Econometrica Supplementary Material

SUPPLEMENT TO "AGENCY MODELS WITH FREQUENT ACTIONS"

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APPENDIX C: ADDITIONAL PROOFS FOR APPENDIX B

PROOF OF LEMMA 6: In each of the above problems, the policy $(a, c, W) = (0, 0, e^{r\Delta}w)$ is an available policy that satisfies all the constraints and delivers a value of at least $F(w) + [\min F'](e^{r\Delta} - 1)\bar{w} = F(w) + O(\Delta)$. Let $\hat{h} = \mathbb{E}^{\Delta}[h(a(z))]$, $\hat{u} = \mathbb{E}^{\Delta}[u(c(\Delta(x+a(z))))]$, and $\hat{W} = \mathbb{E}^{\Delta}[W(\Delta(x+a(z)), z)]$. The promise keeping constraint implies that

$$\hat{W} - w = \tilde{r}\Delta e^{r\Delta}(w + \hat{h} - \hat{u}) = O(\Delta),$$

since $w \in [\underline{w}, \bar{w}]$, $\hat{h} \in [0, h(A)]$, and $\hat{u} \in [0, \bar{u}]$. Therefore, $W(\Delta(x+a(z)), z) - w = (W(\Delta(x + a(z)), z) - \hat{W}) + (\hat{W} - w)$ implies

$$\mathbb{E}^{\Delta}\big[\big(W\big(\Delta\big(x + a(z)\big), z\big) - w\big)^2\big] = \mathbb{V}^{\Delta}\big[W\big(\Delta\big(x + a(z)\big), z\big)\big] + O\big(\Delta^2\big).$$

Consequently, for Y either $\Phi^{\Delta,q}(a, c, W; F, w)$ or $\Phi^{\Delta}(a, c, W; F)$, we have $Y \geq F(w) + O(\Delta)$ and

$$Y \leq \tilde{r}\Delta A + e^{-r\Delta}\bigg(F(w) + \tilde{r}\Delta e^{r\Delta}F'(w)(w + \hat{h} - \hat{u}) + \frac{\max F''}{2}\mathbb{V}^{\Delta}\big[W\big(\Delta\big(x + a(z)\big), z\big)\big]\bigg) + O\big(\Delta^2\big),$$

which, after rearranging terms, gives the result for an appropriate V. *Q.E.D.*

PROOF OF LEMMA 7: Since $G_X(\cdot|z)$ are linearly independent, let $\phi_z(x)$ be the functions bounded by some B such that

$$\int \phi_z(x)g_{X|Z}(x|z) = 0, \quad \int \phi_z(x)g_{X|Z}\big(x|z'\big) < -1 \quad \forall z, z'.$$

Fix some $(\bar{a}, \bar{h})$ and consider the optimal policy $a(\cdot), v(\cdot, \cdot)$ for the problem $\Theta(\bar{a}, \bar{h})$. We define $v^*(x, z) = v(x, z) + \varepsilon\phi_z(x)$ and let $a^*(\cdot)$ be defined by the (FOC$_\Theta$). Note that, for all z,

$$\int_{\mathbb{R}} 2\varepsilon\phi_z(x)g'_{X|Z}(x|z)\,dx = O(\varepsilon),$$

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and so, from (FOC_Θ), $|a(z) - a^*(z)| = O(\varepsilon)$. This implies that, for $\tilde{a} = \mathbb{E}_Z[a^*(z)]$ and $\tilde{h} = \mathbb{E}_Z[h(a^*(z))]$ $|\bar{a} - \tilde{a}|, |\bar{h} - \tilde{h}| = O(\varepsilon)$. On the other hand,

$$\begin{aligned}\mathbb{E}\big[\big|v^*(x,z)^2 - v(x,z)^2\big|\big] &\le \varepsilon^2 M^2 + 2\mathbb{E}\big[\big|\varepsilon\phi_z(x)v(x,z)\big|\big] \\ &\le \varepsilon^2 M^2 + 2\varepsilon M\sqrt{\mathbb{E}\big[v(x,z)^2\big]} \\ &= \varepsilon^2 M^2 + 2\varepsilon M\sqrt{\Theta(\bar{a},\bar{h})}. \qquad \textit{Q.E.D.}\end{aligned}$$

PROOF OF LEMMA 10: (i) Fix $\varepsilon > 0$ and consider a function v that satisfies $\mathbb{E}[v(x,z)^2] \le 1$. For any $\delta > 0$, pick M_δ big enough so that (from Lebesgue's Monotone Convergence Theorem)

$$\int\left[\int_{|v|>M_\delta} v^2(x,z)g_{X|Z}(x|z)\,dx\right]dG_Z(z) \le \delta. \tag{36}$$

From Chebyshev's inequality,

$$\mathbb{P}_Z\left[\int_{|v|>M_\delta} v^2(x,z)g_{X|Z}(x|z)\,dx > \gamma\right] \le \frac{\delta}{\gamma}. \tag{37}$$

Therefore, for all z for which $\int_{|v|>M_\delta} v^2(x,z)g_{X|Z}(x|z)\,dx \le \gamma$,

$$\begin{aligned}&\int_{|v|>M_\delta}\big|v(x,z)g'_{X|Z}(x|z)\big|\,dx \\ &\quad\le \left[\int_{|v|>M_\delta} v^2(x,z)g_{X|Z}(x|z)\,dx \times \int \frac{g'_{X|Z}(x|z)^2}{g_{X|Z}(x|z)}\,dx\right]^{1/2} \le \sqrt{\gamma\bar{M}}.\end{aligned}$$

The result thus follows by picking $\gamma = \varepsilon^2/\bar{M}$ and $\delta = \varepsilon\gamma$.

(ii) Let γ and δ be as in (i) and M_δ be such that (36) holds. For any z for which $\int_{|v|>M_\delta} v^2(x,z)g_{X|Z}(x|z)\,dx \le \gamma$ and any z', we have

$$\begin{aligned}&\int_{|v|>M_\delta}\big|v(x,z)g_{X|Z}\big(x|z'\big)\big|\,dx \\ &\quad\le \left[\int_{|v|>M_\delta} v^2(x,z)g_{X|Z}(x|z)\,dx \times \int \frac{g_{X|Z}\big(x|z'\big)^2}{g_{X|Z}(x|z)}\,dx\right]^{1/2} \le \sqrt{\gamma\bar{M}},\end{aligned}$$

where the last inequality follows from the assumption (A3). The proof then follows from (37). *Q.E.D.*

PROOF OF LEMMA 11: (i) For every x and z, $|g'_{X|Z}(x|z) - g'_{X|Z}(x - \delta(z)|z)| \le \delta|g''_{X|Z}(x - \xi(x,z)|z)|$ for some $\xi(x,z) \in [0,\delta(z)] \subset [0,\hat{\delta}]$. Therefore, with $\bar{\delta}$

and $\bar{M}$ the constants in (A2), for every $\delta \leq \min\{\bar{\delta}, \frac{\varepsilon}{M\bar{M}}\}$ we have that

$$\int_{|v|\leq M} \left|v(x,z)\left[g'_{X|Z}(x|z) - g'_{X|Z}\big(x-\delta(z)|z\big)\right]\right| dx \leq \delta M \int \left|g''_{X|Z}\big(x-\xi(x,z)|z\big)\right| dx \leq \delta M\bar{M} \leq \varepsilon,$$

which establishes (27). The proof of (ii) is analogous and is omitted.

(iii) Similarly, for any $\delta \leq \min\{\bar{\delta}, \varepsilon/[M^2\sqrt{\bar{M}}]\}$, we have that

$$\begin{aligned} &\int\int_{|v|\leq M} \left|v(x,z)^2\big(g(x,z) - g\big(x-a(z),z\big)\big)\right| dx\,dz \\ &\leq \delta M^2 \int\int \left|g'_{X|Z}\big(x-\xi(x,z)|z\big)g_Z(z)\right| dx\,dz \\ &\leq \delta M^2 \int\left[\int \frac{g'_{X|Z}\big(x-\xi(x,z)|z\big)^2}{g_{X|Z}(x|z)}\,dx\right]^{1/2} g_Z(z)\,dz \leq \delta M^2\sqrt{\bar{M}} \leq \varepsilon, \end{aligned}$$

with the second inequality following from the Cauchy–Schwarz inequality, which establishes the lemma. *Q.E.D.*

APPENDIX D: THE HJB EQUATION

The following lemma establishes a property of the variance of continuation values function Θ that will be crucial to all the following results on the properties of the HJB equation.

LEMMA 15: *Suppose* (A2) *holds. Then the variance of continuation values function is bounded away from zero for strictly positive expected effort levels*,

$$\Theta(\bar{a},\bar{h}) \geq \underline{\theta} > 0 \quad \forall \bar{a} > 0, \bar{h}. \tag{38}$$

PROOF: Consider function Θ^n that is defined just as Θ except that the condition (TR_Θ) is dropped. On the one hand, trivially, $\Theta \geq \Theta^n$. On the other hand, from Lemma 1 it follows that

$$\Theta^n \geq \frac{\gamma^2}{\min_z \mathcal{I}_{g_{X|Z}(\cdot|z)}} \geq \frac{\gamma^2}{\bar{M}} > 0,$$

where γ is such that $h'(a) \geq \gamma$ for $a > 0$ and $\bar{M}$ is from assumption (A2). *Q.E.D.*

The following lemma establishes some basic properties of the solution of the HJB equation.

LEMMA 16: *Suppose* $\Theta(\bar{a}, \bar{h}) \geq \underline{\theta} > 0$.

(i) *For any initial conditions* $F(\underline{w})$ *and* $F'(\underline{w})$, *the HJB equation* (7) *has a unique solution* F *in any interval* $[\underline{w}, \bar{w}] \subset \mathbb{R}$.

(ii) F *is twice continuously differentiable and* (F, F') *depends continuously on the initial conditions*.

(iii) F' *is monotone with respect to* $F'(\underline{w})$. *That is, if* F_1 *and* F_2 *are two solutions of the HJB equation in an interval* $[\underline{w}, \bar{w}] \subset \mathbb{R}$ *with* $F_1(\underline{w}) = F_2(\underline{w})$ *and* $F_1'(\underline{w}) > F_2'(\underline{w})$, *then* $F_1'(w) > F_2'(w)$ (*and hence* $F_1(w) > F_2(w)$) *for all* $w > \underline{w}$.

PROOF: See Sannikov (2008). *Q.E.D.*

COROLLARY 2: *The HJB equation* (7) *with the boundary conditions* (8) *and* (9) *has a unique solution* F.

The corollary follows immediately from Lemma 16. Note also that the continuity and monotonicity in the initial slope suggest the natural procedure for computing F.

LEMMA 17: *Suppose* $\Theta(\bar{a}, \bar{h}) \geq \underline{\theta} > 0$. *The solution* F *of the HJB equation* (7) *with the boundary conditions* (8) *and* (9) *is strictly concave*.

PROOF: See Sannikov (2008). *Q.E.D.*

Part (i) of the next lemma establishes that the function F in the statement of Theorem 1 satisfies the HJB equation (22), with the constraint "$\bar{a} > 0$" dropped. Part (ii) shows a related result for the general case from Section 5, which will be used in Appendix F below.

LEMMA 18:

(i) *The function* F *in Theorem* 1 *solves HJB equation* (22).

(ii) *For any* $[\underline{w}, \bar{w}] \subset (0, w_{\mathrm{sp}})$, *there exists* $\gamma > 0$ *such that, for all sufficiently small* ζ, *the* F_ζ *as in Theorem* 3 *solves equation* (19) *on* $[\underline{w}, \bar{w}]$ *with an additional constraint* $\bar{a} \geq \gamma$.

PROOF: (i) For any $\lambda \in \mathbb{R}$, let H_λ be the linear function tangent to the retirement curve $\{(w, \underline{F}(w)) : w \in [0, \bar{u})\}$ with the slope λ (if $\lambda \geq \underline{F}'(0)$, $H_\lambda(w) = \lambda w$). On the one hand, since F and $\underline{F}$ are concave and $F \geq \underline{F}$, for any $w \in I$ we have $F(w) \geq H_{F'(w)}(w)$. On the other hand, for any $w \in I$, the value of the maximization problem in the expression above under constraint $\bar{a} = 0$ is at most $\max_c\{-c + F'(w)(w - u(c))\} = \underline{F}(w') + \underline{F}'(w')(w - w') = H_{F'(w)}(w)$, where w' is such that either $\underline{F}'(w') = F'(w)$ or $w' = 0$ in case $F'(w) > \underline{F}'(0)$. Consequently, choosing $\bar{a} = 0$ in the maximization problem above can never be strictly optimal. Equivalently, since F satisfies the HJB equation (7), it also satisfies the equation (22) with the constraint "$\bar{a} > 0$" dropped.

(ii) We may assume $w_{\text{sp}} > 0$. Note also that, for any $\zeta > 0$ and F_ζ as in Theorem 1, we have

$$\bar{F}'(\bar{w}_{\text{sp}}) \leq F'_\zeta(w) \leq \bar{F}(\underline{w})/\underline{w},$$

for all $w \in [\underline{w}, \bar{w}]$. We will establish that there is $\alpha > 0$ such that, for any ζ and $w \in [\underline{w}, \bar{w}]$, $F_\zeta(w) - H_{F'_\zeta(w)}(w) \geq \alpha$. If not, then let $\{w_n\}$, $\{w'_n\}$, $\{\zeta_n\}$, and $\{\alpha_n\}$ with $w_n \in [\underline{w}, \bar{w}]$, $w'_n \leq w_{\text{sp}}$, $\zeta_n \downarrow 0$, $\alpha_n \downarrow 0$ be such that $F_{\zeta_n}(w_n) - H_{F'_{\zeta_n}(w_n)}(w_n) \leq \alpha_n$ (where w'_n is such that $\underline{F}'(w'_n) = F'_{\zeta_n}(w_n)$). We consider three cases, and in each derive a contradiction.

Case 1: Suppose that, for some $\delta > 0$ and all n, $w'_n \in [\delta, w_{\text{sp}} - \delta]$. The concavity of F_{ζ_n} and $\underline{F}$ implies that $F_{\zeta_n}(w_n) - H_{F'_{\zeta_n}(w_n)}(w_n) \geq F_{\zeta_n}(w'_n) - H_{F'_{\zeta_n}(w_n)}(w'_n) = F_{\zeta_n}(w'_n) - \underline{F}(w'_n)$. But, since F_{ζ_n} is increasing as $\zeta_n \downarrow 0$ (Proposition 1, part (i)), $F_{\zeta_n}(w'_n) - \underline{F}(w'_n) \geq \inf_{w \in [\delta, w_{\text{sp}} - \delta]} F_{\zeta_1}(w) - \underline{F}(w) > 0$, a contradiction.

Case 2: If $w'_n \downarrow 0$ (we might assume so by choosing a subsequence), then we would have $F_{\zeta_n}(w_n) \to H_{F'_{\zeta_n}(w_n)}(w_n) \to \underline{F}'(0) \times w_n$. By concavity of all F_{ζ_n}, this would imply that, first, $F_{\zeta_n}(w) \to \underline{F}'(0) \times w$ for all $w \in [0, w_n]$, and second, that there is a sequence $\{w''_n\}$, $w''_n \in [0, w_n]$, such that $F'_{\zeta_n}(w''_n) \to \underline{F}'(0)$ and $F''_{\zeta_n}(w''_n) \to 0$. But then

$$\begin{aligned} F_{\zeta_n}(w''_n) &\to \max_{a,c}\bigl\{(a - c) + \underline{F}'(0)\bigl(w''_n + h(a) - u(c)\bigr)\bigr\} \\ &= \max_{a}\bigl\{a + \underline{F}'(0)\bigl(w''_n + h(a)\bigr)\bigr\} > \underline{F}'(0)w''_n, \end{aligned}$$

where the equality follows from the fact that $\underline{F}'(0) = \frac{1}{u'(0)}$ and strict concavity of u, while the inequality follows from $h'_+(0) < u'(0)$. This establishes the required contradiction.

Case 3: If $w'_n \uparrow w_{\text{sp}}$, we derive the contradiction in the analogous way as in Case 2.

We have established that, for all ζ and $w \in [\underline{w}, \bar{w}]$, $F_\zeta(w) - H_{F'_\zeta(w)}(w) \geq \alpha > 0$. On the other hand, for any ζ and $w \in [\underline{w}, \bar{w}]$, if we restrict the policy on the right-hand side of equation (22) to satisfy $\bar{a} \leq \gamma$, for sufficiently small $\gamma > 0$, then

$$\begin{aligned} &\sup_{\bar{a} \leq \gamma, \bar{h}, c}\left\{(\bar{a} - c) + F'_\zeta(w)\bigl(w + \bar{h} - u(c)\bigr) + \frac{1}{2}F''_\zeta(w) r \max\left\{\zeta, \Theta(\bar{a}, \bar{h})\right\}\right\} \\ &\quad \leq \max_{c}\left\{-c + F'_\zeta(w)\bigl(w - u(c)\bigr) + \frac{1}{2}F''_\zeta(w) r\zeta\right\} + \frac{\alpha}{2} \\ &\quad \leq H_{F'_\zeta(w)}(w) + \frac{\alpha}{2} \leq F_\zeta(w) - \frac{\alpha}{2}, \end{aligned}$$

where the first inequality follows because F'_ζ are uniformly bounded on $[\underline{w}, \bar{w}]$ and $\bar{h} \leq \frac{\bar{a}}{A} h(A)$. This establishes the lemma. *Q.E.D.*

D.1. *Proof of Proposition 1*

The proposition is based on the following "single crossing" lemma.

LEMMA 19: *Consider two functions $\Theta \geq_{D_+^\Theta} \underline{\Theta} \geq 0$, and suppose that $F^\Theta, F^{\underline{\Theta}}: I \to \mathbb{R}$ solve the corresponding HJB equations* (7) *with $F^{\Theta''} \leq 0$.*

(i) *If for some w, $F^\Theta(w) = F^{\underline{\Theta}}(w)$ and $F^{\Theta\prime}(w') > F^{\underline{\Theta}\prime}(w')$ in a right neighborhood of w, then $F^{\Theta\prime}(w') > F^{\underline{\Theta}\prime}(w')$ for all $w' > w$.*

(ii) *Assume $\Theta >_{D_+^\Theta} \underline{\Theta}$. If for some w, $F^\Theta(w) = F^{\underline{\Theta}}(w)$ and $F^{\Theta\prime}(w) \geq F^{\underline{\Theta}\prime}(w)$, then $F^{\Theta\prime}(w') > F^{\underline{\Theta}\prime}(w')$ for all $w' > w$.*

Note that the precondition of part (i) is implied by (but is not equivalent to) $F^\Theta(w) = F^{\underline{\Theta}}(w)$ and $F^{\Theta\prime}(w) > F^{\underline{\Theta}\prime}(w)$.

PROOF OF LEMMA 19: We prove only part (i) (the proof of part (ii) is analogous). First, by assumption, $F^{\Theta\prime}(w') > F^{\underline{\Theta}\prime}(w')$ for all $w' > w$ sufficiently close to w. Suppose now that there exists $w' > w$ with $F^{\Theta\prime}(w') \leq F^{\underline{\Theta}\prime}(w')$—we now assume that w' is the smallest with this property. Since $F^{\Theta\prime} >_{(w,w')} F^{\underline{\Theta}\prime}$, we have that $F^\Theta(w') > F^{\underline{\Theta}}(w')$. Therefore, it must be the case that $F^{\Theta''}(w') > F^{\underline{\Theta}''}(w')$; otherwise, since $F^{\Theta''}(w') \leq 0$ and $\Theta \geq_{D_+^\Theta} \underline{\Theta}$, every policy $(\bar{a}, \bar{h}, c)$ would yield a weakly higher value of the right-hand side of HJB equation (7) for $F^{\underline{\Theta}}(w')$ than for $F^\Theta(w')$. But then $F^{\Theta''}(w') > F^{\underline{\Theta}''}(w')$ implies that $F^{\Theta\prime}(w'') < F^{\underline{\Theta}\prime}(w'')$ for w'' in a left neighborhood of w', contradicting the minimality of w'. *Q.E.D.*

Given the lemma, the proof of part (i) of Proposition 1 proceeds as follows. Applying part (i) of Lemma 19 to $w = 0$, if $F^{\Theta\prime}(0) > F^{\underline{\Theta}\prime}(0)$, then $F^{\Theta\prime}(w') > F^{\underline{\Theta}\prime}(w')$ for all $w' > 0$. Therefore, $F^{\underline{\Theta}} \geq \underline{F}$ would imply $F^\Theta(w') > \underline{F}(w')$ for all $w' > 0$, violating the boundary conditions for F^Θ. Using the analogous argument, $F^{\underline{\Theta}\prime}(w) \geq F^{\Theta\prime}(w)$ for all $w \in [0, w_{\mathrm{sp}}^\Theta]$, and so $F^{\underline{\Theta}}(w) \geq F^\Theta(w)$, for all $w \in [0, w_{\mathrm{sp}}^\Theta]$, establishing part (i) of the proposition. The proof of part (ii) is analogous.

We note that part (i) of Proposition 1 is immediately applicable to the limit values for the general case defined in Theorem 3 (as it is applicable to the functions F_ζ and weak inequalities are preserved in the limit). The following lemma shows that, under an additional mild constraint, part (ii), that is, strict monotonicity, is applicable to the general case as well.

Consider the following assumption:

(Cont) $\Theta(\bar{a}, \bar{h}) \geq \delta(\bar{a})$ for a continuous δ with $\delta(\bar{a}) > 0$ when $\bar{a} > 0$.

For example, the assumption (Cont) is always satisfied in the pure hidden information case.[37]

[37]Roughly: for $\bar{a} > 0$ it must be the case, from (FOC$_\Theta$-PHI), that v' is bounded below above zero at a range with strictly positive mass (that depends on $\bar{a}$). This implies (Cont), for appropriate δ.

LEMMA 20: *Assume* (Cont) *holds. Then* F *as in Theorem* 3 *solves the HJB equation* (7) *with boundary conditions* (8) *and* (9).

PROOF: Choose any $[\underline{w}, \bar{w}] \subset (0, w_{\text{sp}})$. Part (ii) of Lemma 18 guarantees that, for sufficiently small ζ, all F_ζ satisfy the constraint $\bar{a} \geq \gamma$ on $[\underline{w}, \bar{w}]$, for some $\gamma > 0$. Therefore, for sufficiently small ζ, all F_ζ satisfy, on $[\underline{w}, \bar{w}]$,

$$F''(w) = \inf_{\bar{a} \geq \gamma, \bar{h}, c} \left\{ \frac{F(w) - (\bar{a} - c) - F'(w)\big(w + \bar{h} - u(c)\big)}{r\Theta(\bar{a}, \bar{h})/2} \right\},$$

with the right-hand side Lipschitz continuous in $(w, F(w), F'(w))$, since $\Theta \geq \delta(\gamma) > 0$ for $\bar{a} \geq \gamma$.

Part (i) of Proposition 1 guarantees that F_ζ converge in the supremum norm as $\zeta \downarrow 0$ to a function F. Since F'_ζ are uniformly bounded on $[\underline{w}, \bar{w}]$, it follows that all F''_ζ and F'_ζ are Lipschitz continuous with the same Lipschitz constant, and so F'_ζ converge to F' not only in L^1 but in the supremum norm, by the Arzela–Ascoli Theorem. Uniform Lipschitz continuity guarantees also that $F' = \frac{d}{dw}F$, that $F'' := \lim_{\zeta \downarrow 0} F''_\zeta$ exists, and F satisfies the above equation (all on $[\underline{w}, \bar{w}]$). Since the set $[\underline{w}, \bar{w}]$ is arbitrary, this proves that F solves (7) in $(0, w_{\text{sp}})$, and so establishes proof of the lemma. *Q.E.D.*

D.2. *Proof of Proposition 2*

The proof follows from the following lemma.

LEMMA 21: *For any* $\delta > 0$, *there is* $\varepsilon > 0$ *sufficiently small and* $\tilde{w} \in [0, \bar{w}_{\text{sp}}]$ *such that the following holds*: *If* $r\Theta \leq \varepsilon$, *then the solution* F *of the HJB equation* (7) *with initial conditions*

$$F(\tilde{w}) = \bar{F}(\tilde{w}) - \delta, \quad F'(\tilde{w}) = \bar{F}'(\tilde{w})$$

satisfies

$$F'' \leq_{[0, \bar{w}_{\text{sp}}]} -\frac{2\delta}{\varepsilon}.$$

PROOF: For any $\lambda \in [\bar{F}'(\bar{w}_{\text{sp}}), \infty)$, let G_λ be the linear function tangent to the first best frontier $\{(w, \bar{F}(w)) : w \in [0, \bar{w}_{\text{sp}}]\}$ with the slope λ. We will show that if, for an arbitrary $w \in [0, w_{\text{sp}}]$,

$$G_{F'(w)}(w) - F(w) \geq \delta, \tag{39}$$

then $F''(w) \leq -\frac{2\delta}{\varepsilon}$. Note that then, as long as $-\frac{2\delta}{\varepsilon} \leq \min_{w \in [0, \bar{w}_{\text{sp}}]} \bar{F}''(w)$, the above condition will be satisfied over the whole interval $[0, \bar{w}_{\text{sp}}]$, which will establish the lemma.

The HJB equation (7) takes the form

$$F''(w) \le \min_{a,h,c} \frac{2}{r\Theta(a,h)} \{F(w) - (a-c) - F'(w)(w + h - u(c))\}. \tag{40}$$

Let w' be such that $F'(w) = \bar{F}'(w')$. For the policy $(a(w'), c(w'))$ in the problem (1) at w', we have

$$\begin{aligned} &F(w) - (a(w') - c(w')) - F'(w)(w + h(a(w')) - u(c(w'))) \\ &\quad = \bar{F}(w') - (a(w') - c(w')) - \bar{F}'(w')(w' + h(a(w')) - u(c(w'))) \\ &\qquad + [F(w) - \bar{F}(w') + F'(w)(w' - w)] \\ &\quad = [F(w) - \bar{F}(w') + F'(w)(w' - w)] \le -\delta, \end{aligned}$$

where the last equality follows from (1), while the last inequality follows from (39). Since $(a(w'), h(a(w')), c(w'))$ is an available policy in the problem (40) and $r\Theta \le \varepsilon$, this establishes that $F''(w) \le -\frac{2\delta}{\varepsilon}$. *Q.E.D.*

Given the lemma, for any $\delta > 0$, the solution F of the HJB equation (7) with initial conditions $F(\tilde{w}) = \bar{F}(\tilde{w}) - \delta$, $F'(\tilde{w}) = \bar{F}'(\tilde{w})$ with $\tilde{w} \in [\delta, \bar{w}_{\rm sp}]$ will satisfy $F(\underline{w}) = \underline{F}(\underline{w})$ and $F(\bar{w}) = \underline{F}(\bar{w})$ for some $0 < \underline{w} < \bar{w} < \bar{w}_{\rm sp}$. This together with Proposition 6 and part (ii) of Lemma 5 establishes the proof of the proposition.

APPENDIX E: PROOF OF PROPOSITION 3

Fix period length $\Delta > 0$ and densities g and γ satisfying (14). Fix also a contract[38] $\{c_n\}$ together with action plans $\{a_{g,n}\}, \{a_{\gamma,n}\}$ such that $\{c_n\}, \{a_{g,n}\}$ is incentive compatible under g and $\{c_n\}, \{a_{\gamma,n}\}$ is incentive compatible under γ, and they deliver expected discounted utilities $w_g, w_\gamma \in [0, \bar{u})$ to the agent. In any period n and after any history of public signals $(y_0, \dots, y_{n-1})$, the contract and action plans give rise to a pair of continuation values $w_{g,n}$ and $w_{\gamma,n}$ (with $w_g = w_{g,0}$ and $w_\gamma = w_{\gamma,0}$) as well as a per-period policy $(a_{g,n}, a_{\gamma,n}, c_n(y), W_{g,n}(y), W_{\gamma,n}(y))$, where $W_{g,n}(y)$ and $W_{\gamma,n}(y)$ are the continuation value functions at the end of the period for the respective noise densities. The policies are such that the promise keeping and the incentive compatibility

[38] All the objects introduced in this section also depend on the history of public signals $(y_0, \dots, y_{n-1})$ and so can be treated as random variables. Throughout the section, we will suppress it from the notation.

constraints are satisfied:

$$w_{\phi,n} = \mathbb{E}_{\phi}^{\Delta}\big[\tilde{r}\Delta\big[u\big(c_n\big(\Delta(x+a_{\phi,n})\big)\big) - h(a_{\phi,n})\big] + e^{-r\Delta}W_{\phi,n}\big(\Delta(x+a_{\phi,n})\big)\big], \tag{PK$_2$}$$

$$a_{\phi,n} \in \arg\max_{\hat{a}\in\mathcal{A}} \mathbb{E}_{\phi}^{\Delta}\big[\tilde{r}\Delta\big[u\big(c_n\big(\Delta(x+\hat{a})\big)\big) - h(\hat{a})\big] + e^{-r\Delta}W_{\phi,n}\big(\Delta(x+\hat{a})\big)\big], \tag{IC$_2$}$$

for $\phi \in \{g, \gamma\}$. Let $p = \{(a_{g,n}, a_{\gamma,n}, c_n(y), W_{g,n}(y), W_{\gamma,n}(y))\}_{n\in\mathbb{N}}$ be the complete dynamic policy function. Finally, let $F_{\phi,n}^{\Delta,p}(w_{\phi,n})$ be the principal's continuation value from period n onwards, and for a function $f : [0, \bar{u}) \to \mathbb{R}$, define $T_{\phi,n}^{\Delta,p}(f) = \Phi_{\phi}^{\Delta}(a_{\phi,n}, c_n, W_{\phi,n}; f)$, for $\phi \in \{g, \gamma\}$. Thus $T_{\phi,n}^{\Delta,p}(f)$ is the principal's continuation value if he follows the policy p in period n and the continuation value in period $n+1$ is given by f.

To establish the proposition, we show that if $w_g, w_\gamma \in (0, w_{\text{sp}})$, then there is $\delta > 0$ such that, for sufficiently small Δ, $F_{g,0}^{\Delta,p}(w_g) + F_{\gamma,0}^{\Delta,p}(w_\gamma) \le F(w_g) + F(w_\gamma) - \delta$, where F is as in Theorem 1.

For the proof of the proposition, we use the following five claims. Claim 1 is related to Lemma 5. It shows roughly that for a given contract $\{c_n\}$ and incentive compatible action plans $\{a_{g,n}\}, \{a_{\gamma,n}\}$ and the policies p they give rise to, how far the value of the contracts generated by them falls short of F $(F(w_g) + F(w_\gamma) - F_{g,0}^{\Delta,p}(w_g) - F_{\gamma,0}^{\Delta,p}(w_\gamma))$ can be expressed as a discounted expected sum of how far each policy applied to F falls short of F $(F(w_{g,n}) - T_{g,n}^{\Delta,p}(F) + F(w_{\gamma,n}) - T_{\gamma,n}^{\Delta,p}(F))$. Taking the expectation with respect to the density $\zeta(y) = \min\{g(y), \gamma(y)\}$ provides a lower bound and simplifies the analysis.

The idea behind the construction in the remaining four claims is as follows. For any $\varepsilon > 0$, consider the set $S_\varepsilon = \{(w_g, w_\gamma) \in [\varepsilon, w_{\text{sp}} - \varepsilon]^2 : |w_g - w_\gamma| > \varepsilon, \max\{w_g, w_\gamma\} > w_0 + \varepsilon\}$, where w_0 is such that $F'(w_0) = \underline{F}'(0) = -\frac{1}{u'(0)}$. Claim 2 shows that once the pair of continuation values $(w_{g,n}, w_{\gamma,n})$ are in this set, $F(w_{g,n}) - T_{g,n}^{\Delta,p}(F) + F(w_{\gamma,n}) - T_{\gamma,n}^{\Delta,p}(F)$ must be negative. The reason is that, to achieve $F(w_{g,n}) + F(w_{\gamma,n})$, the wages paid in the separate two optimal policies for each noise distribution must be different (such that $-1/u'(c_g) = F'(w_g)$, and $-1/u'(c_\gamma) = F'(w_\gamma)$), whereas the single contract restricts the per-period policy to have the same wage for each distribution.

Claim 3 shows that if $F(w_{g,n}) - T_{g,n}^{\Delta,p}(F) + F(w_{\gamma,n}) - T_{\gamma,n}^{\Delta,p}(F)$ is to remain small, it must be that the variances (under density ζ) of $W_g - W_\gamma$ must be bounded away from zero, and the variances of continuation values W_g, W_γ not too big. This follows from the results in the paper: for the policy p to fare well, the continuation values for each noise must be approximately linear in likelihood ratio. Also, since the likelihood ratios are linearly independent by assumption, $W_g - W_\gamma$ cannot be too small. Using Claim 3, Claim 4 shows that, under policies p, once the process of continuation values (w_g, w_γ) enters set S_ε, it must stay there for a while with nonnegligible probability (under ζ); Claim 5 shows that, starting at any interior point of continuation values, the process enters S_ε in finite time with nonnegligible probabilities. Those results, together with Claim 2, establish the proposition.

Define $\zeta(y) = \min\{g(y), \gamma(y)\}$ (and accordingly $\zeta^\Delta(y) = \min\{g^\Delta(y), \gamma^\Delta(y)\}$).

CLAIM 1: *For the function F as in Theorem 1 and any $N \in \mathbb{N}$,*

$$\begin{aligned}
&F(w_g) + F(w_\gamma) - F^{\Delta,p}_{g,0}(w_g) - F^{\Delta,p}_{\gamma,0}(w_\gamma)\\
&\quad\geq \mathbb{E}^\Delta_\zeta\Bigg[\sum_{n=0}^{N} e^{-rn\Delta}\big(F(w_{g,n}) - T^{\Delta,p}_{g,n}(F) + F(w_{\gamma,n}) - T^{\Delta,p}_{\gamma,n}(F)\big)\\
&\qquad + e^{-r(N+1)\Delta}\big(F(w_{g,N+1}) - F^{\Delta,p}_{g,N+1}(w_{g,N+1})\\
&\qquad + F(w_{\gamma,N+1}) - F^{\Delta,p}_{\gamma,N+1}(w_{\gamma,N+1})\big)\Bigg].
\end{aligned}$$

PROOF: For any $(w_g, w_\gamma) \in [0, \bar{u})^2$, we have

$$\begin{aligned}
&F(w_g) + F(w_\gamma) - F^{\Delta,p}_{g,0}(w_g) - F^{\Delta,p}_{\gamma,0}(w_\gamma)\\
&\quad= F(w_g) + F(w_\gamma) - T^{\Delta,p}_{g,0}\big(F^{\Delta,p}_{g,1}\big) - T^{\Delta,p}_{\gamma,0}\big(F^{\Delta,p}_{\gamma,1}\big)\\
&\quad= F(w_g) + F(w_\gamma) - T^{\Delta,p}_{g,0}(F) - T^{\Delta,p}_{\gamma,0}(F)\\
&\qquad + T^{\Delta,p}_{g,0}(F) + T^{\Delta,p}_{\gamma,0}(F) - T^{\Delta,p}_{g,0}\big(F^{\Delta,p}_{g,1}\big) - T^{\Delta,p}_{\gamma,0}\big(F^{\Delta,p}_{\gamma,1}\big)\\
&\quad= F(w_g) + F(w_\gamma) - T^{\Delta,p}_{g,0}(F) - T^{\Delta,p}_{\gamma,0}(F)\\
&\qquad + e^{-r\Delta}\mathbb{E}^\Delta_g\big[F(w_{g,1}) - F^{\Delta,p}_{g,1}(w_{g,1})\big]\\
&\qquad + e^{-r\Delta}\mathbb{E}^\Delta_\gamma\big[F(w_{\gamma,1}) - F^{\Delta,p}_{\gamma,1}(w_{\gamma,1})\big]\\
&\quad\geq F(w_g) + F(w_\gamma) - T^{\Delta,p}_{g,0}(F) - T^{\Delta,p}_{\gamma,0}(F)\\
&\qquad + e^{-r\Delta}\mathbb{E}^\Delta_\zeta\big[F(w_{g,1}) - F^{\Delta,p}_{g,1}(w_{g,1}) + F(w_{\gamma,1}) - F^{\Delta,p}_{\gamma,1}(w_{\gamma,1})\big].
\end{aligned}$$

Iterating the inequality yields the proof. *Q.E.D.*

CLAIM 2: *For $\varepsilon > 0$, there is δ_1 such that, for any $(w_{g,n}, w_{\gamma,n}) \in S_\varepsilon$ and a sufficiently small $\Delta > 0$,*

$$F(w_{g,n}) - T^{\Delta,p}_{g,n}(F) + F(w_{\gamma,n}) - T^{\Delta,p}_{\gamma,n}(F) > \delta_1 \Delta.$$

PROOF: Let us define $T^{\Delta,q,p}_{\phi,n}(f) = \Phi^{\Delta,q}_{\phi}(a_{\phi,n}, c_n, W_{\phi,n}; f, w_{\phi,n})$, for $\phi \in \{g, \gamma\}$. Using analogues to Lemmas 12 and 14, we establish that, for any $\delta' > 0$, there is δ such that, for sufficiently small Δ, if $F(w_{\phi,n}) - T^{\Delta,p}_{\phi,n}(F) < \delta\Delta$, then

$$\left|T^{\Delta,q,p}_{\phi,n}(F) - T^{\Delta,p}_{\phi,n}(F)\right| < \delta'\Delta,$$

for $\phi \in \{g, \gamma\}$.

Fix $\varepsilon > 0$. In view of the above bound, it is sufficient to establish that there is δ_1 such that, for $(w_{g,n}, w_{\gamma,n}) \in S_\varepsilon$, we have $F(w_{g,n}) - T^{\Delta,q,p}_{g,n}(F) + F(w_{\gamma,n}) - T^{\Delta,q,p}_{\gamma,n}(F) > \delta_1\Delta$, and so, due to Proposition 6 and Lemmas 12 and 14, to show that

$$T^{\Delta,q}_{g}F(w_{g,n}) - T^{\Delta,q,p}_{g,n}(F) + T^{\Delta,q}_{\gamma}F(w_{\gamma,n}) - T^{\Delta,q,p}_{\gamma,n}(F) > \delta_1\Delta,$$

where $T^{\Delta,q}_{g}$ and $T^{\Delta,q}_{\gamma}$ stand for operator $T^{\Delta,q}$ under the respective noise densities.

We have

$$\begin{aligned}
&T^{\Delta,q}_{\phi}F(w_{\phi,n}) \\
&\quad = \sup_c -\tilde{r}\Delta\big\{c + F'(w_{\phi,n})u(c)\big\} + \sup_{a,W} \Psi^{\Delta}_{\phi}(a, W; F, w_{\phi,n}), \\
&T^{\Delta,q,p}_{g,n}(F) + T^{\Delta,q,p}_{\gamma,n}(F) \\
&\quad = -\tilde{r}\Delta\big\{2c_n + F'(w_{g,n})u(c_n) + F'(w_{\gamma,n})u(c_n)\big\} \\
&\qquad + \Psi^{\Delta}_{g}(a_{g,n}, W_{g,n}; F, w_{g,n}) + \Psi^{\Delta}_{\gamma}(a_{\gamma,n}, W_{\gamma,n}; F, w_{\gamma,n}),
\end{aligned}$$

where

$$\begin{aligned}
\Psi^{\Delta}_{\phi}(a, W; F, w) &= e^{-\Delta r}F(w) + \tilde{r}\Delta\big\{a + F'(w)\big(w + h(a)\big)\big\} \\
&\qquad + e^{-\Delta r}\mathbb{E}^{\Delta}_{\phi}\left[\frac{1}{2}F''(w)\big(W(\Delta x) - w\big)^2\right],
\end{aligned}$$

$\phi \in \{g, \gamma\}$. Thus, we have

$$\begin{aligned}
&T^{\Delta,q}_{g}F(w_{g,n}) - T^{\Delta,q,p}_{g,n}(F) + T^{\Delta,q}_{\gamma}F(w_{\gamma,n}) - T^{\Delta,q,p}_{\gamma,n}(F) \\
&\quad \geq \sup_c -\tilde{r}\Delta\big\{c + F'(w_{g,n})u(c)\big\} + \sup_c -\big\{c + F'(w_{\gamma})u(c)\big\} \\
&\qquad + \tilde{r}\Delta\big\{2c_n + F'(w_{g,n})u(c_n) + F'(w_{\gamma,n})u(c_n)\big\} \\
&\quad > \delta_1,
\end{aligned}$$

for some $\delta_1 > 0$. The second inequality follows from the strict concavity of u and the fact that F'' is bounded away from 0, and so $|F'(w_{g,n}) - F'(w_{\gamma,n})|$ is bounded away from zero as long as $|w_{g,n} - w_{\gamma,n}| > \varepsilon$. *Q.E.D.*

CLAIM 3: *For* $\varepsilon > 0$, *there is* $\delta_2 > 0$ *such that*, *for any* $(w_{g,n}, w_{\gamma,n}) \in [\varepsilon, w_{\mathrm{sp}} - \varepsilon]^2$ *and a sufficiently small* Δ, *if*

$$F(w_{g,n}) - T^{\Delta,p}_{g,n}(F) + F(w_{\gamma,n}) - T^{\Delta,p}_{\gamma,n}(F) < \delta_2\Delta, \tag{41}$$

then

$$\mathbb{V}^{\Delta}_{\zeta}\big[W_{g,n}\big(\Delta(x + a_{g,n})\big) - W_{\gamma,n}\big(\Delta(x + a_{\gamma,n})\big)\big] > \delta_2\Delta, \tag{42}$$

as well as

$$\begin{aligned} &F(w_{g,n}) - T^{\Delta,p}_{g,n}(F) + F(w_{\gamma,n}) - T^{\Delta,p}_{\gamma,n}(F) \\ &\quad > \delta_2\left(\mathbb{V}^{\Delta}_{g}\big[W_{g,n}\big(\Delta(x + a_{g,n})\big)\big] - \Delta\frac{\big(rh'(A)\big)^2}{\mathcal{I}_{gX}}\right) \\ &\qquad + \delta_2\left(\mathbb{V}^{\Delta}_{\gamma}\big[W_{\gamma,n}\big(\Delta(x + a_{\gamma,n})\big)\big] - \Delta\frac{\big(rh'(A)\big)^2}{\mathcal{I}_{\gamma X}}\right). \end{aligned} \tag{43}$$

PROOF: Lemmas 20 and 8 imply that, for certain $\delta_2 > 0$ and sufficiently small Δ, if (41) holds and $(w_{g,n}, w_{\gamma,n}) \in [\varepsilon, w_{\mathrm{sp}} - \varepsilon]^2$, then $a_g, a_\gamma > \gamma > 0$. But then Lemmas 8 and 1 imply that $W_{\phi,n}(\Delta(x_{\phi,n} + a_{\phi,n})) \approx \mathbb{E}^{\Delta}_{\phi}[W_{\phi,n}(\Delta(x + a_{\phi,n}))] + \sqrt{\Delta}D_\phi \frac{g'_\phi(x)}{g_\phi(x)}$ (in $L_2(\phi^\Delta)$ and so in $L_2(\zeta^\Delta)$), for $\phi \in \{g, \gamma\}$. Thus the first inequality follows from (14). On the other hand, F'' bounded away from zero immediately implies the second inequality. *Q.E.D.*

CLAIM 4: *For* $\varepsilon > 0$ *there are* $\delta_3, T > 0$ *such that*, *for any* $(w_{g,0}, w_{\gamma,0}) \in S_\varepsilon$ *and a sufficiently small* Δ,

$$\mathbb{E}^{\Delta}_{\zeta}\left[\sum_{n=0}^{T/\Delta} e^{-rn\Delta}\big(F(w_{g,n}) - T^{\Delta,p}_{g,n}(F) + F(w_{\gamma,n}) - T^{\Delta,p}_{\gamma,n}(F)\big)\right] \le \delta_3$$

implies

$$\mathbb{P}^{\Delta}_{\zeta}\big[(w_{g,n}, w_{\gamma,n}) \in S_{\varepsilon/2}, n = 0, \dots, T/\Delta\big] > \delta_3.$$

PROOF: If the precondition is satisfied, then (43) in Claim 3 implies that, for $\phi \in \{g, \gamma\}$,

$$\mathbb{V}^{\Delta}_{\phi}\big[(w_{\phi,n} - w_{\phi,n'})\big] \le \frac{1}{\delta_2}\left(\frac{T\big(rh'(A)\big)^2}{\mathcal{I}_{gX}} + \delta_3\right) =: C_{T,\delta_3}$$

for $n < n' \le T/\Delta$, with $C_{T,\delta_3} \to 0$ as T, $\delta_3 \to 0$. We also have

$$\mathbb{E}_\phi^\Delta\big[(w_{\phi,n'} - w_{\phi,n})\big]^2 \le D_{T,\delta_3}$$

for $n < n' \le T/\Delta$, with $D_{T,\delta_3} \to 0$ as T, $\delta_3 \to 0$. Thus if τ is the stopping time of the process $|w_{\phi,t} - w_{\phi,0}|$ reaching the set $[\alpha, \infty)$, we have

$$\begin{aligned}
&\mathbb{P}_\phi^\Delta\Big[\max_{n\le T/\Delta} |w_{\phi,n} - w_{\phi,0}| \ge \alpha\Big] \\
&\quad = \mathbb{P}_\phi^\Delta\Big[\max_{n\le T/\Delta} |w_{\phi,n} - w_{\phi,0}| \ge \alpha, |w_{\phi,T/\Delta} - w_{\phi,0}| \ge \alpha/2\Big] \\
&\qquad + \mathbb{P}_\phi^\Delta\Big[\max_{n\le T/\Delta} |w_{\phi,n} - w_{\phi,0}| \ge \alpha, |w_{\phi,T/\Delta} - w_{\phi,0}| < \alpha/2\Big] \\
&\quad \le \mathbb{P}_\phi^\Delta\big[|w_{\phi,T/\Delta} - w_{\phi,0}| \ge \alpha/2\big] + \mathbb{P}_\phi^\Delta\big[|w_{\phi,T/\Delta} - w_{\phi,\tau}| \ge \alpha/2\big] \\
&\quad \le 2\frac{C_{T,\delta_3} + D_{T,\delta_3}}{\alpha^2/4}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\mathbb{P}_\zeta^\Delta\big[(w_{g,n}, w_{\gamma,n}) \notin S_{\varepsilon/2} \text{ for some } n \le T/\Delta | (w_{g,0}, w_{\gamma,0}) \in S_\varepsilon\big] \\
&\quad \le \mathbb{P}_\zeta^\Delta\Big[\max_{n\le T/\Delta} |w_{g,n} - w_{g,0}| \ge \varepsilon/4 \text{ or } \max_{n\le T/\Delta} |w_{\gamma,n} - w_{\gamma,0}| \ge \varepsilon/4\Big] \\
&\quad \le \mathbb{P}_\zeta^\Delta\Big[\max_{n\le T/\Delta} |w_{g,n} - w_{g,0}| \ge \varepsilon/4\Big] + \mathbb{P}_\zeta^\Delta\Big[\max_{n\le T/\Delta} |w_{\gamma,n} - w_{g,0}| \ge \varepsilon/4\Big] \\
&\quad \le \mathbb{P}_g^\Delta\Big[\max_{n\le T/\Delta} |w_{g,n} - w_{g,0}| \ge \varepsilon/4\Big] + \mathbb{P}_\gamma^\Delta\Big[\max_{n\le T/\Delta} |w_{\gamma,n} - w_{g,0}| \ge \varepsilon/4\Big] \\
&\quad \le 8\frac{C_{T,\delta_3} + D_{T,\delta_3}}{\varepsilon^2/16} \to 0,
\end{aligned}$$

as T, $\delta_3 \to 0$. This establishes the claim. *Q.E.D.*

CLAIM 5: *For $\varepsilon > 0$, there are $\delta_4, T > 0$ such that, for any $(w_{g,0}, w_{\gamma,0}) \in [\varepsilon, w_{\text{sp}} - \varepsilon]^2$ and a sufficiently small Δ,*

$$\mathbb{E}_\zeta^\Delta\left[\sum_{n=0}^{T/\Delta-1} e^{-rn\Delta}\big(F(w_{g,n}) - T_{g,n}^{\Delta,p}(F) + F(w_{\gamma,n}) - T_{\gamma,n}^{\Delta,p}(F)\big)\right] \le \delta_3$$

implies

$$\mathbb{P}_\zeta^\Delta\big[(w_{g,T/\Delta}, w_{\gamma,T/\Delta}) \in S_\varepsilon\big] > \delta_4.$$

PROOF: The proof relies on (42) in Claim 3. It is similar to the proof of the previous claim and is omitted. *Q.E.D.*

Given the claims, the rest of the proof is as follows. If $(w_g, w_\gamma) \in S_\varepsilon$, then, for the constants as in the claims,

$$\begin{aligned} &F(w_g) + F(w_\gamma) - F_g^{\Delta,p}(w_g) - F_\gamma^{\Delta,p}(w_\gamma) \\ &\quad \geq \mathbb{E}_\zeta^\Delta\left[\sum_{n=0}^{N} e^{-rn\Delta}\big(F(w_{g,n}) - T_{g,n}^{\Delta,p}(F) + F(w_{\gamma,n}) - T_{\gamma,n}^{\Delta,p}(F)\big)\right] \\ &\quad \geq \min\left\{\delta_3, \frac{1-e^{-rT}}{1-e^{-r\Delta}}\delta_3\delta_1\right\}, \end{aligned}$$

where the first inequality follows from Claim 1 and the second inequality follows from Claims 2 and 4.

If, on the other hand, $(w_g, w_\gamma) \in [\varepsilon, w_{\text{sp}} - \varepsilon]^2 \setminus S_\varepsilon$, then

$$\begin{aligned} &F(w_g) + F(w_\gamma) - F_g^{\Delta,p}(w_g) - F_\gamma^{\Delta,p}(w_\gamma) \\ &\quad \geq \mathbb{E}_\zeta^\Delta\Bigg[\sum_{n=0}^{T/\Delta-1} e^{-rn\Delta}\big(F(w_{g,n}) - T_{g,n}^{\Delta,p}(F) + F(w_{\gamma,n}) - T_{\gamma,n}^{\Delta,p}(F)\big) \\ &\qquad + e^{-rT}\big(F(w_{g,T/\Delta}) - F_{g,T/\Delta}^{\Delta,p}(w_{g,T/\Delta}) + F(w_{\gamma,T/\Delta}) - F_{\gamma,T/\Delta}^{\Delta,p}(w_{\gamma,T/\Delta})\big)\Bigg] \\ &\quad \geq \min\left\{\delta_4, e^{-rT}\delta_4 \min\left\{\delta_3, \frac{1-e^{-rT}}{1-e^{-r\Delta}}\delta_3\delta_1\right\}\right\}, \end{aligned}$$

where the first inequality follows from Claim 1 and the second inequality follows from Claim 5 and the inequalities above. This establishes the proof of the proposition.

We note that the proof can be extended beyond the pure hidden action case and $\mathcal{I}_{gX} = \mathcal{I}_{\gamma X}$. As regards the equality of Fisher information quantities, this guarantees that the limits of the values of contracts F_g and F_γ for two noise distributions are the same function F (Lemma 1). Because of that, as long as the continuation values w_g and w_γ are not the same, the derivatives $F_g'(w_g)$ and $F_\gamma'(w_\gamma)$ differ as well, which is crucial for Claim 2. Dropping the assumption $\mathcal{I}_{gX} = \mathcal{I}_{\gamma X}$, the proof would be analogous, yet the computation of the set of continuation values (w_g, w_γ) for which $F_g'(w_g) \neq F_\gamma'(w_\gamma)$ would be cumbersome.

On the other hand, the assumption of pure hidden action models was also not crucial for the proof: For two different information structures, the proof will work as long as, roughly, the optimal policies in the problem of minimizing variance of continuation values are sufficiently different (see Claim 3).

APPENDIX F: PROOFS FOR SECTION 5.1

In this section, we establish Theorem 3 and the analogue of Theorem 2, which takes the following form (see the definition of simple contract action plan below):

THEOREM 4: *For $\zeta > 0$, let F_ζ be as in Theorem 3 and fix period length Δ, agent's promised value $w \in [0, \bar{u})$, and an approximation error $\varepsilon > 0$. A corresponding simple contract-action plan is incentive compatible by construction and $[O(\varepsilon) + O(\Delta^{1/3}) + O(\zeta)]$-suboptimal.*

The proof of the theorems follows just as in Appendix A from Lemma 5 and the following version of Proposition 6, which is proven in Section F.1.

PROPOSITION 7: *Fix $\zeta \geq 0$ and F_ζ solving the HJB equation (19) on an interval I with $F''_\zeta < 0$. Then $|T_I^\Delta F_\zeta - F_\zeta|_{I^\Delta} = o(\Delta) + O(\zeta\Delta)$. Moreover, for any $\varepsilon > 0$, $\Delta > 0$, and $w \in I^\Delta$, $\Phi^\Delta(a, c, W; F_\zeta) \geq F_\zeta(w) - O(\varepsilon\Delta) - O(\zeta\Delta)$, where (a, c, W) is a simple policy defined for $(F_\zeta, \varepsilon, \Delta, w)$ by (11) and (12).*

The simple contract-action plans are defined almost identically to those in Section 3.2 as follows. First, let us define the appropriate Bellman operators as in Section 3.2. For an interval $I \subset \mathbb{R}$ and any function $f : I \to \mathbb{R}$, define the new function $T_I^\Delta f : I \to \mathbb{R}$ by

$$T_I^\Delta f(w) = \sup_{a,c,W} \Phi^\Delta(a, c, W; f) \tag{44}$$

subject to

$$a(z) \in \mathcal{A} \quad \forall z, \quad c(y) \geq 0 \quad \text{and} \quad W(y) \in I \quad \forall y,$$

$$w = \mathbb{E}^\Delta\big[\tilde{r}\Delta\big[u\big(c\big(\Delta\big(x + a(z)\big)\big)\big) - h\big(a(z)\big)\big] + e^{-r\Delta} W\big(\Delta\big(x + a(z)\big)\big)\big], \tag{PK}$$

$$a(x) \in \arg\max_{\hat{a} \in \mathcal{A}} \tilde{r}\Delta\big[u\big(c\big(\Delta(x + \hat{a})\big)\big) - h(\hat{a})\big] + e^{-r\Delta} W\big(\Delta(x + \hat{a})\big) \quad \forall x. \tag{IC-PHI}$$

We note that the Belman operator T_I^Δ excludes reporting by the agent. However, in the pure hidden information case, this is without loss of generality: With reporting, there may not exist two different noise realizations resulting in the same signal in equilibrium (as incentive compatibility would be violated). Thus, reporting is redundant.

Consider the following definition of simple policies (compare Definitions 1 and 3).

DEFINITION 4: For any $\zeta \geq 0$ and F_ζ solving (19) on an interval I, period length $\Delta > 0$, agent's promised value $w \in I$, and an approximation error $\varepsilon > 0$, define a *simple policy* (a, c, W) as follows. Let $(\bar{a}, \bar{h}, c)$ be an ε-suboptimal policy of (19) at w, and for the corresponding $(\bar{a}, \bar{h})$, let (a, v) be an ε-suboptimal policy of (20).

If $w \in I^\Delta$, let

$$c(y) = c,$$

$$W(y) = C + \sqrt{\Delta}\tilde{r}e^{r\Delta} \times \begin{cases} v(-M) & \text{if } y/\sqrt{\Delta} < -M, \\ v(y/\sqrt{\Delta}) & \text{if } |y/\sqrt{\Delta}| \leq M, \\ v(M) & \text{if } y/\sqrt{\Delta} > M, \end{cases}$$

$a(z)$ is an action that satisfies the (IC) constraint in (44),

where M is such that $\mathbb{P}_X([-M, M]) \geq 1 - \varepsilon$ and C is chosen to satisfy the (PK) constraint in (44). If $w \notin I^\Delta$, define the policy as in (12).

The definition differs from the one in Section 3.2 in that: (i) argument function is F_ζ, not F, (ii) reporting is ignored, (iii) continuation value function must be nondecreasing, (iv) range of signals for which incentives are provided (or M_ε) is readjusted. Given the above definition, simple contract-action plans are defined as in Definition 1.

Notice that, unlike in the model analyzed in the paper, there is no additional incentive compatibility constraint associated with truthful reporting, and so, by construction, simple policies are fully incentive compatible. Also, as before, (PK) is satisfied by construction, and $W(y) \in I$ if Δ is sufficiently small. Thus, simple policies are feasible for the problem (44), and so Proposition 7 verifies only that they are close to optimal.

F.1. *Proof of Proposition 7*

As in the paper, define $T_I^{\Delta,c}$ by restricting the consumption schedule $c(y)$ to be constant. Let us also define $T_I^{\Delta,d} f(w)$ as $T_I^{\Delta,c} f(w)$ with the additional constraints that $a(\cdot)$ is piecewise continuously differentiable and $W(\cdot)$ is continuous. Finally, we modify the simplified operator $T^{\Delta,q}$ defined in (4) by replacing the local (first-order) incentive constraint (FOC$_q$) by [39]

$$\tilde{r}h'\big(a(x)\big) = e^{-r\Delta}W'(\Delta x) \quad \forall x. \tag{FOC$_q$-PHI}$$

[39]When $a(z) = 0$ or $a(z) = A$, at an optimum the inequalities in the (IC) constraint are attained with equality (see, e.g., Edmans and Gabaix (2011)).

The proof of Proposition 7 is established by a sequence of lemmas, similarly as in Appendix A. Regarding the values, the line of the argument can be illustrated as follows:

$$F \underset{\text{Lemma 22}}{\sim} T^{\Delta,q}F \underset{\text{Lemma 23}}{\sim} T_I^{\Delta,d}F \underset{\text{Lemma 25}}{\sim} T_I^{\Delta,c}F \underset{\text{Lemma 14}}{\sim} T_I^{\Delta}F.$$

Note that the last equivalence follows from the same lemma as in the paper. Here we focus on the other three.

First, Lemma 8 extends readily to the current pure hidden information case. Likewise, we extend the definition of *quadratic simple* policies (see Definition 2).[40]

REMARK 1: In the pure hidden information case, the v in the definition of a quadratic simple policy at w is continuous and piecewise twice continuously differentiable (see the definition of Θ). We assume that for any $\varepsilon > 0$, there is a common finite set D such that the set of functions v'' for all $w \in I$ are equicontinuous outside of D, which is without loss of generality.

The following is essentially a corollary of Lemma 8.

LEMMA 22: *Fix* $\zeta \geq 0$ *and* F_ζ *solving the HJB equation* (19) *on an interval* I *with* $F_\zeta'' < 0$. *Then* $|T^{\Delta,q}F_\zeta - F_\zeta|_I = o(\Delta) + O(\zeta\Delta)$. *Moreover, for any* $\varepsilon, \Delta > 0$, $w \in I$, *and corresponding quadratic simple policy* (a_q, c_q, W_q), $\Phi^{\Delta,q}(a_q, c_q, W_q; F_\zeta, w) \geq F_\zeta(w) - O(\Delta\varepsilon) - O(\zeta\Delta)$, *uniformly in* I.

PROOF: From Lemma 8, we have

$$\begin{aligned}
&T^{\Delta,q}F_\zeta(w) - F_\zeta(w) \\
&\quad = \sup_{\bar{a},\bar{h},c} \tilde{r}\Delta\Big\{(\bar{a} - c) + F_\zeta'(w)\big(w + \bar{h} - u(c)\big) \\
&\qquad + e^{r\Delta}\frac{\tilde{r}}{2}F_\zeta''(w)\Theta(\bar{a}, \bar{h}) - F_\zeta(w)\Big\} + O\big(\Delta^2\big) \\
&\quad = O(\zeta\Delta) + O\big(\Delta^2\big).
\end{aligned}$$

The last equality follows because F_ζ satisfies the HJB equation (19). Lemma 8 also yields that $\Phi^{\Delta,q}(a_q, c_q, W_q; F_\zeta, w) \geq F_\zeta(w) - O(\Delta^2) - O(\Delta\varepsilon) - O(\zeta\Delta)$, establishing the proof. *Q.E.D.*

We establish now the crucial Lemma 23, the analogue of Lemma 12 in the paper. First, we extend the general definition of simple policies to the pure hidden information case (compare Definition 3 in the paper).

[40]Note that since the reporting is suppressed, the continuation value functions v in the definition of Θ and W_Δ^q in the definition of quadratic simple policies depend only on a single variable y.

DEFINITION 5: For a twice differentiable function $F : I \to \infty$ with $F'' < 0$, $\varepsilon > 0$, $\Delta > 0$, $w \in I^\Delta$, and quadratic simple policies (a_q, c_q, W_q) in the problem $T^{\Delta,q}F(w)$ based on (a, v), define the *simple policy* (a, c, W) for $T_I^{\Delta,c}F(w)$ as

$$c = c_q,$$

$$W(y) = C + \begin{cases} W_q(-\sqrt{\Delta}M_\varepsilon) & \text{if } \Delta x < -\sqrt{\Delta}M_\varepsilon, \\ W_q(\Delta x) & \text{if } |\Delta x| \le \sqrt{\Delta}M_\varepsilon, \\ W_q(\sqrt{\Delta}M_\varepsilon) & \text{if } \Delta x > \sqrt{\Delta}M_\varepsilon, \end{cases}$$

$a(z)$ is an action that satisfies the (IC) constraint in (44),

where M_ε is such that $\mathbb{P}_X([-M_\varepsilon, M_\varepsilon]) \ge 1 - \varepsilon$ and C is chosen to satisfy the (PK) constraint in (44).

LEMMA 23: *Let $F : I \to \mathbb{R}$ be twice continuously differentiable with $F'' < 0$. Then $|T_I^{\Delta,c}F - T^{\Delta,q}F|_{I^\Delta} = o(\Delta)$. Moreover, for fixed $\varepsilon > 0$, $\Delta > 0$, and $w \in I^\Delta$, consider quadratic simple policy (a_q, c_q, W_q) for $T^{\Delta,q}F(w)$. If Δ and ε are sufficiently small, for the corresponding simple policy (a, c, W), $\Phi^\Delta(a, c, W; F) \ge \Phi^{\Delta,q}(a_q, c_q, W_q; F, w) - O(\varepsilon\Delta) - o(\Delta)$, uniformly in w.*

PROOF: Fix $\varepsilon > 0$, $\Delta > 0$ such that $\sqrt{\Delta} < \delta/A$, for δ as in Lemma 11 (with $M = M_\varepsilon$), and $w \in I^\Delta$.

Step 1: In this step, we show that $\Phi^\Delta(a, c, W; F) \ge \Phi^{\Delta,q}(a_q, c_q, W_q; F, w) - O(\varepsilon\Delta)$, uniformly in w. Since ε is arbitrary, by Lemma 8, this establishes $|T^{\Delta,q}F - T_I^{\Delta,d}F|^+_{I^\Delta} = o(\Delta)$.

First, the inequality (29) holds by the same arguments as before. It will thus be enough to establish (32), (33), and (34).

Given the definition of W, the necessary local version of (IC) takes the following form:[41]

$$\tilde{r}h'\big(a(x)\big) = e^{-r\Delta}W'(y) = \tilde{r}v'\big(\sqrt{\Delta}\big(x + a(x)\big)\big), \tag{45}$$

whereas, given the definition of W_q and (FOC$_q$-PHI), we have

$$\tilde{r}h'\big(a_q(x)\big) = e^{-r\Delta}W_q'(\Delta x) = \tilde{r}v'(\sqrt{\Delta}x).$$

Let D be the finite set of points such that each v in the definition of the policy is twice continuously differentiable on $\mathbb{R} \setminus D$ (see Remark 1) and consider the set

$$N_\varepsilon^\Delta = [-M_\varepsilon/\sqrt{\Delta}, M_\varepsilon/\sqrt{\Delta} - A] \setminus \bigcup_{d \in D} \big\{d/\sqrt{\Delta} + \zeta : \zeta \in [0, A]\big\}.$$

[41]Recall that the W function, just as W_q, is constant in the second argument.

For sufficiently small Δ, $\mathbb{P}^{\Delta}[N_{\varepsilon}^{\Delta}] \geq 1-\varepsilon$. Moreover, for any $x \in N_{\varepsilon}^{\Delta}$, v' is continuously differentiable on $[\sqrt{\Delta}x, \sqrt{\Delta}(x+a(x))]$. Consequently, for all such x, $|h'(a_q(x)-h'(a(x))| \leq \sqrt{\Delta}\max v''$, where the maximum is taken over the set $[-M_{\varepsilon}, M_{\varepsilon}]$, and hence

$$\left|a_q(x)-a(x)\right| \leq \frac{\sqrt{\Delta}\max v''}{\inf h''}.$$

Since $\mathbb{P}^{\Delta}[N_{\varepsilon}^{\Delta}] \geq 1-\varepsilon$, we have that the inequalities (32) and (33) hold. Moreover, by taking the maximum over $\max v''$ over $[-M_{\varepsilon}, M_{\varepsilon}]$ for all w (which is well defined, due to the assumption of equicontinuity), we establish that the bounds in those inequalities are uniform in $w \in I^{\Delta}$. Finally, (34) follows from Lemma 10 just as in the previous case. This establishes the proof.

Step 2: In this step, we show that $|T_I^{\Delta,d}F(w)-T^{\Delta,q}F(w)|_{I^{\Delta}}^{+}=o(\Delta)$.[42]

For a policy (a,c,W) that is $\varepsilon\Delta$-suboptimal in the problem $T_I^{\Delta}, dF(w)$, define (a_q,c_q,W_q) as follows. Let $c_q=c$, $a_q(x)=a(x)$ for $x \in [-M_{\varepsilon}/\sqrt{\Delta}+1, M_{\varepsilon}/\sqrt{\Delta}-1]$, $a_q(x)=0$ for $x \notin [-M_{\varepsilon}/\sqrt{\Delta}, M_{\varepsilon}/\sqrt{\Delta}]$, and a_q piecewise continuously differentiable. W_q is constant in the second argument and is defined by the local IC in (4), continuity, and (PK). The policy (a_q,c_q,W_q) is feasible by construction, and we must prove that $\Phi^{\Delta}, q(a_q,c_q,W_q;F,w) \geq \Phi^{\Delta}(a,c,W;F)-O(\varepsilon\Delta)$.

On the one hand, $\mathbb{P}^{\Delta}[a_q(x)=a(x)] \geq 1-2\varepsilon$ for sufficiently small Δ, which implies the analogues of (32) and (33). On the other hand, for all $\underline{x}, \bar{x} \in [-M_{\varepsilon}/\sqrt{\Delta}, M_{\varepsilon}/\sqrt{\Delta}]$,

$$\begin{aligned}
&W_q(\Delta\bar{x},z)-W_q(\Delta\underline{x},z)\\
&\quad=\tilde{r}e^{r\Delta}\int_{\underline{x}}^{\bar{x}}\Delta h'\big(a_q(x)\big)\,dx=\tilde{r}e^{r\Delta}\int_{\underline{x}}^{\bar{x}}\Delta h'\big(a(x)\big)\,dx\\
&\quad=\tilde{r}e^{r\Delta}\left[\int_{\underline{x}}^{\bar{x}}\Delta h'\big(a(x)\big)\big(1+a'(x)\big)\,dx-\Delta\big(h\big(a(\bar{x})\big)-h\big(a(\underline{x})\big)\big)\right]\\
&\quad=W\big(\Delta\big(\bar{x}+a(\bar{x})\big),\bar{x}\big)-W\big(\Delta\big(\underline{x}+a(\underline{x})\big),\underline{x}\big)+O(\Delta),
\end{aligned}$$

where the last inequality follows from the local neccesary version of (IC-PHI). Consequently, $\mathbb{V}^{\Delta}[W_q(\Delta x,x)] \leq \mathbb{V}^{\Delta}[W(\Delta(x+a(x)),x)\mathbf{1}_{|x|\leq M_{\varepsilon}/\sqrt{\Delta}}]+O(\Delta^2)$. Moreover, since $\mathbb{V}^{\Delta}[W(\Delta(x+a(x)),x)] \leq V\Delta$ (Lemma 6) and $W' \in [0,h'(A)]$, there is K_{ε} such that, for any Δ, $|x| \leq M_{\varepsilon}/\sqrt{\Delta}$ implies $y \in B$, where $B=\{y \mid |W(y)-\mathbb{E}^{\Delta}[W(\Delta(x+a(x)),x)]| \leq \sqrt{\Delta}K_{\varepsilon}\}$. Altogether, $\Phi^{\Delta,q}(a_q,c_q,W_q;F,w)$

[42] In Step 1, we used the fact that the quadratic simple policies, for all Δ, are based on the same set of v functions from the definition of Θ. In particular, the W_q functions have the same number of points of discontinuity, for all Δ. In this step, without additional proofs we cannot assume such uniformity, and so the construction is different.

is equal to

$$\begin{aligned}
&\tilde{r}\Delta\big(\mathbb{E}^{\Delta}\big[a(x)\big]-c\big)+e^{-r\Delta}\bigg[F(w)+F'(w)\mathbb{E}^{\Delta}\big[W\big(\Delta\big(x+a(x)\big),x\big)-w\big] \\
&\quad+\frac{1}{2}F''(w)\mathbb{V}^{\Delta}\big[W\big(\Delta\big(x+a(x)\big),x\big)\mathbf{1}_B\big]\bigg]+O(\varepsilon\Delta) \\
&\le \Phi^{\Delta}(a,c,W;F)+O(\varepsilon\Delta),
\end{aligned}$$

which establishes the lemma. *Q.E.D.*

We move on to establish "$T_I^{\Delta,d}F \underset{\text{Lemma 25}}{\sim} T_I^{\Delta,c}F$." The following Lemma 24 is related to the standard results in the static mechanism design.

LEMMA 24: *Suppose $X \equiv Z$. For any $\Delta > 0$ and $w \in I^{\Delta}$, if (a, c, W) satisfies* (IC) *in $T_I^{\Delta,c}F(w)$, then $x + a(x)$ is nondecreasing. Conversely, if (a, c, W) satisfies the local version of* (IC) *almost everywhere and $x + a(x)$ is nondecreasing, then (a, c, W) satisfies the IC.*

PROOF: The proof is standard, but we provide it for completeness. Suppose first that (a, c, W) is incentive compatible. Therefore, for any $x' > x$,

$$\begin{aligned}
&-\tilde{r}h\big(a(x')\big)+e^{-r\Delta}W\big(\Delta\big(x'+a(x')\big),x'\big) \\
&\quad\ge -\tilde{r}h\big(a(x)-(x'-x)\big)+e^{-r\Delta}W\big(\Delta\big(x+a(x)\big),x\big), \\
&-\tilde{r}h\big(a(x)\big)+e^{-r\Delta}W\big(\Delta\big(x+a(x)\big),x\big) \\
&\quad\ge -\tilde{r}h\big(a(x')+(x'-x)\big)+e^{-r\Delta}W\big(\Delta\big(x'+a(x')\big),x'\big).
\end{aligned}$$

Hence,

$$h\big(a(x')\big)-h\big(a(x)-(x'-x)\big)\le h\big(a(x')+(x'-x)\big)-h\big(a(x)\big).$$

Since h is convex, this implies that $a(x') \ge a(x) - (x' - x)$.

Conversely, we argue by contradiction. Assume that (a, c, W) satisfies the local IC and $x + a(x)$ is nondecreasing. Let

$$V\big(x,x'\big)=-\tilde{r}h\big(a(x')+(x'-x)\big)+e^{r\Delta}W\big(\Delta\big(x'+a(x')\big),x'\big).$$

By local IC, $V_2(x, x) = 0$ for all x. Suppose that for some $x' > x$, we have $0 < V(x, x') - V(x, x)$. Then

$$\begin{aligned}
0 &< \int_x^{x'} V_2(x,s)\,ds=\int_x^{x'}\big[V_2(x,s))-V_2(s,s)\big]\,ds \\
&= -\int_x^{x'}\int_x^{s} V_{12}(z,s)\,dz\,ds.
\end{aligned}$$

But

$$V_{12}(z,s)=\tilde{r}h''\big(a(s)+(s-z)\big)\big(1+a'(s)\big)\geq 0,$$

which is a contradiction. The case $V(x,x')>V(x,x)$ with $x'<x$ is analogous. *Q.E.D.*

LEMMA 25: *Let $Z=X$, and let $F:I\to\mathbb{R}$ be twice continuously differentiable with $F''<0$. Then $|T_I^{\Delta,d}F=T_I^{\Delta,c}F|_{I^\Delta}=o(\Delta)$.*

PROOF: Fix $\Delta,\varepsilon>0$ and consider any Δ-suboptimal policy (a,c,W) for $T^{\Delta,c}F(w)$. Let M_ε be such that $\mathbb{P}_X^\Delta[[-M_\varepsilon/\sqrt{\Delta},M_\varepsilon/\sqrt{\Delta}]]\geq 1-\varepsilon$. We construct a policy (a_d,c_d,W_d) as follows. Below, the function $a_d(\cdot)$ is derived from the function $a(\cdot)$ so that $a_d(\cdot)$ is piecewise continuously differentiable and $x+a_d(x)$ is nondecreasing. Then we let $c_d=c$, and W_d be such that it satisfies the local version of (IC):

$$\tilde{r}h'\big(a_d(x)\big)=e^{-r\Delta}W_d'\big(\Delta\big(x+a_d(x)\big)\big),$$

is continuous, and the constant of integration is adjusted so that it satisfies the PK condition. By Lemma 24, the policy (a_d,c_d,W_d) is feasible by construction.

Below, we will define a_d so that $a_d(x)=0$ if $x\notin[-M_\varepsilon/\sqrt{\Delta},M_\varepsilon/\sqrt{\Delta}+A]$, $x+a_d(x)$ is nondecreasing, and

$$\int_{-M_\varepsilon/\sqrt{\Delta}}^{M_\varepsilon/\sqrt{\Delta}}\big|a_d(x)-a(x)\big|\,dx\leq\varepsilon\quad\text{and}\quad\int_{-M_\varepsilon/\sqrt{\Delta}}^{M_\varepsilon/\sqrt{\Delta}}\big|a_d'(x)-a'(x)\big|\,dx\leq\varepsilon. \tag{46}$$

Recall that if f is nondecreasing, then f is differentiable a.e. and $\int_a^b f'(x)\,dx\leq f(b)-f(a)$.[43] Since

$$\begin{aligned}&h'\big(a_d(x)\big)\big(1+a_d'(x)\big)-h'\big(a(x)\big)\big(1+a'(x)\big)\\&\quad=h'\big(a_d(x)\big)\big(a_d'(x)-a'(x)\big)+\big(h'\big(a_d(x)\big)-h'\big(a(x)\big)\big)\big(1+a'(x)\big),\end{aligned}$$

(46) implies that, for any $\underline{x},\bar{x}\in[-M_\varepsilon/\sqrt{\Delta},M_\varepsilon/\sqrt{\Delta}]$,

$$\begin{aligned}&W_d\big(\Delta\big(\bar{x}+a_d(\bar{x})\big)\big)-W_d\big(\Delta\big(\underline{x}+a_d(\underline{x})\big)\big)\\&\quad=\tilde{r}e^{r\Delta}\Delta\int_{\underline{x}}^{\bar{x}}h'\big(a_d(x)\big)\big(1+a_d'(x)\big)\,dx\\&\quad\leq W\big(\Delta\big(\bar{x}+a(\bar{x})\big)\big)-W\big(\Delta\big(\underline{x}+a(\underline{x})\big)\big)\\&\qquad+\tilde{r}e^{r\Delta}\Delta\bigg[h'(A)\varepsilon+\max h''\bigg[\frac{2M_\varepsilon}{\sqrt{\Delta}}+a(\bar{x})-a(\underline{x})\bigg]\bigg].\end{aligned}$$

[43]See, for example, Theorem 2 in Chapter 5 of Royden (1988).

The rest of the proof will follow as in the last step of Lemma 12 to establish that $\Phi^{\Delta}(a_d, c_d, W_d; F) \geq \Phi^{\Delta}(a, c, W; F) - O(\varepsilon\Delta)$.

We now construct an a_d satisfying (46) and $x + a_d(x)$ is nondecreasing. First, note that since, for any $y > x$, we have $a(x) \geq a(y) - \frac{y-x}{\Delta}$, a may not discontinuously decrease. Therefore, the set of points $D \subset [-M_\varepsilon/\sqrt{\Delta}, M_\varepsilon/\sqrt{\Delta}]$ at which a may be discontinuous is at most countable. Moreover, if $J = \sum_{x\in D}(a(x_+) - a(x_-))$, then

$$J + \int_{-M_\varepsilon/\sqrt{\Delta}}^{M_\varepsilon/\sqrt{\Delta}} \big(1 + a'(x)\big)\, dx = \frac{2M_\varepsilon}{\sqrt{\Delta}} + a(\bar{x}) - a(\underline{x}) \leq A + \frac{2M_\varepsilon}{\sqrt{\Delta}}.$$

Since $1 + a'(x) \geq 0$, this implies that $J \leq A + \frac{2M_\varepsilon}{\sqrt{\Delta}}$. Let D_f be a finite set of points where a is discontinuous such that $\sum_{x\in D_f}(a(x_+) - a(x_-)) \geq J - \varepsilon/2$, and let $\delta = \min_{x\in D_f}(a(x_+) - a(x_-))$.

For any $n \in \mathbb{N}$ and $x \in [-M_\varepsilon/\sqrt{\Delta}, M_\varepsilon/\sqrt{\Delta}]$, let

$$a'_n(x) = \frac{n}{2}\int_{x-1/n}^{x+1/n} a'(s)\, ds.$$

The function a'_n is differentiable, and for any x, $a'_n(x) \geq -1$ (since $a'(x) \geq -1$). From Lebesgue's Density Theorem, it follows that for sufficiently large n, $\int_{-M_\varepsilon/\sqrt{\Delta}}^{M_\varepsilon/\sqrt{\Delta}} |a'_n(x) - a'(x)|\, dx \leq \delta$.

Finally, for $D_f = \{d_1, \dots, \dot{d}_n\}$, $d_0 = -M_\varepsilon/\sqrt{\Delta}$, $d_{n+1} = M_\varepsilon/\sqrt{\Delta}$, and for any $x \in [d_i, d_{i+1})$, let

$$a_d(x) = a(d_i) + \int_{d_i}^{x} a'_n(s)\, ds.$$

The function a_d satisfies (46) and $x + a_d(x)$ is nondecreasing by construction, which establishes the proof. *Q.E.D.*

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