y}(\mathbf{s}) \cdot \mathbf{q}^s)}{\mu_s} \sum_{r \in \Theta} q_r^s \frac{\partial y_r(\mathbf{s})}{\partial s_t},$$

for $s \neq t$, and

$$\frac{\partial y_t(\mathbf{s})}{\partial s_t} = \alpha_t \frac{g'_t(1 - \mathbf{y}(\mathbf{s}) \cdot \mathbf{q}^t)}{\mu_t} + \alpha_t (1 - s_t) \frac{g''_t(1 - \mathbf{y}(\mathbf{s}) \cdot \mathbf{q}^t)}{\mu_t} \sum_{r \in \Theta} q_r^t \frac{\partial y_r(\mathbf{s})}{\partial s_t}.$$

Since $\mathbf{y}(\mathbf{s}) \rightarrow \mathbf{0}$ as $\mathbf{s} \rightarrow_+ \mathbf{0}$, we get

$$\frac{\partial y_s(\mathbf{0})}{\partial s_t} = \alpha_s \mu'_s \sum_{r \in \Theta} q_r^s \frac{\partial y_r(\mathbf{0})}{\partial s_t}, \quad \forall s \neq t, \quad \frac{\partial y_t(\mathbf{0})}{\partial s_t} = \alpha_t + \alpha_t \mu'_t \sum_{r \in \Theta} q_r^t \frac{\partial y_r(\mathbf{0})}{\partial s_t}.$$

Writing Λ_α for a diagonal matrix with $\Lambda_{tt} = \alpha_t$, in matrix notation this gives

$$\frac{\partial \mathbf{y}(\mathbf{0})}{\partial s_t} = \alpha_t \mathbf{e}_t + \Lambda_\alpha M' \Lambda_\alpha^{-1} \frac{\partial \mathbf{y}(\mathbf{0})}{\partial s_t} \quad \implies \quad \frac{\partial \mathbf{y}(\mathbf{0})}{\partial s_t} = \Lambda_\alpha (I - M')^{-1} \mathbf{e}_t.$$

Now differentiate the second equation in (1) to get

$$\frac{\partial \pi_s(\mathbf{s})}{\partial s_t} = \alpha_s (1 - s_s) g'_s(1 - \mathbf{y}(\mathbf{s}) \cdot \mathbf{q}^s) \sum_{r \in \Theta} q_r^s \frac{\partial y_r(\mathbf{s})}{\partial s_t},$$

for $s \neq t$, and

$$\frac{\partial \pi_t(\mathbf{s})}{\partial s_t} = \alpha_t g_t (1 - \mathbf{y}(\mathbf{s}) \cdot \mathbf{q}^t) + \alpha_t (1 - s_t) g_t' (1 - \mathbf{y}(\mathbf{s}) \cdot \mathbf{q}^t) \sum_{r \in \Theta} q_r^t \frac{\partial y_r(\mathbf{s})}{\partial s_t}.$$

Substituting $\mathbf{s} = \mathbf{0}$ and using the above expression for $\mathbf{y}(\mathbf{0})$ gives

$$\frac{\partial \pi_s(\mathbf{0})}{\partial s_t} = \alpha_s \mu_s \sum_{r \in \Theta} q_r^s \frac{\partial y_r(\mathbf{0})}{\partial s_t}, \quad \frac{\partial \pi_t(\mathbf{0})}{\partial s_t} = \alpha_t + \alpha_t \mu_t \sum_{r \in \Theta} q_r^t \frac{\partial y_r(\mathbf{0})}{\partial s_t}, \text{ implying}$$

$$\frac{\partial \boldsymbol{\pi}(\mathbf{0})}{\partial s_t} = \alpha_t \mathbf{e}_t + \Lambda_\alpha M \Lambda_\alpha^{-1} \frac{\partial \mathbf{y}(\mathbf{0})}{\partial s_t} = \alpha_t \mathbf{e}_t + \Lambda_\alpha M (I - M')^{-1} \mathbf{e}_t.$$

Using the definition of the seed multiplier now gives

$$\beta_t(\mathbf{0}) = \frac{1}{p_t} \frac{\partial \pi(\mathbf{0})}{\partial s_t} = \frac{1}{p_t} \mathbf{p}^\top \frac{\partial \boldsymbol{\pi}(\mathbf{0})}{\partial s_t} = \alpha_t + \frac{1}{p_t} \mathbf{p}^\top \Lambda_\alpha M (I - M')^{-1} \mathbf{e}_t$$

The given expression is not quite that in the Theorem statement, but a simple counting argument shows it is equivalent. Entry st of the matrix $M(I - M')^{-1}$ exactly counts the expected number of infectious paths from a given type s individual to type t individuals—multiplying on the left by Λ_α gives us the number that start from a susceptible type s individual. The second term above sums the entries in column t with weights $\frac{p_t}{p_s}$, thereby counting the number of these paths from a given type t individual to type s individuals. We could equivalently sum the entries in row t of $M(I - M')^{-1}$ and multiply by α_t , which yields the expression in the Theorem statement. \square

Proof of Theorem 2

Following the same argument as Theorem 1, at $\mathbf{s} = \mathbf{0}$ we have

$$\frac{\partial y_s(\mathbf{0})}{\partial s_t} = \alpha_s \frac{g_s''(1 - \boldsymbol{\zeta} \cdot \mathbf{q}^s)}{\mu_s} \sum_{r \in \Theta} q_r^s \frac{\partial y_r(\mathbf{0})}{\partial s_t},$$

for $s \neq t$, and

$$\frac{\partial y_t(\mathbf{0})}{\partial s_t} = \alpha_t \frac{g_t'(1 - \boldsymbol{\zeta} \cdot \mathbf{q}^t)}{\mu_t} + \alpha_t \frac{g_t''(1 - \boldsymbol{\zeta} \cdot \mathbf{q}^t)}{\mu_t} \sum_{r \in \Theta} q_r^t \frac{\partial y_r(\mathbf{0})}{\partial s_t}.$$

Recall that

$$\zeta_t = \alpha_t \left(1 - \frac{g_t'(1 - \boldsymbol{\zeta} \cdot \mathbf{q}^t)}{\mu_t} \right) \implies \mu_t = \frac{\alpha_t g_t'(1 - \boldsymbol{\zeta} \cdot \mathbf{q}^t)}{\alpha_t - \zeta_t},$$

so we can rewrite these equations as

$$\frac{\partial y_s(\mathbf{0})}{\partial s_t} = (\alpha_s - \zeta_s) \frac{g_s''(1 - \boldsymbol{\zeta} \cdot \mathbf{q}^s)}{g_s'(1 - \boldsymbol{\zeta} \cdot \mathbf{q}^s)} \sum_{r \in \Theta} q_r^s \frac{\partial y_r(\mathbf{0})}{\partial s_t},$$

for $s \neq t$, and

$$\frac{\partial y_t(\mathbf{0})}{\partial s_t} = \alpha_t - \zeta_t + (\alpha_t - \zeta_t) \frac{g_t''(1 - \boldsymbol{\zeta} \cdot \mathbf{q}^t)}{g_t'(1 - \boldsymbol{\zeta} \cdot \mathbf{q}^t)} \sum_{r \in \Theta} q_r^t \frac{\partial y_r(\mathbf{0})}{\partial s_t}.$$

The generating function for the conditioned degree distribution $D_{\zeta,t}$ is

$$g_{\zeta,t}(s) = \sum_{k=0}^{\infty} \mathbb{P}(D_{\zeta,t} = s) s^k = \sum_{k=0}^{\infty} \frac{\mathbb{P}(D_t = k)(1 - \boldsymbol{\zeta} \cdot \mathbf{q}^t)^k}{g_t(1 - \boldsymbol{\zeta} \cdot \mathbf{q}^t)} s^k = \frac{g_t((1 - \boldsymbol{\zeta} \cdot \mathbf{q}^t)s)}{g_t(1 - \boldsymbol{\zeta} \cdot \mathbf{q}^t)}.$$

From this we can compute

$$\mu_{\zeta,t} = g'_{\zeta,t}(1) = \frac{1 - \boldsymbol{\zeta} \cdot \mathbf{q}^t}{g_t(1 - \boldsymbol{\zeta} \cdot \mathbf{q}^t)} g_t'(1 - \boldsymbol{\zeta} \cdot \mathbf{q}^t), \text{ and}$$

$$\mu'_{\zeta,t} = \frac{g''_{\zeta,t}(1)}{\mu_{\zeta,t}} = \frac{1 - \zeta \cdot \mathbf{q}^t}{g'_t(1 - \zeta \cdot \mathbf{q}^t)} g''_t(1 - \zeta \cdot \mathbf{q}^t).$$

Define $\tilde{y}_t(\mathbf{s}) = \frac{y_t(\mathbf{s})}{1 - \zeta_t}$. Substituting the expression for $\mu'_{\zeta,t}$ into the partial derivatives above gives

$$\frac{\partial y_s(\mathbf{0})}{\partial s_t} = \frac{(\alpha_s - \zeta_s)\mu'_{\zeta,s}}{(1 - \zeta \cdot \mathbf{q}^s)} \sum_{r \in \Theta} q_r^s \frac{\partial y_r(\mathbf{0})}{\partial s_t} \implies \frac{\partial \tilde{y}_s(\mathbf{0})}{\partial s_t} = \alpha_{\zeta,s} \mu'_{\zeta,s} \sum_{r \in \Theta} q_{\zeta,r}^s \frac{\partial \tilde{y}_r(\mathbf{0})}{\partial s_t},$$

for $s \neq t$, and

$$\frac{\partial y_t(\mathbf{0})}{\partial s_t} = \alpha_t - \zeta_t + \frac{(\alpha_t - \zeta_t)\mu'_{\zeta,t}}{(1 - \zeta \cdot \mathbf{q}^t)} \sum_{r \in \Theta} q_r^t \frac{\partial y_r(\mathbf{0})}{\partial s_t},$$

which implies

$$\frac{\partial \tilde{y}_t(\mathbf{0})}{\partial s_t} = \alpha_{\zeta,t} + \alpha_{\zeta,t} \mu'_{\zeta,t} \sum_{r \in \Theta} q_{\zeta,r}^t \frac{\partial \tilde{y}_r(\mathbf{0})}{\partial s_t}.$$

Notice these equations are analogous to the ones we solved in the proof of Theorem 1, and their solution is

$$\frac{\partial \tilde{\mathbf{y}}(\mathbf{0})}{\partial s_t} = \tilde{\alpha}_t \mathbf{e}_t + \Lambda_{\tilde{\alpha}} M'_\zeta \Lambda_{\tilde{\alpha}}^{-1} \frac{\partial \tilde{\mathbf{y}}(\mathbf{0})}{\partial s_t} \implies \frac{\partial \tilde{\mathbf{y}}(\mathbf{0})}{\partial s_t} = \Lambda_{\tilde{\alpha}} (I - M'_\zeta)^{-1} \mathbf{e}_t.$$

Taking a derivative, we have

$$\begin{aligned} \frac{\partial \pi_s(\mathbf{0})}{\partial s_t} &= \alpha_s g'_s(1 - \zeta \cdot \mathbf{q}^s) \sum_{r \in \Theta} q_r^s \frac{\partial y_r(\mathbf{0})}{\partial s_t} \\ &= \alpha_s \mu_{\zeta,s} \frac{g_s(1 - \zeta \cdot \mathbf{q}^s)}{1 - \zeta \cdot \mathbf{q}^s} \sum_{r \in \Theta} q_r^s \frac{\partial y_r(\mathbf{0})}{\partial s_t} \\ &= \alpha_s g_s(1 - \zeta \cdot \mathbf{q}^s) \mu_{\zeta,s} \sum_{r \in \Theta} q_{\zeta,r}^s \frac{\partial \tilde{y}_r(\mathbf{0})}{\partial s_t} \end{aligned}$$

for $s \neq t$, and analogously

$$\frac{\partial \pi_t(\mathbf{0})}{\partial s_t} = \alpha_t g_t (1 - \boldsymbol{\zeta} \cdot \mathbf{q}^t) \left(1 + \mu_{\zeta,t} \sum_{r \in \Theta} q_{\zeta,r}^t \frac{\partial \tilde{y}_r(\mathbf{0})}{\partial s_t} \right)$$

Writing Λ for a diagonal matrix with $\Lambda_{tt} = \alpha_t g_t (1 - \boldsymbol{\zeta} \cdot \mathbf{q}^t)$, in matrix notation this is

$$\frac{\partial \boldsymbol{\pi}(\mathbf{0})}{\partial s_t} = \alpha_t g_t (1 - \boldsymbol{\zeta} \cdot \mathbf{q}^t) \mathbf{e}_t + M \Lambda_{\bar{\alpha}}^{-1} \frac{\partial \tilde{\mathbf{y}}(\mathbf{0})}{\partial s_t} = \alpha_t g_t (1 - \boldsymbol{\zeta} \cdot \mathbf{q}^t) \mathbf{e}_t + \Lambda M_{\zeta} (I - M'_{\zeta})^{-1} \mathbf{e}_t.$$

We therefore have

$$\beta_t(\mathbf{0}) = \frac{1}{p_t} \frac{\partial \pi(\mathbf{0})}{\partial s_t} = \frac{1}{p_t} \mathbf{p}^\top \frac{\partial \boldsymbol{\pi}(\mathbf{0})}{\partial s_t} = \alpha_t g_t (1 - \boldsymbol{\zeta} \cdot \mathbf{q}^t) + \frac{1}{p_t} \mathbf{p}^\top \Lambda M_{\zeta} (I - M'_{\zeta})^{-1} \mathbf{e}_t,$$

and a similar counting argument as in the proof of Theorem 1 shows this is equivalent to the stated expression. \square

Proof of Proposition 1

I prove the following more general result that also applies to the viral case—taking $\boldsymbol{\zeta} = \mathbf{0}$ gives the stated Proposition.

Theorem 1 (3). *Suppose neighbor type distributions follow a simple homophily model with parameter h . Write \mathbf{x} for the vector $(I - M'_{\zeta})^{-1} \mathbf{1}$, and define $\bar{x} = \sum_{t \in \Theta} (\alpha_t - \zeta_t) q_t^* x_t$. Then*

$$\bar{x} = \frac{\sum_{t \in \Theta} \frac{(\alpha_t - \zeta_t)(1 - \boldsymbol{\zeta} \cdot \mathbf{q}^t) q_t^*}{1 - \boldsymbol{\zeta} \cdot \mathbf{q}^t - h \mu'_{\zeta,t} (\alpha_t - \zeta_t)}}{1 - (1 - h) \sum_{t \in \Theta} \frac{(\alpha_t - \zeta_t) \mu'_{\zeta,t} q_t^*}{1 - \boldsymbol{\zeta} \cdot \mathbf{q}^t - h \mu'_{\zeta,t} (\alpha_t - \zeta_t)}},$$

and for each $t \in \Theta$, we have

$$\beta_t = \alpha_t g_t (1 - \boldsymbol{\zeta} \cdot \mathbf{q}^t) \left(1 + \mu_{\zeta,t} \left(\frac{h(\alpha_t - \zeta_t) + (1 - h)\bar{x}}{1 - \boldsymbol{\zeta} \cdot \mathbf{q}^t - h \mu'_{\zeta,t} (\alpha_t - \zeta_t)} \right) \right). \quad (4)$$

Proof. Recall entry ts of M'_ζ is

$$\alpha_{\zeta,s} q_{\zeta,s}^t \mu'_{\zeta,t} = \frac{\alpha_s - \zeta_s}{1 - \zeta_s} \frac{q_s^t (1 - \zeta_s)}{1 - \zeta \cdot \mathbf{q}^t} \mu'_{\zeta,t} = \frac{q_s^t (\alpha_s - \zeta_s)}{1 - \zeta \cdot \mathbf{q}^t} \mu'_{\zeta,t}.$$

We can rewrite $\mathbf{x} = (I - M'_\zeta)^{-1} \mathbf{1}$ as $\mathbf{x} = \mathbf{1} + M'_\zeta \mathbf{x}$. This defines a system of equations in which equation t is

$$x_t = 1 + \frac{\mu'_{\zeta,t}}{1 - \zeta \cdot \mathbf{q}^t} \left(h(\alpha_t - \zeta_t) x_t + (1 - h) \sum_{s \in \Theta} (\alpha_s - \zeta_s) q_s^* x_s \right).$$

Substituting $\bar{x} = \sum_{s \in \Theta} (\alpha_s - \zeta_s) q_s^* x_s$ and solving, we get

$$x_t = \frac{1 - \zeta \cdot \mathbf{q}^t + \mu'_{\zeta,t} (1 - h) \bar{x}}{1 - \zeta \cdot \mathbf{q}^t - h \mu'_{\zeta,t} (\alpha_t - \zeta_t)}.$$

Now multiply by $(\alpha_t - \zeta_t) q_t^*$ and sum over $t \in \Theta$ to get

$$\bar{x} = \sum_{t \in \Theta} \frac{q_t^* (\alpha_t - \zeta_t) (1 - \zeta \cdot \mathbf{q}^t + \mu'_{\zeta,t} (1 - h) \bar{x})}{1 - \zeta \cdot \mathbf{q}^t - h \mu'_{\zeta,t} (\alpha_t - \zeta_t)},$$

which implies

$$\bar{x} \left(1 - (1 - h) \sum_{t \in \Theta} \frac{(\alpha_t - \zeta_t) \mu'_{\zeta,t} q_t^*}{1 - \zeta \cdot \mathbf{q}^t - h \mu'_{\zeta,t} (\alpha_t - \zeta_t)} \right) = \sum_{t \in \Theta} \frac{(\alpha_t - \zeta_t) (1 - \zeta \cdot \mathbf{q}^t) q_t^*}{1 - \zeta \cdot \mathbf{q}^t - h \mu'_{\zeta,t} (\alpha_t - \zeta_t)},$$

and hence

$$\bar{x} = \frac{\sum_{t \in \Theta} \frac{(\alpha_t - \zeta_t) (1 - \zeta \cdot \mathbf{q}^t) q_t^*}{1 - \zeta \cdot \mathbf{q}^t - h \mu'_{\zeta,t} (\alpha_t - \zeta_t)}}{1 - (1 - h) \sum_{t \in \Theta} \frac{(\alpha_t - \zeta_t) \mu'_{\zeta,t} q_t^*}{1 - \zeta \cdot \mathbf{q}^t - h \mu'_{\zeta,t} (\alpha_t - \zeta_t)}}$$

as desired.

To complete the proof, we compute the multipliers:

$$\begin{aligned}
\beta_t &= \alpha_t g_t (1 - \boldsymbol{\zeta} \cdot \mathbf{q}^t) (1 + \mathbf{e}_t^\top M_{\boldsymbol{\zeta}} \mathbf{x}) \\
&= \alpha_t g_t (1 - \boldsymbol{\zeta} \cdot \mathbf{q}^t) \left(1 + \sum_{s \in \Theta} m_{\boldsymbol{\zeta}, ts} x_s \right) \\
&= \alpha_t g_t (1 - \boldsymbol{\zeta} \cdot \mathbf{q}^t) \left(1 + \frac{\mu_{\boldsymbol{\zeta}, t}}{1 - \boldsymbol{\zeta} \cdot \mathbf{q}^t} \sum_{s \in \Theta} q_s^t (\alpha_s - \zeta_s) x_s \right) \\
&= \alpha_t g_t (1 - \boldsymbol{\zeta} \cdot \mathbf{q}^t) \left(1 + \frac{\mu_{\boldsymbol{\zeta}, t}}{1 - \boldsymbol{\zeta} \cdot \mathbf{q}^t} ((1 - h)\bar{x} + h(\alpha_t - \zeta_t)x_t) \right).
\end{aligned}$$

Substituting

$$x_t = \frac{1 - \boldsymbol{\zeta} \cdot \mathbf{q}^t + \mu'_{\boldsymbol{\zeta}, t}(1 - h)\bar{x}}{1 - \boldsymbol{\zeta} \cdot \mathbf{q}^t - h\mu'_{\boldsymbol{\zeta}, t}(\alpha_t - \zeta_t)},$$

we get

$$\begin{aligned}
(1 - h)\bar{x} + h(\alpha_t - \zeta_t)x_t &= (1 - h)\bar{x} + h(\alpha_t - \zeta_t) \frac{1 - \boldsymbol{\zeta} \cdot \mathbf{q}^t + \mu'_{\boldsymbol{\zeta}, t}(1 - h)\bar{x}}{1 - \boldsymbol{\zeta} \cdot \mathbf{q}^t - h\mu'_{\boldsymbol{\zeta}, t}(\alpha_t - \zeta_t)} \\
&= (1 - \boldsymbol{\zeta} \cdot \mathbf{q}^t) \frac{h(\alpha_t - \zeta_t) + (1 - h)\bar{x}}{1 - \boldsymbol{\zeta} \cdot \mathbf{q}^t - h\mu'_{\boldsymbol{\zeta}, t}(\alpha_t - \zeta_t)},
\end{aligned}$$

implying the result. □

Proof of Proposition 2

Write \mathbf{q}^* for the unbiased neighbor type distribution and write $\{\mathbf{q}^t\}_{t \in \Theta}$ for a fixed collection of neighbor type distributions with $q_t^t \geq q_t^*$ and $q_s^t \leq q_s^*$ for $t \neq s$. I first show that $\bar{\beta} = \hat{\beta}$ when $\mathbf{q}^t = \mathbf{q}^*$ for each t . This is exactly the simple homophily model with $h = 0$, so we can substitute into the formula derived in the proof of Proposition 5 to get

$$\frac{1}{\alpha} \beta_t = 1 + \frac{\alpha \mu_t}{1 - \alpha \sum_{s \in \Theta} \mu'_s q_s^*} = 1 + \frac{\alpha \mu_t}{1 - \alpha \bar{\mu}'}$$

Averaging over types with weights p_t exactly gives $\hat{\beta}$ as desired.

Now define $\mathbf{q}^t(x) = x\mathbf{q}^* + (1-x)\mathbf{q}^t$, and write $\beta_t(x)$ for the seed multiplier of type t under the distributions $\{\mathbf{q}^t(x)\}_{t \in \Theta}$, holding fixed the degree distributions and susceptibilities.

Taking a weighted average of the multipliers and differentiating with respect to x gives

$$\begin{aligned} \frac{1}{\alpha} \frac{\partial \bar{\beta}(x)}{\partial x} &= \mathbf{p}^\top \left(\frac{\partial M(x)}{\partial x} (I - M'(x))^{-1} + M(x) (I - M'(x))^{-1} \frac{\partial M'(x)}{\partial x} (I - M'(x))^{-1} \right) \mathbf{1} \\ &= \mathbf{p}^\top \left(\frac{\partial M(x)}{\partial x} + M(x) (I - M'(x))^{-1} \frac{\partial M'(x)}{\partial x} \right) (I - M'(x))^{-1} \mathbf{1}. \end{aligned}$$

Using the consistency condition $p_s \mu_s q_t^s = p_t \mu_t q_s^t$, the vector $\mathbf{p}^\top \frac{\partial M(x)}{\partial x}$ has entry s is equal to

$$\alpha \sum_{t \in \Theta} p_t \mu_t (q_s^t - q_s^*) = \alpha \sum_{t \in \Theta} p_s \mu_s (q_t^s - q_t^*) = 0.$$

We therefore have

$$\frac{1}{\alpha} \frac{\partial \bar{\beta}(x)}{\partial x} = \mathbf{p}^\top M(x) (I - M'(x))^{-1} \frac{\partial M'(x)}{\partial x} (I - M'(x))^{-1} \mathbf{1}.$$

We can equivalently write the right hand side above as $\mathbf{1}^\top DX \mathbf{1}$ in which D is a diagonal matrix with entries $\mathbf{p}^\top M(x)$ and $X = (I - M'(x))^{-1} \frac{\partial M'(x)}{\partial x} (I - M'(x))^{-1}$. To finish the proof, I show that DX is a limit of positive definite matrices, implying the derivative is non-negative.

First, note that if X is positive definite, then DX is as well. In this case, the leading principal minors of X have strictly positive determinants, and since D is diagonal with positive entries, the leading principal minors of DX also have positive determinants. We conclude that DX is positive definite whenever X is. Second, observe that $I - M'(x)$ is positive definite because $\rho(M'(x)) < 1$, implying the eigenvalues of $I - M'(x)$ are all strictly positive. Hence, we know that $(I - M'(x))^{-1}$ is positive definite. The matrix X is then

positive definite if we can show that $\frac{\partial M'(x)}{\partial x}$ is positive definite.

We can readily show that $\frac{\partial M'(x)}{\partial x}$ is positive semi-definite as its eigenvalues must be non-negative. Entry ts of this matrix equals $\alpha\mu'_t(q_s^t - q_s^*)$ —by assumption, the diagonal entries are non-negative, and each row sums to zero. Defining $\lambda = \max_t \alpha\mu'_t(q_t^t - q_t^*)$, we can write

$$\frac{\partial M'(x)}{\partial x} = \lambda I - A$$

for a positive matrix A with row sums bounded by λ . The Perron root of A is therefore bounded by λ , implying all eigenvalues of $\lambda I - A$ are non-negative. The matrix $X_\epsilon = (\lambda + \epsilon)I - A$ is then positive definite for any $\epsilon > 0$, so our above work implies DX_ϵ is positive definite, and $\mathbf{1}^\top DX_\epsilon \mathbf{1} > 0$ for any ϵ . Taking a limit as $\epsilon \rightarrow 0$ shows that $\frac{1}{\alpha} \frac{\partial \bar{\beta}(x)}{\partial x} \geq 0$ for all $x \in [0, 1]$ as desired. In particular, we have $\bar{\beta} = \bar{\beta}(1) \geq \bar{\beta}(0) = \hat{\beta}$. \square

Proof of Proposition 3

With a Poisson degree distribution, the forward distribution is the same as the degree distribution, so we have

$$y_t(\mathbf{s}) = \pi_t(\mathbf{s}) = \alpha_t (1 - (1 - s_t)g_t(1 - \boldsymbol{\pi}(\mathbf{s}) \cdot \mathbf{q}^t)) = \alpha_t \left(1 - (1 - s_t)e^{-\mu_t \boldsymbol{\pi}(\mathbf{s}) \cdot \mathbf{q}^t}\right).$$

Differentiating for each type $s \in \Theta$ with respect to s_t and solving gives

$$\frac{\partial \pi_s(\mathbf{s})}{\partial s_t} = \frac{\alpha_s(1 - s_s)(1 - h) \sum_{r \in \Theta} q_r^* \frac{\partial \pi_r(\mathbf{s})}{\partial s_t}}{1 - \alpha_s h(1 - s_s)e^{-\mu_s \boldsymbol{\pi}(\mathbf{s}) \cdot \mathbf{q}^s}} e^{-\mu_s \boldsymbol{\pi}(\mathbf{s}) \cdot \mathbf{q}^s} \quad \text{for } s \neq t, \quad \text{and}$$

$$\frac{\partial \pi_t(\mathbf{s})}{\partial s_t} = \frac{\alpha_t + \alpha_t(1 - s_t)(1 - h) \sum_{r \in \Theta} q_r^* \frac{\partial \pi_r(\mathbf{s})}{\partial s_t}}{1 - \alpha_t h(1 - s_t)e^{-\mu_t \boldsymbol{\pi}(\mathbf{s}) \cdot \mathbf{q}^t}} e^{-\mu_t \boldsymbol{\pi}(\mathbf{s}) \cdot \mathbf{q}^t}.$$

Weighting by q_r^* and summing over types gives

$$\begin{aligned} \sum_{r \in \Theta} q_r^* \frac{\partial \pi_r(\mathbf{s})}{\partial s_t} &= \frac{\alpha_t q_t^* e^{-\mu_t \boldsymbol{\pi}(\mathbf{s}) \cdot \mathbf{q}^t}}{1 - \alpha_t h(1 - s_t) e^{-\mu_t \boldsymbol{\pi}(\mathbf{s}) \cdot \mathbf{q}^t}} \\ &+ \left(\sum_{r \in \Theta} q_r^* \frac{\partial \pi_r(\mathbf{s})}{\partial s_t} \right) \sum_{r \in \Theta} \frac{\alpha_r (1 - s_r) q_r^* (1 - h) e^{-\mu_r \boldsymbol{\pi}(\mathbf{s}) \cdot \mathbf{q}^r}}{1 - \alpha_r h(1 - s_r) e^{-\mu_r \boldsymbol{\pi}(\mathbf{s}) \cdot \mathbf{q}^r}} \\ &= \frac{\alpha_t q_t^* e^{-\mu_t \boldsymbol{\pi}(\mathbf{s}) \cdot \mathbf{q}^t}}{1 - h(\alpha_t - \pi_t(\mathbf{s}))} + \left(\sum_{r \in \Theta} q_r^* \frac{\partial \pi_r(\mathbf{s})}{\partial s_t} \right) \sum_{r \in \Theta} \frac{q_r^* (1 - h) (\alpha_r - \pi_r(\mathbf{s}))}{1 - h(\alpha_r - \pi_r(\mathbf{s}))}, \end{aligned}$$

in which the last line uses the fixed point equation for $\pi_r(\mathbf{s})$. Solving, we get

$$\sum_{r \in \Theta} q_r^* \frac{\partial \pi_r(\mathbf{s})}{\partial s_t} = \frac{\frac{\alpha_t q_t^* e^{-\mu_t \boldsymbol{\pi}(\mathbf{s}) \cdot \mathbf{q}^t}}{1 - h(\alpha_t - \pi_t(\mathbf{s}))}}{1 - \sum_{r \in \Theta} \frac{q_r^* (1 - h) (\alpha_r - \pi_r(\mathbf{s}))}{1 - h(\alpha_r - \pi_r(\mathbf{s}))}}.$$

Substituting this and simplifying, we obtain

$$\frac{\partial \pi(\mathbf{s})}{\partial s_t} = \sum_{s \in \Theta} p_s \frac{\partial \pi_s(\mathbf{s})}{\partial s_t} = \frac{\alpha_t e^{-\mu_t \boldsymbol{\pi}(\mathbf{s}) \cdot \mathbf{q}^t}}{1 - h(\alpha_t - \pi_t(\mathbf{s}))} (p_t + X),$$

in which

$$X = \frac{(1 - h) \sum_{r \in \Theta} p_r \frac{q_r^* (\alpha_r - \pi_r(\mathbf{s}))}{1 - h(\alpha_r - \pi_r(\mathbf{s}))}}{1 - (1 - h) \sum_{r \in \Theta} \frac{q_r^* (\alpha_r - \pi_r(\mathbf{s}))}{1 - h(\alpha_r - \pi_r(\mathbf{s}))}}.$$

The result follows after substituting $\mathbf{q}^t = h \mathbf{e}_t + (1 - h) \mathbf{q}^*$ and $q_t^* = \frac{p_t \mu_t}{\sum_{s \in \Theta} p_s \mu_s}$. \square

Proof of Proposition 4

First note that if $\rho(M') \leq 1$ with $\mathbf{m} = \mathbf{0}$, this is clearly the solution, and it trivially satisfies the conditions in the proposition. Otherwise, it should be clear that the constraint must be met with equality, so at the optimal \mathbf{m}^* , we have $M' \mathbf{x} = \mathbf{x}$ for some non-zero vector \mathbf{x} —if $h < 1$, the matrix M' is non-negative and irreducible, so the entries of \mathbf{x} are

non-negative. This yields the system of equations

$$x_t = \mu'_t \left(h\alpha_t(1 - m_t^*)x_t + (1 - h) \sum_{s \in \Theta} \alpha_s(1 - m_s^*)q^*sx_s \right).$$

Note in any solution we must have $h\alpha_t\mu'_t(1 - m_t^*) < 1$, since otherwise this implies $x_t > x_t$, and the spectral radius cannot equal 1, and we can always ensure this by choosing m_t^* sufficiently large. We can rearrange the above equation to get

$$x_t = \frac{\mu'_t(1 - h) \sum_{s \in \Theta} \alpha_s(1 - m_s^*)q^*sx_s}{1 - h\alpha_t\mu'_t(1 - m_t^*)}.$$

Weighting each equation by $\alpha_t(1 - m_t^*)q_t^*$ and summing, we get

$$\sum_{t \in \Theta} \alpha_t(1 - m_t^*)q_t^*tx_t = \left(\sum_{s \in \Theta} \alpha_s(1 - m_s^*)q^*sx_s \right) \left(\sum_{t \in \Theta} \frac{\alpha_t(1 - m_t^*)q_t^*\mu'_t(1 - h)}{1 - h\alpha_t\mu'_t(1 - m_t^*)} \right).$$

Since \mathbf{x} is the eigenvector corresponding to the Perron root, the left hand side is strictly positive, so we know the second term in parentheses on the right must equal 1. Substituting the definition of q_t^* yields the first condition. This reduces our problem to minimizing $\mathbf{m} \cdot \mathbf{p}$, subject to the above equation, and the second claim is an immediate consequence of the first order conditions.

Finally, for the case $h = 1$, we need each subgraph to be non-viral, and the condition for that is precisely $\alpha_t\mu'_t(1 - m_t^*) \leq 1$, which is equivalent to $m^* \geq \frac{\alpha_t\mu'_t - 1}{\alpha_t\mu'_t}$. \square