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SUPPLEMENT TO "INDIVIDUAL HETEROGENEITY AND AVERAGE WELFARE"

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BY JERRY A. HAUSMAN AND WHITNEY K. NEWEY

IN THIS SUPPLEMENTAL MATERIAL, we give proofs of the results in the paper along with supplementary results deriving surplus bounds for discrete and continuous choice, some generalized conditions for the bounds under some knowledge of income effects, results on the potential size of income effects in the gasoline demand application, and additional details on the general bounds application. Assumptions, lemmas, and theorems specific to this Supplemental Material are listed with an "A" prefix, for example, Assumption A1, Lemma A1, Theorem A1, etc.

A1. PROOFS OF THEOREMS IN THE PAPER

The following two technical conditions are referred to in the text and used in the proofs.

ASSUMPTION A1: *η belongs to a complete, separable metric space and $q(x, \eta)$ and $\partial q(x, \eta)/\partial x$ are continuous in (x, η).*

ASSUMPTION A2: *$\eta = (u, \varepsilon)$ for scalar ε and Assumption A1 is satisfied for $\eta = (u, \varepsilon)$ for a complete, separable metric space that is the product of a complete separable metric space for u with Euclidean space for ε, $q(x, \eta) = q(x, u, \varepsilon)$ is continuously differentiable in ε, there is $C > 0$ with $\partial q(x, u, \varepsilon)/\partial \varepsilon \geq 1/C$, $\|\partial q(x, \eta)/\partial x\| \leq C$ everywhere, ε_i is continuously distributed conditional on u_i, with conditional p.d.f. $f_\varepsilon(\varepsilon|u)$ that is bounded and continuous in ε.*

Before proving Theorem 1, we give a result on the derivatives of the quantile with respect to x.

LEMMA A1: *If Assumptions 1 and A2 are satisfied, then $q(x, \eta_i)$ is continuously distributed for each $x \in \chi$ and $\Pr(q(x, \eta_i) \leq r)$ and $Q(\tau|x)$ are continuously differentiable in r and x, and for the p.d.f. $f_q(r)$ of $q(x, \eta_i)$ at r,*

$$\frac{\partial \Pr\big(q(x, \eta_i) \leq r\big)}{\partial x} = -f_q(r) E\left[\frac{\partial q(x, \eta_i)}{\partial x}\middle| q(x, \eta_i) = r\right],$$

$$\frac{\partial Q(\tau|x)}{\partial x} = E\left[\frac{\partial q(x, \eta_i)}{\partial x}\middle| q(x, \eta_i) = Q(\tau|x)\right].$$

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PROOF: Let $F_\varepsilon(e|u) = \Pr(\varepsilon \le e|u) = \int_{-\infty}^{e} f_\varepsilon(r|u)\,dr$. Then by the fundamental theorem of calculus, $F_\varepsilon(e|\eta)$ is differentiable in e, and the derivative $f_\varepsilon(e|u)$ is continuous in e by hypothesis. Let $q^{-1}(x, u, r)$ denote the inverse function of $q(x, u, \varepsilon)$ as a function of ε. Then

$$\Pr\big(q(x, \eta_i) \le r\big) = E\big[1\big(\varepsilon_i \le q^{-1}(x, u_i, r)\big)\big] = E\big[F_\varepsilon\big(q^{-1}(x, u_i, r)|u_i\big)\big].$$

By the inverse function theorem, $q^{-1}(x, u, r)$ is continuously differentiable in x and r, with

$$\partial q^{-1}(x, u, r)/\partial r = \big[\partial q\big(x, u, q^{-1}(x, u, r)\big)/\partial\varepsilon\big]^{-1},$$
$$\partial q^{-1}(x, u, r)/\partial x = -\frac{\partial q\big(x, u, q^{-1}(x, u, r)\big)/\partial x}{\partial q\big(x, u, q^{-1}(x, u, r)\big)/\partial\varepsilon}.$$

By Assumption A2, both $\partial q^{-1}(x, u, r)/\partial r$ and $\partial q^{-1}(x, u, r)/\partial x$ are bounded. Then by the chain rule, $F(q^{-1}(x, u_i, r)|u_i)$ is differentiable in r and x with bounded continuous derivatives, so that $E[F_\varepsilon(q^{-1}(x, u_i, r)|u_i)]$ is differentiable in r and x with

$$\begin{aligned}
&\frac{\partial E\big[F_\varepsilon\big(q^{-1}(x, u_i, r)|u_i\big)\big]}{\partial r} \\
&\quad = E\big[f_\varepsilon\big(q^{-1}(x, u_i, r)|u_i\big)\big\{\partial q\big(x, u, q^{-1}(x, u_i, r)\big)/\partial\varepsilon\big\}^{-1}\big] \\
&\quad = E\big[f_r(r|u_i)\big] = f_r(r),
\end{aligned}$$

where $f_r(r)$ and $f_r(r|u)$ are the marginal and conditional p.d.f. of $q(x, \eta_i)$, respectively, and the second equality follows by the change of variables $r = q(x, u_i, \varepsilon_i)$. Similarly,

$$\begin{aligned}
&\frac{\partial E\big[F_\varepsilon\big(q^{-1}(x, u_i, r)|u_i\big)\big]}{\partial x} \\
&\quad = -E\left[f_\varepsilon\big(q^{-1}(x, u_i, r)|u_i\big)\frac{\partial q\big(x, u_i, q^{-1}(x, u_i, r)\big)/\partial x}{\partial q\big(x, u_i, q^{-1}(x, u_i, r)\big)/\partial\varepsilon}\right] \\
&\quad = -\int f_{r,u}(r, u)\left[\frac{\partial q\big(x, u, q^{-1}(x, u, r)\big)}{\partial x}\right] d\mu(u) \\
&\quad = -f_r(r)E\left[\frac{\partial q(x, \eta_i)}{\partial x}\,\middle|\, q(x, \eta_i) = r\right],
\end{aligned}$$

where $f_{r,u}(r, u)$ is a joint p.d.f. with respect to the product of Lebesgue measure and a dominating measure μ for u_i, and the last equality follows by multiplying and dividing by $f_r(r)$. This result gives the first conclusion. The second conclusion follows by the inverse function theorem. *Q.E.D.*

PROOF OF THEOREM 1: Since $q(x, \eta)$ satisfies Assumption 1, we have $\partial q(x, \eta_i)/\partial p + q(x, \eta_i)\partial q(x, \eta_i)/\partial y \leq 0$ for all η_i. Therefore, following Dette, Hoderlein, and Neumeyer (2011), we have by Lemma A1 that, for each τ with $0 < \tau < 1$, the quantile $Q(\tau|x)$ is continuously differentiable in x and

$$\begin{aligned}
&\frac{\partial Q(\tau|x)}{\partial p} + Q(\tau|x)\frac{\partial Q(\tau|x)}{\partial y} \\
&\quad = E\left[\frac{\partial q(x, \eta_i)}{\partial p} + Q(\tau|x)\frac{\partial q(x, \eta_i)}{\partial y}\middle| q(x, \eta_i) = Q(\tau|x)\right] \\
&\quad = E\left[\frac{\partial q(x, \eta_i)}{\partial p} + q(x, \eta_i)\frac{\partial q(x, \eta_i)}{\partial y}\middle| q(x, \eta_i) = Q(\tau|x)\right] \leq 0.
\end{aligned}$$

It is well known that, with two goods and a continuously differentiable demand function, the Slutzky condition for the non numeraire good suffices for the function to be a demand function for $p > 0$, $y > 0$, giving the first conclusion.

For the second conclusion, note that $\tilde{q}(x, \tilde{\eta})$ satisfies Assumption 1 for $0 < \tilde{\eta} < 1$ by hypothesis. Also, $Q(\tau|x) \leq r$ if and only if $\tau \leq F(r|x, q, G)$ by the definition of $Q(\tau|x)$ and the properties of $F(r|x, q, G)$ as a function of r. To show this, suppress the x, q, G arguments in F and Q. Note that by the definition $Q(\tau) = \inf\{\tilde{r} : F(\tilde{r}) \geq \tau\}$, we have $Q(\tau) > r$ implies $\tau > F(r)$. Now suppose $\tau > F(r)$. By $F(r)$ continuous from the right, there is $\varepsilon > 0$ such that $F(\tilde{r}) < \tau$ for $\tilde{r} \in [r, r+\varepsilon)$. Also, by $F(r)$ monotonic increasing, $F(\tilde{r}) \geq \tau$ implies $\tilde{r} \geq r + \varepsilon$. Therefore, $Q(\tau) > r$. It follows that $Q(\tau) > r$ if and only if $\tau > F(r)$. This also implies its contrapositive, $Q(\tau) \leq r$ if and only if $\tau \leq F(r)$. It then follows that

$$\begin{aligned}
\int 1\big(\tilde{q}(x, \tilde{\eta}) \leq r\big)\tilde{G}(d\tilde{\eta}) &= \int_0^1 1\big(Q(\tilde{\eta}|x) \leq r\big)\, d\tilde{\eta} \\
&= \int_0^1 1\big(\tilde{\eta} \leq F(r|x, q, G)\big)\, d\tilde{\eta} \\
&= F(r|x, q, G). \qquad\qquad \textit{Q.E.D.}
\end{aligned}$$

PROOF OF THEOREM 2: In this proof, we proceed by calculating the true average surplus and the quantile average surplus and finding that they are numerically different for the specification given in the statement of Theorem 2. We first consider the true average surplus for the demand specification

$$\begin{aligned}
&q(p, y, \eta) = \eta_1 - p + \eta_2 y, \quad \eta_1 \sim U(0, 1), \\
&\Pr(\eta_2 = 1/3) = \Pr(\eta_3 = 2/3) = 1/2,
\end{aligned}$$

for a price change with $p^0 = 0.1$, $p^1 = 0.2$, and $\bar{y} = 3/4$. Note that for all $\eta_1 \in [0, 1]$, $\eta_2 \in \{1/3, 2/3\}$, and $p \in [0.1, 0.2]$, we have

$$\eta_1 - p + \eta_2 \bar{y} \geq 0, \quad p(\eta_1 - p + \eta_2 \bar{y}) \leq \bar{y},$$
$$-1 + \eta_2(\eta_1 - p + \eta_2 \bar{y}) < 0,$$

so that over the range of η and p, we consider demand is positive, within the budget constraint, and satisfies the Slutzky condition.

Next, for a linear demand function (which has constant income effect η_3) and two goods, we have

$$\begin{aligned} S(\eta) &= \Delta p \int_0^1 q\big(p^0 + t\Delta p, \bar{y}, \eta\big) \exp(-t\eta_3 \Delta p) \\ &= \int_0^1 (A + Bt) \exp(-Ct)\, dt \\ &= \frac{A}{-C}\big[\exp(-Ct)\big]_0^1 + \frac{B}{-C}\big[t \exp(-Ct)\big]_0^1 + \frac{B}{-C^2}\big[\exp(-Ct)\big]_0^1 \\ &= \frac{A}{C} + \frac{B}{C^2} - e^{-C}\left(\frac{A}{C} + \frac{B}{C} + \frac{B}{C^2}\right), \end{aligned}$$
$$A = \Delta p\big(\eta_1 + \eta_2 p^0 + \eta_3 \bar{y}\big), \quad B = (\Delta p)^2 \eta_2, C = \eta_3 \Delta p.$$

Note that A and B are linear in η_1 and η_2. Assuming that (η_1, η_2) is independent of η_3 gives

$$\int S(\eta) G(d\eta_1, d\eta_2, \eta_3) = \frac{\bar{A}}{C} + \frac{\bar{B}}{C^2} - e^{-C}\left(\frac{\bar{A}}{C} + \frac{\bar{B}}{C} + \frac{\bar{B}}{C^2}\right),$$
$$\bar{A} = \Delta p\big(\bar{\eta}_1 + \bar{\eta}_2 p^0 + \eta_3 \bar{y}\big), \quad \bar{B} = (\Delta p)^2 \bar{\eta}_2.$$

For $p^0 = 0.1$, $p^1 = 0.2$, $\bar{y} = 3/4$, $\bar{\eta}_1 = 1/2$, $\bar{\eta}_2 = -1$ and for η_3 equal to 1/3 or 2/3 with probability 0.5, we find that

$$\bar{S} = 0.070673.$$

Next, we derive the quantile demand and then calculate average surplus. For clarity, we do this when $\eta_1 \sim U(0, 1)$, η_2 is a constant β, and when $\eta_3 = \underline{b}$ with probability π and $\eta_3 = \bar{b}$ with probability $(1 - \pi)$, where $\bar{b} > \underline{b}$ and η_3 is independent of η_1. Define

$$r_0 = \beta p + \underline{b} y, \quad r_1 = \beta p + \bar{b} y,$$
$$r_2 = \beta p + 1 + \underline{b} y, \quad r_3 = \beta p + 1 + \bar{b} y.$$

Assuming that $1 + \underline{b}y > \bar{b}y$, we have that

$$
\begin{aligned}
F(r|x) &= \Pr\bigl(q(x, \eta) \le r\bigr) \\
&= \Pr(\eta_1 + \beta p + \eta_3 y \le r) \\
&= \begin{cases} 0, & r < r_0, \\ \pi(r - r_0), & r_0 \le r < r_1, \\ (r - r_1) + F(r_1|x), & r_1 \le r < r_2, \\ (1-\pi)(r - r_2) + F(r_2|x), & r_2 \le r < r_3, \\ 1, & r \ge r_3. \end{cases}
\end{aligned}
$$

Note that this CDF is a mixture, over two values of η_3, of two CDFs for a $U(0,1)$. It has slope π or $1-\pi$ over the ranges where only one CDF is increasing, and slope 1 where both are increasing. Inverting this function as a function of r gives the corresponding quantile function

$$
Q(\tau|p, y) = \begin{cases} r_0 + \dfrac{\tau}{\pi}, \quad 0 < \tau \le \pi(r_1 - r_0), \\ r_1 + \tau - \pi(r_1 - r_0), \\ \qquad\qquad \pi(r_1 - r_0) < \tau < r_2 - r_1 + \pi(r_1 - r_0), \\ r_2 + \dfrac{\tau - \bigl[r_2 - r_1 + \pi(r_1 - r_0)\bigr]}{1-\pi}, \\ \qquad\qquad r_2 - r_1 + \pi(r_1 - r_0) \le \tau < 1. \end{cases}
$$

In terms of the original parameters, the quantile function is given by

$$
Q(\tau|p, y) = \begin{cases} \beta p + \underline{b}y + \dfrac{\tau}{\pi}, \quad 0 < \tau \le \pi(\bar{b} - \underline{b})y, \\ \beta p + \bar{b}y + \tau - \pi(\bar{b} - \underline{b})y, \\ \qquad\qquad \pi(\bar{b} - \underline{b})y < \tau < 1 - (1-\pi)(\bar{b} - \underline{b})y, \\ 1 + \beta p + \underline{b}y + \dfrac{\tau - \bigl[1 - (1-\pi)(\bar{b} - \underline{b})y\bigr]}{1-\pi}, \\ \qquad\qquad 1 - (1-\pi)(\bar{b} - \underline{b})y \le \tau < 1. \end{cases}
$$

Plugging in $\beta = -1$, $\pi = 0.5$, $\underline{b} = 1/3$, and $\bar{b} = 2/3$, we obtain the quantile demand implied by the true model, equaling

$$
Q(\tau|p, y) = \begin{cases} -p + y/3 + 2\tau, & 0 < \tau \le y/6, \\ -p + y/2 + \tau, & y/6 < \tau < 1 - y/6, \\ -p + 2y/3 + 2\tau - 1 & 1 - y/6 \le \tau < 1. \end{cases}
$$

We can rewrite this as a function of y for given $\tau = \tilde{\eta}$ as

$$
\tilde{q}(p, y, \tilde{\eta}) = \begin{cases} -p + 1(y < 6\tilde{\eta})(y/2 + \tilde{\eta}) + 1(y \ge 6\tilde{\eta})(y/3 + 2\tilde{\eta}), \\ \qquad\qquad\qquad\qquad \tilde{\eta} \le 1/2, \\ -p + 1\bigl(y < 6(1-\tilde{\eta})\bigr)(y/2 + \tilde{\eta}) \\ \quad + 1\bigl(y \ge 6(1-\tilde{\eta})\bigr)(2y/3 + 2\tilde{\eta} - 1), \quad \tilde{\eta} > 1/2, \end{cases}
$$

where we have used the fact that we only need to evaluate this demand where $y < 3$. This is the quantile demand function, which is observationally equivalent to the true demand by construction. It is nonlinear in y, with an income effect that varies as y crosses over a threshold.

Note that $y \leq 3/4$ is in the income range relevant for our calculation. When $\tilde{\eta} \in [1/8, 7/8]$, the demand function will be linear income over the evaluation range for the consumer surplus calculation, with income effect equal to $1/2$. For smaller $\tilde{\eta}$ or values of $\tilde{\eta}$ closer to 1, the income effect can change with y. The mix of nonlinearities that is evident in the comparison of this complicated demand function with the simple true linear, varying coefficients specification results in the quantile average surplus being different from the true average surplus.

Because the demand function is nonlinear in y for $\tilde{\eta} \notin [1/8, 7/8]$, we compute surplus numerically for each value of $\tilde{\eta}$ and then average. We do this by drawing 50,000 values of $\tilde{\eta}$ from a $U(0, 1)$, computing the equivalent variation from a price change with $p^0 = 0.1$, $p^1 = 0.2$, $\bar{y} = 3/4$, $\pi = 1/2$, $\underline{b} = 1/3$, $\bar{b} = 2/3$, and averaging across the draws to obtain an average quantile surplus of 0.070774. This value is different than the true average surplus computed above. Therefore, average surplus for the observationally equivalent quantile demand is different than for true demand and hence average surplus is not identified. *Q.E.D.*

PROOF OF THEOREM 3: By $q(p(t), \bar{y} - s, \eta)$ Lipschitz in t and s, we know that the solution $s(t, \eta)$ to the differential equation (3.1) exists and is unique. By condition (i), we have $ds(t, \eta)/dt \leq 0$ so that $s(t, \eta) \geq 0$ for all $t \in [0, 1]$ by $s(1, \eta) = 0$. Let

$$s_B(t, \eta) = e^{Bt} \int_t^1 \left[q\big(p(s), \bar{y}, \eta\big)^T \frac{dp(s)}{ds} \right] e^{-Bs}\, ds$$

be the solution to

$$\text{(A1.1)} \qquad \frac{ds_B(t, \eta)}{dt} = a(t, \eta) + B s_B(t, \eta), \quad s_B(1, \eta) = 0,$$
$$a(t, \eta) = -q\big(p(t), \bar{y}, \eta\big)^T \frac{dp(t)}{dt}.$$

Then, applying the inequality from condition (ii) of Theorem 3, it follows that

$$\text{(A1.2)} \qquad \frac{ds(t, \eta)}{dt}$$
$$= -q\big(p(t), \bar{y} - s(t, \eta), \eta\big)^T dp(t)/dt$$

$$\begin{aligned}&\geq -q\big(p(t),\bar{y},\eta\big)^T\, dp(t)/dt + \underline{B}s(t,\eta)\\&=\frac{ds_{\underline{B}}(t,\eta)}{dt},\end{aligned}$$

where $\dot{s}(t,\eta)$ is a mean value in $[0, s(t,\eta)]$. Since $s(1,\eta)=0=s_{\underline{B}}(1,\eta)$, it follows by this inequality that $s_{\underline{B}}(t,\eta)\geq s(t,\eta)$. It follows similarly that $s_{\bar{B}}(t,\eta)\leq s(t,\eta)$, so that, evaluating at $t=0$, we have

(A1.3) $\quad s_{\bar{B}}(0,\eta)\leq s(0,\eta)=S(\eta)\leq s_{\underline{B}}(0,\eta).$

Evaluating at $t=0$, we have $s_B(0,\eta)=\int_0^1 q(p(t),\bar{y},\eta)^T[dp(t)/dt]e^{-Bt}\,dt$. Also note that $dp(t)/dt$ is bounded by continuity of $dp(t)/dt$ on $[0,1]$, and that, by all the elements of $p(t)$ bounded away from zero and $q(p(t),\bar{y},\eta)^T p(t)\leq \bar{y}$, the demand vector $q(p(t),\bar{y},\eta)$ is bounded uniformly in η and t. Therefore, by the Fubini theorem,

$$E\big[s_B(0,\eta_i)\big]=\int_0^1 \bar{q}\big(p(t),\bar{y}\big)^T\big[dp(t)/dt\big]e^{-Bt}\,dt=\bar{S}_B.$$

Taking expectations of equation (A1.3) then gives

$$\bar{S}_{\bar{B}}=E\big[s_{\bar{B}}(0,\eta_i)\big]\leq\bar{S}\leq E\big[s_{\underline{B}}(0,\eta_i)\big]=\bar{S}_{\underline{B}}. \qquad Q.E.D.$$

PROOF OF COROLLARY 4: It follows by Lemma A1 that $Q_1(\tau|x)$ is continuously differentiable in x and

$$\begin{aligned}\frac{\partial Q_1(\tau|x)}{\partial p}&=E\left[\frac{\partial q_1(x,\eta_i)}{\partial p}\middle| q_1(x,\eta_i)=Q_1(\tau|x)\right],\\ \frac{\partial Q_1(\tau|x)}{\partial y}&=E\left[\frac{\partial q_1(x,\eta_i)}{\partial y}\middle| q_1(x,\eta_i)=Q_1(\tau|x)\right].\end{aligned}$$

As shown by Dette, Hoderlein, and Neumeyer (2011), it follows that the Slutzky condition for the first price is satisfied by the conditional quantile, that is,

$$\frac{\partial Q_1(\tau|x)}{\partial p_1}+Q_1(\tau|x)\frac{\partial Q_1(\tau|x)}{\partial y}\leq 0.$$

Therefore, at each $0<\tau<1$, $Q_1(\tau|x)$ is a demand function as a function of p_1 and y. Furthermore, by $\underline{B}\leq\Delta p_1\partial q_1(x,\eta)/\partial y\leq\bar{B}$, we have

$$\underline{B}\leq E\left[\Delta p_1\frac{\partial q_1(x,\eta_i)}{\partial y}\middle| q_1(x,\eta_i)=Q_1(\tau|x)\right]=\Delta p_1\frac{\partial Q_1(\tau|x)}{\partial y}\leq\bar{B}.$$

Consider the demand process $\tilde{q}_1(x, \tilde{\eta}) = Q_1(\tilde{\eta}|x)$ for $\tilde{\eta} \sim U(0, 1)$. Note that $\int_0^1 S^\tau \, d\tau$ is average surplus for this demand process. Also,

$$\int \tilde{q}_1(x, \tilde{\eta})\tilde{G}(d\tilde{\eta}) = \int_0^1 Q_1(\tilde{\eta}|x) \, d\tilde{\eta} = \bar{q}_1(x).$$

Therefore, the conclusion follows by the conclusion of Theorem 3. *Q.E.D.*

A2. THE EXPENDITURE FUNCTION AND EXACT CONSUMER SURPLUS FOR DISCRETE AND CONTINUOUS CHOICE

Discrete and continuous choice models are important in applications. For instance, gasoline demand could be modeled as gasoline purchases that are made jointly with the purchase of automobiles. In those models, the heterogeneity can influence the discrete choices as well as the demand for a particular commodity; for example, see Dubin and McFadden (1984) and Hausman (1985). Multiple sources of heterogeneity are an integral part of these models, with separate disturbances for discrete and continuous choices. The general heterogeneity we consider allows for such multi-dimensional heterogeneity. Here we consider discrete and continuous choice with general heterogeneity, focusing on the effect of price changes in the continuous demand. Bhattacharya (2015) has recently considered surplus for changes in the prices of the discrete alternatives with general heterogeneity.

We first consider the individual choice problem and the associated expenditure function. We adopt the framework of Dubin and McFadden (1984) and Hausman (1985), extending previous results to the expenditure function. Suppose that the agent is choosing among J discrete choices in addition to choosing q. The consumer choice problem is

$$\max_{j,q,a} U_j(q, a, \eta) \quad \text{s.t.} \quad p^T q + a + r_j \leq y, \tag{A2.1}$$

where r_j is the usage price of choice j relative to the price of the numeraire good a. Here we assume that, for each η and j, the function $U_j(q, a)$ is strictly quasi-concave (preferences are strictly convex) and satisfies local nonsatiation. Let

$$q_j(p, y, \eta) = \arg\max_q U_j(q, a, \eta) \quad \text{s.t.} \quad p^T q + a \leq y$$

be the demand function associated with the jth utility function and let

$$V_j(p, y, \eta) = U_j\big(q_j(p, y, \eta), y - p^T q_j(p, y, \eta), \eta\big)$$

be the associated indirect utility function. The utility maximizing choice of the discrete good will be $\arg\max_j V_j(p, y - r_j, \eta)$ and the indirect utility function will be $V(p, r, y, \eta) = \max_j V_j(p, y - r_j, \eta)$, where $r = (r_1, \dots, r_J)^T$. When there is a unique discrete choice j (depending on p, r, y, and η) that maximizes utility, that is, where $V_j(p, y - r_j, \eta) > V_k(p, y - r_k, \eta)$ for all $k \neq j$, the demand $q(p, y, r, \eta)$ will be

$$q(p, r, y, \eta) = q_j(p, y - r_j, \eta).$$

When there are multiple values of the discrete choice that maximize utility, the demand will generally be a correspondence, containing one point for each value of j that maximizes utility.

In what follows, we will assume that $(V_1(p, y - r_1, \eta), \dots, V_J(p, y - r_J, \eta))$ is continuously distributed and that the probability of ties is zero. Nevertheless, the case with ties is important for us. Surplus is calculated by integrating the demand function as price changes while income is compensated to keep utility constant. As compensated income changes, ties may occur and the demand for q may jump. With gasoline demand, compensated income changes could result in a choice of car with different gas mileage, leading to a jump. Such jumps must be accounted for in the bounds analysis.

Turning to welfare analysis, let $e(p, r, u, \eta)$ denote the expenditure function in this discrete/continuous choice setting, defined as

$$\begin{aligned} &e(p, r, u, \eta) \\ &\quad = \min\Big\{y \text{ s.t. } \max_{j,q,a}\{U_j(q, a, \eta) \text{ s.t. } p^T q + a + r_j \le y\} \ge u\Big\}. \end{aligned}$$

As usual, it is the minimum value of income that allows individual η to attain utility level u. There is a simple, intuitive relationship between this expenditure function, the ones associated with the continuous choice of q for each j, and the indirect utility function $V(p, r, y, \eta) = \max_j V_j(p, y - r_j, \eta)$. Let $e_j(p, u, \eta) = \min_{q,a}\{p^T q + a : U_j(q, a, \eta) \ge u\}$ be the expenditure function for the utility function $U_j(q, a, \eta)$ $(j = 1, \dots, J)$.

LEMMA A2: *If, for each j and η, the utility $U_j(q, a, \eta)$ is strictly quasi-concave and satisfies local nonsatiation, then $e(p, u, \eta) = \min_j\{e_j(p, u, \eta) + r_j\}$, $V(p, r, e(p, r, u, \eta), \eta) = u$, and $e(p, r, V(p, r, y, \eta), \eta) = y$.*

PROOF: For notational convenience, drop the η argument. Define $\bar{e}(p, r, u) = \min_j\{e_j(p, u) + r_j\}$. By the definition of $\bar{e}(p, r, u)$, it follows that $\bar{e}(p, r, u) = e_{j^*}(p, u) + r_{j^*}$ for some j^* that need not be unique. By the definition of $e_{j^*}(p, u)$ and standard results, there is q^* such that $U_{j^*}(q^*, a^*) \ge u$ and $p^T q^* + a^* = e_{j^*}(p, u)$, so $p^T q^* + a^* + r_{j^*} = \bar{e}(p, r, u)$. Since $U_{j^*}(q^*, a^*) \ge u$ and $p^T q^* + a^* + r_{j^*} \le \bar{e}(p, r, u)$, it follows that

$$\max_{q,j}\{U_j(q, a, j) \text{ s.t. } p^T q + a + r_j \le \bar{e}(p, r, u)\} \ge U_j(q^*, a^*) \ge u.$$

It follows that $e(p, r, u) \le \bar{e}(p, r, u)$. Next, consider any $\bar{y} < \bar{e}(p, r, u)$. Then by the definition of $\bar{e}(p, r, u)$, we have $\bar{y} - r_j < e_j(p, u)$ for all $j \in \{1, \ldots, J\}$. Since $e_j(p, u)$ is the expenditure function, it follows that $\max_{q,a}\{U_j(q, a)$ s.t. $p^T q + a \le \bar{y} - r_j\} < u$ for every j, and so $\max_{q,a,j}\{U_j(q, a)$ s.t. $p^T q + a \le \bar{y} - r_j\} < u$. It follows that $\bar{y} < e(p, r, u)$. Since this is true for every $\bar{y} < \bar{e}(p, r, u)$, it follows that $\bar{e}(p, r, u) = e(p, r, u)$.

Next, note that by the definition of the expenditure function $e(p, r, u)$ as the minimum income level that will allow an individual to reach utility u, we have $V(p, r, e(p, r, u)) \ge u$. Also, $V(p, r, y)$ is monotonically increasing in y by $V_j(p, y - r_j)$ monotonically increasing in y for each j and $V_k(p, e_k(p, u)) = u$ by standard results for indirect utility and expenditure functions. By the definition of $V(p, r, y)$ and monotonicity of $V_j(p, y - r_j)$ in y, there is j with

$$V\big(p, r, e(p, r, u)\big) = V_j\big(p, e(p, u) - r_j\big) \le V_j\big(p, e_j(p, u)\big) = u,$$

where the inequality holds by the first conclusion that implies $e(p, u) \le e_j(p, u) + r_j$. Therefore, we have $V(p, e(p, r, u), r) = u$. Similarly, we have $e(p, r, V(p, r, y)) \le y$ by the definitions and there is j such that, by $e_j(p, u)$ increasing in u,

$$\begin{aligned} e\big(p, r, V(p, r, y)\big) &= e_j\big(p, V(p, r, y)\big) + r_j \\ &\ge e_j\big(p, V_j(p, y - r_j)\big) + r_j = y, \end{aligned}$$

so that $e(p, r, V(p, r, y)) = y$. *Q.E.D.*

Turning now to exact surplus for discrete/continuous choice, the equivalent variation for a price change from p^0 to p^1 with income $\bar{y}$ for individual η is $S(\eta) = \bar{y} - e(p^0, r, u^1, \eta)$, where u^1 is the utility at p^1, r, and $\bar{y}$. Consider a price path $p(t)$ as in the body of the paper. Then $s(t, \eta) = \bar{y} - e(p(t), r, u^1, \eta)$ is the equivalent variation for a price change from $p(t)$ to p^1 for income $\bar{y}$, where u^1 is the utility at p^1. The next result gives conditions for $s(t, \eta)$ to satisfy the same differential equation as in the continuous case.

LEMMA A3: *If, for each j and η, the utility $U_j(q, a, \eta)$ is strictly quasi-concave and satisfies local nonsatiation, then at any $p(t)$ and η such that there is j with $V_j(p(t), \bar{y} - s(t, \eta) - r_j, \eta) > V_k(p(t), \bar{y} - s(t, \eta) - r_k, \eta)$ for all $k \neq j$, it follows that $s(t, \eta)$ is differentiable and*

$$\frac{ds(t, \eta)}{dt} = -q\big(p(t), r, \bar{y} - s(t, \eta), \eta\big)^T dp(t)/dt.$$

PROOF: For notational convenience, suppress the η argument and let $p = p(t)$. By definition, we have $s(t, \eta) = \bar{y} - e(p(t), r, u^1, \eta)$. Consider j^* such that

$$V\big(p, r, e\big(p, r, u^1\big)\big) = V_{j^*}\big(p, e\big(p, r, u^1\big) - r_{j^*}\big).$$

For any $k \neq j^*$, it follows by duality that $V_k(p, e_k(p, u^1)) = u^1$. Therefore, we have

$$\begin{aligned} V_k\big(p, e_k\big(p, u^1\big)\big) &= u^1 \\ &= V\big(p, r, e\big(p, r, u^1\big)\big) \\ &= V_{j^*}\big(p, e\big(p, r, u^1\big) - r_{j^*}\big) \\ &> V_k\big(p, e\big(p, r, u^1\big) - r_k\big). \end{aligned}$$

By $V_k(p, y)$ monotonically increasing in y, it follows that $e_k(p, u^1) > e(p, r, u^1) - r_k$. Since this is true for every $k \neq j^*$, we have

$$e\big(p, r, u^1\big) = e_{j^*}\big(p, u^1\big) + r_{j^*} < e_k\big(p, u^1\big) + r_k, \quad \text{for all } k \neq j.$$

Also note that by standard duality results, for the Hicksian demand $h_{j^*}(p, u)$,

$$\begin{aligned} \frac{\partial e_j\big(p, u^1\big)}{\partial p} &= h_{j^*}(p, u) = q_{j^*}\big(p, e_{j^*}(p, u)\big) = q_{j^*}\big(p, e\big(p, r, u^1\big) - r_{j^*}\big) \\ &= q_{j^*}\big(p, \bar{y} - s(t) - r_{j^*}\big) = q\big(p, r, \bar{y} - s(t)\big), \end{aligned}$$

where the last equality follows by the $q(p, r, y) = q_{j^*}(p, y - r_{j^*})$ when $V_{j^*}(p, y - r_{j^*}) > V_k(p, y - r_k)$ for all $k \neq j^*$. Since each $e_k(p, u^1)$ is continuous in p, the previous inequality continues to hold in a neighborhood of p. Therefore, by $e_{j^*}(p, u^1)$ differentiable, Shephard's Lemma, and the chain rule, on that neighborhood $s(t) = y - e(p(t), r, u^1)$ is differentiable and

$$\begin{aligned} \frac{ds(t)}{dt} &= -\frac{de\big(p(t), r, u^1\big)}{dt} \\ &= -\frac{\partial e_j\big(p(t), u^1\big)}{\partial p}^T \frac{dp(t)}{dt} \\ &= -q\big(p(t), r, \bar{y} - s(t)\big)^T \frac{dp(t)}{dt}. \end{aligned}$$

Q.E.D.

The discontinuity of individual demand does affect the bounds for average consumer surplus. The previous bounds depend on income effects. With jumps, we construct bounds that are based on limits on the size of the jump and on the proportion of individuals whose demand would jump as income is compensated along with the price change. For that purpose, we make use of a demand decomposition into continuous and jump components.

ASSUMPTION A3: *There are functions* $\dot{q}(p, r, y, \eta)$, $\check{q}(p, r, y, \eta)$, $A(\eta)$ *and constants* $\underline{B}$, $\bar{B}$ *such that* $\bar{A} = E[A(\eta)]$ *exists and for* $t \in [0, 1]$, *and*

$0 \le s \le s(t, \eta)$,

$$
\begin{aligned}
&q\big(p(t), r, \bar{y}-s, \eta\big) = \dot{q}\big(p(t), r, \bar{y}-s, \eta\big) + \check{q}\big(p(t), r, \bar{y}-s, \eta\big),\\
&\big|\check{q}\big(p(t), r, \bar{y}-s, \eta\big)^T dp(t)/dt\big| \le A(\eta),\\
&\underline{B}s \le \big[\dot{q}\big(p(t), r, \bar{y}, \eta\big) - \dot{q}\big(p(t), r, \bar{y}-s, \eta\big)\big]^T dp(t)/dt \le \bar{B}s.
\end{aligned}
$$

Here we assume that the demand function can be decomposed into a jump component $\check{q}(p, r, y, \eta)$ and a Lipschitz continuous component $\dot{q}(p, r, y, \eta)$, with lower and upper bounds $\underline{B}$ and $\bar{B}$, respectively, on how much $\dot{q}(p(t), r, \bar{y}-s, \eta)^T dp(t)/dt$ may vary with $s > 0$. The term $A(\eta)$ is an individual specific bound on the jump. It will be zero for individuals whose demand function does not jump as income is compensated up to the surplus amount $S(\eta) = s(0, \eta)$. For example, for gasoline demand, it will be zero for individuals who would not change car types over the range of income being compensated.

To describe bounds on average surplus that allow for jumps, let

$$\bar{s}_{a,B}(t) = e^{Bt} \int_t^1 \big[\bar{q}\big(p(s), r, \bar{y}\big)^T dp(s)/ds - a\big] e^{-Bs}\, ds$$

be the solution to the differential equation

$$\frac{d\bar{s}_{a,B}(t)}{dt} = -\bar{q}\big(p(t), r, \bar{y}\big)^T \frac{dp(t)}{dt} + a + B\bar{s}_{a,B}(t), \quad \bar{s}_{a,B}(1) = 0.$$

Letting $\bar{S}_{a,B} = \bar{s}_{a,B}(0)$, we have

$$\bar{S}_{a,B} = \int_0^1 \big[\bar{q}\big(p(t), r, \bar{y}\big)^T dp(t)/dt\big] e^{-Bt} + \frac{a}{B}\big(e^{-B} - 1\big).$$

THEOREM A4: *If Assumptions* 1, A1, *and* A3 *are satisfied, the elements of* $p(t)$ *are bounded away from zero, and with probability* 1 *for all but a finite number of* t *values there is* j *with* $V_j(p(t), \bar{y} - s(t, \eta) - r_j, \eta) > V_k(p(t), \bar{y} - s(t, \eta) - r_k, \eta)$ *for all* $k \ne j$, *it follows that*

$$\bar{S}_{2\bar{A},\bar{B}} \le \bar{S} \le \bar{S}_{-2\bar{A},\underline{B}}.$$

Also, if $q(p(t), r, y - s, \eta)^T dp(t)/dt \le q(p(t), r, y, \eta)^T dp(t)/dt$ *for all* $t \in [0, 1]$ *and* $s \in [0, s(t, \eta)]$, *then* $\bar{S} \le \bar{S}_0 = \int_0^1 [\bar{q}(p(t), r, y)^T dp(t)/dt]\, dt$.

PROOF: For notational convenience, suppress the η argument. Let

$$s_{a,B}(t) = e^{Bt} \int_t^1 \big[q\big(p(s), r, \bar{y}\big)^T dp(s)/ds - a\big] e^{-Bs}\, ds$$

be the solution to the differential equation

$$\frac{ds_{a,B}(t)}{dt} = -q\big(p(t), r, \bar{y}\big)^T dp(t)/dt + a + Bs_{a,B}(t), \quad s_{a,B}(1) = 0.$$

By Assumption A3,

$$\begin{aligned}
&\big[q\big(p(t), r, \bar{y}\big) - q\big(p(t), r, \bar{y} - s\big)\big]^T \frac{dp(t)}{dt} \\
&\quad = \big[\dot{q}\big(p(t), r, \bar{y}\big) - \dot{q}\big(p(t), r, \bar{y} - s\big) + \check{q}\big(p(t), r, \bar{y}\big) \\
&\qquad - \check{q}\big(p(t), r, \bar{y} - s\big)\big]^T \frac{dp(t)}{dt} \\
&\quad \le \bar{B}s + 2A.
\end{aligned}$$

Therefore, by Lemma A3, it follows that at any point where $V_j(p(t), \bar{y} - s(t) - r_j) > V_k(p(t), \bar{y} - s(t) - r_k)$ for all $k \neq j$, $s(t)$ is differentiable and

$$\begin{aligned}
\frac{ds(t)}{dt} &= -q\big(p(t), r, \bar{y} - s(t)\big)^T \frac{dp(t)}{dt} \\
&\le -q\big(p(t), r, \bar{y}\big)^T \frac{dp(t)}{dt} + \bar{B}s(t) + 2A = \frac{ds_{2A,\bar{B}}(t)}{dt}.
\end{aligned}$$

Note that $s(t)$ is continuous by continuity of the expenditure function and $p(t)$. Consider the event $\mathcal{E}$ where there are no ties in the values of the indirect utility functions (i.e., where there is j^* depending on t such that $V_{j^*}(p(t), \bar{y} - s(t, \eta) - r_{j^*}) > V_k(p(t), \bar{y} - s(t, \eta) - r_k)$ for all $k \neq j^*$), at all t except a finite number. When $\mathcal{E}$ occurs, we have

$$s(t) = -\int_t^1 \frac{ds(u)}{du}\, du.$$

Similarly, we have $s_{2A,\bar{B}}(t) = -\int_t^1 [ds_{2A,\bar{B}}(u)/du]\, du$. Then, by $ds(t)/dt \le ds_{2A,\bar{B}}(t)/dt$, it follows that $s(t) \ge s_{2A,\bar{B}}(t)$. Evaluating at $t = 0$, we get $S \ge s_{2A,\bar{B}}(0)$. It follows similarly that $S \le s_{-2A,\underline{B}}(0)$. Thus, adding back the η notation, when the event $\mathcal{E}$ occurs, we have

$$s_{2A,\bar{B}}(0, \eta) \le S(\eta) \le s_{-2A,\underline{B}}(0, \eta).$$

Also, it follows similarly to the proof of Theorem 3 that

$$E\big[s_{a,B}(0, \eta)\big] = \bar{S}_{a,B}.$$

Since $\Pr(\mathcal{E}) = 1$, taking expectations through the previous inequality gives the first conclusion.

For the second conclusion, note that $q(p(t), r, y - s(t, \eta), \eta)^T dp(t)/dt \leq q(p(t), r, y, \eta)^T dp(t)/dt$, so that

$$\begin{aligned}\frac{ds(t,\eta)}{dt} &= -q\big(p(t), r, \bar{y} - s(t), \eta\big)^T \frac{dp(t)}{dt} \\ &\geq -q\big(p(t), r, \bar{y}, \eta\big)^T \frac{dp(t)}{dt}.\end{aligned}$$

The second conclusion then follows similarly to the first one. *Q.E.D.*

These bounds adjust for the possible presence of discontinuity in individual demands by adding $2E[D(\eta)]$ to $-\bar{q}(p, y)$ in the equation for the upper bound and subtracting the same term in the equation for the lower bound. This adjustment will be small when the largest possible jump is small or when the proportion of individuals with a discontinuity is small. One can drop this term for the bound for normal goods.

A3. GENERALIZED CONDITIONS FOR BOUNDS ON EXACT CONSUMER SURPLUS

The purpose of this section is to show that known bounds on income effects are not required for validity of the bounds in Theorem 3. To describe this result, let

$$B_u(\eta) = \max_{t \in [0,1], s \in [0, S(\eta)]} \frac{\partial q\big(p(t), y - s, \eta\big)}{\partial y}^T \frac{dp(t)}{dt}.$$

This bound is an individual specific upper bound for income effects. Such bounds always exist for continuous demand functions. This can be thought of as an individual specific version of the income effect bounds. Also let

$$\begin{aligned}
S_u(\eta) &= \int_0^1 \big[q\big(p(t), \bar{y}, \eta\big)^T dp(t)/dt\big] e^{-B_u(\eta)t}\, dt, \\
\bar{S}_u' &= \int 1\big(B_u(\eta) \geq B\big) S_u(\eta) G(d\eta), \\
\bar{S}_u'' &= \int 1\big(B_u(\eta) < B\big) S_u(\eta) G(d\eta), \\
S_B(\eta) &= \int_0^1 \big[q\big(p(t), \bar{y}, \eta\big)^T dp(t)/dt\big] e^{-Bt}\, dt, \\
\bar{S}_B' &= \int 1\big(B_u(\eta) \geq B\big) s_B(\eta) G(d\eta), \\
\bar{S}_B'' &= \int 1\big(B_u(\eta) < B\big) s_B(\eta) G(d\eta).
\end{aligned}$$

We have the following result:

THEOREM A5: *Suppose that Assumptions* 1 *and* A1 *are satisfied*, (i) $q(x,\eta)^T dp(t)/dt \geq 0$, *and* (ii) *all prices in* $p(t)$ *are bounded away from zero*. *If* $\bar{S}'_B - \bar{S}'_u \leq \bar{S}''_u - \bar{S}''_B$, *then*

$$\bar{S} \geq \bar{S}_B, \quad \bar{D} \geq \bar{S}_B - \bar{q}(p^1, \bar{y})^T \Delta p.$$

Also, *if* $\bar{S}'_B \leq c$, *then*

$$\bar{S} \geq \bar{S}_B - c, \quad \bar{D} \geq \bar{S}_B - c - \bar{q}(p^1, \bar{y})^T \Delta p.$$

PROOF: Let us note that $S_u(\eta) = s_u(0, \eta)$, where

$$s_u(t, \eta) = e^{B_u(\eta)t} \int_t^1 \left[q(p(s), \bar{y}, \eta)^T dp(s)/ds\right] e^{-B_u(\eta)s} \, ds$$

is the solution to

$$\frac{ds_u(t, \eta)}{dt} = a(t, \eta) + B_u(\eta) s_u(t, \eta), \quad s_u(1, \eta) = 0,$$

$$a(t, \eta) = -q(p(t), \bar{y}, \eta)^T \frac{dp(t)}{dt}.$$

It follows exactly in the proof of Theorem 3 that $S(\eta) \geq S_u(\eta)$, so that

$$\bar{S}_u = E[S_u(\eta)] \leq \bar{S}.$$

Also, we have

$$\bar{S}_u = \bar{S}'_u + \bar{S}''_u, \quad \bar{S}_B = \bar{S}'_B + \bar{S}''_B.$$

Therefore, $\bar{S}'_B - \bar{S}'_u \leq \bar{S}''_u - \bar{S}''_B$ if and only if $\bar{S}_B \leq \bar{S}_u$, which implies $\bar{S}_B \leq \bar{S}$.
Now suppose $\bar{S}'_B \leq c$. Note that $\bar{S}''_B \leq \bar{S}''_u$ and $\bar{S}'_u \geq 0$. Then

$$\bar{S}_B - c = \bar{S}''_B + \bar{S}'_B - c \leq \bar{S}''_B \leq \bar{S}''_u \leq \bar{S}_u \leq \bar{S}. \qquad \textit{Q.E.D.}$$

The first conclusion of this result gives a more general condition for validity of the bounds. Although the result is simple, the decomposition helps clarify that the surplus bounds hold over a much wider class of conditions than just bounded income effects. When B is well into the tail of the distribution of $B_u(\eta)$, it should be the case that $\bar{S}'_B - \bar{S}'_u$ is small while $\bar{S}''_u - \bar{S}''_B$ is large, leading to the bounds being satisfied.

The second conclusion gives a more general bound that may sometimes be applicable. For example, suppose that only the price of the first good is changing and let $\bar{Q}_B = \sup_{0 \le t \le 1} E[q_1(p(t), \bar{y}, \eta) | B_u(\eta) \ge B]$. Then by the usual Chebyshev inequality type argument,

$$\begin{aligned}
\bar{S}'_B &= \int 1\big(B_u(\eta) \ge B\big) \left\{ \int_0^1 \big[q\big(p(t), \bar{y}, \eta\big)^T dp(t)/dt\big] e^{-Bt}\, dt \right\} G(d\eta) \\
&= \Delta p_1 \int_0^1 \Pr\big(B_u(\eta) \ge B\big) E\big[q_1\big(p(t), \bar{y}, \eta\big) | B_u(\eta) \ge B\big] e^{-Bt}\, dt \\
&\le \bar{Q}_B \Delta p_1 \left(\int_0^1 e^{-Bt}\, dt \right) \Pr\big(B_u(\eta) \ge B\big) \le \frac{\bar{Q}_B \Delta p_1 E\big[B_u(\eta)^r\big]}{B^r},
\end{aligned}$$

where the last inequality follows by $e^{-Bt} \le 1$, and hence $\int_0^1 e^{-Bt}\, dt \le 1$, and by the Holder inequality.

A4. BOUNDING SURPLUS BOUND ERROR IN GASOLINE APPLICATION

This reasoning just above applies to the justification of the lower bound for surplus in the gasoline demand example. In a linear varying coefficients model, we estimate the bounding term in the above equation for $r = 2$ to be

$$\begin{aligned}
\frac{\bar{Q}_B \Delta p_1 E\big[B_u(\eta)^2\big]}{B^2} &= \frac{\bar{Q}_B \Delta p_1 \big[(0.000726)^2 + (0.00241)^2\big]}{(0.0197)^2} \\
&\le \bar{Q}_B \Delta p_1 (0.015).
\end{aligned}$$

It is reasonable to suppose that average demand for large income effects is not very large relative to overall average demand. If anything, given the essential nature of transportation, we might expect that average demand is smaller for those with high income effects. This makes $\bar{Q}_B \Delta p_1 \le 2\bar{S}_{\bar{B}}$ a very reasonable assumption. Applying the inequality at the end of the last section, we thus find that if the linear random coefficients model were true, $\bar{S}'_B \le (0.03)\bar{S}_B$. Then by the second conclusion of Theorem A5, we have $\bar{S} \ge (0.97)\bar{S}_{\bar{B}}$, so that the lower bound given in the empirical application is very close to correct. We note that this calculation of $(0.97)\bar{S}_{\bar{B}}$ as a lower bound is very conservative, giving us high confidence in the lower bound used in the empirical application.

A5. DETAILS FOR GENERAL BOUNDS ESTIMATION

We used a third order power series in $\ln p$ and $\ln y$ to estimate the quantile of $\ln q$. We also used the same power series for $m_j(x)$, which corresponds to the empirical specification with income effect bounds. We estimated the conditional quantile at 99 evenly spaced values, $\tau \in \{0.01, 0.02, \dots, 0.99\}$.

We imposed the Slutzky condition appropriate for the natural log of demand on the quantiles at 81 values of x corresponding to nine price and income values drawn randomly from the range of the data. We drew one set of $L = 1{,}000$ coefficients, ensuring that each coefficient vector gave a demand at each $\tau \in \{0.01, 0.02, \ldots, 0.99\}$ satisfying the Slutzky condition on a grid that is the 0.01, 0.03, 0.05, 0.07, and 0.09 quantiles of $\ln(p)$ and $\ln(y)$. We evaluated the constraints at five quantile values for r including the median. We calculated the bounds as described in Section 6, using $\hat{F}(r|x) = \sum_{k=1}^{99} \Phi([r - \hat{Q}(0.01k|x)]/0.01)$ in place of $F(r|x)$ in the constraints, where $\Phi(s)$ is the $N(0,1)$ CDF that is used to smooth out $\hat{Q}^{-1}(r|x)$.

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Dept. of Economics, MIT, E52-518, 50 Memorial Drive, Cambridge, MA 02139, U.S.A.; jhausman@mit.edu

and

Dept. of Economics, MIT, E52-318A, 50 Memorial Drive, Cambridge, MA 02139, U.S.A.; wnewey@mit.edu.

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