Supplementary Appendix for "How Well Does Bargaining Work in Consumer Markets? A Robust Bounds Approach"*

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Abstract

This file contains additional material of Freyberger and Larsen (2024), including auxiliary lemmas, proofs of the main theorems, additional discussion of the data and the assumptions, details on estimation and inference, Monte Carlo simulations, and a discussion of two extensive-form bargaining models.

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A Auxiliary Lemmas

We prove two auxiliary lemmas. The first is used to prove that a lower or upper bound is in the identified set and the second to show that any CDF above the lower bound (or below the upper bound) is also in the identified set.

Lemma 1. *Let X be a random variable, Y a random vector, and denote the conditional CDF by $F_{X|Y}(x \mid y)$. Let $g(x,y) : \mathbb{R}^{1+\dim(Y)} \to [0,1]$ be a CDF for all y.*

1. *If $g(x,y) \geq F_{X|Y}(x \mid y)$ for all x and y, then there exists a random variable W such that $W \leq X$ and $F_{W|Y}(x \mid y) = g(x,y)$.*

2. *If $g(x,y) \leq F_{X|Y}(x \mid y)$ for all x and y, then there exists a random variable W such that $W \geq X$ and $F_{W|Y}(x \mid y) = g(x,y)$.*

Proof. We only prove part 1; part 2 follows from analogous arguments. Define $g^{-1}(z,y) = \inf\{x \in \mathbb{R} : g(x,y) \geq z\}$. Let $U \sim U[0,1]$ be independent of (X,Y) and define $\tilde{F}_{X,U|Y}(x,u,y) = P(X < x \mid Y = y) + P(X = x \mid Y = y)u$. Finally, let $W = g^{-1}\left(\tilde{F}_{X,U|Y}(X,U,Y), Y\right)$. We show that $W \leq X$ and $F_{W|Y}(x \mid y) = g(x,y)$.

For the second part, it is sufficient to prove that $\tilde{F}_{X,U|Y}(X,U,Y) \mid Y = y \sim U[0,1]$ for all y. To do so, let $\bar{x}_1(y), \bar{x}_2(y), \ldots, \bar{x}_{M(y)}(y)$ be the mass points of X conditional on y. Then for all $m = 1, 2, \ldots, M(y)$

$$\tilde{F}_{X,U|Y}(X,U,Y) \sim U[P(X < \bar{x}_m(y) \mid Y = y), P(X = \bar{x}_m(y) \mid Y = y)]$$

conditional on $Y = y$ and $X = \bar{x}_m(y)$ and for all $m = 1, \ldots, M(y) - 1$

$$\tilde{F}_{X,U|Y}(X,U,Y) \sim U[P(X \leq \bar{x}_m(y) \mid Y = y), P(X < \bar{x}_{m+1}(y) \mid Y = y)]$$

conditional on $Y = y$ and $X \in (\bar{x}_m(y), \bar{x}_{m+1}(y))$. Finally if $P(X < \bar{x}_1(y) \mid Y = y) > 0$, then $\tilde{F}_{X,U|Y}(X,U,Y) \mid Y = y, X < \bar{x}_1(y) \sim U[0, P(X < \bar{x}_1(y) \mid Y = y)]$ and if $P(X > \bar{x}_{M(y)}(y) \mid Y = y) > 0$, then $\tilde{F}_{X,U|Y}(X,U,Y) \mid Y = y, X > \bar{x}_{M(y)}(y) \sim U[P(X \leq \bar{x}_{M(y)}(y) \mid Y = y), 1]$. Since the supports of the intervals of these uniforms only overlap at the boundaries, the union of the supports is $[0,1]$, and the difference of the upper and lower bound is equal to the probability that X is in the respective set (that is either $X = \bar{x}_m(y)$ or $X \in (\bar{x}_m(y), \bar{x}_{m+1}(y))$), it follows that $\tilde{F}_{X,U|Y}(X,U,Y) \mid Y = y \sim U[0,1]$ for all y.

To show that $W \leq X$, notice that, since $g(x,y) \geq F_{X|Y}(x \mid y)$ for all x and y, it holds that

$$g^{-1}(z,y) = \inf\{x \in \mathbb{R} : g(x,y) \geq z\} \leq \inf\{x \in \mathbb{R} : F_{X|Y}(x \mid y) \geq z\} = F_{X|Y}^{-1}(z \mid y)$$

Next, notice that, if $Y = y$ and $X = \bar{x}_m(y)$, then $\tilde{F}_{X,U|Y}(x,u,y) \leq P(X = \bar{x}_m(y) \mid Y = y)$ for all $u \in [0,1]$ and

$$W = g^{-1}\left(\tilde{F}_{X,U|Y}(\bar{x}_m(y),U,y),y\right) \leq F_{X|Y}^{-1}(P(X = \bar{x}_m(y) \mid Y = y) \mid y) = \bar{x}_m(y)$$

Finally, for all $Y = y$ and $X = x \notin \{\bar{x}_1(y), \bar{x}_2(y), \ldots, \bar{x}_{M(y)}(y)\}$,
$W = g^{-1}\left(\tilde{F}_{X,U|Y}(x,U,y),y\right) \leq F_{X|Y}^{-1}(P(X \leq x \mid Y = y) \mid y) \leq x.$ □

Lemma 2. *Let X be a random variable, Y a random vector, and $g(x) : \mathbb{R} \to [0,1]$ a CDF.*

1. *If $g(x) \geq F_X(x)$ for all x, then there exists a random variable W such that $W \leq X$ and $F_W(x) = g(x)$. Moreover, if $F_{X|Y}(x \mid y)$ is either weakly increasing, weakly decreasing, or constant in an element of y for all x, then $F_{W|Y}(x \mid y)$ shares this property.*

2. *If $g(x) \leq F_X(x)$ for all x, then there exists a random variable W such that $W \geq X$ and $F_W(x) = g(x)$. Moreover, if $F_{X|Y}(x \mid y)$ is either weakly increasing, weakly decreasing, or constant in an element of y for all x, then $F_{W|Y}(x \mid y)$ shares this property.*

Proof. Define $g^{-1}(z) = \inf\{x \in \mathbb{R} : g(x) \geq z\}$. Let $U \sim U[0,1]$ be independent of (X,Y) and define $\tilde{F}_{X,U}(x,u) = P(X < x) + P(X = x)u$. Finally, let $W = g^{-1}\left(\tilde{F}_{X,U}(X,U)\right)$. The first two parts of the lemma follow immediately from the proof of Lemma 1. To show that $F_{X|Y}(x \mid y)$ and $F_{W|Y}(x \mid y)$ share the same monotonicity properties, define $p_X(x) = P(X = x)$

and notice that

$$P(W \leq w \mid Y = y) = P\left(g^{-1}\left(\tilde{F}_{X,U}(X,U)\right) \leq w \mid Y = y\right)$$
$$= \int_0^1 P\left(g^{-1}\left(F_X(X) + p_X(X)(u-1)\right) \leq w \mid Y = y\right) du$$

The function $\tilde{F}_{X,U}(x,u) = P(X < x) + P(X = x)u$ is equal to $F_X(x)$ at every continuity point of $F_X(x)$. At a discontinuity point, $\tilde{F}_{X,U}(x,u)$ is equal to a point in the interval $[P(X < x), P(X \leq x)]$, depending on the value of u. Hence, $\tilde{F}_{X,U}(x,u)$ is weakly increasing in x for all u. Since g^{-1} is also weakly increasing, $h(x,u) = g^{-1}(F_X(x) + p_X(x)(u-1))$ is weakly increasing in x for all u. It follows that there exists $x(u,w)$ such that $h(X,u) \leq w$ is equivalent to to $X \leq x(u,w)$ or $X < x(u,w)$. Hence,

$$P(h(X,u) \leq w \mid Y = y) = P(X \leq x(u) \mid Y = y)$$

or

$$P(h(X,u) \leq w \mid Y = y) = P(X < x(u) \mid Y = y)$$

If the right hand side is weakly increasing/decreasing/constant in y for all u, it follows that $P(W \leq w \mid Y = y)$ shares the same monotonicity property. □

B Proofs of Main Theorems

Proof of ***Theorem 1*** *(Unconditional Bounds).* $X_Q^S \leq S \leq X_{AC}^S \Rightarrow P(X_{AC}^S \leq x) \leq P(S \leq x) \leq P(X_Q^S \leq x)$. Similarly, $X_{AC}^B \leq B \leq X_Q^B \Rightarrow P(X_Q^B \leq x) \leq P(B \leq x) \leq P(X_{AC}^B \leq x)$.

If $S = X_{AC}^S$, then $P(X_{AC}^S \leq x) = P(S \leq x)$. Hence, the lower bound can be attained. Moreover, it follows from Lemma 2 (with $X = X_{AC}^S$) that any CDF $F(x)$ with $F(x) \geq P(X_{AC}^S \leq x)$ for all x is also in the identified set. Similar arguments imply sharpness of the upper bound and the buyer bounds. □

Proof of ***Theorem 2*** *(Monotonicity).* Conditional on $P_1^S = y$, $P_2^B = z$, $X_{AC}^{B*}(y,z) \leq B \leq X_Q^{B*}(y,z)$

and $X_{AC}^{B*}(y,z)$ and $X_{Q}^{B*}(y,z)$ are non-random. Then (2) and $X_{AC}^{B} \leq B \leq X_{Q}^{B} \Rightarrow P(B \leq x) \geq \int \mathbf{1}(X_{Q}^{B*}(y,z) \leq x) dF_{P_1^S,P_2^B}(y,z)$. The buyer upper bound is analogous. For the seller, conditional on $P_1^S = y$, we have $X_Q^{S*}(y) \leq S \leq X_{AC}^{S*}(y)$ and $X_Q^{S*}(y)$ and $X_{AC}^{S*}(y)$ are non-random. Combining this with (1) implies $\int \mathbf{1}(X_{AC}^{S*}(y) \leq x) dF_{P_1^S}(y) \leq P(S \leq x) \leq \int \mathbf{1}(X_{Q}^{S*}(y) \leq x) dF_{P_1^S}(y)$.

The seller lower bound is attained with $S = X_{AC}^{S*}(P_1^S)$, in which case S is increasing in P_1^S. We can apply a modification of Lemma 2 (with $X = X_{AC}^{S*}(P_1^S)$) to show that any CDF $F(x)$ with $F(x) \geq P(X_{AC}^{S*}(P_1^S) \leq x)$ for all x is also in the identified set. To do so, let $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_L$ be the mass points of $X_{AC}^{S*}(P_1^S)$. Let $\tilde{U} \sim U[0,1]$ be independent of $X_{AC}^{S*}(P_1^S)$ and

$$U = \begin{cases} F_{P_1^S \mid X_{AC}^{S*}(P_1^S) = \bar{x}_l}(P_1^S) & \text{if } X_{AC}^{S*}(P_1^S) = \bar{x}_l \qquad \forall l = 1,2,\ldots,L \\ \tilde{U} & \text{if } X_{AC}^{S*}(P_1^S) \in (\bar{x}_l, \bar{x}_u) \end{cases}$$

and define $\tilde{F}_{X_{AC}^{S*}(P_1^S),U}(x,u) = P\left(X_{AC}^{S*}(P_1^S) < x\right) + P(X_{AC}^{S*}(P_1^S) = x)u$ Finally, let $F^{-1}(z) = \inf\{x \in \mathbb{R} : F(x) \geq z\}$ and $W = F^{-1}\left(\tilde{F}_{X,U}(X_{AC}^{S*}(P_1^S), U)\right)$.

Since $U \mid X_{AC}^{S*}(P_1^S) \sim U[0,1]$, the arguments from the proof of Lemma 2 imply $P(W \leq x) = F(x)$. The construction ensures that W is increasing in P_1^S and hence, the CDF is attained when $S = W$. Sharpness of the upper bound follows analogously.

Similarly, the lower bound for the buyer is attained when $B = X_Q^{B*}(P_1^S, P_2^B) \leq X_Q^B$ and, just like above, a modification of Lemma 2 implies that any CDF $F(x)$ with $F(x) \geq P(X_Q^{B*}(P_1^S, P_2^B) \leq x)$ for all x is also in the identified set. □

Proof of ***Theorem 3*** *(Independence).* First, note $P(S \leq x) = \int P(S \leq x \mid P_1^S = y) dF_{P_1^S}(y)$, which, by A3.i, is $\int \max_z P(S \leq x \mid P_1^S = y, P_2^B = z) dF_{P_1^S}(y)$. The lower bound follows from $S \leq X_{AC}^S$. The seller upper bound is analogous. For the buyer, $P(B \leq x) = \max_y P(B \leq x \mid P_1^S = y)$ and $P(B \leq x) = \min_y P(B \leq x \mid P_1^S = y)$. The bounds follow from $X_{AC}^B \leq B \leq X_Q^B$.

Attainment of the seller lower bounds follows from Lemma 1 with $X = X_{AC}^S$ and $Y = (P_1^S, P_2^B)$, $g(x, \{y,z\}) = \max_{z'} P(X_{AC}^S \leq x \mid P_1^S = y, P_2^B = z')$, and $W = S$. Notice that in this application of Lemma 1, Y is two-dimensional, and consequently so is the second argument of $g(\cdot,\cdot)$: here, that second argument is $\{y,z\}$. Since g does not depend on z, the implied distribution of S is independent of P_2^B conditional on P_1^S. Any CDF above the lower bound being in the identified set follows from Lemma 2. Sharpness of the upper bound and the buyer bounds follows from analogous arguments. □

Proof of ***Theorem 4*** *(Stochastic Monotonicity).* By (2), $P(B \le x) = \int P(B \le x \mid P_1^S = y, P_2^B = z) dF_{P_1^S,P_2^B}(y,z)$. By A4 this can then be written $\int \max_{z' \ge z} P(B \le x \mid, P_1^S = y, P_2^B = z') dF_{P_1^S,P_2^B}(y,z)$ or $\int \min_{z' \le z} P(B \le x \mid P_1^S = y, P_2^B = z') dF_{P_1^S,P_2^B}(y,z)$. The bounds follow from $X_{AC}^B \le B \le X_Q^B$. For the seller, we have, by (1) and A4, $P(S \le x) = \int \max_{y' \ge y} P(S \le x \mid P_1^S = y') dF_{P_1^S}(y)$ and $P(S \le x) = \int \min_{y' \le y} P(S \le x \mid P_1^S = y') dF_{P_1^S}(y)$. The bounds follow from $X_Q^S \le S \le X_{AC}^S$.

Attainment of the seller lower bounds follows from Lemma 1 with $X = X_{AC}^S$, $Y = P_1^S$, $g(x,y) = \max_{y' \ge y} P(X_{AC}^S \le x \mid P_1^S = y')$, and $W = S$. Since g is weakly increasing y for all x, the implied conditional distribution of S satisfies A4.i. Any CDF above the lower bound being in the identified set follows from Lemma 2. Sharpness of the upper bound and the buyer bounds follows from analogous arguments. □

Proof of ***Theorem 5*** *(Positive Correlation).* First, note $P(S \le x) = \int P(S \le x \mid P_1^S = y, P_2^B = z) dF_{P_1^S,P_2^B}(y,z)$, which, by A5, is $\int \max_{z' \ge z} P(S \le x \mid P_1^S = y, P_2^B = z') dF_{P_1^S,P_2^B}(y,z)$. The lower bound follows from $S \le X_{AC}^S$. The seller upper bound is analogous. For the buyer, we have $P(B \le x) = \int \max_{y' \ge y} P(B \le x \mid P_1^S = y') dF_{P_1^S}(y)$ and $P(B \le x) = \int \min_{y' \le y} P(B \le x \mid P_1^S = y') dF_{P_1^S}(y)$. The bounds follow from $X_{AC}^B \le B \le X_Q^B$. The sharpness arguments follow from similar arguments as those in the proof of Theorem 4. □

Proof of ***Theorem 6*** *(Surplus Stochastic Monotonicity and Buyer Monotonicity).* Note $P(B - S \ge x) = \int P(B - S \ge x \mid P_1^S = y, P_2^B = z) dF_{P_1^S,P_2^B}(y,z)$. By A6, this can be written $\int \max_{z' \le z} P(B - S \ge x \mid P_1^S = y, P_2^B = z') dF_{P_1^S,P_2^B}(y,z)$. The lower bound follows from A1.i and A2.ii, implying $X_{AC}^{B^*}(P_1^S, P_2^B) - X_{AC}^S \le B - S$. The upper bound is analogous.

Lemma 1 applied with $X = X_{AC}^S - X_{AC}^{B^*}(P_1^S, P_2^B)$, $Y = (P_1^S, P_2^B)$, and $g(x, \{y,z\}) = \max_{z' \le z} \Pr(X_{AC}^S - X_{AC}^{B^*}(y,z) \le x \mid P_1^S = y, P_2^B = z')$ implies that there exists a random variable W such that

$W \geq X_{AC}^{B^*}(P_1^S, P_2^B) - X_{AC}^S$ and

$$P(W \geq x \mid P_1^S = y, P_2^B = z) = \int \max_{z' \leq z} P(X_{AC}^{B^*}(y,z) - X_{AC}^S \geq x \mid, P_1^S = y, P_2^B = z') dF_{P_1^S, P_2^B}(y,z)$$

Letting $B = X_{AC}^{B^*}(P_1^S, P_2^B)$ and $S = B - W$ implies that all assumptions are satisfied and the lower bound is attained. Any CDF above the lower bound being in the identified set follows modifying Lemma 2 as in the proof of Theorem 2. □

Proof of ***Theorem 7*** *(Surplus Weak Monotonicity and Buyer Monotonicity).* By A1 and A2, $S \leq X_{AC}^S$ and $B \geq X_{AC}^{B*}(y,z)$ conditional on $P_1^S = y$ and $P_2^B = z$. Define $X_{AC}^{B*-S}(y,z) = \overline{\text{supp}}(X_{AC}^{B*}(y,z) - X_{AC}^S : P_2^B \leq z, P_1^S = y)$. The assumptions imply that $B - S \geq X_{AC}^{B*-S}(y,z)$ conditional on $P_1^S = y$ and $P_2^B = z$ and thus, $P(B - S \geq x) \geq \int \mathbf{1}(X_{AC}^{B*-S}(y,z) \geq x) dF_{P_1^S, P_2^B}(y,z)$. The upper bound is analogous.

The lower bound is attained when $B = X_{AC}^{B*}(P_1^S, P_2^B)$ and $S = X_{AC}^{B*}(P_1^S, P_2^B) - X_{AC}^{B*-S}(P_1^S, P_2^B)$ in which case both monotonicity assumptions hold. That any CDF above the lower bound is also in the identified set follows from modifying Lemma 2, as in the Theorem 2 proof. □

C Bounds Based on Combined Assumptions

B.1. Derivation of Bounds Combining Assumptions.

Theorem 8. *(Independence + Monotonicity.) (5) gives a sharp lower bound for F_S under A1.i, A2.i, and A3.i and a sharp upper bound for F_S under A1.ii, A2.i, and A3.i. The following inequalities give a sharp lower bound for F_B under A1.iv, A2.ii, and A3.ii and a sharp upper bound for F_B under A1.iii, A2.ii, and A3.ii:*

$\max_y P(X_Q^{B*}(P_1^S, P_2^B) \leq x \mid P_1^S = y) \leq F_B(x) \leq \min_y P(X_{AC}^{B*}(P_1^S, P_2^B) \leq x \mid P_1^S = y)$

Proof. Sharpness of the seller lower bound follows from the proof of Theorem 2 and the observation that $X_{AC}^{S*}(P_1^S)$ is independent of P_2^B conditional of $P_1^S = y$ (because $X_{AC}^{S*}(y)$ is deterministic).

The lower bound follows from $P(B \le x) = \max_y P(B \le x \mid P_1^S = y)$ and $B \le X_Q^{B*}(P_1^S, P_2^B)$. The upper bound is analogous. Attainment of the lower bound follows from modifying Lemma 2. In particular, denote the lower and upper mass point of $X_Q^{B*}(P_1^S, P_2^B)$ conditional on $P_1^S = y$ by $\tilde{x}_1(y), \ldots, \tilde{x}_{L(y)}(y)$ and let $U = F_{P_2^B \mid P_1^S = y, X_Q^{B*}(y, P_2^B) = \bar{x}_u(y)}(P_2^B, P_1^S)$ if $P_1^S = y$ and $X_Q^{B*}(P_1, P_2^B) = \bar{x}_l(y)$ and $U \sim U[0,1]$ independent of all other random variables if $P_1^S = y$ and $X_Q^{B*}(P_1, P_2^B) \neq \bar{x}_l(y)$ for all $l = 1, 2, \ldots L(y)$. Then $U \mid P_1^S = y, X_Q^{B*}(P_1, P_2^B) = x \sim U[0,1]$ for x and y. Using this random variable U, Lemma 1 applied with $X = X_Q^{B*}(P_1^S, P_2^B)$, $Y = P_1^S$, $g(x,y) = \max_{y'} P(X_Q^{B*}(P_1^S, P_2^B) \le x \mid P_1^S = y')$, and $W = B$ implies that the lower bound can be attained. The construction also ensures that W is weakly increasing in P_2^B conditional on P_S^1. Any CDF above the lower bound being in the identified set follows from a similar modification of Lemma 2. □

Theorem 9. *(Positive Correlation + Monotonicity.) (5) gives a sharp lower bound for F_S under A1.i, A2.i, and A5.i and a sharp upper bound for F_S under A1.ii, A2.i, and A5.i. The following inequalities give a sharp lower bound for F_B under A1.iv, A2.ii, and A5.ii and a sharp upper bound for F_B under A1.iii, A2.ii, and A5.ii:*

$$\int \max_{y' \ge y} P(X_Q^{B*}(P_1^S, P_2^B) \le x \mid P_1^S = y') dF_{P_1^S}(y) \le F_B(x) \le \int \max_{y' \ge y} P(X_{AC}^{B*}(P_1^S, P_2^B) \le x \mid P_1^S = y') dF_{P_1^S}(y)$$

Proof. Sharpness of the seller lower bound follows from the proof of Theorem 2 and the observation that $X_{AC}^{S*}(P_1^S)$ is independent of P_2^B conditional of $P_1^S = y$ (because $X_{AC}^{S*}(y)$ is deterministic).

The lower bound follows from $P(B \le x) = \int \max_{y' \ge y} P(B \le x \mid P_1^S = y') dF_{P_1^S}(y)$ and $B \le X_Q^{B*}(P_1^S, P_2^B)$. The upper bound is analogous. Sharpness follows from the same arguments as those in the proof of Theorem 8. □

Theorem 10. *(Independence + Stochastic Monotonicity.) The following inequalities give a sharp lower bound for F_S under A1.i, A4.i, and A3.i; a sharp upper bound for F_S under A1.ii, A4.i, and A3.i; a sharp lower bound for F_B under A1.iv, A4.ii, and A3.ii; and a sharp upper bound for F_B under A1.iii, A4.ii, and A3.ii:*

$$\int \max_{y' \geq y} \max_{z} m_{AC}^{S}(x,y',z)dF_{P_1^S}(y) \leq F_S(x) \leq \int \min_{y' \leq y} \min_{z} m_{Q}^{S}(x,y',z)dF_{P_1^S}(y)$$
$$\max_{y} \int \max_{z' \geq z} m_{Q}^{B}(x,y,z')dF_{P_2^B|P_1^S}(z|y) \leq F_B(x) \leq \min_{y} \int \max_{z' \geq z} m_{AC}^{B}(x,y,z')dF_{P_2^B|P_1^S}(z|y)$$

Proof. Note $P(S \leq x) = \int \max_{y' \geq y} P(S \leq x \mid P_1^S = y')dF_{P_1^S}(y)$, and $P(S \leq x \mid P_1^S = y') = \max_z P(S \leq x \mid P_1^S = y', P_2^B = z)$. The lower bound follows from $S \leq X_{AC}^S$. For the buyer, $P(B \leq x) = \max_y P(B \leq x \mid P_1^S = y)$. Applying (2) and A4 yields $P(B \leq x \mid P_1^S = y) = \int \max_{z' \geq z} P(B \leq x \mid P_1^S = y, P_2^B = z)dF_{P_2^B|P_1^S}(z|y)$. The lower bound follows from $B \leq X_Q^B$. Analogous arguments yield the upper bounds. Sharpness follows from the same arguments as those with the corresponding single assumption case using the structure of the bounds implied by the combination of the independence and stochastic monotonicity assumptions, along with Lemmas 1 and 2. □

Theorem 11. *(Positive Correlation + Stochastic Monotonicity.) The following inequalities give a sharp lower bound for F_S under A1.i, A4.i, and A5.i; a sharp upper bound for F_S under A1.ii, A4.i, and A5.i; a sharp lower bound for F_B under A1.iv, A4.ii, and A5.ii; and a sharp upper bound for F_B under A1.iii, A4.ii, and A5.ii:*

$$F_S(x) \geq \int \max_{y' \geq y} \int \max_{z' \geq z} m_{AC}^{S}(x,y',z')dF_{P_2^B|P_1^S}(z|y')dF_{P_1^S}(y)$$
$$F_S(x) \leq \int \min_{y' \leq y} \int \min_{z' \geq z} m_{Q}^{S}(x,y',z')dF_{P_2^B|P_1^S}(z|y')dF_{P_1^S}(y)dy$$
$$F_B(x) \geq \int \max_{y' \geq y} \int \max_{z' \geq z} m_{Q}^{B}(x,y',z')dF_{P_2^B|P_1^S}(z|y')dF_{P_1^S}(y)$$
$$F_B(x) \leq \int \min_{y' \leq y} \int \min_{z' \geq z} m_{AC}^{B}(x,y',z')dF_{P_2^B|P_1^S}(z|y')dF_{P_1^S}(y)$$

Proof. Note $P(S \leq x) = \int \max_{y' \geq y} P(S \leq x \mid P_1^S = y')dF_{P_1^S}(y)$ and
$$\begin{aligned} P(S \leq x \mid P_1^S = y') &= \int P(S \leq x \mid P_1^S = y', P_2^B = z)dF_{P_2^B|P_1^S}(z|y') \\ &= \int \max_{z' \geq z} P(S \leq x \mid P_1^S = y', P_2^B = z)dF_{P_2^B|P_1^S}(z|y') \end{aligned}$$

The lower bound follows from $S \leq X_{AC}^S$. For the buyer, $P(B \leq x) = \int \max_{y' \geq y} P(B \leq x \mid P_1^S = y')dF_{P_1^S}(y)$ and

$$
\begin{aligned}
P(B \le x \mid P_1^S = y') &= \int P(B \le x \mid P_1^S = y', P_2^B = z) dF_{P_2^B|P_1^S}(z|y') \\
&= \int \max_{z' \ge z} P(B \le x \mid P_1^S = y', P_2^B = z) dF_{P_2^B|P_1^S}(z|y')
\end{aligned}
$$

The lower bound follows from $B \le X_Q^B$. Upper bounds follow analogously. Sharpness follows from arguments as those with the corresponding single assumption case using the structure of the bounds implied by the combination of the positive correlation and stochastic monotonicity assumptions, along with Lemmas 1 and 2. □

D Additional Discussion of Data and Assumptions

D.1. Sample Restrictions. As discussed in Section 2, what we refer to as our *original data* consists of all eBay Best-Offer-enabled listings from June 2012 through May 2013 satisfying the following: a buyer makes an offer, the item has a product identifier, and the product's reference price is computed based on at least ten non-Best-Offer posted price sales. We impose several sample restrictions on this dataset to obtain our estimation sample. These restrictions are described in the notes to Table A1, which shows the number of observations from the original sample that are dropped due to each restriction.[1] Several restrictions are data cleaning steps that drop only a small fraction of observations. Our major restrictions are the following. First, we remove negotiations in which an agent is involved in other negotiations simultaneously, dropping 54.3% of observations. Second, after imposing this restriction, we then keep only the first seller with whom a given buyer interacts for a given product, and the first buyer with whom a given seller negotiates, dropping 13.27%. Third, we limit to products for which we have at least 200 observations, dropping 26.60%.

We now consider how our results change when we modify step 5 of the sample restrictions listed in Table A1. Table A2 replicates Table 3 from the body of the paper but where, instead of step 5, we keep the *last* (in panel A) or a *random* (in panel B) seller among

[1]The precise fraction of observations dropped due to each restriction depends on the order in which the restrictions are imposed. The order we followed is the order in which they are listed in Table A1.

Table A1: Data Cleaning

1) Fraction incomplete sequences	0.0226
2) Fraction overlapping sequences	0.5434
3) Fraction additional incomplete sequences	0.0003
4) Fraction extreme outlier offers/prices	0.0296
5) Fraction dropped by keeping only first seller/buyer	0.1327
6) Fraction with fewer than 200 negotiations per product	0.2660

Notes: Table shows the order in which our additional sample restrictions are enforced on the original data, and the fraction of observations in the original data that are dropped at each step. First row shows a small fraction are dropped due to incomplete or nonsensical bargaining data, including observations where (i) an offer or the final price is higher than the Buy-It-Now price, (ii) more than one offer arrives at the same time from the same buyer, (iii), additional actions take place after one party accepts, (iv) one or both parties make more than three offers in a given sequence, (v) the data indicates a counteroffer takes place but the offer itself is not recorded, or (vi) agents make non-monotonic offers (e.g. a buyer offers more than the seller has asked for or a seller asks for less than the buyer has offered). Second row shows the fraction of observations dropped due to overlapping negotiations. Third row shows an additional small fraction of incomplete/nonsensical observations more easily identified after overlapping sequences are dropped in step 2. Fourth row shows observations dropped because of offers or auto accept/decline prices being greater than 2.5 times the reference price. Fifth row shows fraction dropped when we keep only the first seller a buyer negotiates with and vice versa. Final row shows fraction of the data dropped because the product had fewer than 200 negotiations.

those with whom a given buyer interacts, and similarly for sellers interacting with multiple buyers. The results are similar to those in Table 3. Table A3 replicates Table 4 with these two samples and also shows similar results to the body of the paper. Let the *last version* of the data denote the sample used in panel A and the *random version* denote the sample used in panel B. Recall that the inefficient impasse lower bound for the median product is 0.373 in our main sample. This number decreases to 0.307 when we use the last version of the data and to 0.342 when we use the random version. Thus, the implications for inefficient impasse are similar in these samples as in our main sample, albeit slightly lower.

As described in Section 3.1, the body of the paper defines a seller *quitting* as a situation where the seller chooses to *decline* an offer and the buyer takes no further action. Table A4 replicates columns 1–3 of Tables 3–4 using alternative definitions of a seller quitting. In panel A, all seller declines (even those followed by additional buyer activity) are considered seller quits. Panel B defines seller quits as seller declines occurring after the buyer has exhausted the three-offer limit. Panel C expands the definition from panel B to also treat

Table A2: Bounds Crossing with Different Bilateral Bargaining Pairs

	Seller Bounds			Buyer Bounds		
	Frac. Cross	Frac. Reject	IVE	Frac. Cross	Frac. Reject	IVE
A. Restricting to last seller a buyer negotiates with, and vice versa						
Unconditional (A1)	0	0	0	0	0	0
Monotonicity (A2)	1.00	1.00	0.24	0	0	0
Independence (A3)	0.14	0	0.00	0.43	0.09	0.01
Stochastic Monotonicity (A4)	0	0	0	0	0	0
Positive Correlation (A5)	0	0	0.00	0	0	0
Mon. + Indep. (A2 + A3)	1.00	1.00	0.24	0.63	0.46	0.06
Mon. + Pos. Corr (A2 +A5)	1.00	1.00	0.24	0	0	0
Stoch. Mon. + Indep. (A4 + A3)	0.31	0	0.00	0.43	0.06	0.006
Stoch. Mon. + Pos. Corr (A4 + A5)	0	0	0.00	0	0	0
B. Restricting to random seller a buyer negotiates with, and vice versa						
Unconditional (A1)	0	0	0	0	0	0
Monotonicity (A2)	1.00	1.00	0.23	0	0	0
Independence (A3)	0.08	0	0.00	0.41	0.11	0.01
Stochastic Monotonicity (A4)	0	0	0	0	0	0
Positive Correlation (A5)	0	0	0.00	0	0	0
Mon. + Indep. (A2 + A3)	1.00	1.00	0.23	0.54	0.46	0.07
Mon. + Pos. Corr (A2 +A5)	1.00	1.00	0.23	0	0	0
Stoch. Mon. + Indep. (A4 + A3)	0.19	0	0.00	0.41	0.14	0.007
Stoch. Mon. + Pos. Corr (A4 + A5)	0	0	0.00	0	0	0

Notes: Table replicates Table 3 using different samples. The main sample in the paper restricts to the first seller a given buyer negotiates with and the first buyer a given seller negotiates with. Panel A above uses the *last* seller and *last* buyer. Panel B selects a *random* seller among those a given buyer negotiates with, and a *random* buyer among those that a given seller negotiates with.

seller auto-declines as quits. Under any of these definitions, the results are qualitatively similar to the main results. Relative to the body of the paper, the definition in panel A increases the number of observations with a nonzero X_Q^S, leading to a slight increase in the number of crossings involving independence and small narrowing of the bounds. The definition in panel B has the opposite effect, classifying fewer cases as seller quits (and hence fewer observations with a nonzero X_Q^S), leading to fewer crossings and wider bounds than in the main results. Results in panel C lie between those of panels A and B.[2]

[2]Note that the first-best trade probability lower bound is unaffected by the definition of seller quits; it only depends on accept or counter actions. The upper bound is affected, but only slightly: the average (across products) of the upper bound in Figure 6.B is 0.94, and this number changes to 0.92, 0.99, and 0.95 under the definitions of seller quits considered in Table A4 panels A, B, and C, respectively.

Table A3: Widths of Bounds with Different Bilateral Bargaining Pairs

	Seller Bounds			Buyer Bounds		
	Min	Mean	Max	Min	Mean	Max
A. Restricting to last seller a buyer negotiates with, and vice versa						
Unconditional (A1)	0.369	0.434	0.519	0.410	0.430	0.461
Monotonicity (A2)	–	–	–	0.310	0.376	0.442
Independence (A3)	0.190	0.300	0.415	0.109	0.240	0.354
Stochastic Monotonicity (A4)	0.362	0.430	0.512	0.399	0.420	0.455
Positive Correlation (A5)	0.280	0.375	0.504	0.383	0.414	0.433
Mon. + Indep. (A2 + A3)	–	–	–	0.058	0.183	0.331
Mon. + Pos. Corr (A2 +A5)	–	–	–	0.263	0.352	0.400
Stoch. Mon. + Indep. (A4 + A3)	0.200	0.292	0.381	0.118	0.248	0.360
Stoch. Mon. + Pos. Corr (A4 + A5)	0.276	0.374	0.496	0.377	0.408	0.428
B. Restricting to random seller a buyer negotiates with, and vice versa						
Unconditional (A1)	0.346	0.425	0.526	0.412	0.429	0.460
Monotonicity (A2)	–	–	–	0.303	0.374	0.434
Independence (A3)	0.187	0.289	0.434	0.139	0.244	0.348
Stochastic Monotonicity (A4)	0.342	0.422	0.516	0.397	0.419	0.453
Positive Correlation (A5)	0.274	0.373	0.513	0.367	0.413	0.434
Mon. + Indep. (A2 + A3)	–	–	–	0.081	0.184	0.333
Mon. + Pos. Corr (A2 +A5)	–	–	–	0.249	0.351	0.410
Stoch. Mon. + Indep. (A4 + A3)	0.182	0.281	0.379	0.147	0.251	0.349
Stoch. Mon. + Pos. Corr (A4 + A5)	0.268	0.372	0.502	0.361	0.407	0.428

Notes: Table replicates Table 4 using different samples. The main sample in the paper restricts to the first seller a given buyer negotiates with and the first buyer a given seller negotiates with. Panel A above uses the *last* seller and *last* buyer. Panel B selects a *random* seller among those a given buyer negotiates with, and a *random* buyer among those that a given seller negotiates with.

D.2. Bargaining Costs. We do not explicitly model bargaining costs in this paper. Such costs could take many forms; we highlight only a few here and show how they would fit into our framework. Suppose the buyer faces an additive disutility, call it χ, when making or accepting an offer, and the buyer's gross utility and outside option are V and μ (as in Section 3.2). The buyer's value (willingness to pay) for this trade would be $B = V - \mu - \chi$. The object B is the buyer's value net of bargaining costs, just as it is net of any outside option. An alternative cost form is the fee τ that a seller pays eBay when a sale occurs; with this fee, the seller's willingness to sell is $S \equiv \breve{S} + \tau$, where $\breve{S}$ is the least the seller would be willing to accept absent eBay fees.[3] Costs could also include a shipping cost ψ a

[3]In practice, this fee is a percentage commission (typically 10%) but in this discussion we consider it to be additive for simplicity.

Table A4: Seller Bounds with Alternative Definitions of Quits

	Crossings			Widths		
	Frac. Cross	Frac. Reject	IVE	Min	Mean	Max
A. Define all seller declines as quits						
Unconditional (A1)	0	0	0	0.281	0.401	0.520
Monotonicity (A2)	1.00	1.00	0.23	–	–	–
Independence (A3)	0.08	0	0.00	0.134	0.259	0.425
Stochastic Monotonicity (A4)	0	0	0	0.275	0.397	0.512
Positive Correlation (A5)	0	0	0.00	0.244	0.356	0.507
Mon. + Indep. (A2 + A3)	1.00	1.00	0.23	–	–	–
Mon. + Pos. Corr (A2 +A5)	1.00	1.00	0.23	–	–	–
Stoch. Mon. + Indep. (A4 + A3)	0.22	0.03	0.00	0.131	0.247	0.370
Stoch. Mon. + Pos. Corr (A4 + A5)	0	0	0.00	0.246	0.356	0.495
B. Define last-round seller declines as quits						
Unconditional (A1)	0	0	0	0.488	0.532	0.577
Monotonicity (A2)	1.00	1.00	0.17	–	–	–
Independence (A3)	0.03	0	0.00	0.391	0.460	0.524
Stochastic Monotonicity (A4)	0	0	0	0.483	0.530	0.574
Positive Correlation (A5)	0	0	0.00	0.454	0.505	0.555
Mon. + Indep. (A2 + A3)	1.00	1.00	0.17	–	–	–
Mon. + Pos. Corr (A2 +A5)	1.00	1.00	0.17	–	–	–
Stoch. Mon. + Indep. (A4 + A3)	0.03	0	0.00	0.372	0.448	0.517
Stoch. Mon. + Pos. Corr (A4 + A5)	0	0	0.00	0.441	0.503	0.554
C. Define last-round seller declines or auto-declines as quits						
Unconditional (A1)	0	0	0	0.306	0.450	0.544
Monotonicity (A2)	1.00	1.00	0.22	–	–	–
Independence (A3)	0.06	0	0.00	0.171	0.336	0.469
Stochastic Monotonicity (A4)	0	0	0	0.294	0.443	0.536
Positive Correlation (A5)	0	0	0.00	0.279	0.414	0.528
Mon. + Indep. (A2 + A3)	1.00	1.00	0.22	–	–	–
Mon. + Pos. Corr (A2 +A5)	1.00	1.00	0.22	–	–	–
Stoch. Mon. + Indep. (A4 + A3)	0.11	0	0.00	0.163	0.323	0.416
Stoch. Mon. + Pos. Corr (A4 + A5)	0	0	0.00	0.274	0.411	0.518

Notes: Table replicates columns 1–3 of Tables 3–4 using alternative definitions of seller quits. The body of the paper defines a seller quit as a seller decline followed by no additional buyer activity. Panel A reports results when all seller declines are considered seller quits. Panel B defines seller quits as seller declines occurring in the last period of the game. Panel C defines seller quits as last-period declines or any auto-decline.

buyer pays, resulting in buyer value of $B = V - \mu - \psi$. Finally, costs might take the form of discounting, such that, for a discount factor δ, the buyer receives $\delta(V - P)$ by accepting an offer P and μ otherwise. The buyer's willingness to pay that could be bounded by observing accept or quit decisions would then be $B = V - \mu/\delta$.[4]

E Estimation and Inference

E.1. Inference and Median Unbiased Estimation with Single Assumptions

Here we focus only on lower bound estimators. Upper bound estimators are analogous. Let $z_{1-\alpha}$ be the $1-\alpha$ quantile from a standard normal distribution. The unconditional lower bound estimators are empirical distribution functions and pointwise, one-sided $1-\alpha$ confidence bands are therefore $\frac{1}{n}\sum_{i=1}^{n}\mathbf{1}(X_{AC,i}^{S} \leq x) - z_{1-\alpha}\sqrt{\frac{\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{1}(X_{AC,i}^{S}\leq x)\right)\left(1-\frac{1}{n}\sum_{i=1}^{n}\mathbf{1}(X_{AC,i}^{S}\leq x)\right)}{n}}$ and $\frac{1}{n}\sum_{i=1}^{n}\mathbf{1}(X_{Q,i}^{B} \leq x) - z_{1-\alpha}\sqrt{\frac{\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{1}(X_{Q,i}^{B}\leq x)\right)\left(1-\frac{1}{n}\sum_{i=1}^{n}\mathbf{1}(X_{Q,i}^{B}\leq x)\right)}{n}}$. Similarly, we calculate confidence bands under monotonicity by replacing $X_{AC,i}^{S}$ with $\hat{X}_{AC}^{S*}(P_{1,i}^{S})$ and $X_{Q,i}^{B}$ with $\tilde{X}_{Q}^{B*}(P_{1,i}^{S}, P_{2,i}^{B})$

The large-sample distributions of the stochastic monotonicity bounds are nonstandard because the bounds are only directionally differentiable functions of conditional mean functions. Similar inference problems arise in Chernozhukov et al. (2013) and Fang and Santos (2018). However, neither paper applies in our setting because Chernozhukov et al. (2013) focus on maxima and minima of conditional mean functions and Fang and Santos (2018) is not applicable with nonparametric estimators. Developing a (nonstandard) bootstrap procedure, as in Fang and Santos (2018), while allowing for nonparametric estimators, is beyond the scope of our paper. We therefore use subsampling, which is known to be consistent under weak assumptions (Politis and Romano 1994). We use a subsample size of $b_n = n^{-3/4}$ and bandwidths of $b_n^{-1/3}$ and $b_n^{-1/4}$ for one- and two-dimensional functions, respectively. Due to the smaller sample size, we undersmooth relatively more in the subsamples than

[4] If bargaining costs were heterogeneous across agents — for example, if each faced distinct additive disutilities or discount rates — other assumptions we work with, such as monotonicity, could be violated.

in our main estimation sample to ensure that the finite sample biases of the nonparametric estimators in the subsamples do not dominate those in the original sample. Confidence bands for the independence and positive correlation bounds are also based on subsampling.

The estimated stochastic monotonicity, independence, and positive correlation bounds are generally inward biased due to the maxima and minima. As explained in Chernozhukov et al. (2013) , a *half-median-unbiased estimator* is given by simply constructing a 50% one-sided confidence interval, which we calculate using our subsampling procedure.

E.2. Estimation and Inference of Bounds Combining Assumptions. Here we focus only on lower bound estimators. Upper bound estimators are analogous. We first consider the bounds in Theorem 8. For the buyer, we write the lower bound as

$$\begin{aligned}\max_y \left(\int \mathbf{1}(X_Q^{B*}(y,z) \leq x) dF_{P_2^B|P_1^S}(z|y) \right) &= \max_y \left(P(X_Q^{B*}(y,P_2^B) \leq x \mid P_1^S = y) \right) \\ &= \max_y P(X_Q^{B*}(P_1^S,P_2^B) \leq x \mid P_1^S = y)\end{aligned}$$

We estimate $P(X_Q^{B*}(P_1^S,P_2^B) \leq x \mid P_1^S = y)$ using $\tilde{X}_Q^{B*}(P_1^S,P_2^B)$ instead of $X_Q^{B*}(P_1^S,P_2^B)$ and the Nadaraya-Watson kernel estimator with an Epanechnikov kernel and bandwidth $n^{-1/4}$.

We write the buyer lower bound from Theorem 9 as

$$\int \max_{y' \geq y} P(X_Q^{B*}(y',P_2^B) \leq x \mid P_1^S = y') dF_{P_1^S}(y) = \int \max_{y' \geq y} P(X_Q^{B*}(P_1^S,P_2^B) \leq x \mid P_1^S = y') dF_{P_1^S}(y)$$

Using the estimator $\widehat{P}(\tilde{X}_Q^{B*}(P_1^S,P_2^B) \leq x \mid P_1^S = y')$ from above, we estimate the lower bound by $\frac{1}{n}\sum_{i=1}^n \max_{y' \in \omega_1(P_{1,i}^S)} \widehat{P}(\tilde{X}_Q^{B*}(P_1^S,P_2^B) \leq x \mid P_1^S = y')$.

We estimate the seller lower bound in Theorem 10 using the sample analog $\frac{1}{n}\sum_{i=1}^n \max_{y' \in \omega_1(P_{1,i}^S)} \max_z \hat{m}_{AC}^S(x,y',z)$. For the buyer, define $g_Q^B(x,y,z) = \max_{z' \geq z} m_Q^B(x,y,z')$. Then we can write the lower bound as $\max_y \left(E[g_Q^B(x,P_1^S,P_2^B) \mid P_1^S = y] \right)$. For each x, we estimate $E[g_Q^B(x,P_1^S,P_2^B) \mid P_1^S = y]$ using the Nadaraya-Watson kernel estimator with an Epanechnikov kernel function and bandwidth $n^{-1/4}$. Let $\hat{E}[g_Q^B(x,P_1^S,P_2^B) \mid P_1^S = y]$ denote the estimator. Our estimator is $\max_{y \in Q_{0.05}(P_{1,i}^S)} \left(\hat{E}[g_Q^B(x,P_1^S,P_2^B) \mid P_1^S = y] \right)$, where

$g_Q^B(x,y,P_2^B) = \max_{z' \in \omega_2(P_{2,i}^B)} m_Q^B(x,y,z')$.

For the bounds from Theorem 11, we estimate the seller lower bound by $\frac{1}{n}\sum_{i=1}^n \max_{y' \in \omega_1(P_{1,i}^S)} \left(\hat{E}[g_{AC}^S(x,P_1^S,P_2^B) \mid P_1^S = y']\right)$ as well as the buyer lower bound by $\frac{1}{n}\sum_{i=1}^n \max_{y' \in \omega_1(P_{1,i}^S)} \left(\hat{E}[g_Q^B(x,P_1^S,P_2^B) \mid P_1^S = y']\right)$.

Inference is based on subsampling in all of these cases, as explained in Section E.1. Using subsampling, we then also obtain median unbiased estimators as described in the previous subsection.

E.3. Estimation and Inference of Bounds on First-Best Trade Probability. For the lower bound in Theorem 6, we first estimate $P(X_{AC}^{B*} - X_{AC}^S \geq x \mid P_1^S = y, P_2^B = z)$ by replacing $X_{AC}^{B*} - X_{AC}^S$ with $\hat{X}_{AC}^{B*}(y,z) - \hat{X}_{AC}^S(y)$ and using the Nadaraya-Watson kernel estimator with an Epanechnikov kernel function and bandwidth $n^{-1/5}$. Denote the estimator by $\hat{m}^{B-S}(y,z)$. The estimated lower bound is then $\frac{1}{n}\sum_{i=1}^n \max_{z' \in \omega_2(P_{2,i}^B)} \hat{m}^{B-S}(P_{1,i}^S, z')$. For the lower bound in Theorem 7, define $\hat{X}_{AC}^{B*-S}(y,z) \equiv \min_{i: P_{2,i}^B \leq z, P_{1,i}^S \in N(y)} (X_{AC,i}^B - X_{AC,i}^S)$, where the neighborhood $N(y)$ is as in Section 4. The estimated lower bound is then $\frac{1}{n}\sum_{i=1}^n \mathbf{1}(\hat{X}_{AC}^{B*-S}(P_{1,i}^S, P_{2,i}^B) \geq x)$. The estimator that is based on both buyer and seller monotonicity is $\frac{1}{n}\sum_{i=1}^n \mathbf{1}(\hat{X}_{AC}^{B*}(P_{1,i}^S) - \hat{X}_{AC}^{S*}(P_{1,i}^S) \geq x)$.

For the confidence bands of the marginal distributions, we explain in Section E.1 that we use conservative estimators and one-sided confidence bands. To construct the two-sided confidence intervals for the lower bound in Theorem 6 (shown in Table 5), we first use the previously described estimators and subsampling to approximate the quantiles of the centered distribution. We then use these quantiles as well as conservative point estimators to construct confidence intervals. In particular, for the lower endpoint of the confidence interval, we use $\hat{X}_{AC}^{B*}(y,z) + \eta_n - \hat{X}_{AC}^S(y)$ instead of $\hat{X}_{AC}^{B*}(y,z) - \hat{X}_{AC}^S(y)$, and for the upper endpoint we use $\hat{X}_{AC}^{B*}(y,z) - \eta_n - \hat{X}_{AC}^S(y)$, where $\eta_n = n^{-1/2}$ as in Section 4.1. The confidence interval for the sale probability is simply based on the large sample distribution of the empirical distribution function. The confidence interval for the lower bound in Theorem 7 is also based on the empirical distribution function but using conservative estimates of X_{AC}^{B*-S}, as this estimator suffers from the same potential inward bias as the buyer monotonicity bounds. In particular, for the lower and upper endpoints of the confidence interval, we use $\tilde{X}_{AC}^{B*-S}(y,z)+\eta_n$ and $\tilde{X}_{AC}^{B*-S}(y,z)-\eta_n$, respectively, where again $\eta_n = n^{-1/2}$.

E.4. Testing. Let $g_l(x)$ and $g_u(x)$ denote a lower and upper bound derived under some assumptions. To test the null hypothesis that the imposed assumptions are true, we use a scaled estimated version of $\frac{1}{J}\sum_{j=1}^{J}\min\{g_u(x_j)-g_l(x_j),0\}$, which is equal to 0 under the null hypothesis and negative if the bounds cross (at one of the J grid points). We use an equally spaced grid on $[0,2.5]$ with $J=25$. Note that bounds might not cross even if the assumptions do not hold.

The test statistic is based on the estimators discussed in Sections 4 and E.2. For the unconditional and the monotonicity bounds, we use the large sample distribution of the empirical distribution functions to approximate the distribution function of the test statistic. For all other assumptions, we use subsampling, as described in sections E.1 and E.2, to approximate the distribution of the test statistic. We reject the null if the test statistic is smaller than the α quantile of that distribution (essentially using a one-sided test).

F Monte Carlo Simulations

We present a Monte Carlo study of the buyer and seller distribution bounds. There is naturally a great deal of flexibility in how to simulate two-sided bargaining; here we simply simulate outcome data consistent with our assumptions. We do not simulate actual equilibrium play of a two-sided bargaining game, as the equilibria focused on in previous work (Perry 1986; Grossman and Perry 1986; Cramton 1992) do not result in multiple offers by a given party that vary with the party's value.

F.1. Algorithm for Simulating Bargaining Data. The primary parameters we vary in this exercise are α_b and α_s, which we refer to as *shade factors*; the probability a buyer and seller accept/decline; and the means of buyer and seller value distributions. Shade factors allow us to vary how aggressive agents' offers are: a buyer with value b and shade factor α_b makes the same offers as a buyer with value $b+\alpha_b$ and shade factor 0, and in this sense shade factors set a minimum level of offer shading. The probability a buyer or seller accepts or quits (instead of making a counteroffer) allows us to investigate how countering frequency affects bound tightness. Varying mean values allows us to adjust the potential surplus. The DGP is described in Table A5.

Table A5: Algorithm for Simulating Bargaining Data

0. **Initialize:** Draw $B \sim F_B$ and $S \sim F_S$. Set shade factors α_B and α_S, and set cap $T_{\max}$ on number of rounds. Set functions $p_{BQ}(k)$, $p_{BA}(k)$, $p_{SQ}(k)$, and $p_{SA}(k)$ specifying probabilities, in round k, of buyer quitting, buyer accepting, seller quitting, or seller accepting
1. **Round 1:** Seller offers $P_1^S = g_1(S, \alpha_s, U_1)$ where $U_1 \sim U[0,1]$ and g_1 is a function weakly increasing in all arguments (we vary g_1 in our illustrations)
2. **Round 2:** Buyer offers $P_2^B = U_2(B - \alpha_b)$ if $P_1^S > B$ and $P_2^B = U_2 \min\{P_1^S, B - \alpha_b\}$ if $P_1^S \leq B$, where $U_2 \in (0,1)$ is random or fixed depending on specific setup
3. **Round 3**$\geq k < T_{\max}$ **,** k **odd:** Seller responds to buyer's last offer
   - **Case 1.** $P_{k-1}^B < S$: Seller quits with probability $p_{SQ}(k)$, or else makes counteroffer $P_k^S = U_3 P_{k-2}^S + (1 - U_3)(S + \alpha_S)$, where $U_3 \in (0,1)$ and its distribution depends on specific setup
   - **Case 2.** $P_{k-1}^B \geq S$: Seller accepts with fixed probability $p_{SA}(k)$, or else makes counteroffer $P_k^S = U_3 P_{k-2}^S + (1 - U_3)\max(P_{k-1}^B, S + \alpha_S)$
4. **Round 4**$\geq k < T_{\max}$**,** k **even:** Buyer responds to seller's last offer P_{k-1}^S
   - **Case 1.** $P_{k-1}^S > B$: Buyer quits with probability $p_{BQ}(k)$, or else makes counteroffer $P_k^B = U_4 P_{k-2}^B + (1 - \lambda)(B - \alpha_B)$, where $U_4 \in (0,1)$ and its distribution depends on specific setup
   - **Case 2.** $P_{k-1}^S \leq B$: Buyer accepts with probability $p_{BA}(k)$, or else makes counteroffer $P_k^B = U_4 P_{k-2}^B + (1 - U_4)\min(P_{k-1}^S, b - \alpha_B)$
5. **Round** $T_{\max}$**:** Terminate with no trade occurring

F.2. Results of Monte Carlo Exercise. Figure A1 illustrates several of our bounds estimated using this simulated data. For each panel, we simulate 100 replications of the DGP and then report the true distribution, the true bounds as well as the estimated bounds along with 95% one-sided, pointwise confidence bands averaged across these replications. In each example we set $n = 200$ and $T_{max} = 8$. We draw the values from a Beta distribution, which has support on $[0,1]$. The Beta distribution has two parameters, α and β. We set $\alpha = 2$ and set β depending on which mean value we want to achieve. We then add the maximum shading factor to ensure that bids are always non-negative. We vary the parameters of the DGP in each panel in order to illustrate what features will lead to bounds that are loose (panels on the left) or tight (panels on the right). We focus only on three sets of bounds — the seller unconditional bounds, seller monotonicity bounds, and buyer independence bounds — to conserve space and because the intuition gained by these three cases extends to the other bounds in the paper. In each panel, lower bounds are shown with solid lines and upper bounds with dashed lines. The true CDF as well as the true bounds, both approximated using a sample size of $200,000$, are shown with a dot-dash line. We obtain the true bounds by estimating the bounds on a very large sample (1 million observations). Comparing the estimated bounds on small samples to these true bounds allows us to evaluate bias in the estimated bounds.

We show the seller unconditional bounds in panels A and B. For the seller unconditional bounds to be relatively tight, it must be the case that sellers quit at prices close to their values and also counter at prices close to their values. As an example, consider a setting where buyer and seller values are highly correlated and have a similar mean. Suppose the typical play of the game is that the seller offers a price a little above her value, the buyer counters at a price a little below the seller's value (and also below the buyer's value, naturally), and the seller then quits. This sequence of play is consistent with the weak revealed preference assumptions that the unconditional bounds are built on (A1) and it yields the tight bounds on seller values illustrated in panel B.[5] We can also easily generate wide unconditional bounds. For example, consider a case where the seller typically makes offers far above her value and rarely quits. Such bounds are illustrated in panel A.[6] Here, the correlation structure between buyer and seller values plays no role.

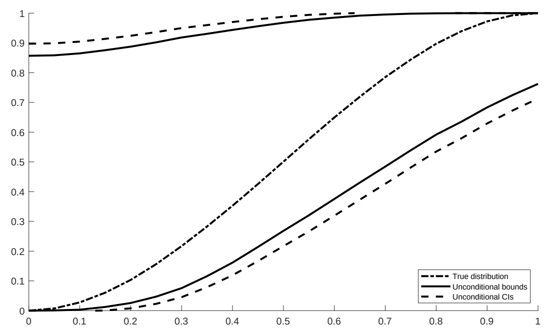
We illustrate the seller monotonicity bounds in panels C and D. The monotonicity bounds will improve upon the unconditional bounds when there is some probability that sellers who start with relatively low first offers end the game at relatively high final accept/counter or quit prices. This can occur due to randomness in the value of the buyer to whom the seller

[5]For this case, B and S have mean 0 and correlation 0.999. We set $p_{BQ}(k) = p_{BA}(k) = p_{SA}(k) = 0$, $p_{SQ}(k) = 0.95$, $g_1(S, \alpha_s, U_1) = 1.1(S + \alpha_s)$, and $U_2 = 0.9$, $U_3 \sim U[0, 0.5]$, $U_4 \sim U[0, 0.5]$.

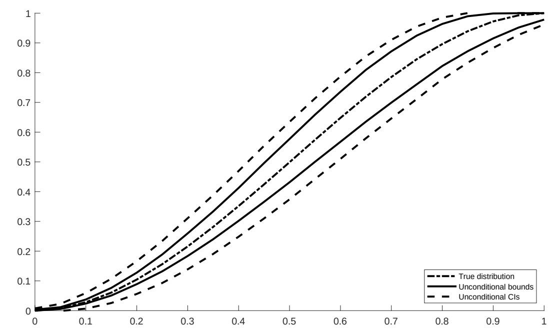
[6]For this case, B and S have mean 0 and correlation 0. We set $p_{BQ}(k) = p_{BA}(k) = 1$, $p_{SA}(k) = 0$, $p_{SQ}(k) = 0.25$, $g_1(S, \alpha_s, U_1) = 1.5(S + \alpha_s) + 0.5$, and $U_2 = 0.9$, $U_3 \sim U[0, 0.5]$, $U_4 \sim U[0, 0.5]$.

Figure A1: Simulation Results

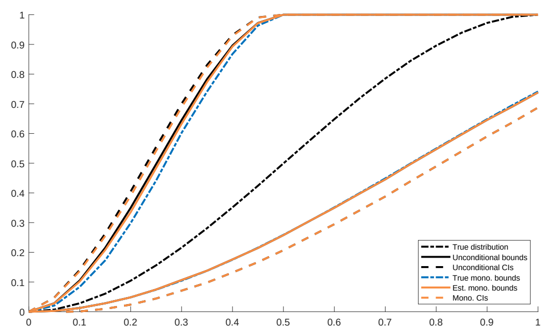
(A) Unconditional Bounds, Seller (Wide)



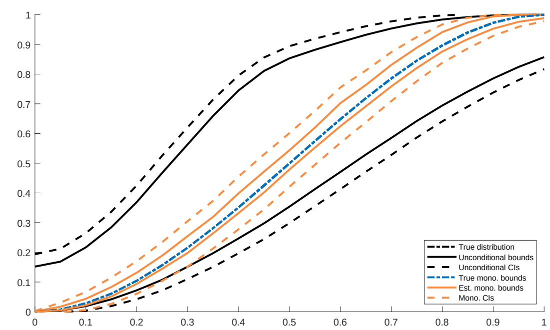
(B) Unconditional Bounds, Seller (Tight)



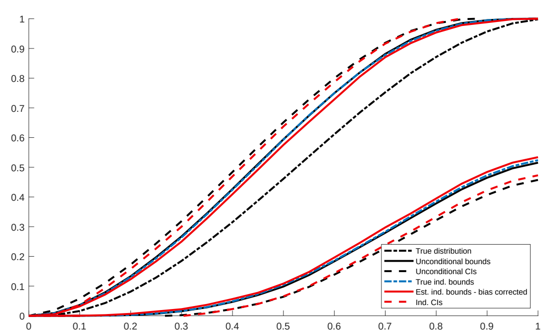
(C) Monotonicity Bounds, Seller (Wide)



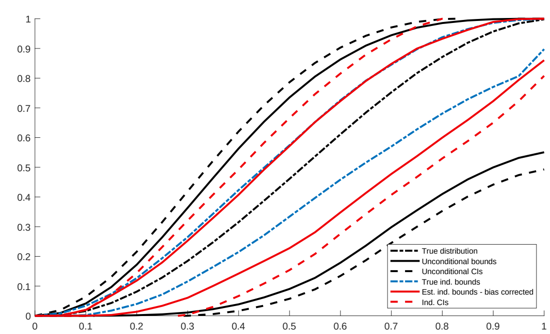
(D) Monotonicity Bounds, Seller (Tight)



(E) Independence Bounds, Buyer (Wide)



(F) Independence Bounds, Buyer (Tight)



Notes: The figure shows bounds estimated from simulated data under cases where bounds are wide (on left) vs. narrow (on right). Panels A and B show unconditional seller bounds. Panels C and D show seller monotonicity bounds. Panels E and F show buyer independence bounds. Estimated bounds are shown with solid lines, confidence bands with dashed lines, and true CDF and true bounds with a dot-dash line. Panels C–F also show unconditional bounds for comparison.

is matched and due to features of bargaining at later rounds of the game. We illustrate such a case in panel D.[7] If, however, the final accept/counter and quit prices of a seller are, like P_1^S, deterministically mononotonic in the seller's value, the monotonicity assumption will do nothing to improve upon the unconditional bounds (because $X_{AC}^{S*} = X_{AC}^S$ and $X_Q^{S*} = X_Q^S$ in that case, and the unconditional bounds will equal the monotonicity bounds). We illustrate this situation in panel C, where the monotonicity bounds are equally as wide as the unconditional bounds.[8]

Finally, we illustrate the buyer independence bounds in panels E and F. Recall that these bounds are obtained by combining $P(B \leq x|P_1^S = y) = P(B \leq x)$ (buyer independence) with weak rationality on the part of the buyer ($X_{AC}^B \leq B$ for the buyer upper bound). The buyer independence assumption will therefore yield no improvement over the buyer unconditional upper bounds if X_{AC}^B and X_Q^B are, like B, independent of P_1^S. This case is illustrated in panel E.[9] It is easy to generate a case in which X_{AC}^B *does* depend on P_1^S, and this yields a much tighter upper bound. To do so, we generate data such that B is independent of P_1^S, but X_{AC}^B and X_Q^B are not because bids in later rounds directly depend on P_1^S.[10]

F.3. Bias Correction. Estimators of unconditional bounds are unbiased and, as explained in Section 4, estimators of the monotonicity bounds have an outward bias. While we use half-median-unbiased estimators for the independence bounds, this estimator can still be biased in finite samples, which is particularly noticeable in panel F of Figure A1, where the estimate of the lower bound lies below the true bound. The estimator (and the other estimators that involve minima and maxima) have two main sources of biases that go in opposite directions. First, there is a inward bias that arises from taking the minimum (or maximum) of an estimated function. This bias term is handled by using the half-median-

[7]For this case, B and S have mean 0 and correlation 0.999. We set $p_{BQ}(k) = p_{BA}(k) = p_{SA}(k) = 0$, $p_{SQ}(k) = 0.5$, $g_1(S, \alpha_s, U_1) = 1.5(S + \alpha_s)$, and $U_2 = 0.5$, $U_2 \sim U[0, 0.5]$, $U_3 \sim U[0, 0.5]$.

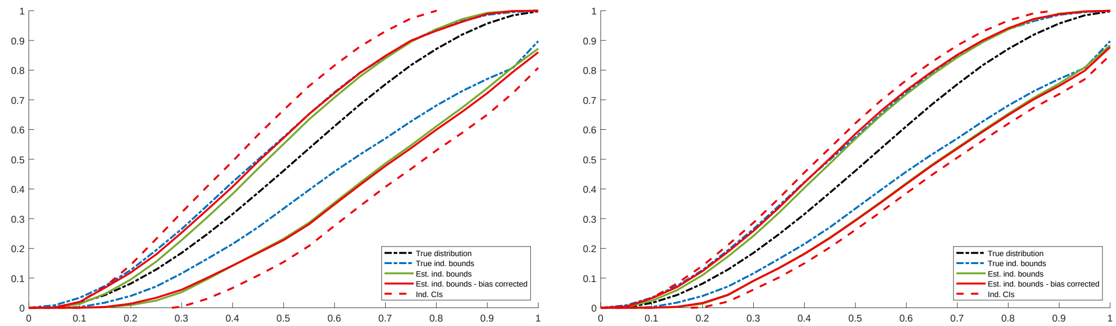
[8]For this case, B and S have mean 0 and correlation 0.999. We set $p_{BQ}(k) = p_{SQ}(k) = 1$, $p_{BA}(k) = p_{SA}(k) = 0$, $g_1(S, \alpha_s, U_1) = 1.5(S + \alpha_s)$, and $U_2 = 0.5$, $U_3 \sim U[0, 0.5]$, $U_4 \sim U[0, 0.5]$.

[9]For this case, B and S have mean 0 and correlation 0. We set $p_{BQ}(k) = 0.95$, $p_{SQ}(k) = p_{BA}(k) = p_{SA}(k) = 0$, $g_1(S, \alpha_s, U_1) = U_1$ with $U_1 \sim [1, 1.5]$, and $U_2 \sim U[0.75, 1]$, and $U_3 = U_4 = U_1$.

[10]For this case, B and S have mean 0 and correlation 0. We set $p_{BQ}(k) = 0.95$, $p_{SQ}(k) = p_{BA}(k) = p_{SA}(k) = 0$, $g_1(S, \alpha_s, U_1) = U_1(S + \alpha_s)$ with $U_1 \sim [1, 1.5]$, and $U_2 \sim U[0.75, 1]$, and $U_3 = U_4 = \max\{1 - P_1^S, 0\}$.

Figure A2: Bias Comparisons

(A) Independence Bounds, Buyer, $n = 200$ (B) Independence Bounds, Buyer, $n = 2000$



Notes: The figure shows buyer independence bounds estimated from simulated data and two sample sizes under cases where bounds are narrow.

unbiased estimator. Second, the Nadaraya-Watson estimator is biased and, at the maximum, the estimator of the function is downward biased (and upward biased at the minimum). This bias term is handled by using undersmoothing, but still results in an outward bias in finite samples. In panel F, the second bias term dominates.

Figure A2 shows the buyer independence bounds again, but also includes the non-bias-corrected estimator. For the upper bound, where the first bias term dominates, the half-median-unbiased estimator is closer to the true bound, but the bias-adjustment has almost no effect for the lower bound. As the sample sizes increases, the biases decrease, as can be seen from panel B, where the sample size is $2,000$.

G Related Extensive-Form Models

In this section we consider two extensive-form bargaining models: Cramton (1992) and Perry (1986). In each case, when discussing unobserved heterogeneity, our notation here differs slightly from the body of the paper. Here we write a seller's value with additively separable unobserved heterogeneity included as $\tilde{S} = S + W$ (in the body of the paper, we instead write $S = \tilde{S} + W$). Similarly, for the multiplicative case, we write $\tilde{S} = SW$. We apply this notation to the buyer's value and to buyer and seller offers as well. We adopt this change so that variables without $(\tilde{\cdot})$ always represent those *absent* unobserved heterogeneity.

G.1. Cramton (1992). This model studies a setting similar to ours, where a seller and buyer with independent private values engage in bargaining. One possible outcome in the Cramton (1992) equilibrium is for the seller to make the first offer, P_1^S, which completely reveals the seller's value S. The buyer then either accepts, quits, or makes a counteroffer P_2^B that completely reveals her value B.[11] These first two offers are $P_1^S = \frac{\delta S+\gamma(S)}{1+\delta}$ and $P_2^B = \frac{\delta B+S}{1+\delta}$, where δ is a discount factor and the object $\gamma(S)$ is the buyer type indifferent between accepting and rejecting the seller's offer of P_1^S given that the seller has revealed her type to be S and the buyer's value is bounded above by some $\overline{b}$. In the equilibrium studied in Cramton (1992), the function $\gamma(\cdot)$ is given by the following:

$$F_B(\overline{b}) - F_B(\gamma) - (1-\delta^2)(\gamma - s) f_B(\gamma) = \int_s^{\gamma} \delta^3 \left(\frac{b-s}{\gamma-s}\right)^{1+\delta} dF_B(b) \tag{17}$$

This object is quite complex, depending on the CDF and density of buyer values, F_B and f_B. If buyer and seller values are uniformly distributed, $\gamma(s)$ has a closed-form solution $\gamma(s) = \alpha - (2\alpha - 1)s$, where α is defined by $1 - 2\alpha = \frac{-\delta}{2+\delta-\delta^2}$. For our arguments here, we assume $\gamma(s)$ is differentiable with $\gamma'(s) \in (-\delta, 0)$. It is possible to show that $\gamma'(s) \in (-\delta, 0)$ is satisfied with slack in the uniform case, which we state as the following lemma:[12]

Lemma 3. *In the Cramton (1992) model, if buyer and seller values are uniformly distributed, the function $\gamma(\cdot)$ satisfies (with slack) $\gamma'(\cdot) \in (-\delta, 0)$.*

Proof. Note that $\alpha \in (\frac{1}{2}, \frac{3}{4})$ for $\delta \in (0,1]$, so $\gamma'(s) \in [-.5, 0)$ for $\delta \in (0,1]$. Therefore, $\gamma'(s) < 0$ is satisfied with slack. Now note $\gamma'(s) = 1 - 2\alpha$. Setting $\gamma'(s) \geq -\delta$ yields $\frac{-\delta}{2+\delta-\delta^2} \geq -\delta \iff \delta^2 \leq 2$, and thus $\gamma'(s) > -\delta$ is satisfied with slack. □

[11] If the buyer chooses to make a counteroffer, P^B, the buyer exploits this knowledge of the seller's type and makes an offer that corresponds to the Rubinstein (1982) equilibrium offer for the case where the buyer and seller know each others' values. Note that, in the Cramton (1992) equilibrium, the timing of these offers is also important in revealing an agent's value, but the level of the offers is sufficient for our purposes.

[12] It is also possible to derive sufficient conditions for these properties outside of the uniform case; these would be similar to the assumption referred to as "$(F\delta)$" in Cramton (1992). These conditions are cumbersome. Like Cramton, therefore, we instead show they are satisfied with slack in the uniform case and thus they do not appear to be overly restrictive.

An immediate result of this property is that the equilibrium offers, $P_1^S = \frac{\delta S+\gamma(S)}{1+\delta}$ and $P_2^B = \frac{\delta B+S}{1+\delta}$, satisfy A2 (strictly, in fact): P_1^S is strictly monotone in S because $\gamma'(s) > -\delta$, and hence P_2^B is also strictly monotone in B conditional on P_1^S.

Now consider a modified setting where the equilibrium of Cramton (1992) is played, but in a given realization of the game B and S are both shifted additively by a common amount, W, that is independent of B and S. Specifically, a buyer's value is given by $B+W$ and a seller's by $S+W$, where $W = w$ is known to both agents but not to the econometrician. Cramton's model assumes, without loss of generality, that buyer values are distributed on $[0,1]$. In our modification, we instead have values distributed on $[w, 1+w]$ and equilibrium offers simply shift additively by w, becoming $P_1^S + w$ and $P_2^B + w$, as we now show:

Lemma 4. *Suppose seller and buyer values in the Cramton (1992) setting are given by $S+W$ and $B+W$. If, when $W=0$, the first two offers are given by $P_1^S = p_1^S$ and $P_2^B = p_2^B$, then, when $W = w$, these offers are given by $p_1^S + w$ and $p_2^B + w$.*

Proof. We first prove the following claim: The function $\gamma(\cdot)$ satisfies additive separability. Let $\tilde{\gamma}(s,w)$ represent the value of γ in a game in which $W=w$; thus $\gamma(s) = \tilde{\gamma}(s,0)$. We will show that $\tilde{\gamma}(s,w) = \gamma(s)+w$. To see this, let $\tilde{\overline{b}} = \overline{b}+w$, and $\tilde{s} = s+w$, and let $\tilde{F}_{\tilde{B}}$ and $\tilde{f}_{\tilde{B}}$ be the distribution and density of $\tilde{B}$.

The condition defining $\tilde{\gamma}$ is given by modifying (17) to become $\tilde{F}_{\tilde{B}}(\tilde{\overline{b}}) - \tilde{F}_{\tilde{B}}(\tilde{\gamma}) - (1-\delta^2)(\tilde{\gamma}-\tilde{s})\tilde{f}_{\tilde{B}}(\tilde{\gamma}) = \int_{\tilde{s}}^{\tilde{\gamma}} \delta^3 \left(\frac{x-\tilde{s}}{\tilde{\gamma}-\tilde{s}}\right)^{1+\delta} d\tilde{F}_{\tilde{B}}(x)$. Note that, for any number x, $\tilde{F}_{\tilde{B}}(\tilde{x}) = F_B(\tilde{x}-w)$ and $\tilde{f}_{\tilde{B}}(\tilde{x}) = f_B(\tilde{x}-w)$. We now apply a change of variables from x to $y = x-w$ in the integral, yielding $\int_s^{\tilde{\gamma}-w} \delta^3 \left(\frac{y+w-\tilde{s}}{\tilde{\gamma}-\tilde{s}}\right)^{1+\delta} dF_B(y)$. Combining these results yields

$$F_B(b) - F_B(\tilde{\gamma}-w) - (1-\delta^2)(\tilde{\gamma}-\tilde{s})f_B(\tilde{\gamma}-w) = \int_s^{\tilde{\gamma}-w} \delta^3 \left(\frac{y-s}{\tilde{\gamma}-\tilde{s}}\right)^{1+\delta} dF_B(y) \qquad (18)$$

Comparing (17) to (18) demonstrates that, if γ is the solution to the former then $\gamma+w$ is the solution to the latter, proving the claim.

Now consider the equilibrium conditional on a realization of W. Offers will be given by $\tilde{p}_1^S = \frac{\delta\tilde{s}+\tilde{\gamma}(s,w)}{1+\delta} = \frac{\delta s+\gamma(s)}{1+\delta} + w$ and $\tilde{p}_2^B = \frac{\delta\tilde{b}+\tilde{s}}{1+\delta} = \frac{\delta b+s}{1+\delta} + w$, satisfying additivity. □

We now demonstrate that unobserved heterogeneity can lead to a violation of the monotonicity assumption even while stochastic monotonicity is satisfied. We show that monotonicity of the seller's first offer $\tilde{P}^S$ in the seller's value $\tilde{S}$ is violated in this setting, and we prove an analogous result for the buyer. Note that here we are considering what this setting would look like to the econometrician, who would see observations of different instances of the game and where realizations of W may vary across these observations.

Lemma 5. *The Cramton (1992) equilibrium offers in the game with additive unobserved heterogeneity can violate A2, but A4 is still satisfied.*

Proof. Suppose s increases by 1 and w decreases by $\eta < 1$ (so $\tilde{s}$ increases overall). Because $\gamma(s)' \in (-\delta, 0)$, the change in $\tilde{p}_1^S$ due to the change in s is at most an increase of $\frac{\delta}{1+\delta}$, and the change in $\tilde{p}_1^S$ due to the change in w is a decrease of η. For any $\eta \in \left(\frac{\delta}{1+\delta}, 1\right)$, $\tilde{p}_1^S$ decreases even though $\tilde{s}$ increases, violating seller monotonicity.

For buyer monotonicity, suppose s increases by η_s and w decreases by η_w such that $\tilde{p}_1^S$ does not change. It then holds that $0 < \eta_w < \eta_s$. Next suppose b increases by $\eta_b \in (\eta_w < \eta_s)$. Then $\tilde{b} = b + w$ decreases, but since $\frac{\delta(b+\eta_b)+(s+\eta_s)}{1+\delta} + w - \eta_w > \frac{\delta b + s}{1+\delta} + w$, $\tilde{p}_2^B$ increases.

To see that stochastic monotonicity is satisfied for the seller, let $g(s) \equiv \frac{s-\gamma(s)}{1+\delta}$, which is strictly increasing under our assumption that $\gamma(\cdot)$ is strictly decreasing. Then we have

$$
\begin{aligned}
P(\tilde{S} \leq x | \tilde{P}_1^S = y) &= P\left(S \leq x - W | P_1^S + W = y\right) \\
&= \int P\left(S \leq x - y + f(S) | w = y - f(S), W = w\right) f_{W|w=y-f(S)}(w) dw \\
&= \int P\left(S \leq x - y + \frac{\delta S + \gamma(S)}{1+\delta} \middle| w = y - f(S), W = w\right) f_W(w) dw \\
&= \int P\left(g(S) \leq x - y | f(S) = y - w, W = w\right) f_W(w) dw = \int P\left(g(S) \leq x - y | f(S) = y - w\right) f_W(w) dw \\
&= \int P\left(g(f^{-1}(y-w)) \leq x - y\right) f_W(w) dw = \int \mathbf{1}\left(g(f^{-1}(y-w)) \leq x - y\right) f_W(w) dw
\end{aligned}
$$

Since $g(f^{-1}(\cdot))$ is a strictly increasing function, $P(\tilde{S} \leq x | \tilde{P}_1^S = y)$ is strictly decreasing in y. In the third and fifth line, we use that W and S are independent. For the buyer, write

$$
\begin{aligned}
P(\tilde{B} \leq x | \tilde{P}_1^S = y, \tilde{P}_2^B = z) &= P\left(B \leq x - W \middle| f(S) + W = y, \frac{\delta B + S}{1+\delta} + W = z\right) \\
&= \int P\left(B \leq x - w \middle| f(S) + w = y, \frac{\delta B + S}{1+\delta} + w = z\right) f_W(w) dw \\
&= \int P\left(B \leq x - w \middle| S = f^{-1}(y-w), \frac{\delta B + f^{-1}(y-w)}{1+\delta} = z - w\right) f_W(w) dw \\
&= \int P\left(B \leq x - w \middle| S = f^{-1}(y-w), B = \frac{1}{\delta}\left((1+\delta)(z-w) - f^{-1}(y-w)\right)\right) f_W(w) dw \\
&= \int \mathbf{1}\left(\frac{1}{\delta}\left((1+\delta)(z-w) - f^{-1}(y-w)\right) \leq x - w\right) f_W(w) dw
\end{aligned}
$$

which is decreasing in z. In the third line, we used that W is independent of (S,B). □

The Cramton model assumes independence of B and S, so seller independence holds: S is independent of P_2^B conditional on P_1^S because P_1^S reveals S to the buyer, and thus, conditional on P_1^S, S does not vary. However, even maintaining the independence of B and S, additive unobserved heterogeneity would make $B+W$ be correlated with $P_1^S + W$, violating buyer independence. The proof of Lemma 6 focuses on the uniform case, showing that seller independence can also be violated without violating positive correlation.

Lemma 6. *The Cramton (1992) equilibrium offers in a game with additive unobserved heterogeneity can violate A3.i for the seller and A3.ii for the buyer.*

Proof. In the Cramton model *with unobserved heterogeneity*, clearly $\tilde{B}$ is correlated with $\tilde{P}_1^S$ through W, so buyer independence (A3.ii) is violated. For seller independence (A3.i), note from Lemma 5 that $\tilde{P}_2^B$ can be written as follows, where $\tilde{P}_1^S$ is fixed at y: $\tilde{P}_2^B = \frac{\delta B + f^{-1}(y-W)}{1+\delta} + W$. Now consider a change in $\tilde{S}$. Holding $\tilde{P}_1^S$ fixed at y, this change in $\tilde{S}$ must also correspond to a change in W (or else $\tilde{P}_1^S$ could not remain constant).

The change in W will necessarily affect $\tilde{P}_2^B$ unless the terms in $\tilde{P}_2^B$ depending on W offset one another; that is, unless $\frac{d}{dw}\left(\frac{f^{-1}(y-w)}{1+\delta} + w\right) = 0$. To see that this is not the case, note $\gamma' \in (-\delta, 0)$ implies $f' \in (0, \frac{\delta}{1+\delta})$, and, by the inverse function theorem, $f^{-1\prime} \in (\frac{1+\delta}{\delta}, \infty)$. This implies $\frac{d}{dw}\frac{f^{-1}(y-w)}{1+\delta} + w \in (-\infty, -1/\delta + 1)$. For any $\delta < 1$, this derivative is non-zero, and thus variation in W also leads to variation in $\tilde{P}_2^B$, violating seller independence. $\square$

G.2. Perry (1986). Perry (1986) omits discounting. Agents face a per-offer additive cost, c_S for sellers or c_B for buyers. While players alternate offers, the equilibrium that Perry focuses on has only one offer, which the opponent accepts or rejects. One outcome in the Perry game is for the seller to make this offer, P_1^S, given by $P_1^S = \frac{1-F_B(P_1^S)}{f_B(P_1^S)} + S$, where f_B is the density of buyer values. In this equilibrium, P_1^S clearly satisfies monotonicity (A2.i), and hence also satisfies stochastic monotonicity (A4.i).

In a version of this model with additively separable unobserved heterogeneity, the seller's offer will also be additively separable in the unobserved heterogeneity. Specifically, $\tilde{P}_1^S = \frac{1-F_{\tilde{B}}(\tilde{P}_1^S)}{f_{\tilde{B}}(\tilde{P}_1^S)} + \tilde{S} = \frac{1-F_B(\tilde{P}_1^S - W)}{f_B(\tilde{P}_1^S - W)} + S + W = P_1^S + W$. Thus, $\tilde{P}_1^S = P_1^S + W$.

In this modified version of the model, seller monotonicity (A2.i) can be violated. To show this, we re-write $P_1^S = \frac{1-F_B(P_1^S)}{f_B(P_1^S)} + S$ as $\phi(p_1^S) = s$, where $\phi(p_1^S) \equiv p_1^S - \frac{1-F_B(p_1^S)}{f_B(p_1^S)}$ is the buyer's *virtual value* function. Implicit differentiation of $\phi(p_1^S) = s$ with respect to s yields $\frac{dp_1^S}{ds} = \frac{1}{\phi'(p_1^S)}$. Consider now a case where s increases by 1 and w decreases by $\eta < 1$, and hence $\tilde{s}$ increases overall. The object $\tilde{p}_1^S$ will increase by $\frac{1}{\phi'(p_1^S)} - \eta$. For any distribution F_B with $\phi'(\cdot) > 1$, there exists an $\eta < 1$ such that p_1^S will increase by less than when η when s increases by 1, and, in such a case, $\tilde{p}_1^S$ will decrease overall. The uniform distribution on [0,1] is one such example, where this condition is satisfied with slack, with $\phi'(\cdot) = 2$.

Consider now a case in which agents play the equilibrium of Perry (1986), but in a given realization of the game B and S are both scaled *multiplicatively* (rather than shifted additively) by some amount W which, again, is common knowledge to both agents. Thus, $\tilde{S} = WS$ and $\tilde{B} = WB$. In this case, it can be shown that $\tilde{P}_1^S = WP_1^S = W\frac{1-F_B(\tilde{P}_1^S/W)}{f_B(\tilde{P}_1^S/W)} + WS$, where P_1^S is the offer the seller would make if the realization of W were 1. The presence of this multiplicative heterogeneity can lead to violations of weak monotonicity across instances of the game. Consider, for simplicity, the case where $B \sim U[0,1]$. In this case, the expression for $\tilde{P}_1^S$ simplifies to $2p_1^S = w + ws$. Suppose s increases by 1 and w decreases, scaling down by some factor $\eta \in (\frac{1}{2}, 1)$; overall, $\tilde{s}$ increases by a factor of $2\eta > 1$. However, the expression for $\tilde{P}_1^S$ then implies that $\tilde{p}_1^S = \eta p_1^S$, and thus $\tilde{p}_1^S$ decreases.

Independence of B and P_1^S (A3.ii) is satisfied in this model absent unobserved heterogeneity. However, once unobserved heterogeneity is included, $\tilde{P}_1^S$ and $\tilde{B}$ will be correlated through W in both the additively or multiplicatively separable unobserved heterogeneity models. The Perry model cannot serve for studying monotonicity of the buyer's first offer or independence of the P_2^B from S because only one offer occurs in equilibrium.

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