Econometrica Supplementary Material

SUPPLEMENT TO "AGGREGATE DYNAMICS IN LUMPY ECONOMIES"

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APPENDIX A: PROOFS OF MAIN RESULTS

A.1. *Auxiliary Theorems*

AUXILIARY THEOREM 1: LET X BE A (SUB)MARTINGALE *on the filtered space* $(\Omega, \mathbb{P}, \mathcal{F})$ *and let* τ *be a stopping time. If* $(\{X_t\}_t, \tau)$ *is a well-defined stopping process, then*

$$\mathbb{E}[X_\tau]\,(\geq) = \mathbb{E}[X_0]. \tag{A.1}$$

This result is known as the Optional Sampling Theorem (OST). See Appendix B.1 for definitions and Theorem 4.4 in Stokey (2009) for the proof.

AUXILIARY THEOREM 2: *Let* g *be a real-valued function of a Brownian motion* x_t, F *the ergodic distribution of* x, *and* τ *a stopping time. Assume that* $\int g(x)\,\mathrm{d}F(x) = \lim_{T\to\infty} T^{-1}\int_0^T g(x_t)\,\mathrm{d}t$ *for all initial* x_0 *and a constant reset state* $x_\tau = x^*$. *The following relationship holds*:

$$\underbrace{\mathbb{E}\left[\int_0^\tau g(x_t)\,\mathrm{d}t \,\middle|\, x_0 = x^*\right]}_{\text{occupancy measure}} = \underbrace{\int g(x)\,\mathrm{d}F(x)}_{\text{steady-state mass}}\ \underbrace{\mathbb{E}\big[\tau | x_0 = x^*\big]}_{\text{proportionality constant}}. \tag{A.2}$$

This result establishes the equivalence between the steady-state distribution and the occupancy measure. The occupancy measure (LHS) is the average time an agent's state spends at a given value. It is proportional to the stationary mass of agents at that particular state (RHS), with a proportionality constant equal to the expected time between adjustments. For example, if $g(x) = x^m$, then $\mathbb{E}[\int_0^\tau x_t^m\,\mathrm{d}t | x_0 = x^*] = \mathbb{E}[x^m]\mathbb{E}[\tau | x_0 = x^*]$. For notation clarity, we use $\overline{\mathbb{E}}[\cdot] = \mathbb{E}[\cdot | x_0 = x^*]$. See Appendix B.2 for the proof and Stokey (2009) for more details.

A.2. *Proof of Lemma 1*

The proof has four steps. First, we characterize the value function $V(k, z, e)$ and the optimal policy (k^-, k^*, k^+) as a function of capital and productivities (k, z, e). Second, we reduce the dimensionality of the state space by defining a single state variable, the capital-to-productivity ratio $\hat{k} \equiv k/(ze)$, and rewrite the value function as $v(\hat{k})$ and the policy as $(\hat{k}^-, \hat{k}^*, \hat{k}^+)$. Third, we re-express all objects as functions of log capital-to-productivity

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ratios $\hat{x} = \log(\hat{k})$, obtaining the value function $\mathcal{V}(\hat{x})$ and the policy $(\hat{x}^-, \hat{x}^*, \hat{x}^+)$. Fourth, we express the optimal policy as a system of equations in fundamental parameters.

Step 1. Characterizing the Value Function and Policy in Terms of (k, z, e). The investment policy is characterized by two inaction thresholds $k^-(z, e)$ and $k^+(z, e)$ that determine the inaction region, together with a reset value $k^*(z, e)$ where capital is set upon an adjustment. The firms' policies satisfy the following conditions:

1. In the interior of the inaction region, that is, $k \in (k^-(z, e), k^+(z, e))$, $V(k, e, z)$ solves the HJB equation:

$$\begin{aligned} rV(k,e,z) = {} & ez\left(\frac{k}{ez}\right)^{\alpha} + \mu_z z\frac{\partial V(k,e,z)}{\partial z} - \zeta k\frac{\partial V(k,e,z)}{\partial k} \\ & + \frac{\sigma^2}{2}e\frac{\partial V(k,e,z)}{\partial e} + \frac{(\sigma e)^2}{2}\frac{\partial^2 V(k,e,z)}{\partial e^2} \\ & + \lambda\big(V\big(k^*,e,z\big) - V(k,e,z) - \big(k^* - k\big)\big). \end{aligned} \tag{A.3}$$

2. The value-matching conditions for all (e, z) are

$$V\big(k^*,e,z\big) + \big(k^+ - k^*\big) - \theta^+ ez = V\big(k^+,e,z\big), \tag{A.4}$$

$$V\big(k^*,e,z\big) + \big(k^- - k^*\big) - \theta^- ez = V\big(k^-,e,z\big). \tag{A.5}$$

3. The optimality and smooth-pasting conditions are

$$\frac{\partial V(k,e,z)}{\partial k} = 1 \quad \text{for } k \in \{k^-, k^*, k^+\}, \tag{A.6}$$

$$\left.\frac{\partial V(k,e,z)}{\partial e}\right|_{k=k^*} = \theta^s z + \left.\frac{\partial V(k,e,z)}{\partial e}\right|_{k=k^s} \quad \text{for } s \in \{+,-\}, \tag{A.7}$$

$$\left.\frac{\partial V(k,e,z)}{\partial z}\right|_{k=k^*} = \theta^s e + \left.\frac{\partial V(k,e,z)}{\partial z}\right|_{k=k^s} \quad \text{for } s \in \{+,-\}. \tag{A.8}$$

For additional details about the sufficiency of HJB equations, value-matching, optimality, and smooth-pasting conditions, see Oksendal (2007) and Baley and Blanco (2019).

Step 2. Rewriting the System in Terms of $\hat{k}$. We rewrite the system from (A.3) to (A.8) in terms of the capital-to-productivity ratio $\hat{k} \equiv k/(ze)$. To simplify notation, we let $\rho \equiv (r + \lambda - \mu_z - \sigma^2/2)$ and $\nu \equiv \zeta + \mu_z$. We guess a new value function $v(\hat{k}) \equiv V(ze\hat{k}, z, e)/ze$ and three real numbers $(\hat{k}^-, \hat{k}^*, \hat{k}^+) \equiv ze \times (k^-, k^*, k^+)$, and we verify that $v(\hat{k})$ and $(\hat{k}^-, \hat{k}^*, \hat{k}^+)$ satisfy the following three conditions to establish equivalence with the system above:

1. In the interior of the inaction region, that is, for $\hat{k} \in (\hat{k}^-, \hat{k}^+)$, $v(\hat{k})$ solves the HJB:

$$\rho v(\hat{k}) = \hat{k}^{\alpha} + \left(-\nu - \frac{\sigma^2}{2}\right)\hat{k}v'(\hat{k}) + \frac{\sigma^2}{2}\hat{k}^2 v''(\hat{k}) + \lambda\big(v\big(\hat{k}^*\big) - \big(\hat{k}^* - \hat{k}\big)\big). \tag{A.9}$$

2. At the boundaries of the inaction region $\hat{k} \in \{\hat{k}^-, \hat{k}^+\}$, $v(\hat{k})$ satisfies the value-matching conditions:

$$v(\hat{k}^+) = v(\hat{k}^*) - \theta^+ - (\hat{k}^* - \hat{k}^+), \tag{A.10}$$

$$v(\hat{k}^-) = v(\hat{k}^*) - \theta^- - (\hat{k}^* - \hat{k}^-). \tag{A.11}$$

3. At the boundaries of the inaction region and the reset state, $v(\hat{k})$ satisfies the smooth-pasting and the optimality conditions:

$$v'(\hat{k}) = 1 \quad \text{for } \hat{k} \in \{\hat{k}^-, \hat{k}^*, \hat{k}^+\}. \tag{A.12}$$

Given the guess $V(k, z, e) = zev(k/(ze))$, the following relationships hold:

$$\frac{\partial V(k, e, z)}{\partial k} = v'(\hat{k}), \tag{A.13}$$

$$\frac{\partial V(k, e, z)}{\partial e} = zv(\hat{k}) - \frac{k}{e} v'(\hat{k}), \tag{A.14}$$

$$\frac{\partial^2 V(k, e)}{\partial e^2} = \frac{k^2}{ze^3} v''(\hat{k}), \tag{A.15}$$

$$\frac{\partial V(k, e, z)}{\partial z} = ev(\hat{k}) - \frac{k}{z} v'(\hat{k}). \tag{A.16}$$

First, we show the optimality condition for $\hat{k}^*$. From (A.6) and (A.13), we obtain

$$\left.\frac{\partial V(k, e, z)}{\partial k}\right|_{k=k^*} = 1 \quad \Longleftrightarrow \quad \left. v'\left(\frac{k}{ez}\right)\right|_{k=k^*} = 1.$$

Second, we rewrite the HJB equation in terms of $\hat{k}$. Using the guess and substituting (A.13) to (A.16) into (A.3):

$$\begin{aligned}(r + \lambda)zev(\hat{k}) = {} & ze\hat{k}^\alpha - \zeta k v'(\hat{k}) + \mu_z z\left(ev(\hat{k}) - \frac{k}{z} v'(\hat{k})\right) \\ & + \frac{\sigma^2}{2} e\left(zv(\hat{k}) - \frac{k}{e} v'(\hat{k})\right) + \frac{(\sigma e)^2}{2}\left(\frac{k^2}{ze^3} v''(\hat{k})\right) \\ & + \lambda\left(zev(\hat{k}^*) - (k^* - k)\right).\end{aligned} \tag{A.17}$$

Dividing both sides by ze, rearranging terms, and using the definitions of ρ and ν:

$$\rho v(\hat{k}) = \hat{k}^\alpha + \left(-\nu - \frac{\sigma^2}{2}\right)\hat{k} v'(\hat{k}) + \frac{\sigma^2}{2}\hat{k}^2 v''(\hat{k}) + \lambda\left(v(\hat{k}^*) - (\hat{k}^* - \hat{k})\right).$$

Third, we rewrite the value-matching conditions in terms of $\hat{k}$. Substituting the guess into (A.4) and (A.5), and dividing by ze:

$$\begin{aligned} & V(k^*, z, e) - (k^* - k^+) - \theta^+ ze = V(k^+, z, e) \\ \Longleftrightarrow \quad & v(\hat{k}^*) - (\hat{k}^* - \hat{k}^+) - \theta^+ = v(\hat{k}^+), \end{aligned} \tag{A.18}$$

$$V(k^*, z, e) - (k^* - k^-) - \theta^- ze = V(k^-, z, e)$$
$$\iff \quad v(\hat{k}^*) - (\hat{k}^* - \hat{k}^-) - \theta^- = v(\hat{k}^-). \tag{A.19}$$

Fourth, we verify the three smooth-pasting conditions for the lower adjustment threshold k^- (the arguments are equivalent for k^+). From (A.6) and (A.13), we verify the smooth pasting for k:

$$\left.\frac{\partial V(k, e, z)}{\partial k}\right|_{k=k^-} = 1 \quad \iff \quad v'\left(\frac{k}{ez}\right)\Big|_{k=k^-} = 1 \quad \iff \quad v'(\hat{k}^-) = 1.$$

From (A.7) and (A.16), we verify smooth pasting for z:

$$\left.\frac{\partial V(k, e, z)}{\partial z}\right|_{k=k^*} = e\theta^- + \left.\frac{\partial V(k, e, z)}{\partial z}\right|_{k=k^-} \quad \iff \quad v(\hat{k}^-) - \hat{k}^- = v(\hat{k}^*) - \theta^- - \hat{k}^*,$$

where we substituted the smooth-pasting and optimality conditions $v'(\hat{k}^-) = v'(\hat{k}^*) = 1$. Analogously, from (A.8) and (A.14), one verifies smooth pasting for e.

Step 3. Rewriting the System in Terms of $\hat{x}$. Now we rewrite the system from (A.9) to (A.12) in terms of the log capital-to-productivity ratios $\hat{x} \equiv \log(k/(ze))$. We guess and verify a new value function $\mathcal{V}(\hat{x}) = v(\exp(\hat{k}))$ and policies $(\hat{x}^-, \hat{x}^*, \hat{x}^+)$. If $\mathcal{V}(\hat{x})$ and the values $(\hat{x}^-, \hat{x}^*, \hat{x}^+)$ satisfy the following conditions, then $\mathcal{V}(\hat{x}) = v(\exp(\hat{k}))$ and the optimal policy is $(\hat{x}^-, \hat{x}^*, \hat{x}^+)$:

1. In the interior of the inaction region, that is, $\hat{x} \in (\hat{x}^-, \hat{x}^+)$, $\mathcal{V}(\hat{x})$ solves the HJB:

$$\rho\mathcal{V}(\hat{x}) = \exp(\alpha\hat{x}) - \nu\mathcal{V}'(\hat{x}) + \frac{\sigma^2}{2}\mathcal{V}''(\hat{x}) + \lambda\big(\mathcal{V}(\hat{x}^*) - (\exp(\hat{x}^*) - \exp(\hat{x}))\big). \tag{A.20}$$

2. At the boundaries of the inaction region, $\hat{x} \in \{\hat{x}^-, \hat{x}^+\}$, $v(\hat{x})$ satisfies the value-matching conditions:

$$\mathcal{V}(\hat{x}^+) = \mathcal{V}(\hat{x}^*) - \theta^+ - (\exp(\hat{x}^*) - \exp(\hat{x}^+)), \tag{A.21}$$
$$\mathcal{V}(\hat{x}^-) = \mathcal{V}(\hat{x}^*) - \theta^- - (\exp(\hat{x}^*) - \exp(\hat{x}^-)). \tag{A.22}$$

3. At the boundaries of the inaction region and the reset state, $\mathcal{V}(\hat{x})$ satisfies the smooth-pasting and the optimality conditions:

$$\mathcal{V}'(\hat{x}) = \exp(\hat{x}) \quad \text{for } \hat{x} \in \{\hat{x}^-, \hat{x}^*, \hat{x}^+\}. \tag{A.23}$$

The proof is similar to that of Step 2. To show that (A.20) to (A.23) $\iff$ (A.9) to (A.12), substitute the guess and rewrite in terms of $\hat{x}$. Then verify the validity of the conditions.

Step 4. Policy as System of Equations. We characterize the value function and the optimal policy as a solution of a system of equations. The homogeneous solution of (A.20) is $\mathcal{V}^h(\hat{x}) = A_1 e^{\pi_1\hat{x}} + A_2 e^{\pi_2\hat{x}}$, where A_2, A_1 are functions to be determined and the roots π_1 and π_2 are given by

$$\pi_1 = \frac{\nu}{\sigma^2} - \sqrt{\frac{\nu^2}{\sigma^4} + \frac{2\rho}{\sigma^2}}, \qquad \pi_2 = \frac{\nu}{\sigma^2} + \sqrt{\frac{\nu^2}{\sigma^4} + \frac{2\rho}{\sigma^2}}. \tag{A.24}$$

Using the method of undetermined coefficients to find the nonhomogeneous solution:

$$\mathcal{V}(\hat{x}) = A_1 e^{\pi_1 \hat{x}} + A_2 e^{\pi_2 \hat{x}} + C(\alpha) e^{\alpha \hat{x}} + C(1)\lambda e^{\hat{x}} + E, \tag{A.25}$$

with the function $C(\alpha)$ and the constant E given by

$$C(\alpha) = \left(\rho + \alpha\nu/\sigma^2 - \sigma^2\alpha^2/2\right)^{-1}, \tag{A.26}$$

$$E = \frac{\lambda}{\rho - \lambda}\left(A_1 e^{\pi_1 \hat{x}^*} + A_2 e^{\pi_2 \hat{x}^*} + C(\alpha) e^{\alpha \hat{x}^*} + C(1) e^{\hat{x}^*} - e^{\hat{x}^*}\right). \tag{A.27}$$

The value-matching conditions in (A.21) and (A.22) imply

$$A_1\left(e^{\pi_1 \hat{x}^-} - e^{\pi_1 \hat{x}^*}\right) + A_2\left(e^{\pi_2 \hat{x}^-} - e^{\pi_2 \hat{x}^*}\right) = B\left(\hat{x}^-\right),$$

$$A_1\left(e^{\pi_1 \hat{x}^+} - e^{\pi_1 \hat{x}^*}\right) + A_2\left(e^{\pi_2 \hat{x}^+} - e^{\pi_2 \hat{x}^*}\right) = B\left(\hat{x}^+\right),$$

where the functions $B(\hat{x}^-)$ and $B(\hat{x}^+)$ are defined as

$$B\left(\hat{x}^-\right) \equiv -\theta^- + C(\alpha)\left(e^{\alpha \hat{x}^*} - e^{\alpha \hat{x}^-}\right) + \left(\lambda C(1) - 1\right)\left(e^{\hat{x}^*} - e^{\hat{x}^-}\right), \tag{A.28}$$

$$B\left(\hat{x}^+\right) \equiv -\theta^+ + C(\alpha)\left(e^{\alpha \hat{x}^*} - e^{\alpha \hat{x}^+}\right) + \left(\lambda C(1) - 1\right)\left(e^{\hat{x}^*} - e^{\hat{x}^+}\right). \tag{A.29}$$

Using Cramer's rule to solve the previous system for A_1 and A_2 yields

$$A_1 = \frac{B\left(\hat{x}^-\right)\left(e^{\pi_2 \hat{x}^+} - e^{\pi_2 \hat{x}^*}\right) - B\left(\hat{x}^+\right)\left(e^{\pi_2 \hat{x}^-} - e^{\pi_2 \hat{x}^*}\right)}{D}, \tag{A.30}$$

$$A_2 = \frac{B\left(\hat{x}^-\right)\left(e^{\pi_1 \hat{x}^+} - e^{\pi_1 \hat{x}^*}\right) - B\left(\hat{x}^+\right)\left(e^{\pi_1 \hat{x}^-} - e^{\pi_1 \hat{x}^*}\right)}{D}, \tag{A.31}$$

where the determinant D is

$$D = \left(e^{\pi_1 \hat{x}^-} - e^{\pi_1 \hat{x}^*}\right)\left(e^{\pi_2 \hat{x}^+} - e^{\pi_2 \hat{x}^*}\right) - \left(e^{\pi_1 \hat{x}^+} - e^{\pi_1 \hat{x}^*}\right)\left(e^{\pi_2 \hat{x}^-} - e^{\pi_2 \hat{x}^*}\right). \tag{A.32}$$

Finally, the last conditions that need to be satisfied by the value function and optimal policy are optimality and the two smooth-pasting conditions in (A.23). Summarizing, the value function and optimal policy are given by the system from (A.23) to (A.32).

A.3. *Proof of Lemma 2*

We extend the result in Álvarez, Le Bihan, and Lippi (2016) for higher-order moments and an arbitrary policy. Start from the definition of the CIR in (24) and fix an $m \in \mathbb{N}$:

$$\begin{aligned}
\mathrm{CIR}_m(\delta) &\equiv \int_0^\infty \left(\mathbb{E}_t\left[x^m\right] - \mathbb{E}\left[x^m\right]\right) \mathrm{d}t \\
&= \mathbb{E}\left[\int_0^\infty \left(x_t^m - \mathbb{E}\left[x^m\right]\right) \mathrm{d}t\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\int_0^\infty \left(x_t^m - \mathbb{E}\left[x^m\right]\right) \mathrm{d}t \middle| \mathcal{F}_0\right]\right] \\
&= \int_{x^-}^{x^+} \mathbb{E}\left[\int_0^\infty \left(x_t^m - \mathbb{E}\left[x^m\right]\right) \mathrm{d}t \middle| x\right] \mathrm{d}F_0(x).
\end{aligned} \tag{A.33}$$

In the second equality, we use Fubini's theorem to exchange the time and state integration; in the third equality, we use the law of iterated expectations to condition on the initial filtration (see definition of the probability space in Appendix B.1); and in the fourth equality, we use the strong Markov property of x to express the CIR as a function of the initial distribution F_0.

Next, we show that the CIR can be expressed as the sum of average deviations from the steady-state value up to the first adjustment plus a correction term. Let $\eta(x_t) \equiv x_t^m - \mathbb{E}[x^m]$ denote the time-t deviation from the steady-state moment; let $\{T_i\}_{i=1}^{\infty}$ be a sequence of stopping times; and let $H(\mathcal{T})$ be a counter of the number of adjustments before a deterministic date $\mathcal{T}$. Each step is numbered and corresponds to an equality below.

In equality (1), we follow the strategy in Alexandrov (2020) to approach the CIR in (A.33) as the limit of a finite-horizon problem:[30]

$$\mathrm{CIR}_m(\delta) \overset{(1)}{=} \int_{x^-}^{x^+} \mathbb{E}\left[\lim_{\mathcal{T}\to\infty} \int_0^{\mathcal{T}} \eta(x_t)\,\mathrm{d}t \middle| x\right] \mathrm{d}F_0(x). \tag{A.34}$$

In (2), we write the CIR as the sum of three terms: the expected cumulative deviations between time $t = 0$ and the first stopping time T_1, between T_1 and between $T_{H(\mathcal{T})}$, and $T_{H(\mathcal{T})}$ and $\mathcal{T}$:

$$\begin{aligned}&\overset{(2)}{=} \int_{x^-}^{x^+} \left\{\mathbb{E}\left[\int_0^{T_1} \eta(x_t)\,\mathrm{d}t \middle| x\right] + \mathbb{E}\left[\lim_{\mathcal{T}\to\infty} \sum_{h=1}^{H(\mathcal{T})-1} \int_{T_h}^{T_{h+1}} \eta(x_t)\,\mathrm{d}t \middle| x\right]\right\} \mathrm{d}F_0(x) \\ &\quad + \int_{x^-}^{x^+} \mathbb{E}\left[\lim_{\mathcal{T}\to\infty} \int_{T_{H(\mathcal{T})}}^{\mathcal{T}} \eta(x_t)\,\mathrm{d}t \middle| x\right] \mathrm{d}F_0(x).\end{aligned} \tag{A.35}$$

In (3), we use $H(\mathcal{T})$ times the law of iterated expectations to condition on the information set $\mathcal{F}_{T_h}$; here we also identify the tail term and bring it out of the conditional expectation as it does not depend on initial conditions:

$$\begin{aligned}&\overset{(3)}{=} \int_{x^-}^{x^+} \left\{\mathbb{E}\left[\int_0^{T_1} \eta(x_t)\,\mathrm{d}t \middle| x\right] + \mathbb{E}\left[\lim_{\mathcal{T}\to\infty} \sum_{h=1}^{H(\mathcal{T})-1} \mathbb{E}\left[\int_{T_h}^{T_{h+1}} \eta(x_t)\,\mathrm{d}t | \mathcal{F}_{T_h}\right] \middle| x\right]\right\} \mathrm{d}F_0(x) \\ &\quad + \underbrace{\mathbb{E}\left[\lim_{\mathcal{T}\to\infty} \int_{T_{H(\mathcal{T})}}^{\mathcal{T}} \eta(x_t)\,\mathrm{d}t\right]}_{\text{tail}}.\end{aligned} \tag{A.36}$$

In (4), we use the strong Markov property of x, the assumption of an homogeneous reset state, and that x^* is constant for $h \geq 1$ to change the conditioning information from $\mathcal{F}_{T_h}$ to x^*. Additionally, we substitute η for its definition, write in terms of inaction duration $\tau = T_{h+1} - T_h$, and use the relationship between steady-state moments and occupancy

[30] Alexandrov's (2020) finite-horizon strategy is key to reveal the tail term.

measure in Auxiliary Theorem 2 to conclude that every element inside the sum is zero:

$$\overset{(4)}{=} \int_{x^-}^{x^+} \left\{ \mathbb{E}\left[\int_0^{T_1} \eta(x_t)\,\mathrm{d}t \middle| x \right] + \mathbb{E}\left[\lim_{\mathcal{T}\to\infty} \sum_{h=1}^{H(\mathcal{T})-1} \underbrace{\mathbb{E}\left[\int_0^{\tau} x_t^m \,\mathrm{d}t \middle| x^* \right] - \mathbb{E}\left[x^m\right]\mathbb{E}\left[\tau | x^*\right]}_{=0} \right] \right\} \mathrm{d}F_0(x)$$
$$+ \text{tail}. \tag{A.37}$$

In (5), we substitute $F_0(x) = F(x - \delta)$ since the initial distribution is a marginal δ-translation of the steady-state distribution:

$$\overset{(5)}{=} \int_{x^-}^{x^+} \mathbb{E}\left[\int_0^{\tau} \eta(x_t)\,\mathrm{d}t \middle| x \right] \mathrm{d}F(x-\delta) + \text{tail}. \tag{A.38}$$

Since $\mathrm{CIR}_m(0) = 0$, it implies that the tail term is equal to

$$\text{tail} \equiv \mathbb{E}\left[\lim_{\mathcal{T}\to\infty} \int_{T_{H(\mathcal{T})}}^{\mathcal{T}} \eta(x_t)\,\mathrm{d}t \right] = -\int_{x^-}^{x^+} \mathbb{E}\left[\int_0^{\tau} \eta(x_t)\,\mathrm{d}t \middle| x \right] \mathrm{d}F(x). \tag{A.39}$$

Together, (A.34) and (A.39) imply

$$\mathrm{CIR}_m(\delta) = \int_{x^-}^{x^+} \mathbb{E}\left[\int_0^{\tau} \eta(x_t)\,\mathrm{d}t \middle| x \right] \mathrm{d}\big(F(x-\delta) - F(x)\big). \tag{A.40}$$

Finally, defining the value $v_m(x) \equiv \mathbb{E}[\int_0^{\tau} \eta(x_t)\,\mathrm{d}t | x]$ (conditional on a particular initial condition x), we obtain the result: the CIR can be written as the average cumulative deviation until the first adjustment, up to a correction term (the tail):

$$\mathrm{CIR}_m(\delta) = \int_{x^-}^{x^+} v_m(x)\,\mathrm{d}\big(F(x-\delta) - F(x)\big), \quad v_m(x) \equiv \mathbb{E}\left[\int_0^{\tau} \big(x_t^m - \mathbb{E}\left[x^m\right]\big)\,\mathrm{d}t \right]. \tag{A.41}$$

A.4. *Proof of Proposition 1*

The logic behind the mapping between the CIR and steady-state moments in Proposition 1 is simple: All objects are spanned by the same finite basis

$$\mathcal{B}_m \equiv \left\{ \left\{x^i\right\}_{i=0}^{m+2}, e^{\xi_1 x}, e^{\xi_2 x}, e^{\xi_1 x} x, e^{\xi_2 x} x \right\}, \tag{A.42}$$

where ξ_1, ξ_2 are the characteristic roots of the HJB equation satisfied by the steady-state moments. The outline of the proof is as follows. (i) We conduct a first-order approximation of the CIR around $\delta \approx 0$; (ii) we define three functions $\{W, U, Q\}$ and characterize them in terms of the common basis $\mathcal{B}_m$ through Auxiliary Lemmas 1 to 4; (iii) we use the occupancy measure to show how these auxiliary functions, evaluated at the reset state x^*, generate the steady-state moments and CIR; and finally, (iv) we match coefficients of the linear projections on $\mathcal{B}_m$ to obtain the result.

(i) First-Order Approximation of the CIR. We depart from the CIR expression in Lemma 2:

$$\mathrm{CIR}_m(\delta) = \int_{x^-}^{x^+} v_m(x)\big[f(x-\delta) - f(x)\big]\,\mathrm{d}x.$$

Doing a first-order Taylor approximation around $\delta \approx 0$ (noting that $f'(x)$ exist for all $x \in [x^-, x^+] \setminus \{x^*\}$) and integrating by parts (using that no mass exists at the borders of inaction, i.e., $f(x^-) = f(x^+) = 0$), we obtain an expression in terms of $v'_m(x)$:

$$\begin{aligned}\mathrm{CIR}_m(\delta) &= -\delta \int_{x^-}^{x^+} v_m(x) f'(x)\,\mathrm{d}x + o(\delta^2) \\ &= -\delta\left[f(x)v_m(x)\big|_{x^-}^{x^+} - \int_{x^-}^{x^+} v'_m(x) f(x)\,\mathrm{d}x \right] + o(\delta^2) \\ &= \delta \int_{x^-}^{x^+} v'_m(x) f(x)\,\mathrm{d}x + o(\delta^2). \end{aligned} \tag{A.43}$$

(ii) Auxiliary Functions. Define the following three functions:

$$W_m(x) \equiv \mathbb{E}\left[\int_0^\tau x_t^m \,\mathrm{d}t \,\middle|\, x\right], \qquad W_0(x) = \mathbb{E}[\tau|x], \tag{A.44}$$

$$Q_m(x) \equiv \mathbb{E}\left[\int_0^\tau t x_t^m \,\mathrm{d}t \,\middle|\, x\right], \qquad Q_0(x) = \frac{\mathbb{E}[\tau^2|x]}{2}, \tag{A.45}$$

$$U_m(x) \equiv \mathbb{E}\left[\int_0^\tau W'_m(x_t) \,\mathrm{d}t \,\middle|\, x\right], \qquad U_0(x) = \mathbb{E}\left[\int_0^\tau \frac{\mathrm{d}\mathbb{E}[\tau|x]}{\mathrm{d}x} \,\mathrm{d}t \,\middle|\, x\right]. \tag{A.46}$$

Auxiliary Lemmas 1 to 4 below show that $W_m(x)$, $Q_m(x)$, $U_m(x)$ are spanned by the same finite basis $\mathcal{B}_m$.

(iii) CIR and Steady-State Moments in Terms of Auxiliary Functions. Using the equivalence between the occupancy measure and the steady-state distribution in Auxiliary Theorem 2, we express steady-state moments in terms of the functions $\{W_m, Q_m\}$, evaluated at x^*:

$$\mathbb{E}[x^m] = \frac{W_m(x^*)}{W_0(x^*)}, \tag{A.47}$$

$$\mathbb{C}ov[x^m, a] = \mathbb{E}[x^m a] - \mathbb{E}[x^m]\mathbb{E}[a] = \frac{Q_m(x^*)}{W_0(x^*)} - \frac{W_m(x^*)}{W_0(x^*)}\frac{Q_0(x^*)}{W_0(x^*)}. \tag{A.48}$$

Thus, the exact linear combination of steady-state moments that equals the first-order approximation of the CIR is

$$\frac{1}{\sigma^2}\left(\mathbb{E}[x^{m+1}] + \nu \mathbb{C}ov[x^m, a]\right) = \frac{1}{\sigma^2}\left(\frac{W_{m+1}(x^*)}{W_0(x^*)} + \nu\left[\frac{Q_m(x^*)}{W_0(x^*)} - \frac{W_m(x^*)Q_0(x^*)}{W_0(x^*)^2}\right)\right]. \tag{A.49}$$

Analogously, we express the CIR in terms of $\{W_m, U_m\}$. Starting from the definition of $v_m(x)$ in (A.41):

$$v_m(x) = \mathbb{E}\left[\int_0^\tau x_t^m \,\mathrm{d}t \,\middle|\, x\right] - \mathbb{E}[x^m]\mathbb{E}\left[\int_0^\tau 1 \,\mathrm{d}t \,\middle|\, x\right] = W_m(x) - \mathbb{E}[x^m] W_0(x), \tag{A.50}$$

we combine (A.43) and (A.50), and use Auxiliary Theorem 2 to express the CIR as

$$\frac{\mathrm{CIR}_m(\delta)}{\delta} = \int_X \big(W_m'(x) - \mathbb{E}[x^m] W_0'(x)\big)\,\mathrm{d}F(x) = \frac{U_m(x^*)}{W_0(x^*)} - \frac{W_m(x^*)U_0(x^*)}{W_0(x^*)^2} + o(\delta). \quad \text{(A.51)}$$

(iv) Matching Coefficients. Once we have the expressions for the CIR_m and the steady-state moments in terms of the functions $\{W_m, U_m, Q_m\}$, we show that

$$\underbrace{\frac{U_m(x^*)}{W_0(x^*)} - \frac{W_m(x^*)U_0(x^*)}{W_0(x^*)^2}}_{\text{CIR in (A.51)}} = \underbrace{\frac{1}{\sigma^2}\left(\frac{W_{m+1}(x^*)}{W_0(x^*)} + \nu\left[\frac{Q_m(x^*)}{W_0(x^*)} - \frac{W_m(x^*)Q_0(x^*)}{W_0(x^*)^2}\right]\right)}_{\text{Steady-state moments in (A.49)}}. \quad \text{(A.52)}$$

To do this, we make use of the following auxiliary lemmas. First, we establish notation.

Notation. For the next auxiliary lemmas, we use the following notation:

$$\phi \equiv \sigma^2/2\nu; \quad \text{(A.53)}$$

$$\xi_1(\varphi) \equiv \frac{\nu}{\sigma^2} - \sqrt{\frac{\nu^2}{\sigma^4} + \frac{2(\lambda-\varphi)}{\sigma^2}}; \qquad \xi_2(\varphi) \equiv \frac{\nu}{\sigma^2} + \sqrt{\frac{\nu^2}{\sigma^4} + \frac{2(\lambda-\varphi)}{\sigma^2}}; \quad \text{(A.54)}$$

$$\mathcal{D}(\varphi) \equiv e^{\xi_1(\varphi)x^- + \xi_2(\varphi)x^+} - e^{\xi_2(\varphi)x^- + \xi_1(\varphi)x^+}; \quad \text{(A.55)}$$

$$\begin{aligned} \overline{a}_1(\varphi) &\equiv \frac{e^{\xi_1(\varphi)x^+}}{\mathcal{D}(\varphi)}; \qquad \underline{a}_1(\varphi) \equiv \frac{e^{\xi_1(\varphi)x^-}}{\mathcal{D}(\varphi)}; \\ \overline{a}_2(\varphi) &\equiv \frac{e^{\xi_2(\varphi)x^+}}{\mathcal{D}(\varphi)}; \qquad \underline{a}_2(\varphi) \equiv \frac{e^{\xi_2(\varphi)x^-}}{\mathcal{D}(\varphi)}; \end{aligned} \quad \text{(A.56)}$$

$$b_{i,m}(\varphi) = \frac{m!}{i!}\left[\frac{\xi_1(\varphi)+1/\phi}{\xi_1(\varphi)-\xi_2(\varphi)}\xi_1(\varphi)^{i-m} + \frac{\xi_2(\varphi)+1/\phi}{\xi_2(\varphi)-\xi_1(\varphi)}\xi_2(\varphi)^{i-m}\right]; \quad \text{(A.57)}$$

$$\kappa_m(x,\varphi) \equiv \sum_{i=0}^{m} b_{i,m}(\varphi)x^i; \quad \text{(A.58)}$$

$$\underline{\kappa}_m(\varphi) = \kappa_m(x^-,\varphi); \qquad \overline{\kappa}_m(\varphi) \equiv \kappa_m(x^+,\varphi); \qquad \kappa_m^*(\varphi) \equiv \kappa_m(x^*,\varphi). \quad \text{(A.59)}$$

We also define the following parameters (by evaluating the functions above at $\varphi = 0$):

$$\xi_1 \equiv \xi_1(0); \qquad \xi_2 \equiv \xi_2(0); \qquad \mathcal{D} = \mathcal{D}(0); \quad \text{(A.60)}$$

$$\begin{aligned} \overline{a}_1 = \overline{a}_1(0); \qquad \underline{a}_1 \equiv \underline{a}_1(0); \qquad \overline{a}_2 \equiv \overline{a}_2(0); \\ \underline{a}_2 \equiv \underline{a}_2(0); \qquad b_{i,m} \equiv b_{i,m}(0); \end{aligned} \quad \text{(A.61)}$$

$$\begin{aligned} \kappa_m(x) \equiv \kappa_m(x,0); \qquad \overline{\kappa}_m = \kappa_m(x^+,0); \\ \underline{\kappa}_m = \kappa_m(x^-,0); \qquad \kappa_m^* = \kappa_m(x^*,0). \end{aligned} \quad \text{(A.62)}$$

AUXILIARY LEMMA 1: *Let $W_m(x) \equiv \mathbb{E}[\int_0^\tau x_t^m \,\mathrm{d}t \,|\, x]$. It is equal to*

$$\lambda W_m(x) = e^{\xi_1 x}[\underline{a}_2\overline{\kappa}_m - \overline{a}_2\underline{\kappa}_m] + e^{\xi_2 x}[\overline{a}_1\underline{\kappa}_m - \underline{a}_1\overline{\kappa}_m] + \kappa_m(x). \quad \text{(A.63)}$$

AUXILIARY LEMMA 2: *Let* $Q_m(x) \equiv \mathbb{E}[\int_0^\tau t x_t^m \,\mathrm{d}t | x]$. *It is equal to*

$$\lambda Q_m(x) = W_m(x) + \frac{e^{\xi_1 x} x}{\nu - \sigma^2 \xi_1}[\underline{a}_2 \overline{\kappa}_m - \overline{a}_2 \underline{\kappa}_m] + \frac{e^{\xi_2 x} x}{\nu - \sigma^2 \xi_2}[\overline{a}_1 \underline{\kappa}_m - \underline{a}_1 \overline{\kappa}_m] + \kappa_m'(x, 0)$$
$$+ e^{\xi_1 x}\big[\underline{a}_2 \overline{\kappa}_m'(0) + \underline{a}_2'(0)\overline{\kappa}_m - \overline{a}_2 \underline{\kappa}_m'(0) - \overline{a}_2'(0)\underline{\kappa}_m\big]$$
$$+ e^{\xi_2 x}\big[\underline{a}_1 \overline{\kappa}_m'(0) + \underline{a}_1'(0)\overline{\kappa}_m - \overline{a}_1 \underline{\kappa}_m'(0) - \overline{a}_1'(0)\underline{\kappa}_m\big]. \tag{A.64}$$

AUXILIARY LEMMA 3: *Let* $V_j(x) \equiv \mathbb{E}[\int_0^\tau e^{\xi_j x_t} \,\mathrm{d}t | x]$, *for* $j = 1, 2$. *They are equal to*

$$V_1(x) = \left(\frac{\mathcal{D}}{\nu - \sigma^2 \xi_1}\right)\left[e^{\xi_1 x}\big(\underline{a}_1 \overline{a}_2 x^- - \overline{a}_1 \underline{a}_2 x^+\big) + e^{\xi_2 x}\underline{a}_1 \overline{a}_1\big(x^+ - x^-\big) + \frac{e^{\xi_1 x} x}{\mathcal{D}}\right], \tag{A.65}$$

$$V_2(x) = \left(\frac{\mathcal{D}}{\nu - \sigma^2 \xi_2}\right)\left[e^{\xi_2 x}\big(\underline{a}_1 \overline{a}_2 x^+ - \overline{a}_1 \underline{a}_2 x^-\big) + e^{\xi_1 x}\underline{a}_2 \overline{a}_2\big(x^+ - x^-\big) + \frac{e^{\xi_2 x} x}{\mathcal{D}}\right]. \tag{A.66}$$

AUXILIARY LEMMA 4: *Let* $U_m(x) \equiv \mathbb{E}[\int_0^\tau W_m'(x_t) \,\mathrm{d}t | x]$ *be equal to*

$$\lambda U_m(x) = \xi_1 V_1(x)[\underline{a}_2 \overline{\kappa}_m - \overline{a}_2 \underline{\kappa}_m] + \xi_2 V_2(x)[\overline{a}_1 \underline{\kappa}_m - \underline{a}_1 \overline{\kappa}_m] + \sum_{i=1}^m i b_{i,m} W_{i-1}(x). \tag{A.67}$$

AUXILIARY LEMMA 5: *The following relationship holds*:

$$\frac{\mathrm{CIR}_m(\delta)}{\delta} = \frac{\mathbb{E}[x^{m+1}]}{\sigma^2} + \frac{\nu}{\sigma^2}\frac{1}{\lambda}\left[\sum_{i=0}^m b_{i,m}\mathbb{E}[x^i] - \mathbb{E}[x^m] + \frac{\big(\mathcal{K}_{2m} - \mathbb{E}[x^m]\mathcal{K}_{20}\big)}{\overline{\mathbb{E}}[\tau]}\right], \tag{A.68}$$

where we define the following constants:

$$\mathcal{K}_{1m} \equiv \mathcal{L}_1[\overline{a}_2 \underline{\kappa}_m - \underline{a}_2 \overline{\kappa}_m] + \mathcal{L}_2[\underline{a}_1 \overline{\kappa}_m - \overline{a}_1 \underline{\kappa}_m] + \sigma^2 \sum_{i=1}^m i W_{i-1}\big(x^*\big) b_{i,m}, \tag{A.69}$$

$$\mathcal{K}_{2m} \equiv \frac{\mathcal{L}_1}{\nu - \sigma^2 \xi_1}[\underline{a}_2 \overline{\kappa}_m - \overline{a}_2 \underline{\kappa}_m] + \frac{\mathcal{L}_2}{\nu - \sigma^2 \xi_2}[\overline{a}_1 \underline{\kappa}_m - \underline{a}_1 \overline{\kappa}_m], \tag{A.70}$$

$$\mathcal{L}_1 \equiv \mathcal{D}\big[-e^{\xi_1 x^*}\big(\underline{a}_1 \overline{a}_2 x^- - \overline{a}_1 \underline{a}_2 x^+\big) - e^{\xi_2 x^*}\underline{a}_1 \overline{a}_1\big(x^+ - x^-\big)\big] - e^{\xi_1 x^*} x^*, \tag{A.71}$$

$$\mathcal{L}_2 \equiv \mathcal{D}\big[-e^{\xi_2 x^*}\big(\overline{a}_2 \underline{a}_1 x^+ - \underline{a}_2 \overline{a}_1 x^-\big) - e^{\xi_1 x^*}\underline{a}_2 \overline{a}_2\big(x^- - x^+\big)\big] - e^{\xi_2 x^*} x^*. \tag{A.72}$$

AUXILIARY LEMMA 6: *The following relationship holds*:

$$\mathbb{C}ov[x^m, a] = \frac{1}{\lambda}\left[\sum_{i=0}^m b_{i,m}\mathbb{E}[x^i] - \mathbb{E}[x^m] + \frac{\mathcal{K}_{2m} - \mathbb{E}[x^m]\mathcal{K}_{20}}{\sigma^2 \overline{\mathbb{E}}[\tau]}\right]. \tag{A.73}$$

Final Step. Subtracting (A.73) from (A.68) finishes the proof of Proposition 1.

A.5. *Proof of Corollary 1*

Evaluate Proposition 1 at $m = 2$.

A.6. *Proofs of Corollary 2*

The proof of Corollary 2 as stated in the main text is trivial, because as the drift goes to infinity ($\nu \to \infty$), firms speed up their adjustments so that expected duration converges to zero ($\overline{\mathbb{E}}[\tau] \to 0$); this implies that the CIR_1 mechanically converges to zero. We show a stronger result: Even after scaling the fixed costs such that expected duration remains strictly positive as the drift goes to infinity ($\overline{\mathbb{E}}[\tau] \to \mathbb{T}^* > 0$), the CIR_1 goes to zero as well. To do this, we show that when the drift goes to infinity, the steady-state cross-sectional moments in the Bernoulli fixed-cost model converge to those generated by a model without idiosyncratic shocks and free adjustment opportunities ($\sigma^2 = \lambda = 0$) and a small discount factor ($\rho \to 0$), as in Caplin and Spulber (1987) (CS from now on).

As a first step, we compute the steady-state moments in the CS model. The policy consists of a one-sided inaction region and the steady-state distribution $F(x)$ is Uniform over the support $[x^-, x^*]$ (note that the upper bound equals the reset point, i.e., $x^+ = x^*$). Therefore, the steady-state variance equals $\mathbb{V}ar[x] = \frac{1}{12}(x^* - x^-) > 0$. To find the covariance, note that gaps are generated by $x_t = x^* - \nu a_t$. Multiplying both sides by x_t and taking expectations, we have that $\mathbb{V}ar[x] = -\nu \mathbb{C}ov[x, a]$ or $\mathbb{C}ov[x, a] = -\frac{1}{12\nu}(x^* - x^-) < 0$.

As a second step, we show that, as $\nu \to \infty$, the original Bernoulli fixed-cost economy collapses to the CS model characterized above. We use the following notation: $\mathbb{T} = \overline{\mathbb{E}}[\tau]$, $\mathbb{V} = \mathbb{V}[x]$, $\mathbb{C} = \mathbb{C}ov[x, a]$, and consider the following vector of six structural parameters (note the inverse of the fixed costs):

$$\mathcal{P} \equiv \left(\rho, \lambda, \sigma^2, \nu, \frac{1}{\theta^-}, \frac{1}{\theta^+}\right). \tag{A.74}$$

Abusing notation, we denote the firm's policy and steady-state moments as functions of $\mathcal{P}$: $\hat{x}^-(\mathcal{P})$, $\hat{x}^*(\mathcal{P})$, and $\hat{x}^+(\mathcal{P})$, $\mathbb{T}(\mathcal{P})$, $\mathbb{V}(\mathcal{P})$, and $\mathbb{C}(\mathcal{P})$. Staring at the HJB equation satisfied by the value function and the policies in (14), the HJB equations satisfied by steady-state moments in (B.2) and (B.16) (in Appendix B), as well as the KFE satisfied by the steady-state distribution in (19), one easily concludes that

(i) $\hat{x}^-(\mathcal{P})$, $\hat{x}^*(\mathcal{P})$, $\hat{x}^+(\mathcal{P})$, and $\mathbb{V}(\mathcal{P})$ are homogeneous of degree 0 in $\mathcal{P}$, for example, $\mathbb{V}(\mathcal{P}) = \mathbb{V}(\alpha\mathcal{P})$.

(ii) $\mathbb{T}(\mathcal{P})$ and $\mathbb{C}(\mathcal{P})$ are homogeneous of degree -1 in $\mathcal{P}$, for example, $\mathbb{T}(\mathcal{P}) = \alpha\mathbb{T}(\alpha\mathcal{P})$.

Rescaling the Fixed Costs. As a third step, we rescale the Bernoulli fixed costs to keep expected duration between 0 and infinity. For any given drift ν and positive number $\mathbb{T}^* > 0$, one can always find fixed costs that yield an expected duration of $\nu\mathbb{T}^*$ in CS. In particular, assuming that fixed costs vary with the drift and satisfy the following implicit function:

$$\lim_{\rho \downarrow 0} \mathbb{T}\underbrace{\left(\rho, 0, 0, 1, \frac{1}{\theta^-(\nu)\nu}, \frac{1}{\theta^+(\nu)\nu}\right)}_{\mathcal{P} \text{ in CS}} = \nu\mathbb{T}^*, \tag{A.75}$$

we obtain that expected duration in the Bernoulli model converges to $\mathbb{T}^*$ as the drift goes to infinity:

$$\begin{aligned}\lim_{\nu\to\infty}\mathbb{T}(\mathcal{P}) &= \lim_{\nu\to\infty}\frac{1}{\nu}\mathbb{T}\left(\frac{\rho}{\nu},\frac{\lambda}{\nu},\frac{\sigma^2}{\nu},1,\frac{1}{\theta^-(\nu)\nu},\frac{1}{\theta^+(\nu)\nu}\right)\\ &= \lim_{\nu\to\infty}\frac{1}{\nu}\lim_{\rho\downarrow 0}\mathbb{T}\left(\rho,0,0,1,\frac{1}{\theta^-(\nu)\nu},\frac{1}{\theta^+(\nu)\nu}\right) = \mathbb{T}^*. \qquad \text{(A.76)}\end{aligned}$$

Convergence of Moments. As a fourth step, we use the homogeneity of degree 0 of $\mathbb{V}$ and the homogeneity of degree -1 of $\mathbb{C}$ to show that the limit of steady-state moments in the original economy converge to those in CS:

$$\begin{aligned}\lim_{\nu\to\infty}\mathbb{V}(\mathcal{P}) &= \lim_{\nu\to\infty}\mathbb{V}\left(\frac{1}{\nu}\mathcal{P}\right) = \lim_{\nu\to\infty}\lim_{\rho\downarrow 0}\mathbb{V}\left(\rho,0,0,1,\frac{1}{\theta^-(\nu)\nu},\frac{1}{\theta^+(\nu)\nu}\right)\\ &= +\frac{1}{12}\left(x^*-x^-\right), \qquad \text{(A.77)}\\ \lim_{\nu\to\infty}\nu\mathbb{C}(\mathcal{P}) &= \lim_{\nu\to\infty}\mathbb{C}\left(\frac{1}{\nu}\mathcal{P}\right) = \lim_{\nu\to\infty}\lim_{\rho\downarrow 0}\mathbb{C}\left(\rho,0,0,1,\frac{1}{\theta^-(\nu)\nu},\frac{1}{\theta^+(\nu)\nu}\right)\\ &= -\frac{1}{12}\left(x^*-x^-\right). \qquad \text{(A.78)}\end{aligned}$$

With all the elements above, we conclude that

$$\lim_{\nu\to\infty,\sigma^2>0}\mathrm{CIR}_1(\delta) = \lim_{\nu\to\infty,\sigma^2>0}\frac{\mathbb{V}(\mathcal{P})+\nu\mathbb{C}(\mathcal{P})}{\sigma^2} = 0,$$

where the limit is taken under the condition $\sigma^2 > 0$ to guarantee the CIR's differentiability.

A.7. *Logic Behind the Mappings From Data to Steady-State Moments*

Before diving into the proofs, we explain the logic of the proofs of Propositions 2 and 3 using a simple case. Let us assume zero drift and zero reset gap, that is, $\nu = x^* = 0$. We focus on the proof that recovers $\mathbb{E}[x^m]$. In the first step, we apply Itō's lemma to x_t^{m+2}:

$$\mathrm{d}x_t^{m+2} = \sigma(m+2)x_t^{m+1}\,\mathrm{d}W_t + \frac{\sigma^2(m+1)(m+2)}{2}x_t^m\,\mathrm{d}t.$$

In the second step, we integrate between two contiguous adjustment dates, $t = 0$ and $t' = \tau$, taking into account that the initial condition is $x_0 = x^* = 0$, and use $x_\tau = -\Delta x$ to obtain

$$\underbrace{(-\Delta x)^{m+2}}_{\text{investment}} = \sigma(m+2)\underbrace{\int_0^\tau x_t^{m+1}\,\mathrm{d}W_t}_{\text{noise}} + \frac{\sigma^2(m+1)(m+2)}{2}\underbrace{\int_0^\tau x_t^m\,\mathrm{d}t.}_{\text{history of gaps}}$$

An investment on the LHS is related to the history of capital gaps plus a noise term on the RHS. While we cannot recover each individual history, we can recover the average history between adjustments. For this, in the third step we take the expectation and note that the noise term $\int_0^t x_t^{m+1}\,\mathrm{d}W_t$ is a martingale with zero initial condition. Thus, by Auxiliary Theorem 1 (Doob's optional sampling theorem)—which establishes that the expected value of a martingale at a stopping time is equal to its initial expected value—its expectation is zero:

$$\underbrace{\overline{\mathbb{E}}\big[(-\Delta x)^{m+2}\big]}_{\text{moments of investment}} = \frac{\sigma^2}{2}(m+1)(m+2) \underbrace{\overline{\mathbb{E}}\left[\int_0^\tau x_t^m\,\mathrm{d}t\right]}_{\text{average gap during inaction}}.$$

The fourth step applies the equivalence between the occupancy measure and cross-sectional distribution in Auxiliary Theorem 2, which allows us to express the average capital gap between adjustments for one agent as the average capital gap across all agents at the stationary distribution. Finally, we use $\sigma^2 = \overline{\mathbb{E}}[\Delta x^2]/\overline{\mathbb{E}}[\tau]$ from (32) and obtain

$$\underbrace{\frac{\overline{\mathbb{E}}\big[(-\Delta x)^{m+2}\big]}{\overline{\mathbb{E}}\big[\Delta x^2\big]}}_{\text{moments of investment}} = \underbrace{\frac{(m+1)(m+2)}{2}\mathbb{E}\big[x^m\big]}_{\text{average gap across agents}}.$$

This expression structurally links observed cross-sectional moments of investment rates with the unobserved moments of cumulative capital gaps. With analogous steps, one obtains the expressions for the parameters of the stochastic process, the reset gap, and other steady-state moments.

A.8. *Proof of Proposition 2*

Let x_t follow a Brownian motion with nonzero drift and reset state x^*. For the zero-drift case, see Appendix K.2.

Drift: Evaluating the law of motion of gaps $x_t = x^* - \nu t + \sigma W_t$ at $t = \tau$, we find the following equalities:

$$\sigma W_\tau = \nu\tau + x_\tau - x^* = \nu\tau - \Delta x. \tag{A.79}$$

Taking expectations on both sides conditional on adjustment, we have $\sigma\overline{\mathbb{E}}[W_\tau] = \nu\overline{\mathbb{E}}[\tau] - \overline{\mathbb{E}}[\Delta x]$. Since W_τ is a martingale, $\overline{\mathbb{E}}[W_\tau] = W_0 = 0$ by the OST. Rearranging, we obtain the result:

$$\nu = \overline{\mathbb{E}}[\Delta x]/\overline{\mathbb{E}}[\tau]. \tag{A.80}$$

Reset state: We exploit the fact that the cross-sectional average of capital gaps is zero, that is, $\mathbb{E}[x] = 0$. We use Auxiliary Theorem 2 to substitute $\mathbb{E}[x] = \int_X x f(x)\,\mathrm{d}x = \overline{\mathbb{E}}[\int_0^\tau x_t\,\mathrm{d}t]/\overline{\mathbb{E}}[\tau]$ and then write law of motion $x_t = x^* - \nu t + \sigma W_t$:

$$\begin{aligned} 0 = \frac{\overline{\mathbb{E}}\left[\int_0^\tau x_t\,\mathrm{d}t\right]}{\overline{\mathbb{E}}[\tau]} &= \frac{\overline{\mathbb{E}}\left[\int_0^\tau \big(x^* - \nu t + \sigma W_t\big)\,\mathrm{d}t\right]}{\overline{\mathbb{E}}[\tau]} \\ &= x^* - \frac{1}{\overline{\mathbb{E}}[\tau]}\left(\overline{\mathbb{E}}\left[\int_0^\tau \nu t\,\mathrm{d}t\right] + \overline{\mathbb{E}}\left[\int_0^\tau \sigma W_t\,\mathrm{d}t\right]\right). \end{aligned} \tag{A.81}$$

We compute each of the integrals in (A.81). The first integral is straightforward:

$$\overline{\mathbb{E}}\left[\int_0^\tau \nu t \, \mathrm{d}t\right] = \frac{\nu}{2}\overline{\mathbb{E}}\left[\tau^2\right]. \tag{A.82}$$

To compute the second integral, we first apply Itō's lemma to $Y_t \equiv \sigma t W_t$; we have $\mathrm{d}Y_t = \sigma(W_t \, \mathrm{d}t + t \, \mathrm{d}W_t)$ or $\sigma W_t \, \mathrm{d}t = \mathrm{d}Y_t - \sigma t \, \mathrm{d}W_t$. Integrate on both sides from 0 to τ and use that $Y_0 = 0$ to obtain $\int_0^\tau \sigma W_t \, \mathrm{d}t = \sigma \tau W_\tau - \int_0^\tau \sigma t \, \mathrm{d}W_t$. Now take expectations with initial condition x^* to get $\overline{\mathbb{E}}[\int_0^\tau \sigma W_t \, \mathrm{d}t] = \overline{\mathbb{E}}[\sigma \tau W \tau] - \overline{\mathbb{E}}[\int_0^\tau \sigma t \, \mathrm{d}W_t]$ and apply the OST so set $\overline{\mathbb{E}}[\int_0^\tau \sigma t \, \mathrm{d}W_t] = 0$ to zero. Finally, we note that $\sigma W_\tau = \nu \tau - \Delta x$. Thus the second integral in (A.81) equals

$$\overline{\mathbb{E}}\left[\int_0^\tau \sigma W_t \, \mathrm{d}t\right] = \overline{\mathbb{E}}[\sigma \tau W \tau] = \overline{\mathbb{E}}\big[\tau(\nu\tau - \Delta x)\big] = \nu \overline{\mathbb{E}}\big[\tau^2\big] - \overline{\mathbb{E}}[\tau \Delta x]. \tag{A.83}$$

Substituting (A.82) and (A.83) into (A.81):

$$\begin{aligned} x^* &= \frac{1}{\overline{\mathbb{E}}[\tau]}\left(\overline{\mathbb{E}}\left[\int_0^\tau \nu t \, \mathrm{d}t\right] - \overline{\mathbb{E}}\left[\int_0^\tau \sigma W_t \, \mathrm{d}t\right]\right) \\ &= \frac{\nu}{2}\frac{\overline{\mathbb{E}}\big[\tau^2\big]}{\overline{\mathbb{E}}[\tau]} - \left(\frac{\nu\overline{\mathbb{E}}\big[\tau^2\big] - \overline{\mathbb{E}}[\tau\Delta x]}{\overline{\mathbb{E}}[\tau]}\right) \\ &= \frac{\overline{\mathbb{E}}[\tau\Delta x]}{\overline{\mathbb{E}}[\tau]} - \frac{\nu}{2}\frac{\overline{\mathbb{E}}\big[\tau^2\big]}{\overline{\mathbb{E}}[\tau]}. \end{aligned} \tag{A.84}$$

To get the alternative expression for x^* in (33), we rewrite the first term in (A.84) as a covariance and use the expression for the drift: $\overline{\mathbb{E}}[\tau\Delta x]/\overline{\mathbb{E}}[\tau] = \overline{\mathbb{C}ov}[\tau, \Delta x]/\overline{\mathbb{E}}[\tau] + \nu\overline{\mathbb{E}}[\tau]$; then multiply and divide the second term in (A.84) by $\overline{\mathbb{E}}[\tau]$ and rewrite it as the coefficient of variation squared: $\frac{\overline{\mathbb{E}}[\tau^2]}{\overline{\mathbb{E}}[\tau]^2} = 1 + \overline{\mathbb{CV}}^2[\tau]$. Substituting these expressions and rearranging:

$$x^* = \frac{\overline{\mathbb{C}ov}[\tau, \Delta x]}{\overline{\mathbb{E}}[\tau]} + \nu\overline{\mathbb{E}}[\tau] - \nu \underbrace{\frac{\overline{\mathbb{E}}[\tau]}{2}\left(1 + \overline{\mathbb{CV}}^2[\tau]\right)}_{\mathbb{E}[a]} = \frac{\overline{\mathbb{C}ov}[\tau, \Delta x]}{\overline{\mathbb{E}}[\tau]} + \nu\left(\overline{\mathbb{E}}[\tau] - \mathbb{E}[a]\right), \tag{A.85}$$

where we recognize the expression for average age $\mathbb{E}[a] = (\overline{\mathbb{E}}[\tau]/2)(1 + \overline{\mathbb{CV}}^2[\tau])$.

Idiosyncratic volatility: We apply Itô's lemma to x_t^2 to obtain

$$\mathrm{d}x_t^2 = 2x_t \, \mathrm{d}x_t + (\mathrm{d}x_t)^2 = \left(-2\nu x_t + \sigma^2\right) \mathrm{d}t + 2\sigma x_t \, \mathrm{d}W_t. \tag{A.86}$$

We integrate both sides from 0 to τ, and take expectations with respect to $x_0 = x^*$. Then use the OST to set $\overline{\mathbb{E}}[\int_0^\tau x_s dW_s] = 0$ and Auxiliary Theorem 2 to set $\overline{\mathbb{E}}[\int_0^\tau x_s \, \mathrm{d}s] = \mathbb{E}[x]\overline{\mathbb{E}}[\tau] = 0$ and obtain

$$\overline{\mathbb{E}}\big[x_\tau^2\big] - x^{*2} = -2\nu\overline{\mathbb{E}}\left[\int_0^\tau x_s \, \mathrm{d}s\right] + \sigma^2\overline{\mathbb{E}}[\tau] = \sigma^2\overline{\mathbb{E}}[\tau]. \tag{A.87}$$

Since $x_\tau = x^* - \Delta x$, then $\overline{\mathbb{E}}[x_\tau^2] = \overline{\mathbb{E}}[\Delta x^2] - 2x^*\overline{\mathbb{E}}[\Delta x] + x^{*2}$. Thus (A.87) becomes

$$\overline{\mathbb{E}}\big[\Delta x^2\big] - 2x^*\overline{\mathbb{E}}[\Delta x] = \sigma^2\overline{\mathbb{E}}[\tau]. \tag{A.88}$$

Solving for σ^2 and substituting the drift ν in (A.80), we obtain the result:

$$\sigma^2 = \frac{\overline{\mathbb{E}}[\Delta x^2] - 2x^*\overline{\mathbb{E}}[\Delta x]}{\overline{\mathbb{E}}[\tau]} = \frac{\overline{\mathbb{E}}[\Delta x^2]}{\overline{\mathbb{E}}[\tau]} - 2\nu x^*. \tag{A.89}$$

A.9. *Proof of Proposition 3*

Let x_t follow a Brownian motion with nonzero drift and reset state x^*. For the zero-drift case, see Appendix K.2.

Steady-state moments of x: Apply Itô's lemma to x^{m+1}:

$$\mathrm{d}x_t^{m+1} = -(m+1)x_t^m \nu\,\mathrm{d}t + (m+1)x_t^m \sigma\,\mathrm{d}W_t + \frac{\sigma^2}{2}m(m+1)x_t^{m-1}\,\mathrm{d}t. \tag{A.90}$$

Integrating from 0 to τ with initial condition x^*, using the OST to eliminate martingales, and rearranging:

$$\overline{\mathbb{E}}\left[\int_0^\tau x_t^m\,\mathrm{d}t\right] = \frac{1}{\nu(m+1)}\left(x^{*m+1} - \overline{\mathbb{E}}\left[x_\tau^{m+1}\right]\right) + \frac{\sigma^2 m}{2\nu}\overline{\mathbb{E}}\left[\int_0^\tau x_t^{m-1}\,\mathrm{d}t\right]. \tag{A.91}$$

Divide by $\overline{\mathbb{E}}[\tau]$ and substitute $\mathbb{E}[x^m] = \overline{\mathbb{E}}[\int_0^\tau x_t^m\,\mathrm{d}t]/\overline{\mathbb{E}}[\tau]$ and $\overline{\mathbb{E}}[\Delta x] = \nu\overline{\mathbb{E}}[\tau]$:

$$\mathbb{E}\left[x^m\right] = \frac{1}{m+1}\left(\frac{x^{*m+1} - \overline{\mathbb{E}}\left[\left(x^* - \Delta x\right)^{m+1}\right]}{\overline{\mathbb{E}}[\Delta x]}\right) + \frac{\sigma^2 m}{2\nu}\mathbb{E}\left[x^{m-1}\right]. \tag{A.92}$$

Joint steady-state moments of x and age: Apply Itō's lemma to $Y_t \equiv x_t^{m+1}t$:

$$\mathrm{d}Y_t = x_t^{m+1}\,\mathrm{d}t - (m+1)\nu x_t^m t\,\mathrm{d}t + (m+1)x_t^m\sigma\,\mathrm{d}W_t + \frac{\sigma^2 m(m+1)}{2}x_t^{m-1}t\,\mathrm{d}t. \tag{A.93}$$

Integrating from 0 to τ with initial condition x^*, using the OST to eliminate martingales:

$$\begin{aligned}\overline{\mathbb{E}}\left[\tau\left(x^* - \Delta x\right)^{m+1}\right] = {} & \overline{\mathbb{E}}\left[\int_0^\tau x_t^{m+1}\,\mathrm{d}t\right] - (m+1)\nu\overline{\mathbb{E}}\left[\int_0^\tau x_t^m t\,\mathrm{d}t\right] \\ & + \frac{\sigma^2 m(m+1)}{2}\overline{\mathbb{E}}\left[\int_0^\tau x_t^{m-1}t\,\mathrm{d}t\right].\end{aligned} \tag{A.94}$$

Divide by $\overline{\mathbb{E}}[\tau]$. Using the occupancy measure, solving for $\mathbb{E}[x^m a]$, substituting $\nu = \mathbb{E}[\Delta x]/\overline{\mathbb{E}}[\tau]$, we get the following recursion:

$$\begin{aligned}\mathbb{E}\left[x^m a\right] &= \frac{\mathbb{E}\left[x^{m+1}\right]}{\nu(m+1)} - \frac{\overline{\mathbb{E}}\left[\tau\left(x^* - \Delta x\right)^{m+1}\right]}{\nu(m+1)\overline{\mathbb{E}}[\tau]} + \frac{\sigma^2 m}{2\nu}\mathbb{E}\left[x^{m-1}a\right] \\ &= \frac{\overline{\mathbb{E}}[\tau]}{m+1}\left(\frac{\mathbb{E}\left[x^{m+1}\right]}{\overline{\mathbb{E}}[\Delta x]} - \frac{\overline{\mathbb{E}}\left[\tau\left(x^* - \Delta x\right)^{m+1}\right]}{\overline{\mathbb{E}}[\Delta x]\overline{\mathbb{E}}[\tau]}\right) + \frac{\sigma^2 m}{2\nu}\mathbb{E}\left[x^{m-1}a\right] \\ &= \frac{\overline{\mathbb{E}}[\tau]}{m+1}\left(\frac{\mathbb{E}\left[x^{m+1}\right] - \overline{\mathbb{E}}\left[\tilde{\tau}\left(x^* - \Delta x\right)^{m+1}\right]}{\overline{\mathbb{E}}[\Delta x]}\right) + \frac{\sigma^2 m}{2\nu}\mathbb{E}\left[x^{m-1}a\right].\end{aligned} \tag{A.95}$$

A.10. *Proof of Corollary 3*

We express the sufficient statistics for the CIR_1—$\mathbb{V}ar[x]$ and $\mathbb{C}ov[x, a]$—as moments of $(\Delta x, \tau)$. The expressions assume the reset point is zero, that is, $x^* = 0$, so that $x_\tau = -\Delta x$. We recover these moments for the case with and without drift. Recall that $\mathbb{E}[x] = 0$ by definition of gaps. We define the generalized coefficient of variation as $\overline{\mathbb{CV}}^\psi[y] \equiv \overline{\mathbb{E}}[y^\psi]/\overline{\mathbb{E}}[y]^\psi - 1$.

Zero drift. To obtain $\mathbb{V}ar[x] = \mathbb{E}[x^2]$, evaluate (35) at $m = 2$. Multiply and divide by $\overline{\mathbb{E}}[\Delta x^2]$ and substitute the generalized coefficient of variation for $y = \Delta x^2$ and $\psi = 2$:

$$\mathbb{V}ar[x] = \frac{1}{6}\frac{\overline{\mathbb{E}}[\Delta x^4]}{\overline{\mathbb{E}}[\Delta x^2]} = \frac{1}{6}\overline{\mathbb{E}}[\Delta x^2]\frac{\overline{\mathbb{E}}[\Delta x^4]}{\overline{\mathbb{E}}[\Delta x^2]^2} = \frac{1}{6}\overline{\mathbb{E}}[\Delta x^2]\left(1 + \overline{\mathbb{CV}}^2[\Delta x^2]\right). \tag{A.96}$$

To obtain $\mathbb{C}ov[x, a] = \mathbb{E}[xa]$, evaluate (36) at $m = 1$. Multiply and divide by $\overline{\mathbb{E}}[\tau]$, substitute $\sigma^2\overline{\mathbb{E}}[\tau] = \overline{\mathbb{E}}[\Delta x^2]$ from (32), and define $\tilde{\tau} \equiv \tau/\overline{\mathbb{E}}[\tau]$:

$$\begin{aligned}
\mathbb{C}ov[x, a] &= \frac{1}{3}\left(\frac{\overline{\mathbb{E}}[\tau x_\tau^3]}{\overline{\mathbb{E}}[\Delta x^2]} - \frac{\mathbb{E}[x^3]}{\sigma^2}\right) \\
&= \frac{\overline{\mathbb{E}}[\tau]}{3}\left(\frac{\overline{\mathbb{E}}[(\tau/\overline{\mathbb{E}}[\tau])x_\tau^3]}{\overline{\mathbb{E}}[\Delta x^2]} - \frac{\mathbb{E}[x^3]}{\sigma^2\overline{\mathbb{E}}[\tau]}\right) \\
&= \frac{\overline{\mathbb{E}}[\tau]}{3}\left(\frac{\overline{\mathbb{E}}[\tilde{\tau}x_\tau^3] - \mathbb{E}[x^3]}{\overline{\mathbb{E}}[\Delta x^2]}\right).
\end{aligned} \tag{A.97}$$

Nonzero drift. To obtain $\mathbb{V}ar[x] = \mathbb{E}[x^2]$, evaluate (37) at $m = 2$. Multiply and divide by $\overline{\mathbb{E}}[\Delta x]^2$ and substitute the generalized coefficient of variation for $y = \Delta x$ and $\psi = 3$:

$$\mathbb{V}ar[x] = \frac{\overline{\mathbb{E}}[\Delta x^3]}{3\overline{\mathbb{E}}[\Delta x]} = \frac{1}{3}\overline{\mathbb{E}}[\Delta x]^2\left(1 + \overline{\mathbb{CV}}^3[\Delta x]\right). \tag{A.98}$$

To obtain $\mathbb{C}ov[x, a] = \mathbb{E}[xa]$, evaluate (38) at $m = 1$. Substitute $\sigma^2 = \overline{\mathbb{E}}[\Delta x^2]/\overline{\mathbb{E}}[\tau]$ from (32) and define $\tilde{\tau} \equiv \tau/\overline{\mathbb{E}}[\tau]$. Then substitute $\nu = \overline{\mathbb{E}}[\Delta x]/\overline{\mathbb{E}}[\tau]$ from (31):

$$\begin{aligned}
\mathbb{C}ov[x, a] &= \frac{\mathbb{E}[x^2]}{2\nu} - \frac{\sigma^2}{2\nu}\frac{\overline{\mathbb{E}}[\tau x_\tau^2]}{\overline{\mathbb{E}}[\Delta x^2]} + \frac{\sigma^2}{2\nu}\mathbb{E}[a] \\
&= \frac{\mathbb{E}[x^2]}{2\nu} - \frac{\overline{\mathbb{E}}[(\tau/\overline{\mathbb{E}}[\tau])x_\tau^2]}{2\nu} + \frac{\sigma^2}{2\nu}\mathbb{E}[a] \\
&= \frac{\overline{\mathbb{E}}[\tau]}{2}\left(\frac{\mathbb{E}[x^2] - \overline{\mathbb{E}}[\tilde{\tau}x_\tau^2]}{\overline{\mathbb{E}}[\Delta x]}\right) + \frac{\sigma^2}{2\nu}\mathbb{E}[a].
\end{aligned} \tag{A.99}$$

A.11. *Proof of Corollary 4*

Assume the reset point is zero, that is, $x^* = 0$, so that $x_\tau = -\Delta x$.

Zero drift. Since $\nu = x^* = 0$, the CIR_1 is equal to the ratio of cross-sectional variance $\mathbb{V}ar[x]$ to idiosyncratic volatility σ^2. Substitute their expressions and the definition of kurtosis $\overline{\mathbb{K}ur}[y] = \overline{\mathbb{E}}[y^4]/\overline{\mathbb{E}}[y]^2$:

$$\begin{aligned}\frac{\text{CIR}_1(\delta)}{\delta} &= \frac{1}{\sigma^2}\mathbb{V}ar[x] + o(\delta) = \frac{\overline{\mathbb{E}}[\tau]}{\overline{\mathbb{E}}[\Delta x^2]}\frac{1}{6}\frac{\overline{\mathbb{E}}[\Delta x^4]}{\overline{\mathbb{E}}[\Delta x^2]} + o(\delta) \\ &= \frac{\overline{\mathbb{E}}[\tau]}{2}\frac{\overline{\mathbb{K}ur}[\Delta x]}{3} + o(\delta). \end{aligned} \tag{A.100}$$

Nonzero drift and asymmetric policy. The CIR_1 is equal to the convex combination of $\mathbb{V}ar[x]$ and $\mathbb{C}ov[x, a]$, with weights $1/\sigma^2$ and ν/σ^2. First, note the following expression derived from Proposition 3 by substituting $\sigma^2 = \overline{\mathbb{E}}[\Delta x^2]/\overline{\mathbb{E}}[\tau]$:

$$\frac{\nu}{\sigma^2}\mathbb{C}ov[x, a] = \frac{1}{2}\left(\frac{\mathbb{V}ar[x]}{\sigma^2} + \mathbb{E}[a] - \frac{\overline{\mathbb{E}}[\tau x_\tau^2]}{\overline{\mathbb{E}}[\Delta x^2]}\right). \tag{A.101}$$

Substituting the previous expression into the CIR_1, and after some manipulations:

$$\begin{aligned}&\frac{\text{CIR}_1(\delta)}{\delta} \\ &= \frac{\mathbb{V}ar[x]}{\sigma^2} + \frac{\nu}{\sigma^2}\mathbb{C}ov[x, a] + o(\delta) \\ &= \frac{\mathbb{V}ar[x]}{\sigma^2} + \frac{1}{2}\left(\frac{\mathbb{V}ar[x]}{\sigma^2} + \mathbb{E}[a] - \frac{\overline{\mathbb{E}}[\tau x_\tau^2]}{\overline{\mathbb{E}}[\Delta x^2]}\right) + o(\delta) \\ &= \frac{1}{2}\left(\frac{\overline{\mathbb{E}}[\Delta x^3]}{\overline{\mathbb{E}}[\Delta x]\sigma^2} + \mathbb{E}[a] - \frac{\overline{\mathbb{E}}[\tau x_\tau^2]}{\overline{\mathbb{E}}[\Delta x^2]}\right) + o(\delta) \\ &= \frac{\overline{\mathbb{E}}[\tau]}{2}\left(\frac{\overline{\mathbb{E}}[\Delta x^3]}{\overline{\mathbb{E}}[\Delta x]\overline{\mathbb{E}}[\Delta x^2]} + \frac{\mathbb{E}[a]}{\overline{\mathbb{E}}[\tau]} - \frac{\overline{\mathbb{E}}[\tau x_\tau^2]}{\overline{\mathbb{E}}[\Delta x^2]\overline{\mathbb{E}}[\tau]}\right) + o(\delta) \\ &= \frac{\overline{\mathbb{E}}[\tau]}{2}\left(\frac{\overline{\mathbb{E}}\left[\left(\Delta x/\overline{\mathbb{E}}[\Delta x]\right)^3\right]}{\overline{\mathbb{E}}\left[\left(\Delta x/\overline{\mathbb{E}}[\Delta x]\right)^2\right]} + \frac{\mathbb{C}\mathbb{V}^2[\tau] + 1}{2} - \frac{\overline{\mathbb{C}ov}[\tau, x_\tau^2]}{\overline{\mathbb{E}}[\tau]\overline{\mathbb{E}}[\Delta x^2]} - 1\right) + o(\delta). \end{aligned} \tag{A.102}$$

Letting $\widetilde{\Delta x} \equiv \Delta x/\overline{\mathbb{E}}[\Delta x]$, we get the result:

$$\frac{\text{CIR}_1(\delta)}{\delta} = \frac{\overline{\mathbb{E}}[\tau]}{2}\left[\frac{\overline{\mathbb{C}\mathbb{V}}^2[\tau] - 1}{2} + \frac{\overline{\mathbb{E}}[\widetilde{\Delta x}^3]}{\overline{\mathbb{E}}[\widetilde{\Delta x}^2]} - \overline{\mathbb{C}ov}[\tilde{\tau}, \widetilde{\Delta x}^2]\right] + o(\delta). \tag{A.103}$$

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