Econometrica Supplementary Material

SUPPLEMENT TO "ROBUST ESTIMATION AND INFERENCE FOR JUMPS IN NOISY HIGH FREQUENCY DATA: A LOCAL-TO-CONTINUITY THEORY FOR THE PRE-AVERAGING METHOD"

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THIS SUPPLEMENT INCLUDES two appendices. Appendix S.A gives the proofs of the results in the main text. Appendix S.B provides simulation results in support of the theory in the main text.

APPENDIX S.A: PROOFS

In this appendix, we prove the results in the main text. The appendix is organized as follows. Section S.A.1 introduces some notation and decompositions. Section S.A.2 provides some estimates. These preliminary results are repeatedly used throughout the proofs. Section S.A.3 proves Theorem 1 and Corollary 1. Theorems 2–5 are respectively proved in Sections S.A.4–S.A.7.

S.A.1. *Notations*

As in the main text, we denote the paper of Jacod, Podolskij, and Vetter (2010) by JPV, and the paper of Aït-Sahalia, Jacod, and Li (2012) by AJL.

We use the same notations as in the main text. We sometimes write Z_t^η in place of Z_t (so $Z_t^\eta = X_t^\eta + \chi_t$) so as to emphasize the dependence of the observed price on η. We also need some new notation. We denote the continuous part of the efficient price by

$$X_t^* = X_0 + \int_0^t b_s\,ds + \int_0^t \sigma_s\,dW_s.$$

With any predictable function $\tilde{\delta}$, we associate two processes $\tilde{\delta} \star (\mu - \nu)$ and $\tilde{\delta} \star \mu$ as

$$\tilde{\delta} \star (\mu - \nu)_t = \int_0^t \int_E \tilde{\delta}(s, z)(\mu - \nu)(ds, dz),$$

$$\tilde{\delta} \star \mu_t = \int_0^t \int_E \tilde{\delta}(s, z)\mu(ds, dz),$$

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provided that they are well defined. Then we can write the efficient price X_t^η, for $\eta \in [0,1]$, as

$$\begin{aligned} X_t^\eta &= X_t^* + J_t \\ &= X_t^* + \eta\delta 1_{\{|\delta|\le 1\}} \star (\mu-\nu)_t + \eta\delta 1_{\{|\delta|>1\}} \star \mu_t. \end{aligned}$$

We also denote the continuous part of the noisy price as

$$Z_t^* = X_t^* + \chi_t.$$

Below, we introduce two decompositions of X_t^η. For each $q \ge 1$, we define

$$\text{(S.A.1)}\quad \begin{aligned} B_t^q &= \int_0^t b_s^q\,ds, \quad \text{with} \quad b_s^q = -\int_{\{\gamma(z)>1/q,|\delta(s,z)|\le 1\}} \delta(s,z)\lambda(dz), \\ M_t^q &= \delta 1_{\{\gamma\le 1/q\}} \star (\mu-\nu)_t, \\ J_t'^q &= B_t^q + M_t^q, \\ J_t^q &= \delta 1_{\{\gamma>1/q\}} \star \mu_t. \end{aligned}$$

We can then decompose X_t^η as

$$\text{(S.A.2)}\quad X_t^\eta = X_t^* + \eta J_t'^q + \eta J_t^q.$$

Under Assumption H-1, the jumps of X_t^η have finite variation. We extend the notations above by setting

$$\begin{aligned} B_t^\infty &= \int_0^t b_s^\infty\,ds, \quad \text{with} \quad b_t^\infty = -\int \delta(t,z)1_{\{|\delta(t,z)|\le 1\}}\lambda(dz), \\ J_t^\infty &= \delta \star \mu_t. \end{aligned}$$

In this case, we sometimes use an alternative decomposition:

$$\text{(S.A.3)}\quad X_t^\eta = X_t^* + \eta B_t^\infty + \eta J_t^\infty.$$

Recall from the main text that for any process Y and weight function g, we define

$$\begin{aligned} \bar{Y}(g)_i^n &= \sum_{j=1}^{k_n-1} g_j^n \Delta_{i+j}^n Y = -\sum_{j=1}^{k_n} g_j'^n Y_{(i+j-1)\Delta_n}, \\ \hat{Y}(g)_i^n &= \sum_{j=1}^{k_n} \bigl(g_j'^n \Delta_{i+j}^n Y\bigr)^2. \end{aligned}$$

We often use the following property without further mention: if Y is a semimartingale, $\bar{Y}(g)_i^n$ can be represented in integral form as (see (5.4) in JPV)

$$\bar{Y}(g)_i^n = \int_{i\Delta_n}^{i\Delta_n+k_n\Delta_n} g_n(s-i\Delta_n)\,dY_s,$$

where

$$g_n(s) = \sum_{j=1}^{k_n-1} g_j^n 1_{((j-1)\Delta_n, j\Delta_n]}(s).$$

For the sake of notational simplicity, we write $\bar{Y}_i^n$ and $\hat{Y}_i^n$ in place of $\bar{Y}(g)_i^n$ and $\hat{Y}(g)_i^n$ whenever there is no ambiguity about the weight function involved in these definitions. For example, we write $\bar{X}_i^{*n}$, $\bar{B}_i^{q,n}$, $\bar{M}_i^{q,n}$, $\bar{J}_i^{\prime q,n}$, and $\bar{J}_i^{q,n}$ for $\bar{Y}(g)_i^n$ when $Y_t = X_t^*$, B_t^q, M_t^q, $J_t^{\prime q}$, and J_t^q, respectively. Moreover, when $Y_t = X_t^{\eta_n}$ or $Z_t^{\eta_n}$, we further simplify our notations by writing $\bar{X}_i^n$, $\hat{X}_i^n$, $\bar{Z}_i^n$, and $\hat{Z}_i^n$ in place of $\bar{X}^{\eta_n}(g)_i^n$, $\hat{X}^{\eta_n}(g)_i^n$, $\bar{Z}^{\eta_n}(g)_i^n$, and $\hat{Z}^{\eta_n}(g)_i^n$ respectively; because the sequence η_n is always fixed in our proofs, this shorthand notation should not raise any ambiguity.

Throughout the proof, K denotes a constant that may change from line to line; the constant does not depend on the asymptotic stage n or the summation index i. We sometimes emphasize its dependence on some parameter q by writing K_q. As in the main text, we use $\xrightarrow{\mathbb{P},\eta_n}$ to indicate the convergence in probability and use $\xrightarrow{\mathcal{L}\text{-}s,\eta_n}$ to indicate the stable convergence in law under a drifting sequence η_n. For any nonrandom sequence $b_n > 0$, we denote by $o_{p,\eta_n}(b_n)$ a generic sequence of variables ξ_n that satisfies $\xi_n/b_n \xrightarrow{\mathbb{P},\eta_n} 0$, and we denote by $O_{p,\eta_n}(b_n)$ a generic sequence of variables ξ_n such that ξ_n/b_n is stochastically bounded. For notational simplicity, we suppress the dependence of these stochastic symbols on η_n whenever the distribution of the relevant random variables does not depend on η_n.

S.A.2. *Some Useful Estimate*

We first recall some standard estimates for jump increments from Lemmas 2.1.5 and 2.1.7 of Jacod and Protter (2012).

LEMMA 1—Jacod and Protter: *Let $(\omega,t,z)\mapsto \tilde{\delta}(\omega,t,z)$ be a predictable function on $\Omega\times\mathbb{R}_+\times E$.*

(a) *Suppose that $\int_0^t ds\int \tilde{\delta}(s,z)^2\lambda(dz) < \infty$ for all t. Then the process $Y = \tilde{\delta}\star(\mu-\nu)$ is a locally square integrable martingale, and for all finite stopping* *times* τ, *for* $s > 0$ *and* $k \geq 2$, *we have*

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{u\in[0,s]} |Y_{\tau+u} - Y_\tau|^k \Big|\mathcal{F}_\tau\Big] \\
&\quad \leq Ks\mathbb{E}\bigg[\frac{1}{s}\int_\tau^{\tau+s} du \int \big|\tilde{\delta}(u,z)\big|^k \lambda(dz)\Big|\mathcal{F}_\tau\bigg] \\
&\qquad + Ks^{k/2}\mathbb{E}\bigg[\bigg(\frac{1}{s}\int_\tau^{\tau+s} du \int \tilde{\delta}(u,z)^2 \lambda(dz)\bigg)^{k/2} \Big|\mathcal{F}_\tau\bigg].
\end{aligned}$$

(b) *Suppose that* $\int_0^t ds \int |\tilde{\delta}(s,z)|\lambda(dz) < \infty$ *for all* t. *Then the process* $Y = \tilde{\delta} \star \mu$ *is of locally integrable variation, and for all finite stopping times* τ, *for* $s > 0$ *and* $k \geq 1$, *we have*

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{u\in[0,s]} |Y_{\tau+u} - Y_\tau|^k \Big|\mathcal{F}_\tau\Big] \\
&\quad \leq Ks\mathbb{E}\bigg[\frac{1}{s}\int_\tau^{\tau+s} du \int \big|\tilde{\delta}(u,z)\big|^k \lambda(dz)\Big|\mathcal{F}_\tau\bigg] \\
&\qquad + Ks^{k}\mathbb{E}\bigg[\bigg(\frac{1}{s}\int_\tau^{\tau+s} du \int \big|\tilde{\delta}(u,z)\big| \lambda(dz)\bigg)^{k} \Big|\mathcal{F}_\tau\bigg].
\end{aligned}$$

As is often the case in this kind of problem, with the help of a standard localization argument (Jacod (2008)), we can strengthen Assumptions H-r, K, and N as follows without loss of generality:

ASSUNMPTION S.H-r: *We have Assumption* H-r, $\sup_{\omega^{(0)},t} |\delta(\omega^{(0)}, t, z)| \leq \gamma(z)$, *and the processes* b_t, σ_t, *and* X_t *are bounded.*

ASSUNMPTION S.K: *We have Assumption* K, *and further the processes* $\tilde{b}_t$, a_t, a'_t, *and* $\tilde{\sigma}_t$ *are bounded.*

ASSUNMPTION S.N: *We have Assumption* N *and, for any* $q > 0$, *the process* $\int Q_t(\omega^{(0)}, dz)|z|^q$ *is bounded.*

We now collect some estimates that are used repeatedly throughout the proofs.

LEMMA 2: *Suppose that Assumptions* S.H-2 *and* S.N *hold. For any* $u \geq 0$, *there exists* $K > 0$ *such that*

(S.A.4) $\mathbb{E}\big[\big|\bar{Z}_i^{*n}\big|^u |\mathcal{F}_{i\Delta_n}\big] \leq K\Delta_n^{u/4}$,

(S.A.5) $\mathbb{E}\big[\big|\hat{\chi}_i^n\big|^u |\mathcal{F}_{i\Delta_n}\big] \leq K\Delta_n^{u/2}$,

(S.A.6) $\mathbb{E}\big[\big|\hat{Z}_i^{*n} - \hat{\chi}_i^n\big|^u |\mathcal{F}_{i\Delta_n}\big] \leq K\Delta_n^{u}$.

PROOF: Inequalities (S.A.4) and (S.A.5) follow from (5.39) and (5.3) in JPV; (S.A.6) follows from (5.43) of that paper and Jensen's inequality. *Q.E.D.*

LEMMA 3: *Suppose that Assumptions* S.H-2 *and* S.N *hold. Let* η_n *be a sequence in* $[0, 1]$ *and let* $u \geq 0$, $v \geq 1$, *and* $w \geq 2$ *be real numbers. For any* $q \geq 1$, *we have*

$$\mathbb{E}\big[|\bar{B}_i^{q,n}|^u|\mathcal{F}_{i\Delta_n}\big] \leq K_q\Delta_n^{u/2}, \tag{S.A.7}$$

$$\mathbb{E}\big[|\bar{M}_i^{q,n}|^w|\mathcal{F}_{i\Delta_n}\big] \leq K\Delta_n^{1/2}\int \gamma(z)^2 1_{\{\gamma(z)\leq 1/q\}}\lambda(dz), \tag{S.A.8}$$

$$\mathbb{E}\big[|\bar{J}_i'^{q,n}|^w|\mathcal{F}_{i\Delta_n}\big] \leq K_q\Delta_n^{w/2} + K\Delta_n^{1/2}\int \gamma(z)^2 1_{\{\gamma(z)\leq 1/q\}}\lambda(dz), \tag{S.A.9}$$

$$\mathbb{E}\big[|\bar{J}_i^{q,n}|^v|\mathcal{F}_{i\Delta_n}\big] \leq K_q\Delta_n^{1/2}, \tag{S.A.10}$$

$$\mathbb{E}\big[|\bar{Z}_i^n - \bar{Z}_i^{*n}|^w|\mathcal{F}_{i\Delta_n}\big] \leq K\eta_n^w\Delta_n^{1/2}, \tag{S.A.11}$$

$$\mathbb{E}\big[|\bar{Z}_i^n|^w|\mathcal{F}_{i\Delta_n}\big] \leq K\Delta_n^{w/4} + K\eta_n^w\Delta_n^{1/2}, \tag{S.A.12}$$

$$\mathbb{E}\big[|\hat{Z}_i^n - \hat{Z}_i^{*n}|^v|\mathcal{F}_{i\Delta_n}\big] \leq K\eta_n^v\Delta_n^{1/2+v/2}, \tag{S.A.13}$$

$$\mathbb{E}\big[|\hat{Z}_i^n - \hat{\chi}_i^n|^v|\mathcal{F}_{i\Delta_n}\big] \leq K\Delta_n^v + K\eta_n^v\Delta_n^{1/2+v/2}. \tag{S.A.14}$$

PROOF: Note that $\bar{B}_i^{q,n} = \int_{i\Delta_n}^{i\Delta_n+k_n\Delta_n} g_n(s - i\Delta_n)b_s^q\,ds$. Since $g_n(\cdot)$ is bounded, we have $|\bar{B}_i^{q,n}| \leq K\int_{i\Delta_n}^{i\Delta_n+k_n\Delta_n}|b_s^q|\,ds$. By (S.A.1), $|b_s^q|$ is bounded by $\int_{\{\gamma(z)>1/q\}}\lambda(dz)$, which is finite under Assumption H-2. Hence $|\bar{B}_i^{q,n}| \leq K_qk_n\Delta_n$, which implies (S.A.7).

Note that

$$\begin{aligned}\bar{M}_i^{q,n} &= \int_{i\Delta_n}^{i\Delta_n+k_n\Delta_n} g_n(s - i\Delta_n)\,dM_s^q \\ &= \int_{i\Delta_n}^{i\Delta_n+k_n\Delta_n}\int_E g_n(s - i\Delta_n)\delta(s, z)1_{\{\gamma(z)\leq 1/q\}}(\mu - \nu)(ds, dz).\end{aligned}$$

Since $g_n(\cdot)$ is bounded, $|\delta(s, z)| \leq \gamma(z)$, and $w \geq 2$, we can use Lemma 1(a) to derive

$$\begin{aligned}\mathbb{E}\big[|\bar{M}_i^{q,n}|^w|\mathcal{F}_{i\Delta_n}\big] &\leq Kk_n\Delta_n\int_E \gamma(z)^w 1_{\{\gamma(z)\leq 1/q\}}\lambda(dz) \\ &\quad + K(k_n\Delta_n)^{w/2}\left(\int_E \gamma(z)^2 1_{\{\gamma(z)\leq 1/q\}}\lambda(dz)\right)^{w/2}.\end{aligned}$$

Because $\int_E \gamma(z)^2\lambda(dz) < \infty$, the above display implies (S.A.8).

Recall that $J_t'^q = B_t^q + M_t^q$. We derive (S.A.9) by combining (S.A.7) and (S.A.8).

To see (S.A.10), note that

$$\begin{aligned}\bar{J}_i^{q,n} &= \int_{i\Delta_n}^{i\Delta_n+k_n\Delta_n} g_n(s-i\Delta_n)\,dJ_s^q \\ &= \int_{i\Delta_n}^{i\Delta_n+k_n\Delta_n}\int_E g_n(s-i\Delta_n)\delta(s,z)1_{\{\gamma(z)>1/q\}}\mu(ds,dz).\end{aligned}$$

Hence, by Lemma 1(b),

$$\begin{aligned}\mathbb{E}\big[\big|\bar{J}_i^{q,n}\big|^v|\mathcal{F}_{i\Delta_n}\big] &\le Kk_n\Delta_n\int_E \gamma(z)^v 1_{\{\gamma(z)>1/q\}}\lambda(dz) \\ &\quad + K(k_n\Delta_n)^v\left(\int_E \gamma(z)1_{\{\gamma(z)>1/q\}}\lambda(dz)\right)^v \\ &\le K_q\Delta_n^{1/2}.\end{aligned}$$

By (S.A.2), $\bar{Z}_i^n - \bar{Z}_i^{*n} = \eta_n(\bar{J}_i'^{q,n} + \bar{J}_i^{q,n})$. Taking $q=1$, we derive (S.A.11) by combining (S.A.9) and (S.A.10). We then combine (S.A.11) and (S.A.4) to derive (S.A.12).

We now consider (S.A.13). Denote $X_t' = \delta 1_{\{|\delta|\le 1\}} \star (\mu-\nu)_t + \delta 1_{\{|\delta|>1\}} \star \mu_t$. By Lemma 1, we have $\mathbb{E}[(\Delta_i^n X')^{2v}|\mathcal{F}_{i\Delta_n}] \le K\Delta_n$. Hence,

$$\begin{aligned}\mathbb{E}\big[\big(\hat{X}_i'^n\big)^v|\mathcal{F}_{i\Delta_n}\big] &= \mathbb{E}\left[\left(\sum_{j=1}^{k_n}\big(g_j'^n\Delta_{i+j}^n X'\big)^2\right)^v\middle|\mathcal{F}_{i\Delta_n}\right] \\ &\le Kk_n^{v-1}\mathbb{E}\left[\sum_{j=1}^{k_n}\big(g_j'^n\Delta_{i+j}^n X'\big)^{2v}\middle|\mathcal{F}_{i\Delta_n}\right] \\ &\le K\Delta_n^{1+v/2},\end{aligned} \tag{S.A.15}$$

where the first inequality follows from Hölder's inequality; the second inequality holds because $|g_j'^n| \le K/k_n$ and $\mathbb{E}[(\Delta_i^n X')^{2v}|\mathcal{F}_{i\Delta_n}] \le K\Delta_n$.

Now note that $Z_t^{\eta_n} = Z_t^* + \eta_n X_t'$. By the c_r-inequality and the Cauchy–Schwarz inequality, we have

$$\begin{aligned}&\big|\hat{Z}_i^n - \hat{Z}_i^{*n}\big|^v \\ &\quad = \left|\sum_{j=1}^{k_n}\big(g_j'^n\big)^2\big[\big(\Delta_{i+j}^n Z^* + \eta_n\Delta_{i+j}^n X'\big)^2 - \big(\Delta_{i+j}^n Z^*\big)^2\big]\right|^v\end{aligned}$$

$$= \left| \eta_n^2 \sum_{j=1}^{k_n} (g_j'^n)^2 (\Delta_{i+j}^n X')^2 + 2 \sum_{j=1}^{k_n} (g_j'^n)^2 (\Delta_{i+j}^n Z^*)(\eta_n \Delta_{i+j}^n X') \right|^v$$

$$\leq K \left(\eta_n^2 \sum_{j=1}^{k_n} (g_j'^n)^2 (\Delta_{i+j}^n X')^2 \right)^v$$

$$+ K \left(\left(\sum_{j=1}^{k_n} (g_j'^n)^2 (\Delta_{i+j}^n Z^*)^2 \right)^{1/2} \left(\sum_{j=1}^{k_n} (g_j'^n)^2 (\eta_n \Delta_{i+j}^n X')^2 \right)^{1/2} \right)^v$$

$$= K \eta_n^{2v} (\hat{X}_i'^n)^v + K \eta_n^v (\hat{Z}_i^{*n})^{v/2} (\hat{X}_i'^n)^{v/2}.$$

Hence,

$$\begin{aligned} \mathbb{E}\big[|\hat{Z}_i^n - \hat{Z}_i^{*n}|^v | \mathcal{F}_{i\Delta_n}\big] &\leq K \eta_n^{2v} \mathbb{E}\big[(\hat{X}_i'^n)^v | \mathcal{F}_{i\Delta_n}\big] \\ &\quad + K \eta_n^v \mathbb{E}\big[(\hat{Z}_i^{*n})^v | \mathcal{F}_{i\Delta_n}\big]^{1/2} \mathbb{E}\big[(\hat{X}_i'^n)^v | \mathcal{F}_{i\Delta_n}\big]^{1/2} \\ &\leq K \eta_n^{2v} \Delta_n^{1+v/2} + K \eta_n^v \Delta_n^{v/4} \Delta_n^{1/2+v/4} \\ &\leq K \eta_n^v \Delta_n^{1/2+v/2}, \end{aligned}$$

where the first inequality follows from the Cauchy–Schwarz inequality; the second inequality is due to (S.A.15) and $\mathbb{E}[(\hat{Z}_i^{*n})^v | \mathcal{F}_{i\Delta_n}] \leq K \Delta_n^{v/2}$, which in turn follows from (S.A.5) and (S.A.6); the last inequality holds because η_n is bounded. This finishes the proof of (S.A.13).

Combining (S.A.13) with (S.A.6), we get (S.A.14). *Q.E.D.*

LEMMA 4: *Suppose that Assumptions* S.H-1 *and* S.N *hold*. *Let* η_n *be a sequence in* $[0, 1]$. *For any* $u \geq 0$ *and* $v \geq 1$, *there exists* $K > 0$ *such that*

$$\text{(S.A.16)} \quad \mathbb{E}\big[|\bar{Z}_i^n - \bar{Z}_i^{*n}|^v | \mathcal{F}_{i\Delta_n}\big] \leq K \eta_n^v \Delta_n^{1/2},$$

$$\text{(S.A.17)} \quad \mathbb{E}\big[|\bar{Z}_i^n|^v | \mathcal{F}_{i\Delta_n}\big] \leq K \Delta_n^{v/4} + K \eta_n^v \Delta_n^{1/2},$$

$$\text{(S.A.18)} \quad \mathbb{E}\big[|\bar{J}_i'^{q,n}|^v | \mathcal{F}_{i\Delta_n}\big] \leq K \Delta_n^{v/2} + K \Delta_n^{1/2} \int_E \gamma(z) 1_{\{\gamma(z) \leq 1/q\}} \lambda(dz).$$

PROOF: By (S.A.3), we have $\bar{Z}_i^n = \bar{Z}_i^{*n} + \eta_n \bar{B}_i^{\infty,n} + \eta_n \bar{J}_i^{\infty,n}$, so

$$\mathbb{E}\big[|\bar{Z}_i^n - \bar{Z}_i^{*n}|^v | \mathcal{F}_{i\Delta_n}\big] \leq K \eta_n^v \mathbb{E}\big[|\bar{B}_i^{\infty,n}|^v | \mathcal{F}_{i\Delta_n}\big] + K \eta_n^v \mathbb{E}\big[|\bar{J}_i^{\infty,n}|^v | \mathcal{F}_{i\Delta_n}\big].$$

Since $g_n(\cdot)$ and b^∞ are bounded, we have

$$\text{(S.A.19)} \quad \mathbb{E}\big[|\bar{B}_i^{\infty,n}|^v | \mathcal{F}_{i\Delta_n}\big] = \mathbb{E}\left[\left| \int_{i\Delta_n}^{i\Delta_n + k_n \Delta_n} g_n(s - i\Delta_n) b_s^\infty \, ds \right|^v \middle| \mathcal{F}_{i\Delta_n} \right] \leq K \Delta_n^{v/2}.$$

Moreover, using a similar argument as in the proof of (S.A.10), we have

$$
\begin{aligned}
\text{(S.A.20)}\quad & \mathbb{E}\big[\big|\bar{J}_i^{\infty,n}\big|^v|\mathcal{F}_{i\Delta_n}\big] \\
&= \mathbb{E}\bigg[\bigg|\int_{i\Delta_n}^{i\Delta_n+k_n\Delta_n}\int_E g_n(s-i\Delta_n)\delta(s,z)\mu(ds,dz)\bigg|^v\bigg|\mathcal{F}_{i\Delta_n}\bigg] \\
&\leq Kk_n\Delta_n\int\gamma(z)^v\lambda(dz)+K(k_n\Delta_n)^v\bigg(\int\gamma(z)\lambda(dz)\bigg)^v \\
&\leq K\Delta_n^{1/2},
\end{aligned}
$$

where the first inequality is obtained by applying Lemma 1(b); the second inequality holds because $v\geq 1$, and $\gamma(\cdot)$ is bounded and integrable with respect to $\lambda(dz)$.

Combining (S.A.19) and (S.A.20), we readily have (S.A.16). By (S.A.16) and (S.A.4), we have (S.A.17).

To see (S.A.18), first note that under Assumption S.H-1, $J_t^{\prime q}=B_t^\infty+\delta 1_{\{\gamma\leq 1/q\}}\star\mu_t$. Hence,

$$
\bar{J}_i^{\prime q,n}=\bar{B}_i^{\infty,n}+\int_{i\Delta_n}^{i\Delta_n+k_n\Delta_n}\int_E g_n(s-i\Delta_n)\delta(s,z)1_{\{\gamma(z)\leq 1/q\}}\mu(ds,dz).
$$

By Lemma 1(b),

$$
\mathbb{E}\big[\big|\bar{J}_i^{\prime q,n}-\bar{B}_i^{\infty,n}\big|^v|\mathcal{F}_{i\Delta_n}\big]\leq K\Delta_n^{1/2}\int_E\gamma(z)1_{\{\gamma(z)\leq 1/q\}}\lambda(dz).
$$

Using the c_r-inequality, we combine this estimate with (S.A.19) to derive (S.A.18). *Q.E.D.*

The following estimates are elementary consequences of the definitions of d_n and a_n; recall that these are the normalizing factors in the LLN and CLT, respectively. If η_n satisfies $\Delta_n^{-r^*}\eta_n\to h\in[0,\infty]$, we have

$$
\text{(S.A.21)}\quad u\in[0,p]\quad\Rightarrow\quad d_n\eta_n^u=O\big(\Delta_n^{1-p/4+r^*u}\big);
$$

if η_n satisfies $\Delta_n^{-\bar{r}}\eta_n\to h\in[0,\infty]$, we have

$$
\text{(S.A.22)}\quad u\in[0,p-1]\quad\Rightarrow\quad a_n\eta_n^u=O\big(\Delta_n^{3/4-p/4+\bar{r}u}\big).
$$

S.A.3. *Proofs of Theorem 1 and Corollary 1*

LEMMA 5: *Suppose that Assumptions S.H-2 and S.N hold. Let $(\eta_n)_{n\geq 1}\subset[0,1]$ be a sequence that satisfies $\Delta_n^{-r^*}\eta_n\to h$ for some $h\in[0,\infty]$. Then, for any $\varepsilon>0$,*

$$
\lim_{q\to\infty}\limsup_{n\to\infty}\mathbb{P}\big(d_nV\big(\eta_nJ^{\prime q},g,p,0\big)_t^n>\varepsilon\big)=0.
$$

PROOF: Note that

$$\begin{aligned}
&\mathbb{E}\big[d_n V\big(\eta_n J'^q, g, p, 0\big)_t^n\big] \\
&\quad = d_n \eta_n^p \mathbb{E}\left[\sum_{i=1}^{\lfloor t/\Delta_n\rfloor - k_n} \big|\bar{J}_i'^{q,n}\big|^p\right] \\
&\quad \le d_n \eta_n^p \Delta_n^{-1}\left(K_q \Delta_n^{p/2} + K\Delta_n^{1/2}\int \gamma(z)^2 1_{\{\gamma(z)\le 1/q\}}\lambda(dz)\right) \\
&\quad \le K_q \Delta_n^{(p-1)/2} + K\int \gamma^2(z) 1_{\{\gamma(z)\le 1/q\}}\lambda(dz),
\end{aligned}$$

where the first inequality follows from (S.A.9) and the second inequality follows from (S.A.21). Note that $p \ge 2$ and $\lim_{q\to\infty}\int \gamma(z)^2 1_{\{\gamma(z)\le 1/q\}}\lambda(dz) = 0$. Hence,

$$\lim_{q\to\infty}\limsup_{n\to\infty}\mathbb{E}\big|d_n V\big(\eta_n J'^q, g, p, 0\big)_t^n\big| = 0.$$

The claim then follows from Markov's inequality. *Q.E.D.*

We specify an exhausting sequence (T_m) for the jumps of X_t^η as follows. For $q \ge 1$, let $(T(q,m) : m \ge 1)$ be the successive jump times of the Poisson process $1_{\{1/q<\gamma\le 1/(q-1)\}} \star \mu_t$, where $\gamma(\cdot)$ is the function that occurs in Assumption S.H-2. These stopping times have pairwise disjoint graphs as m and q vary, and $(T_m)_{m\ge 1}$ denotes any reordering of the double sequence $(T(q,m) : q, m \ge 1)$. We denote by P_q the collection of m such that $T_m = T(q', m')$ for some $q' \le q$ and $m' \ge 1$. Note that $\{T_m : m \in P_q\}$ exhausts the jumps of the pure jump process J^q.

For a weight function g and an even integer $p \ge 2$, we define, for $q \ge 1$,

$$U(g, p, q)_t = \theta \bar{g}(p) \sum_{m\in P_q : T_m \le t} |\Delta X_{T_m}|^p,$$

and for $l \in \{0, \dots, p/2\}$,

$$V(g, p, l)_t = \theta^{-p/2}\int_0^t \big(2\alpha_s^2 \bar{g}'(2)\big)^l m_{p-2l}(g; \theta\sigma_s, \alpha_s)\, ds. \tag{S.A.23}$$

LEMMA 6: *Suppose that Assumptions S.H-2 and S.N hold. Let $(\eta_n) \subset [0,1]$ be a sequence that satisfies $\Delta_n^{-r^*}\eta_n \to h$ for some $h \in [0,\infty]$. Then for each $q \ge 1$,*

$$\begin{aligned}
&d_n V\big(Z^* + \eta_n J^q, g, p, 0\big)_t^n \\
&\quad \xrightarrow{\mathbb{P},\eta_n} \frac{1}{1+h^p} V(g, p, 0)_t + \frac{h^p}{1+h^p} U(g, p, q)_t.
\end{aligned}$$

PROOF: Let $\Omega_n(t, q)$ be the collection of sample paths on which $|T_m - T_{m'}| > 2k_n\Delta_n$, $2k_n\Delta_n < T_m < t - 2k_n\Delta_n$, and T_m is not a multiple of Δ_n whenever $T_m, T_{m'} \leq t$ for some $m, m' \in P_q$. Since X is càdlàg with no fixed time of discontinuity, $\Omega_n(t, q) \to \Omega$ almost surely as $n \to \infty$. Therefore, for the purpose of proving the claim of this lemma, we can and will restrict our calculation on the set $\Omega_n(t, q)$ without loss of generality. We denote $I_m^n = \lfloor T_m/\Delta_n \rfloor$.

On $\Omega_n(t, q)$, we have the decomposition

$$\text{(S.A.24)}\quad d_n V\big(Z^* + \eta_n J^q, g, p, 0\big)_t^n = d_n V\big(Z^*, g, p, 0\big)_t^n + d_n \sum_{m \in P_q : T_m \leq t} \sum_{i = I_m^n - k_n + 2}^{I_m^n} \big|\eta_n \bar{J}_i^{q,n}\big|^p + d_n \sum_{m \in P_q : T_m \leq t} \left[\sum_{i = I_m^n - k_n + 2}^{I_m^n} \big(\big|\bar{Z}_i^{*n} + \eta_n \bar{J}_i^{q,n}\big|^p - \big|\bar{Z}_i^{*n}\big|^p - \big|\eta_n \bar{J}_i^{q,n}\big|^p \big) \right].$$

Note that

$$\text{(S.A.25)}\quad d_n \sum_{m \in P_q : T_m \leq t} \sum_{i = I_m^n - k_n + 2}^{I_m^n} \big|\eta_n \bar{J}_i^{q,n}\big|^p = d_n \eta_n^p k_n \left(\frac{1}{k_n} \sum_{j=1}^{k_n} \big(g_j^n\big)^p \right) \sum_{m \in P_q : T_m \leq t} |\Delta X_{T_m}|^p \to \frac{h^p}{1 + h^p} U(g, p, q)_t,$$

where the equality holds because, in the restriction to $\Omega_n(t, q)$, the sample path of J^q is a step function with at most one jump on any interval with length $2k_n\Delta_n$; the convergence holds because $d_n \eta_n^p k_n \to \theta h^p/(1 + h^p)$ and $k_n^{-1} \sum_{j=0}^{k_n} (g_j^n)^p \to \bar{g}(p)$.

In the proof of Lemma 2 of AJL, it is shown (by a straightforward extension of Theorem 3.3 in JPV) that

$$\text{(S.A.26)}\quad l \in \{0, \ldots, p/2\} \quad \Rightarrow \quad \Delta_n^{1-p/4} V\big(Z^*, g, p - 2l, l\big)_t^n \xrightarrow{\mathbb{P}} V(g, p, l)_t.$$

In particular, by taking $l = 0$, we derive

$$\text{(S.A.27)}\quad d_n V\big(Z^*, g, p, 0\big)_t^n \xrightarrow{\mathbb{P}} \frac{1}{1 + h^p} V(g, p, 0)_t.$$

Let $\mathcal{H}_t = \mathcal{F}_t \vee \sigma\{T_m : m \in P_q\}$. Note that the Wiener process W is also a Wiener process relative to the filtration $(\mathcal{H}_t)_{t \geq 0}$, because stopping times

$\{T_m : m \in P_q\}$ are independent of W. Then a mild extension of (S.A.4) yields $\mathbb{E}[|\bar{Z}_i^{*n}|^p|\mathcal{H}_0] \leq K\Delta_n^{p/4}$. Since I_m^n is $\mathcal{H}_0$-measurable, we use repeated conditioning to get

$$\mathbb{E}\left[d_n \sum_{m\in P_q:T_m\leq t} \sum_{i=I_m^n-k_n+2}^{I_m^n} \left|\bar{Z}_i^{*n}\right|^p\right] \leq K d_n k_n \Delta_n^{p/4} \leq K\Delta_n^{1/2},$$

which implies

$$d_n \sum_{m\in P_q:T_m\leq t} \sum_{i=I_m^n-k_n+2}^{I_m^n} \left|\bar{Z}_i^{*n}\right|^p = o_p(1). \tag{S.A.28}$$

Note that for any $\beta > 0$, there exists some $K_\beta > 0$ such that $||x+y|^p - |x|^p| \leq K_\beta|y|^p + \beta|x|^p$ for all $x, y \in \mathbb{R}$. For such β and K_β, we can bound the third term on the right-hand side of (S.A.24) by

$$(K_\beta + 1)d_n \sum_{m\in P_q:T_m\leq t} \sum_{i=I_m^n-k_n+2}^{I_m^n} \left|\bar{Z}_i^{*n}\right|^p + \beta d_n \sum_{m\in P_q:T_m\leq t} \sum_{i=I_m^n-k_n+2}^{I_m^n} \left|\eta_n \bar{J}_i^{q,n}\right|^p.$$

Since β can be arbitrarily chosen, (S.A.25) and (S.A.28) imply that the third term on the right-hand side of (S.A.24) is $o_{p,\eta_n}(1)$. The claim then follows from (S.A.24), (S.A.25), and (S.A.27). *Q.E.D.*

LEMMA 7: *Suppose that Assumptions S.H-2 and S.N hold. Let $(\eta_n)_{n\geq 1} \subset [0, 1]$ be a sequence satisfying $\Delta_n^{-r^*}\eta_n \to h$ for some $h \in [0, \infty]$. For each $l \in \{1, \dots, p/2\}$, we have*

$$d_n \sum_{i=0}^{\lfloor t/\Delta_n\rfloor - k_n} \left(\left|\bar{Z}_i^n\right|^{p-2l} - \left|\bar{Z}_i^{*n}\right|^{p-2l}\right)\left(\hat{\chi}_i^n\right)^l = o_{p,\eta_n}(1). \tag{S.A.29}$$

PROOF: When $l = p/2$, the claim holds trivially, because the left-hand side of (S.A.29) is zero. We hence fix some $p \geq 4$ and $l \in \{1, \dots, p/2 - 1\}$. By applying Hölder's inequality with index $m \in (1, p/(p-2))$ and then using (S.A.5), we have

$$\mathbb{E}\left|\left(\left|\bar{Z}_i^n\right|^{p-2l} - \left|\bar{Z}_i^{*n}\right|^{p-2l}\right)\left(\hat{\chi}_i^n\right)^l\right| \leq K\Delta_n^{l/2}\left\{\mathbb{E}\left[\left|\left|\bar{Z}_i^n\right|^{p-2l} - \left|\bar{Z}_i^{*n}\right|^{p-2l}\right|^m\right]\right\}^{1/m}. \tag{S.A.30}$$

It is easy to see that, for any $k \geq 1$, there exists $K > 0$ such that for all $x, y \in \mathbb{R}$,

$$\left||x+y|^k - |x|^k\right| \leq K|y|^k + K|x|^{k-1}|y|. \tag{S.A.31}$$

Hence,

$$\text{(S.A.32)}\quad \{\mathbb{E}[||\bar{Z}_i^n|^{p-2l} - |\bar{Z}_i^{*n}|^{p-2l}|^m]\}^{1/m}$$
$$\leq K\{\mathbb{E}[|\bar{Z}_i^n - \bar{Z}_i^{*n}|^{(p-2l)m}]\}^{1/m}$$
$$+ K\{\mathbb{E}[|\bar{Z}_i^{*n}|^{(p-2l-1)m}|\bar{Z}_i^n - \bar{Z}_i^{*n}|^m]\}^{1/m}$$
$$\leq K\eta_n^{p-2l}\Delta_n^{1/(2m)} + K\Delta_n^{(p-2l-1)/4}\{\mathbb{E}[|\bar{Z}_i^n - \bar{Z}_i^{*n}|^2]\}^{1/2}$$
$$\leq K\eta_n^{p-2l}\Delta_n^{1/(2m)} + K\eta_n\Delta_n^{(p-2l)/4},$$

where the first inequality is obtained by using (S.A.31) with $k = p - 2l$ and then the c_r-inequality; the second inequality follows from (S.A.11), Hölder's inequality with index $2/m$, and (S.A.4); the third inequality follows from (S.A.11) with $w = 2$.

Therefore,

$$d_n\mathbb{E}\left|\sum_{i=0}^{\lfloor t/\Delta_n\rfloor - k_n}(|\bar{Z}_i^n|^{p-2l} - |\bar{Z}_i^{*n}|^{p-2l})(\hat{\chi}_i^n)^l\right|$$
$$\leq Kd_n\Delta_n^{-1}\eta_n^{p-2l}\Delta_n^{1/(2m)}\Delta_n^{l/2} + Kd_n\Delta_n^{-1}\eta_n\Delta_n^{(p-2l)/4}\Delta_n^{l/2}$$
$$\leq K\Delta_n^{(1/2)(1/m-(p-2)/p)} + K\Delta_n^{(p-2)/(4p)} \to 0,$$

where the first inequality follows from (S.A.30) and (S.A.32); the second inequality follows (S.A.21); the convergence is due to our choice of m. The claim (S.A.29) readily follows. *Q.E.D.*

LEMMA 8: *Suppose that Assumptions* S.H-2 *and* S.N *hold*. *Let* η_n *be a sequence in* $[0, 1]$ *and* $l \in \{1, \ldots, p/2\}$.

(a) *If* $\Delta_n^{-\bar{r}}\eta_n \to h$ *for some* $h \in [0, \infty]$, *then*

$$\text{(S.A.33)}\quad a_n\sum_{i=0}^{\lfloor t/\Delta_n\rfloor - k_n}\mathbb{E}[|\bar{Z}_i^n|^{p-2l}|(\hat{Z}_i^n)^l - (\hat{\chi}_i^n)^l|] \to 0.$$

(b) *If* $\Delta_n^{-r^*}\eta_n \to h$ *for some* $h \in [0, \infty]$, *then* (S.A.33) *holds with* a_n *replaced by* d_n.

PROOF: *Step 1*. We prove part (a) in this step. Fix $l \in \{1, \ldots, p/2\}$. By Hölder's inequality,

$$\text{(S.A.34)}\quad \mathbb{E}[|\bar{Z}_i^n|^{p-2l}|(\hat{Z}_i^n)^l - (\hat{\chi}_i^n)^l|]$$
$$\leq \{\mathbb{E}[|\bar{Z}_i^n|^p]\}^{(p-2l)/p}\{\mathbb{E}[|(\hat{Z}_i^n)^l - (\hat{\chi}_i^n)^l|^{p/(2l)}]\}^{2l/p}.$$

Moreover,

$$
\begin{aligned}
\text{(S.A.35)}\quad & \mathbb{E}\big[\big|(\hat{Z}_i^n)^l - (\hat{\chi}_i^n)^l\big|^{p/(2l)}\big] \\
& \le K\mathbb{E}\big[\big|\hat{Z}_i^n - \hat{\chi}_i^n\big|^{p/2}\big] + K\mathbb{E}\big[\big|\hat{\chi}_i^n\big|^{(l-1)p/(2l)}\big|\hat{Z}_i^n - \hat{\chi}_i^n\big|^{p/(2l)}\big] \\
& \le K\mathbb{E}\big[\big|\hat{Z}_i^n - \hat{\chi}_i^n\big|^{p/2}\big] + K\Delta_n^{(l-1)p/(4l)}\big\{\mathbb{E}\big[\big|\hat{Z}_i^n - \hat{\chi}_i^n\big|^{p/2}\big]\big\}^{1/l} \\
& \le K\Delta_n^{p/2} + K\eta_n^{p/2}\Delta_n^{1/2+p/4} + K\Delta_n^{(l-1)p/(4l)}\big(\Delta_n^{p/2} + \eta_n^{p/2}\Delta_n^{1/2+p/4}\big)^{1/l} \\
& \le K\Delta_n^{p/2} + K\eta_n^{p/2}\Delta_n^{1/2+p/4} + K\Delta_n^{p/4+p/(4l)} + K\eta_n^{p/(2l)}\Delta_n^{p/4+1/(2l)} \\
& \le K\Delta_n^{p/4+p/(4l)} + K\eta_n^{p/(2l)}\Delta_n^{p/4+1/(2l)},
\end{aligned}
$$

where the first inequality follows from Taylor's theorem and the c_r-inequality; the second inequality is obtained by using Hölder's inequality and (S.A.5); the third inequality is obtained by applying (S.A.14) with $v = p/2$; the fourth inequality follows the c_r-inequality; the last inequality holds because $l \ge 1$ and η_n is bounded.

Using (S.A.12) with $w = p$, we have $\mathbb{E}[|\bar{Z}_i^n|^p] \le K\Delta_n^{p/4} + K\eta_n^p\Delta_n^{1/2}$. This estimate, together with (S.A.34) and (S.A.35), implies that

$$
\begin{aligned}
\text{(S.A.36)}\quad & a_n \sum_{i=0}^{\lfloor t/\Delta_n\rfloor - k_n} \mathbb{E}\big[\big|\bar{Z}_i^n\big|^{p-2l}\big|(\hat{Z}_i^n)^l - (\hat{\chi}_i^n)^l\big|\big] \\
& \le K a_n \Delta_n^{-1}\big(\Delta_n^{p/4} + \eta_n^p\Delta_n^{1/2}\big)^{(p-2l)/p}\big(\Delta_n^{p/4+p/(4l)} + \eta_n^{p/(2l)}\Delta_n^{p/4+1/(2l)}\big)^{2l/p} \\
& \le K a_n \Delta_n^{-1}\big(\Delta_n^{(p-2l)/4} + \eta_n^{p-2l}\Delta_n^{(p-2l)/(2p)}\big)\big(\Delta_n^{l/2+1/2} + \eta_n\Delta_n^{l/2+1/p}\big) \\
& \le K a_n \Delta_n^{-1}\big(\Delta_n^{p/4+1/2} + \eta_n^{p-2l}\Delta_n^{1-l/p+l/2} + \eta_n\Delta_n^{p/4+1/p} \\
& \qquad + \eta_n^{p-2l+1}\Delta_n^{1/2-l/p+l/2+1/p}\big) \\
& \le K\Delta_n^{1/4} + K\Delta_n^{(3p-4)/(4p^2-4p)} \to 0,
\end{aligned}
$$

where the first three inequalities are obvious, and the last inequality holds because of (S.A.22) and the fact that the terms vanish to zero at the slowest rate when $l = p/2$. This finishes the proof of (S.A.33).

Step 2. We now prove part (b). Under the condition $\Delta_n^{-r^*}\eta_n \to h$, by (S.A.21), we have $d_n\eta_n^u = O(\Delta_n^{1-p/4+r^*u})$ for each $u \in [0, p-1]$. Since $r^*u + 1/4 \ge \bar{r}u$ for all $u \in [0, p-1]$, we also have $d_n\eta_n^u = O(\Delta_n^{3/4-p/4+\bar{r}u})$. By exactly the same calculation as in (S.A.36),

$$
d_n \sum_{i=0}^{\lfloor t/\Delta_n\rfloor - k_n} \mathbb{E}\big[\big|\bar{Z}_i^n\big|^{p-2l}\big|(\hat{Z}_i^n)^l - (\hat{\chi}_i^n)^l\big|\big] \le K\Delta_n^{1/4} + K\Delta_n^{(3p-4)/(4p(p-1))} \to 0,
$$

which implies the claim in part (b). *Q.E.D.*

PROOF OF THEOREM 1: By localization, we can and will suppose that Assumptions S.H-2 and S.N hold. To simplify notation, we set

$$A_t = \frac{1}{1+h^p} V(g,p,0)_t + \frac{h^p}{1+h^p} U(g,p)_t,$$
$$A(q)_t = \frac{1}{1+h^p} V(g,p,0)_t + \frac{h^p}{1+h^p} U(g,p,q)_t, \quad q \geq 1.$$

By dominated convergence, we have

$$A(q)_t \xrightarrow{\mathbb{P}} A_t \quad \text{as} \quad q \to \infty. \tag{S.A.37}$$

Fix any $\varepsilon > 0$. For each $q \geq 1$, we use the triangle inequality to derive

$$\begin{aligned}
&\limsup_{n\to\infty} \mathbb{P}\big(\big|d_n V\big(Z^{\eta_n}, g, p, 0\big)_t^n - A_t\big| > \varepsilon\big) \\
&\quad \leq \limsup_{n\to\infty} \mathbb{P}\big(d_n \big|V\big(Z^{\eta_n}, g, p, 0\big)_t^n - V\big(Z^* + \eta_n J^q, g, p, 0\big)_t^n\big| > \varepsilon/3\big) \\
&\qquad + \limsup_{n\to\infty} \mathbb{P}\big(\big|d_n V\big(Z^* + \eta_n J^q, g, p, 0\big)_t^n - A(q)_t\big| > \varepsilon/3\big) \\
&\qquad + \mathbb{P}\big(\big|A(q)_t - A_t\big| > \varepsilon/3\big).
\end{aligned} \tag{S.A.38}$$

For any $\varepsilon' > 0$, there exists $K' > 0$ such that $\mathbb{P}(A_t > K') \leq \varepsilon'$. Let $\beta = \frac{\varepsilon}{12K'}$. There exists $K_\beta > 0$ such that $||x+y|^p - |x|^p| \leq K_\beta |y|^p + \beta |x|^p$ for any $x, y \in \mathbb{R}$. Hence,

$$\begin{aligned}
&\big|V\big(Z^{\eta_n}, g, p, 0\big)_t^n - V\big(Z^* + \eta_n J^q, g, p, 0\big)_t^n\big| \\
&\quad \leq K_\beta V\big(\eta_n J'^q, g, p, 0\big)_t^n + \beta V\big(Z^* + \eta_n J^q, g, p, 0\big)_t^n.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\limsup_{n\to\infty} \mathbb{P}\big(d_n \big|V\big(Z^{\eta_n}, g, p, 0\big)_t^n - V\big(Z^* + \eta_n J^q, g, p, 0\big)_t^n\big| > \varepsilon/3\big) \\
&\quad \leq \limsup_{n\to\infty} \mathbb{P}\big(K_\beta d_n V\big(\eta_n J'^q, g, p, 0\big)_t^n > \varepsilon/12\big) \\
&\qquad + \limsup_{n\to\infty} \mathbb{P}\big(\beta \big|d_n V\big(Z^* + \eta_n J^q, g, p, 0\big)_t^n - A(q)_t\big| > \varepsilon/12\big) \\
&\qquad + \mathbb{P}\big(\beta\big|A_t - A(q)_t\big| > \varepsilon/12\big) + \mathbb{P}(\beta A_t > \varepsilon/12) \\
&\quad \leq \limsup_{n\to\infty} \mathbb{P}\big(K_\beta d_n V\big(\eta_n J'^q, g, p, 0\big)_t^n > \varepsilon/12\big) \\
&\qquad + \mathbb{P}\big(\beta\big|A_t - A(q)_t\big| > \varepsilon/12\big) + \varepsilon',
\end{aligned}$$

where the first inequality follows from the triangle inequality, and the second inequality follows from Lemma 6 and our choice of K' and β. Because ε' is arbitrary, we use Lemma 5 and (S.A.37) to derive

$$\lim_{q\to\infty}\limsup_{n\to\infty}\mathbb{P}\big(d_n\big|V\big(Z^{\eta_n},g,p,0\big)_t^n - V\big(Z^*+\eta_n J^q,g,p,0\big)_t^n\big| > \varepsilon/3\big)=0. \tag{S.A.39}$$

Note that Lemma 6 implies that the second term on the right-hand side of (S.A.38) is zero. Sending $q\to\infty$ on both sides of (S.A.38), (S.A.37), and (S.A.39) implies

$$d_n V\big(Z^{\eta_n},g,p,0\big)_t^n \xrightarrow{\mathbb{P},\eta_n} A_t. \tag{S.A.40}$$

We now turn to the behavior of $V(Z^{\eta_n},g,p-2l,l)_t^n$ for $l\in\{1,\dots,p/2\}$. By the triangle inequality, we have

$$d_n\big|V\big(Z^{\eta_n},g,p-2l,l\big)_t^n - V\big(Z^*,g,p-2l,l\big)_t^n\big| \le \sum_{j=1}^{3}\zeta(j)_n, \tag{S.A.41}$$

where

$$\zeta(1)_n = d_n\sum_{i=0}^{\lfloor t/\Delta_n\rfloor-k_n}\big|\bar Z_i^n\big|^{p-2l}\big|\big(\hat Z_i^n\big)^l-\big(\hat\chi_i^n\big)^l\big|,$$

$$\zeta(2)_n = d_n\sum_{i=0}^{\lfloor t/\Delta_n\rfloor-k_n}\big|\bar Z_i^{*n}\big|^{p-2l}\big|\big(\hat Z_i^{*n}\big)^l-\big(\hat\chi_i^n\big)^l\big|,$$

$$\zeta(3)_n = d_n\sum_{i=0}^{\lfloor t/\Delta_n\rfloor-k_n}\big(\big|\bar Z_i^n\big|^{p-2l}-\big|\bar Z_i^{*n}\big|^{p-2l}\big)\big(\hat\chi_i^n\big)^l.$$

We use Lemma 8(b) to get $\zeta(1)_n = o_{p,\eta_n}(1)$. By taking the sequence η_n in Lemma 8(b) to be identically zero, we can use that lemma to derive $\zeta(2)_n = o_p(1)$. By Lemma 7, we also have $\zeta(3)_n = o_{p,\eta_n}(1)$. Therefore, both sides of (S.A.41) are $o_{p,\eta_n}(1)$. Combining this with (S.A.26), we derive

$$d_n V\big(Z^{\eta_n},g,p-2l,l\big)_t^n \xrightarrow{\mathbb{P},\eta_n} \frac{1}{1+h^p}V(g,p,l)_t. \tag{S.A.42}$$

Finally, note that $\rho(p)_0=1$ and $\sum_{l=0}^{p/2}\rho(p)_l V(g,p,l)_t = V(g,p)_t$. The claim then follows from (S.A.40) and (S.A.42). *Q.E.D.*

PROOF OF COROLLARY 1: Recall that for two sequences of strictly positive numbers x_n and y_n, we denote $x_n \sim y_n$ if and only if $\lim_{n\to\infty} x_n/y_n = 1$. Theorem 1 implies that for any weight function g,

$$\frac{1+h^p}{h^p} d_n \eta_n^p \bar{V}\big(Z^{\eta_n}, g, p\big)_t^n = \frac{\eta_n^p V(g,p)_t}{h^p} + \theta \bar{g}(p) \sum_{s\le t} |\eta_n \Delta X_s|^p + o_{p,\eta_n}\big(\eta_n^p\big).$$

Observe $(1+h^p) d_n \eta_n^p / h^p \sim \Delta_n^{1/2}$ and $\Delta J = \eta_n \Delta X$. Hence,

$$\Delta_n^{1/2} \bar{V}\big(Z^{\eta_n}, g, p\big)_t^n = \frac{\eta_n^p V(g,p)_t}{h^p} + \theta \bar{g}(p) \sum_{s\le t} |\Delta J_s|^p + o_{p,\eta_n}\big(\eta_n^p\big).$$

The second assertion readily follows. By the definition of $V(g_i, p)_t$ and κ_i, we have $\sum_{i=1}^d \kappa_i V(g_i, p)_t = 0$ and $\theta \sum_{i=1}^d \kappa_i \bar{g}_i(p) = 1$. The first assertion then follows from the above display and the definition of $\hat{H}_t^n$. The claim in Comment (ii) can be proved similarly. *Q.E.D.*

S.A.4. *Proof of Theorem 2*

Throughout the proof, let $g(\cdot)$ be a generic weight function and let η_n be a sequence in $[0,1]$ that satisfies $\Delta_n^{-\bar{r}} \eta_n \to h$ for some $h \in [0, \infty]$. Recall that

$$\tilde{V}(g,p)_t^n = a_n\big(\bar{V}\big(Z^{\eta_n}, g, p\big)_t^n - \Delta_n^{p/4-1} V(g,p)_t - \eta_n^p \Delta_n^{-1/2} U(g,p)_t\big).$$

The key to the proof is the decomposition

$$\tilde{V}(g,p)_t^n = VC(g,p)_t^n + VJ(g,p,q)_t^n + VJ'(g,p,q)_t^n + \sum_{l=1}^{p/2} \rho(p)_l D(g,p,l)_t^n,$$

where

$$\begin{aligned}
VC(g,p)_t^n &= a_n\big(\bar{V}\big(Z^*, g, p\big)_t^n - \Delta_n^{p/4-1} V(g,p)_t\big),\\
VJ(g,p,q)_t^n &= a_n\big(V\big(Z^{\eta_n}, g, p, 0\big)_t^n - V\big(Z^* + \eta_n J'^q, g, p, 0\big)_t^n \\
&\quad - \Delta_n^{-1/2} \eta_n^p U(g,p,q)_t\big),\\
VJ'(g,p,q)_t^n &= a_n\bigg(V\big(Z^* + \eta_n J'^q, g, p, 0\big)_t^n - V\big(Z^*, g, p, 0\big)_t^n \\
&\quad - \Delta_n^{-1/2} \theta \bar{g}(p) \eta_n^p \sum_{s\le t} \big|\Delta J_s'^q\big|^p\bigg),
\end{aligned}$$

and, for $l = 1, \dots, p/2$,

$$D(g, p, l)_t^n = a_n\big(V\big(Z^{\eta_n}, g, p-2l, l\big)_t^n - V\big(Z^*, g, p-2l, l\big)_t^n\big).$$

The terms $VJ(g, p, q)_t^n$ and $VC(g, p)_t^n$ serve as the leading terms in the central limit theorem, contributed respectively by "big jumps" and the continuous part. The terms $D(g, p, l)_t^n$, $l = 1, \dots, p/2$, are asymptotically negligible; so is the term $VJ'(g, p, q)_t^n$ when q is large.

We consider d weight functions $(g_i)_{1\le i\le d}$ as in the main text. So as to describe the convergence of $(VJ(g_i, p, q)_t^n + VC(g_i, p)_t^n)_{1\le i\le d}$ for fixed q, we set up some notation. Since $\Psi_\pm$ and $\Psi'_\pm$ defined in the main text are positive semidefinite, we can consider four independent sequences of independent and identically distributed (i.i.d.) d-dimensional variables $(U_{m-})_{m\ge1}$, $(U_{m+})_{m\ge1}$, $(U'_{m-})_{m\ge1}$, and $(U'_{m+})_{m\ge1}$ that are defined on an extension of the original probability space, independently of $\mathcal{F}$, such that for each m, the d-dimensional variables U_{m-}, U_{m+}, U'_{m-}, and U'_{m+} are centered Gaussian vectors with respective covariances Ψ_-, Ψ_+, Ψ'_-, and Ψ'_+. Let $(T_m)_{m\ge1}$ be the exhausting sequence of stopping times described in Section S.A.3. The following d-dimensional process is well defined:

$$\bar{U}(p)_t = \theta p \sum_{m\ge1: T_m\le t} (\Delta X_{T_m})^{p-1} \times \left(\sqrt{\theta}\sigma_{T_m-}U_{m-} + \sqrt{\theta}\sigma_{T_m}U_{m+} + \frac{\alpha_{T_m-}}{\sqrt{\theta}}U'_{m-} + \frac{\alpha_{T_m}}{\sqrt{\theta}}U'_{m+}\right).$$

For each $q \ge 1$, we associate P_q with the variable

$$\bar{U}(p, q)_t = \theta p \sum_{m\in P_q: T_m\le t} (\Delta X_{T_m})^{p-1} \times \left(\sqrt{\theta}\sigma_{T_m-}U_{m-} + \sqrt{\theta}\sigma_{T_m}U_{m+} + \frac{\alpha_{T_m-}}{\sqrt{\theta}}U'_{m-} + \frac{\alpha_{T_m}}{\sqrt{\theta}}U'_{m+}\right).$$

We also construct a d-dimensional variable $\bar{V}(p)_t$ on the same extension that, conditional on $\mathcal{F}$, is independent of $(U_{m\pm})_{m\ge1}$ and $(U'_{m\pm})_{m\ge1}$, and is a centered Gaussian variable with covariance matrix Σ_C.

The asymptotic distribution of $(VC(g_i, p)_t^n + VJ(g_i, p, q)_t^n)_{1\le i\le d}$ is described by the following lemma.

LEMMA 9: *Suppose that Assumptions S.H-1, S.K, and S.N hold. Then*

$$\big(VC(g_i, p)_t^n + VJ(g_i, p, q)_t^n\big)_{1\le i\le d} \xrightarrow{\mathcal{L}\text{-}s,\eta_n} \frac{1}{1+h^{p-1}}\bar{V}(p)_t + \frac{h^{p-1}}{1+h^{p-1}}\bar{U}(p, q)_t.$$

The asymptotic behaviors of $VJ'(g,p,q)_t^n$ and $D(g,p,l)_t^n$ are given by the following two lemmas. The notation $\mathbb{P}_{\eta_n}$ emphasizes the dependence of the data generating process on η_n.

LEMMA 10: *Suppose that Assumptions S.H-1 and S.N hold. For any* $\varepsilon > 0$,

$$\lim_{q\to\infty}\limsup_{n\to\infty}\mathbb{P}_{\eta_n}\big(|VJ'(g,p,q)_t^n| > \varepsilon\big) = 0.$$

LEMMA 11: *Suppose that Assumptions S.H-1 and S.N hold. For each* $l \in \{1,\dots,p/2\}$,

$$D(g,p,l)_t^n = o_{p,\eta_n}(1).$$

PROOF OF THEOREM 2: By localization, we can suppose that Assumptions S.H-1, S.K, and S.N hold without loss of generality. Note that $\bar{U}(p,q)_t$ converges in probability to $\bar{U}(p)_t$ as $q\to\infty$. Combining this with Lemmas 9 and 10, we derive (note that the first line below does not depend on q)

$$\begin{aligned}\text{(S.A.43)}\quad &\big(VC(g_i,p)_t^n + VJ(g_i,p,q)_t^n + VJ'(g_i,p,q)_t^n\big)_{1\le i\le d}\\ &\xrightarrow{\mathcal{L}\text{-}s,\eta_n} \frac{1}{1+h^{p-1}}\bar{V}(p)_t + \frac{h^{p-1}}{1+h^{p-1}}\bar{U}(p)_t.\end{aligned}$$

By Lemma 11, the difference between $(\tilde{V}(g_i,p)_t^n)_{1\le i\le d}$ and the first line of (S.A.43) is $o_{p,\eta_n}(1)$. The claim readily follows. *Q.E.D.*

We now prove Lemmas 9, 10, and 11. To simplify notation, let $f(x) = |x|^p$ and

$$\begin{aligned}\text{(S.A.44)}\quad F(x,y) &= f(x+y) - f(x) - f'(x)y,\\ G(x,y) &= f(x+y) - f(x) - f(y),\\ H(x,y) &= f(x,y) - f(x) - f(y) - f'(x)y.\end{aligned}$$

The following inequalities are elementary consequences of the Taylor expansion: for even $p\ge 2$,

$$\text{(S.A.45)}\quad \left.\begin{aligned}&\big|F(x,y)\big| \le K\big(|y|^p + y^2|x|^{p-2}\big),\\ &\big|F\big(x+x',y\big) - F(x,y)\big|\\ &\quad\le K\big(|x|^{p-3}\big|x'\big||y|^2 + \big|x'\big||y|^{p-1} + \big|x'\big|^{p-2}|y|^2\big),\\ &\big|G(x,y)\big| \le K\big(|x||y|^{p-1} + |y||x|^{p-1}\big),\\ &\big|H(x,y)\big| \le K\big(|x||y|^{p-1} + y^2|x|^{p-2}\big),\end{aligned}\right\}$$

where the second inequality requires $p\ge 4$.

PROOF OF LEMMA 9: By Theorem 4.1 of JPV,

$$\Delta_n^{3/4-p/4}\big(\bar{V}\big(Z^*, g_i, p\big)_t^n - \Delta_n^{p/4-1}V(g_i, p)_t\big)_{1\le i\le d} \xrightarrow{\mathcal{L}\text{-}s} \bar{V}(p)_t. \tag{S.A.46}$$

Hence,

$$\big(VC(g_i, p)_t^n\big)_{1\le i\le d} \xrightarrow{\mathcal{L}\text{-}s} \frac{1}{1+h^{p-1}}\bar{V}(p)_t. \tag{S.A.47}$$

Let $\Omega_n(t,q)$ and I_m^n be defined in the same way as in the proof of Lemma 6. Since $\Omega_n(t,q)\to\Omega$ almost surely as $n\to\infty$, we can and will restrict our calculation below on $\Omega_n(t,q)$ without loss of generality. For a generic weight function $g(\cdot)$, we have the decomposition

$$\begin{aligned} VJ(g,p,q)_t^n &= a_n\Delta_n^{-1/4}\eta_n^{p-1}\sum_{m\in P_q:T_m\le t} f'(\Delta X_{T_m})\zeta(g)_m^n \\ &\quad + a_n\sum_{m\in P_q:T_m\le t}\sum_{j=1}^{k_n-1} H\big(g_j^n\eta_n\Delta X_{T_m}, \bar{Z}^{*n}_{I_m^n+1-j} + \eta_n\bar{J}'^{q,n}_{I_m^n+1-j}\big) \\ &\quad + R_t^n, \end{aligned} \tag{S.A.48}$$

where

$$\zeta(g)_m^n = \Delta_n^{1/4}\sum_{j=1}^{k_n-1}\big(g_j^n\big)^{p-1}\big(\bar{Z}^{*n}_{I_m^n+1-j} + \eta_n\bar{J}'^{q,n}_{I_m^n+1-j}\big), \quad m\ge 1,$$

$$R_t^n = a_n\eta_n^p\Delta_n^{-1/2}\left(\Delta_n^{1/2}\sum_{j=1}^{k_n-1}\big(g_j^n\big)^p - \theta\bar{g}(p)\right)\sum_{m\in P_q:T_m\le t}|\Delta X_{T_m}|^p.$$

Note that $k_n^{-1}\sum_{j=1}^{k_n}(g_j^n)^p = \bar{g}(p) + O(k_n^{-1})$. Because $k_n\Delta_n^{1/2} = \theta + o(\Delta_n^{1/4})$ by assumption, we have $\Delta_n^{1/2}\sum_{j=1}^{k_n}(g_j^n)^p - \theta\bar{g}(p) = o(\Delta_n^{1/4})$. Since $a_n\eta_n^{p-1} = O(\Delta_n^{1/4})$ and η_n is bounded, we have

$$R_t^n = o_{p,\eta_n}(\eta_n) = o_{p,\eta_n}(1). \tag{S.A.49}$$

Next, we show the negligibility of the second term on the right-hand side of equation (S.A.48). Since $H(\cdot,\cdot)\equiv 0$ when $p=2$, we can suppose $p\ge 4$ without loss. Let $\mathcal{H}_0 = \mathcal{F}_0\vee\sigma\{T_m : m\in P_q\}$. Because the stopping times $\{T_m : m\in P_q\}$ are independent of the Wiener process W and the Poisson measure μ restricted on $\mathbb{R}_+\times\{z:\gamma(z)\le 1/q\}$, the same argument that leads to (S.A.18) yields $\mathbb{E}[|\bar{J}'^{q,n}_{I_m^n+1-j}|^v|\mathcal{H}_0]\le K\Delta_n^{1/2}$ for every $v\ge 1$. Hence,

$$v\ge 1 \quad\Rightarrow\quad \mathbb{E}\big[\big|\bar{Z}^{*n}_{I_m^n+1-j} + \eta_n\bar{J}'^{q,n}_{I_m^n+1-j}\big|^v|\mathcal{H}_0\big]\le K\Delta_n^{v/4} + K\eta_n^v\Delta_n^{1/2}. \tag{S.A.50}$$

Hence,

$$\begin{aligned}
&\mathbb{E}\left[a_n \sum_{m\in P_q:T_m\le t}\sum_{j=1}^{k_n-1}\left|H\left(g_j^n\eta_n\Delta X_{T_m}, \bar{Z}^{*n}_{I_m^n+1-j}+\eta_n\bar{J}'^{q,n}_{I_m^n+1-j}\right)\right|\right]\\
&\le K\mathbb{E}\left[a_n \sum_{m\in P_q:T_m\le t}\sum_{j=1}^{k_n-1}\left|g_j^n\eta_n\Delta X_{T_m}\right|\left|\bar{Z}^{*n}_{I_m^n+1-j}+\eta_n\bar{J}'^{q,n}_{I_m^n+1-j}\right|^{p-1}\right]\\
&\quad + K\mathbb{E}\left[a_n \sum_{m\in P_q:T_m\le t}\sum_{j=1}^{k_n-1}\left|g_j^n\eta_n\Delta X_{T_m}\right|^{p-2}\left|\bar{Z}^{*n}_{I_m^n+1-j}+\eta_n\bar{J}'^{q,n}_{I_m^n+1-j}\right|^{2}\right]\\
&\le Ka_nk_n\eta_n\left(\Delta_n^{(p-1)/4}+\eta_n^{p-1}\Delta_n^{1/2}\right)+Ka_nk_n\eta_n^{p-2}\left(\Delta_n^{1/2}+\eta_n^2\Delta_n^{1/2}\right)\\
&\le Ka_nk_n\eta_n\Delta_n^{(p-1)/4}+Ka_nk_n\eta_n^{p-2}\Delta_n^{1/2}\\
&\le K\Delta_n^{(p-2)/(4(p-1))}+K\Delta_n^{1/(4(p-1))}\to 0,
\end{aligned}$$

where the first inequality follows from (S.A.45); the second inequality is obtained by noting that jumps are bounded under Assumption S.H-1 and using (S.A.50) with $v = p-1$ and $v = 2$, and the law of iterated expectations; the third inequality holds because η_n is bounded; we obtain the last inequality by using (S.A.22). Therefore, we have

$$\text{(S.A.51)}\quad a_n \sum_{m\in P_q:T_m\le t}\sum_{j=1}^{k_n-1}H\left(g_j^n\eta_n\Delta X_{T_m}, \bar{Z}^{*n}_{I_m^n+1-j}+\eta_n\bar{J}'^{q,n}_{I_m^n+1-j}\right)=o_{p,\eta_n}(1).$$

For each $m \ge 1$, define

$$\zeta_m = \sqrt{\theta}\sigma_{T_m-}U_{m-}+\sqrt{\theta}\sigma_{T_m}U_{m+}+\frac{\alpha_{T_m-}}{\sqrt{\theta}}U'_{m-}+\frac{\alpha_{T_m}}{\sqrt{\theta}}U'_{m+}.$$

A straightforward adaptation of Lemma 5.13 in JPV shows that

$$\text{(S.A.52)}\quad \left(\zeta(g_i)_m^n\right)_{1\le i\le d, m\in P_q}\xrightarrow{\mathcal{L}\text{-}s}(\theta\zeta_m)_{m\in P_q}.$$

(To be exact, we make the following modifications. First, replace Z' in JPV with $Z^*+\eta_nJ'^q$, replace X'' with X^*, replace M with $\eta_nJ'^q$, and replace X' with $X^*+\eta_nJ'^q$. Second, A_n defined in JPV (p. 1592) still satisfies $A_n/\sqrt{k_n\Delta_n}\xrightarrow{\mathbb{P}}0$ after these modifications because η_n considered in the present paper is bounded. Third, other calculations in JPV are valid without change and, in particular, (5.89) in JPV is still valid. Finally, note that $\zeta(g)_m^n$ defined here is $k_n\Delta_n^{1/2}$ times $\eta(q,g)_m^n$ in JPV and $k_n\Delta_n^{1/2}\to\theta$ by assumption.)

Since $a_n\Delta_n^{-1/4}\eta_n^{p-1} \to \frac{h^{p-1}}{1+h^{p-1}}$, by the properties of stable convergence, (S.A.52) implies that

$$\left(a_n\Delta_n^{-1/4}\eta_n^{p-1}\sum_{m\in P_q:T_m\le t} f'(\Delta X_{T_m})\zeta(g_i)_m^n\right)_{1\le i\le d} \xrightarrow{\mathcal{L}\text{-}s,\eta_n} \frac{h^{p-1}}{1+h^{p-1}}\bar{U}(p,q)_t. \tag{S.A.53}$$

Combining (S.A.48), (S.A.49), (S.A.51), and (S.A.53), we get

$$\left(VJ(g_i,p,q)_t^n\right)_{1\le i\le d} \xrightarrow{\mathcal{L}\text{-}s,\eta_n} \frac{h^{p-1}}{1+h^{p-1}}\bar{U}(p,q)_t. \tag{S.A.54}$$

By using the same argument as in the proof of Lemma 5.8 in Jacod (2008), we can show that the marginal convergences in (S.A.46) and (S.A.52) also hold jointly; hence (S.A.47) and (S.A.54) also hold jointly. The claim then readily follows. *Q.E.D.*

We now turn to the proof of Lemma 10. For each $q\ge 1$, we denote by Σ_t^q the pth power variation process of the process J'^q, that is, $\Sigma_t^q = \sum_{s\le t}|\Delta J_s'^q|^p$. The proof relies on the decomposition

$$VJ'(g,p,q)_t^n = \sum_{i=0}^{\lfloor t/\Delta_n\rfloor-k_n}\Gamma(q)_i^n + R(q)_t^n, \tag{S.A.55}$$

where

$$\Gamma(q)_i^n = a_n\left(\left|\bar{Z}_i^{*n}+\eta_n\bar{J}_i'^{q,n}\right|^p - \left|\bar{Z}_i^{*n}\right|^p - \eta_n^p\sum_{j=1}^{k_n-1}\left(g_j^n\right)^p\Delta_{i+j}^n\Sigma^q\right),$$

$$R(q)_t^n = a_n\eta_n^p\left(\left(\sum_{i=0}^{\lfloor t/\Delta_n\rfloor-k_n}\sum_{j=1}^{k_n-1}\left(g_j^n\right)^p\Delta_{i+j}^n\Sigma^q\right) - \Delta_n^{-1/2}\theta\bar{g}(p)\Sigma_t^q\right).$$

We first prove a lemma that describes the behavior of $\Gamma(q)_i^n$.

LEMMA 12: *Suppose that Assumptions S.H-1, S.K, and S.N hold. Let η_n be any sequence in $[0,1]$ such that $\Delta_n^{-\bar{r}}\eta_n\to h$ for some $h\in[0,\infty]$. Then we can find a sequence $\varphi(q)$ going to 0 as $q\to\infty$, $r_1>1$ and $r_2>3/2$, with the following property: for any $q\ge 1$ and $i\ge 1$, we have a decomposition $\Gamma(q)_i^n = \Gamma'(q)_i^n + \Gamma''(q)_i^n$, where $\Gamma'(q)_i^n$ and $\Gamma''(q)_i^n$ are $\mathcal{F}_{(i+k_n)\Delta_n}$-measurable, $\mathbb{E}[\Gamma''(q)_i^n|\mathcal{F}_{i\Delta_n}]=0$, and*

$$\mathbb{E}\left|\Gamma'(q)_i^n\right| \le K\Delta_n^{r_1} + K\varphi(q)\Delta_n,$$

$$\mathbb{E}\left[\left|\Gamma''(q)_i^n\right|^2\right] \le K\Delta_n^{r_2} + K\varphi(q)\Delta_n^{3/2}.$$

PROOF: *Step 1*. In this step, we provide a decomposition of $\Gamma(q)_i^n$ via Itô's lemma. Denote $f(x) = |x|^p$ and recall the notation in (S.A.44). We define $b'_s = b_{s+i\Delta_n}$, $\sigma'_s = \sigma_{s+i\Delta_n}$, $\delta'(s,z) = \delta(s + i\Delta_n, z)$, $W'_s = W_{s+i\Delta_n}$, $\mu'([0,s] \times B) = \mu([i\Delta_n, i\Delta_n + s] \times B)$ for each $s \geq 0$ and $B \in \mathcal{E}$, and $b''_s = -\int \delta'(s,z) \times 1_{\{|\delta'(s,z)| \leq 1\}} \lambda(dz)$.

To simplify notation, we define two processes, Y^* and Y, as

$$\text{(S.A.56)} \quad Y_t^* = \int_0^t g_n(s) b'_s \, ds + \int_0^t g_n(s) \sigma'_s \, dW'_s - \sum_{j=1}^{\lfloor t/\Delta_n \rfloor} g_j'^n \chi_{(i+j-1)\Delta_n},$$

$$Y_t = Y_t^* + \eta_n \int_0^t g_n(s) b''_s \, ds + \eta_n \int_0^t \int g_n(s) \delta'(s,z) 1_{\{\gamma(z) \leq 1/q\}} \mu'(ds, dz).$$

We then have $\bar{Z}_i^{*n} = Y_{k_n\Delta_n}^*$ and $\bar{Z}_i^{*n} + \eta_n \bar{J}_i'^{q,n} = Y_{k_n\Delta_n}$. Also observe that

$$\eta_n^p \sum_{j=1}^{k_n - 1} \left(g_j^n\right)^p \Delta_{i+j}^n \Sigma^q = \int_0^{k_n\Delta_n} \int_E f\big(\eta_n g_n(s) \delta'(s,z)\big) 1_{\{\gamma(z) \leq 1/q\}} \mu'(ds, dz).$$

Hence, we can rewrite $\Gamma(q)_i^n$ as

$$\text{(S.A.57)} \quad \Gamma(q)_i^n = a_n\big(f(Y_{k_n\Delta_n}) - f\big(Y_{k_n\Delta_n}^*\big)\big) - a_n \int_0^{k_n\Delta_n} \int_E f\big(\eta_n g_n(s) \delta'(s,z)\big) 1_{\{\gamma(z) \leq 1/q\}} \mu'(ds, dz).$$

Recall that under Assumption S.H-1, jumps have finite variation. Applying Itô's formula to $f(Y_t)$ and $f(Y_t^*)$, we have

$$\text{(S.A.58)} \quad f(Y_t) = \int_0^t f'(Y_s) g_n(s) b'_s \, ds + \frac{1}{2} \int_0^t f''(Y_s) g_n(s)^2 \left(\sigma'_s\right)^2 ds + \eta_n \int_0^t f'(Y_s) g_n(s) b''_s \, ds + \int_0^t f'(Y_s) g_n(s) \sigma'_s \, dW'_s + \sum_{j=1}^{\lfloor t/\Delta_n \rfloor} \left[f\big(Y_{j\Delta_n-} - g_j'^n \chi_{(i+j-1)\Delta_n}\big) - f(Y_{j\Delta_n-})\right]$$

$$+\int_0^t \int \big[f\big(Y_{s-}+\eta_n g_n(s)\delta'(s,z)\big)-f(Y_{s-})\big] \times 1_{\{\gamma(z)\le 1/q\}}\mu'(ds,dz)$$

and

$$f\big(Y_t^*\big)=\int_0^t f'\big(Y_s^*\big)g_n(s)b'_s\,ds + \frac{1}{2}\int_0^t f''\big(Y_s^*\big)g_n(s)^2\big(\sigma'_s\big)^2\,ds + \int_0^t f'\big(Y_s^*\big)g_n(s)\sigma'_s\,dW'_s + \sum_{j=1}^{\lfloor t/\Delta_n\rfloor}\big[f\big(Y^*_{j\Delta_n-}-g_j'^n\chi_{(i+j-1)\Delta_n}\big)-f\big(Y^*_{j\Delta_n-}\big)\big]. \tag{S.A.59}$$

Plug (S.A.58) and (S.A.59) into (S.A.57). After some algebra, we can decompose $\Gamma(q)_i^n$ as

$$\Gamma(q)_i^n=\sum_{k=1}^{9}\zeta(q,k)_i^n, \tag{S.A.60}$$

where

$$\zeta(q,1)_i^n=a_n\int_0^{k_n\Delta_n}\big[f'(Y_s)-f'\big(Y_s^*\big)\big]g_n(s)\big(b'_s+\eta_n b''_s\big)\,ds,$$

$$\zeta(q,2)_i^n=a_n\eta_n\int_0^{k_n\Delta_n} f'\bigg(\int_0^s g_n(u)b'_u\,du\bigg)g_n(s)b''_s\,ds,$$

$$\zeta(q,3)_i^n=\frac{1}{2}a_n\int_0^{k_n\Delta_n}\big[f''(Y_s)-f''\big(Y_s^*\big)\big]g_n(s)^2\big(\sigma'_s\big)^2\,ds,$$

$$\zeta(q,4)_i^n=a_n\int_0^{k_n\Delta_n}\int_E G\big(Y_s,\eta_n g_n(s)\delta'(s,z)\big)1_{\{\gamma(z)\le 1/q\}}\nu(ds,dz),$$

$$\zeta(q,5)_i^n=a_n\sum_{j=1}^{k_n}\big[F\big(Y_{j\Delta_n-},-g_j'^n\chi_{(i+j-1)\Delta_n}\big)-F\big(Y^*_{j\Delta_n-},-g_j'^n\chi_{(i+j-1)\Delta_n}\big)\big],$$

$$\zeta(q,6)_i^n=a_n\eta_n\int_0^{k_n\Delta_n}\bigg[f'\big(Y_s^*\big)-f'\bigg(\int_0^s g_n(u)b'_u\,du\bigg)\bigg]g_n(s)b''_s\,ds,$$

$$\zeta(q,7)_i^n=a_n\int_0^{k_n\Delta_n}\big[f'(Y_s)-f'\big(Y_s^*\big)\big]g_n(s)\sigma'_s\,dW'_s,$$

$$\zeta(q,8)_i^n = a_n \int_0^{k_n\Delta_n} \int_E G\big(Y_{s-}, \eta_n g_n(s)\delta'(s,z)\big) \times 1_{\{\gamma(z)\le 1/q\}}\big(\mu' - \nu\big)(ds,dz),$$

$$\zeta(q,9)_i^n = -a_n \sum_{j=1}^{k_n} \big[f'(Y_{j\Delta_n-}) - f'\big(Y^*_{j\Delta_n-}\big)\big] g_j'^n \chi_{(i+j-1)\Delta_n}.$$

Step 2. In this step, we collect some preliminary estimates. We set

$$\tilde\varphi(q) = \int \gamma(z) 1_{\{\gamma(z)\le 1/q\}} \lambda(dz)$$

and $\varphi(q) = \tilde\varphi(q)^{1/2}$. Since $\int_E \gamma(z)\lambda(dz) < \infty$ under Assumption S.H-1, $\varphi(q) \to 0$ as $q \to \infty$ by dominated convergence. Since $\gamma(\cdot)$ is bounded,

$$u \ge 1, v \ge 1/2 \quad \Rightarrow \quad \left(\int \gamma(z)^u 1_{\{\gamma(z)\le 1/q\}} \lambda(dz)\right)^v \le K\tilde\varphi(q)^v \le K\varphi(q);$$

we will use this simple result repeatedly without further mention.

Let $w \ge 1$. With a straightforward extension of (S.A.18) and (S.A.4), we have for any $s \in [0, k_n\Delta_n]$,

$$\text{(S.A.61)} \quad \mathbb{E}\big|Y_s - Y_s^*\big|^w \le K\eta_n^w \Delta_n^{w/2} + K\eta_n^w \Delta_n^{1/2}\tilde\varphi(q),$$

$$\text{(S.A.62)} \quad \mathbb{E}\big|Y_s^*\big|^w \le K\Delta_n^{w/4},$$

where $K > 0$ does not depend on s. By the c_r-inequality, these estimates further imply

$$\text{(S.A.63)} \quad \mathbb{E}|Y_s|^w \le K\Delta_n^{w/4} + K\eta_n^w \Delta_n^{1/2}\tilde\varphi(q).$$

Moreover, if $w \ge 1$ is also an integer, then for every $v \ge 1$ and $m > 1$,

$$\text{(S.A.64)} \quad \mathbb{E}\big[\big|(Y_s)^w - \big(Y_s^*\big)^w\big|^v\big] \le K\eta_n^{wv}\Delta_n^{wv/2} + K\eta_n^{wv}\Delta_n^{1/2}\tilde\varphi(q) + K\eta_n^v \Delta_n^{(w+1)v/4} + K\eta_n^v \Delta_n^{(w-1)v/4+1/(2m)}\tilde\varphi(q)^{1/m}.$$

This estimate is derived as

$$\begin{aligned}
&\mathbb{E}\big[\big|(Y_s)^w - \big(Y_s^*\big)^w\big|^v\big] \\
&\quad \le K\mathbb{E}\big|Y_s - Y_s^*\big|^{wv} + K\mathbb{E}\big[\big|Y_s^*\big|^{(w-1)v}\big|Y_s - Y_s^*\big|^v\big] \\
&\quad \le K\mathbb{E}\big|Y_s - Y_s^*\big|^{wv} + K\Delta_n^{(w-1)v/4}\big(\mathbb{E}\big|Y_s - Y_s^*\big|^{mv}\big)^{1/m}
\end{aligned}$$

$$\leq K\eta_n^{wv}\Delta_n^{wv/2} + K\eta_n^{wv}\Delta_n^{1/2}\tilde{\varphi}(q) + K\eta_n^{v}\Delta_n^{(w-1)v/4}\big(\Delta_n^{mv/2} + \Delta_n^{1/2}\tilde{\varphi}(q)\big)^{1/m},$$

where the first inequality holds because, by Taylor's expansion, for each integer $q \geq 1$, $|(x+y)^q - x^q| \leq K|y|^q + K|x|^{q-1}|y|$ for any $x, y \in \mathbb{R}$; the second inequality is obtained by applying Hölder's inequality with index m and then using (S.A.62); the third inequality is obtained by applying (S.A.61) with indices wv and mv; (S.A.64) then follows the c_r-inequality.

Step 3. This step consists of 9 substeps. We show the following relationships: if $k \in \{1, \dots, 5\}$ or $k = 6$ and $p \geq 4$, then

$$\text{(S.A.65)}\quad \mathbb{E}\big|\zeta(q,k)_i^n\big| \leq K\Delta_n^{7/6} + K\Delta_n\varphi(q);$$

if $k \in \{7, 8, 9\}$ or $k = 6$ and $p = 2$, then

$$\text{(S.A.66)}\quad \mathbb{E}\big[\zeta(q,k)_i^n|\mathcal{F}_{i\Delta_n}\big] = 0,$$
$$\mathbb{E}\big[\big(\zeta(q,k)_i^n\big)^2\big] \leq K\Delta_n^2 + K\Delta_n^{3/2}\varphi(q).$$

Below, we prove (S.A.65) and (S.A.66) for each k.

Step 3(i). We observe

$$\begin{aligned}\mathbb{E}\big|\zeta(q,1)_i^n\big| &\leq Ka_n\int_0^{k_n\Delta_n}\mathbb{E}\big|f'(Y_s) - f'\big(Y_s^*\big)\big|\,ds\\ &\leq Ka_n\Delta_n^{1/2}\big(\eta_n^{p-1}\Delta_n^{(p-1)/2} + \eta_n^{p-1}\Delta_n^{1/2}\tilde{\varphi}(q)\\ &\quad + \eta_n\Delta_n^{p/4} + \eta_n\Delta_n^{(p-1)/4}\tilde{\varphi}(q)^{1/2}\big)\\ &\leq K\Delta_n^{(2p+1)/4} + K\Delta_n^{5/4}\tilde{\varphi}(q) + K\Delta_n^{(6p-7)/(4p-4)}\\ &\quad + K\Delta_n^{(5p-6)/(4p-4)}\tilde{\varphi}(q)^{1/2}\\ &\leq K\Delta_n^{5/4} + K\Delta_n\varphi(q),\end{aligned}$$

where the first inequality holds because $g_n(\cdot)$ and $b' + \eta_n b''$ are bounded; the second inequality is obtained by using (S.A.64) with $w = p-1$, $v = 1$, and $m = 2$; the third inequality follows from (S.A.22); the last inequality holds because when $p \geq 2$, $\min\{\frac{2p+1}{4}, \frac{6p-7}{4p-4}\} \geq 5/4$ and $\frac{5p-6}{4p-4} \geq 1$. We hence verify (S.A.65) for $k = 1$.

Step 3(ii). Since $|f'(x)| = O(|x|^{p-1})$, and g_n, b', and b'' are bounded, $\mathbb{E}|\zeta(q,2)_i^n| \leq K a_n \eta_n \Delta_n^{p/2}$. Observe that $a_n = O(\Delta_n^{3/4-p/4})$ and $\eta_n = O(1)$. Hence, when $p \geq 2$, $\mathbb{E}|\zeta(q,2)_i^n| \leq K\Delta_n^{3/4+p/4} \leq K\Delta_n^{5/4}$. We thus have (S.A.65) for $k = 2$.

Step 3(iii). Note that when $p = 2$, $f''(\cdot) \equiv 2$ and $\zeta(q,3)_i^n$ is identically zero, which trivially implies (S.A.65). When $p \geq 4$,

$$
\begin{aligned}
&\mathbb{E}\big|\zeta(q,3)_i^n\big| \\
&\quad \leq K a_n \int_0^{k_n\Delta_n} \mathbb{E}\big|f''(Y_s) - f''\big(Y_s^*\big)\big|\,ds \\
&\quad \leq K a_n \Delta_n^{1/2}\big(\eta_n^{p-2}\Delta_n^{(p-2)/2} + \eta_n^{p-2}\Delta_n^{1/2}\tilde{\varphi}(q) \\
&\qquad + \eta_n \Delta_n^{(p-1)/4} + \eta_n \Delta_n^{(p-3)/4+1/3}\tilde{\varphi}(q)^{2/3}\big) \\
&\quad \leq K\Delta_n^{(2p^2-4p+3)/(4p-4)} + K\Delta_n^{(4p-3)/(4p-4)}\varphi(q) \\
&\qquad + K\Delta_n^{(5p-6)/(4p-4)} + K\Delta_n^{(13p-16)/(12p-12)}\varphi(q) \\
&\quad \leq K\Delta_n^{7/6} + K\Delta_n\varphi(q),
\end{aligned}
$$

where the first inequality holds because $g_n(\cdot)$ and σ' are bounded; the second inequality is obtained by applying (S.A.64) with $w = p-2$, $v = 1$, and $m = 3/2$; the third inequality follows from (S.A.22); the last inequality holds because when $p \geq 4$, we have $\frac{2p^2-4p+3}{4p-4} \geq 19/12$, $\frac{5p-6}{4p-4} \geq 7/6$, and $\frac{13p-16}{12p-12} \geq 1$. We hence have (S.A.65) for $k = 3$.

Step 3(iv). Observe

$$
\begin{aligned}
\mathbb{E}\big|\zeta(q,4)_i^n\big| &\leq K a_n \eta_n \varphi(q) \int_0^{k_n\Delta_n} \mathbb{E}\big[|Y_s|^{p-1}\big]\,ds \\
&\quad + K a_n \eta_n^{p-1}\varphi(q) \int_0^{k_n\Delta_n} \mathbb{E}\big[|Y_s|\big]\,ds \\
&\leq K a_n \eta_n \varphi(q)\Delta_n^{p/4+1/4} + K a_n \eta_n^{p-1}\varphi(q)\Delta_n^{3/4} \\
&\leq K\Delta_n^{(5p-6)/(4p-4)}\varphi(q) + K\Delta_n\varphi(q) \\
&\leq K\Delta_n\varphi(q),
\end{aligned}
$$

where the first inequality follows from (S.A.45) and the fact that $g_n(\cdot)$ is bounded and $\delta'(s,z)$ is bounded by $\gamma(z)$; the second inequality is obtained by applying (S.A.63) with $w = p-1$ and $w = 1$; the third inequality follows from (S.A.22); the last inequality holds because $\frac{5p-6}{4p-4} \geq 1$ when $p \geq 2$. We hence verify (S.A.65) for $k = 4$.

Step 3(v). When $p = 2$, $\zeta(q,5)_i^n$ is identically zero and (S.A.65) is trivially true. Now suppose $p \geq 4$. By the second inequality of (S.A.45),

$$\text{(S.A.67)}\quad \mathbb{E}\big|\zeta(q,5)_i^n\big| \leq K\sum_{l=1}^{3}\mathbb{E}\big[\zeta(q,5,l)_i^n\big],$$

where

$$\zeta(q,5,1)_i^n = a_n\sum_{j=1}^{k_n}\big|Y_{j\Delta_n-} - Y^*_{j\Delta_n-}\big|\big|g_j'^n\chi_{(i+j-1)\Delta_n}\big|^{p-1},$$

$$\zeta(q,5,2)_i^n = a_n\sum_{j=1}^{k_n}\big|Y_{j\Delta_n-} - Y^*_{j\Delta_n-}\big|^{p-2}\big|g_j'^n\chi_{(i+j-1)\Delta_n}\big|^{2},$$

$$\zeta(q,5,3)_i^n = a_n\sum_{j=1}^{k_n}\big|Y^*_{j\Delta_n-}\big|^{p-3}\big|Y_{j\Delta_n-} - Y^*_{j\Delta_n-}\big|\big|g_j'^n\chi_{(i+j-1)\Delta_n}\big|^{2}.$$

By Fatou's lemma, (S.A.61)–(S.A.64) still hold if we replace Y_s and Y_s^* there with Y_{s-} and Y_{s-}^*; this will be used below without further mention.

We bound $\mathbb{E}[\zeta(q,5,1)_i^n]$ as

$$\begin{aligned}\text{(S.A.68)}\quad \mathbb{E}\big[\zeta(q,5,1)_i^n\big] &\leq Ka_nk_n^{-(p-1)}\sum_{j=1}^{k_n}\mathbb{E}\big[\big|Y_{j\Delta_n-} - Y^*_{j\Delta_n-}\big|\big] \\ &\leq Ka_n\Delta_n^{(p-2)/2}\eta_n\Delta_n^{1/2} \\ &\leq K\Delta_n^{(p^2+p-3)/(4(p-1))} \\ &\leq K\Delta_n^{17/12},\end{aligned}$$

where the first inequality follows from $|g_j'^n| \leq K/k_n$ and the law of iterated expectations; the second inequality follows from (S.A.61); the third inequality follows (S.A.22); the last inequality holds because $\frac{p^2+p-3}{4(p-1)} \geq \frac{17}{12}$ when $p \geq 4$.

For $\mathbb{E}[\zeta(q,5,2)_i^n]$, we have

$$\begin{aligned}\text{(S.A.69)}\quad \mathbb{E}\big[\zeta(q,5,2)_i^n\big] &\leq Ka_nk_n^{-2}\sum_{j=1}^{k_n}\mathbb{E}\big[\big|Y_{j\Delta_n-} - Y^*_{j\Delta_n-}\big|^{p-2}\big] \\ &\leq Ka_n\Delta_n^{1/2}\eta_n^{p-2}\big(\Delta_n^{(p-2)/2} + \Delta_n^{1/2}\tilde{\varphi}(q)\big) \\ &\leq K\Delta_n^{(2p^2-4p+3)/(4(p-1))} + K\Delta_n^{(4p-3)/(4p-4)}\tilde{\varphi}(q) \\ &\leq K\Delta_n^{19/12} + K\Delta_n\varphi(q),\end{aligned}$$

where the first inequality follows from $|g_j'^n| \le K/k_n$ and the law of iterated expectations; the second inequality follows from (S.A.61) with $w = p-2$; the third inequality is due to (S.A.22); the last inequality holds because when $p \ge 4$, $\frac{2p^2-4p+3}{4(p-1)} \ge 19/12$.

For $\mathbb{E}[\zeta(q,5,3)_i^n]$, we have

$$
\begin{aligned}
\mathbb{E}\big[\zeta(q,5,3)_i^n\big] &\le K a_n k_n^{-2} \sum_{j=1}^{k_n} \mathbb{E}\big[\big|Y^*_{j\Delta_n-}\big|^{p-3}\big|Y_{j\Delta_n-} - Y^*_{j\Delta_n-}\big|\big] \\
&\le K a_n \Delta_n^{1/2} \Delta_n^{(p-3)/4} \big\{\mathbb{E}\big[\big|Y_{j\Delta_n-} - Y^*_{j\Delta_n-}\big|^{3/2}\big]\big\}^{2/3} \\
&\le K a_n \Delta_n^{p/4-1/4} \eta_n \big(\Delta_n^{1/2} + \Delta_n^{1/3} \tilde{\varphi}(q)^{2/3}\big) \\
&\le K \Delta_n^{(5p-6)/(4p-4)} + K \Delta_n^{(13p-16)/(12p-12)} \tilde{\varphi}(q)^{2/3} \\
&\le K \Delta_n^{7/6} + K \Delta_n \varphi(q),
\end{aligned}
\tag{S.A.70}
$$

where the first inequality follows from $|g_j'^n| \le K/k_n$ and the law of iterated expectations; the second inequality is obtained by applying Hölder's inequality with index $3/2$ and then (S.A.62); the third inequality is obtained by applying (S.A.61) with $w = 3/2$ and then the c_r-inequality; the fourth inequality is due to (S.A.22); the last inequality holds because when $p \ge 4$, $\frac{5p-6}{4p-4} \ge 7/6$ and $\frac{13p-16}{12p-12} \ge 1$.

Combining (S.A.67)–(S.A.70), we verify (S.A.65) for $k = 5$.

Step 3(vi). We consider $\zeta(q,6)_i^n$ for $p = 2$ first, so $f'(x) = 2x$. Under Assumption S.N, the $\mathcal{F}_{i\Delta_n}$-conditional mean of

$$
\begin{aligned}
&f'\big(Y_s^*\big) - f'\bigg(\int_0^s g_n(u) b_u' \, du\bigg) \\
&\quad = 2\Bigg(\int_0^s g_n(u) \sigma_u' \, dW_u' - \sum_{j=1}^{\lfloor s/\Delta_n \rfloor} g_j'^n \chi_{(i+j-1)\Delta_n}\Bigg)
\end{aligned}
$$

is zero; by Fubini's theorem, $\mathbb{E}[\zeta(q,6)_i^n | \mathcal{F}_{i\Delta_n}] = 0$. Moreover,

$$
\begin{aligned}
&\mathbb{E}\big[\big|\zeta(q,6)_i^n\big|^2\big] \\
&\quad \le K a_n^2 \eta_n^2 \Delta_n^{1/2} \\
&\qquad \times \mathbb{E}\Bigg[\int_0^{k_n\Delta_n} \Bigg(\int_0^s g_n(u) \sigma_u' \, dW_u' - \sum_{j=1}^{\lfloor s/\Delta_n \rfloor} g_j'^n \chi_{(i+j-1)\Delta_n}\Bigg)^2 ds\Bigg] \\
&\quad \le K a_n^2 \eta_n^2 \Delta_n^{3/2} \\
&\quad \le K \Delta_n^2,
\end{aligned}
$$

where the first inequality follows from the Cauchy–Schwarz inequality and the fact that $g_n(\cdot)$ and b'' are bounded; the second inequality is obtained by applying Fubini's theorem and the Burkholder–Davis–Gundy inequality, and using the fact $|g_j'^n| \le K/k_n$; the third inequality holds because η_n is bounded and $a_n = O(\Delta_n^{1/4})$ when $p = 2$. We hence verify (S.A.66) for $k = 6$.

Next we suppose $p \ge 4$. Note that

$$\begin{aligned}\mathbb{E}\left|a_n\eta_n \int_0^{k_n\Delta_n} f'(Y_s^*)g_n(s)b_s''\,ds\right| &\le K a_n\eta_n \int_0^{k_n\Delta_n} \mathbb{E}\left|f'(Y_s^*)\right|ds \\ &\le K a_n\eta_n\Delta_n^{1/2}\Delta_n^{(p-1)/4} \\ &\le K\Delta_n^{(5p-6)/(4p-4)} \\ &\le K\Delta_n^{7/6},\end{aligned}$$

where the first inequality holds because $g_n(\cdot)$ and b'' are bounded; the second inequality follows from (S.A.62) with $w = p-1$; the third inequality follows (S.A.22); the last inequality holds because when $p \ge 4$, $\frac{5p-6}{4p-4} \ge 7/6$. Combine this estimate with the estimates in Step 3(ii), we verify (S.A.65) for $k = 6$.

Step 3(vii). By the martingale property of stochastic integrals, it is clear that $\mathbb{E}[\zeta(q,7)_i^n|\mathcal{F}_{i\Delta_n}] = 0$. We consider two cases, $p = 2$ and $p \ge 4$, separately. We start with $p = 2$, so $a_n = O(\Delta_n^{1/4})$ and $f'(Y_s) - f'(Y_s^*) = 2(Y_s - Y_s^*)$. We then have

$$\begin{aligned}&\mathbb{E}\left[\left|\zeta(q,7)_i^n\right|^2\right] \\ &\quad\le K a_n^2 \int_0^{k_n\Delta_n} \mathbb{E}\left[\left|Y_s - Y_s^*\right|^2\right]ds \\ &\quad\le K a_n^2\Delta_n^{1/2}\eta_n^2\left(\Delta_n + \Delta_n^{1/2}\tilde{\varphi}(q)\right) \\ &\quad\le K\Delta_n^2 + K\Delta_n^{3/2}\varphi(q),\end{aligned}$$

where the first inequality is obtained by applying the Burkholder–Davis–Gundy inequality and using the fact that $g_n(\cdot)$ and σ' are bounded; the second inequality follows from (S.A.61) with $w = 2$; the last inequality follows $a_n = O(\Delta_n^{1/4})$ and the boundedness of η_n.

When $p \ge 4$, we have

$$\begin{aligned}\mathbb{E}\left[\left|\zeta(q,7)_i^n\right|^2\right] &\le K a_n^2 \int_0^{k_n\Delta_n} \mathbb{E}\left[\left|Y_s^{p-1} - \left(Y_s^*\right)^{p-1}\right|^2\right]ds \\ &\le K a_n^2\Delta_n^{1/2}\left(\eta_n^{2(p-1)}\Delta_n^{p-1} + \eta_n^{2(p-1)}\Delta_n^{1/2}\tilde{\varphi}(q)\right. \\ &\qquad \left. + \eta_n^2\Delta_n^{p/2} + \eta_n^2\Delta_n^{(p-2)/2+1/4}\tilde{\varphi}(q)^{1/2}\right) \\ &\le K\Delta_n^p + K\Delta_n^{3/2}\varphi(q) + K\Delta_n^{(5p-6)/(2p-2)}\end{aligned}$$

$$
\begin{aligned}
&+ K\Delta_n^{(7p-9)/(4p-4)}\varphi(q) \\
&\le K\Delta_n^{7/3} + K\Delta_n^{3/2}\varphi(q),
\end{aligned}
$$

where the first inequality follows the Burkholder–Davis–Gundy inequality and the boundedness of $g_n(\cdot)$ and σ'; the second inequality is obtained by applying (S.A.64) with $w = p-1$, $v = 2$, and $m = 2$; the third inequality follows (S.A.22); the last inequality holds because when $p \ge 4$, $\frac{5p-6}{2p-2} \ge 7/3$ and $\frac{7p-9}{4p-4} \ge 19/12$.

Combining the two cases together, we verify (S.A.66) for $k = 7$.

Step 3(viii). By construction, $\zeta(q,8)_i^n$ is a martingale increment, so $\mathbb{E}[\zeta(q,8)_i^n|\mathcal{F}_{i\Delta_n}] = 0$. Moreover,

$$
\begin{aligned}
&\mathbb{E}\bigl[\bigl|\zeta(q,8)_i^n\bigr|^2\bigr] \\
&\quad\le K a_n^2 \mathbb{E}\biggl[\int_0^{k_n\Delta_n}\int_E G\bigl(Y_s, \eta_n g_n(s)\delta'(s,z)\bigr)^2 1_{\{\gamma(z)\le 1/q\}}\nu(ds,dz)\biggr] \\
&\quad\le K a_n^2 \eta_n^2 \mathbb{E}\biggl[\int_0^{k_n\Delta_n}\int_E |Y_s|^{2(p-1)}\gamma(z)^2 1_{\{\gamma(z)\le 1/q\}}\nu(ds,dz)\biggr] \\
&\qquad + K a_n^2 \eta_n^{2(p-1)} \mathbb{E}\biggl[\int_0^{k_n\Delta_n}\int_E |Y_s|^2\gamma(z)^{2(p-1)} 1_{\{\gamma(z)\le 1/q\}}\nu(ds,dz)\biggr] \\
&\quad\le K a_n^2\eta_n^2\Delta_n^{p/2}\varphi(q) + K a_n^2\eta_n^{2(p-1)}\Delta_n\varphi(q) \\
&\quad\le K\Delta_n^{(4p-5)/(2p-2)}\varphi(q) + K\Delta_n^{3/2}\varphi(q) \\
&\quad\le K\Delta_n^{3/2}\varphi(q),
\end{aligned}
$$

where the first inequality follows the Burkholder–Davis–Gundy inequality; the second inequality is due to (S.A.45), $|\delta'(s,z)| \le \gamma(z)$, and the boundedness of $g_n(\cdot)$; the third inequality is obtained by applying (S.A.63) with $w = 2(p-1)$ and $w = 2$; the fourth inequality is due to (S.A.22); the last inequality holds because $\frac{4p-5}{2p-2} \ge \frac{3}{2}$ when $p \ge 2$. We hence verify (S.A.66) for $k = 8$.

Step 3(ix). Let $\mathcal{G}_t = \mathcal{F}^{(0)} \vee \sigma\{\chi_s : s < t\}$. Then $Y_{j\Delta_n-}$ and $Y^*_{j\Delta_n-}$ are $\mathcal{G}_{(i+j-1)\Delta_n}$-measurable and $\mathbb{E}[\chi_{(i+j-1)\Delta_n}|\mathcal{G}_{(i+j-1)\Delta_n}] = 0$ (the noise is conditionally mean 0). It is then obvious that $\zeta(q,9)_i^n$ is a sum of martingale differences. By repeated conditioning, $\mathbb{E}[\zeta(q,9)_i^n|\mathcal{F}_{i\Delta_n}] = 0$. Moreover, by repeated conditioning, we get

$$
\mathbb{E}\bigl[\bigl|\zeta(q,9)_i^n\bigr|^2\bigr] \le a_n^2 k_n^{-2}\sum_{j=1}^{k_n}\mathbb{E}\bigl[\bigl|f'(Y_{j\Delta_n-}) - f'\bigl(Y^*_{j\Delta_n-}\bigr)\bigr|^2\bigr].
$$

With essentially the same argument as in Step 3(vii), we derive

$$
\mathbb{E}\bigl[\bigl|\zeta(q,9)_i^n\bigr|^2\bigr] \le K\Delta_n^2 + K\Delta_n^{3/2}\varphi(q),
$$

which verifies (S.A.66) for $k=9$.

Step 4. We prove the claims of the lemma in this step. When $p=2$, we define $\Gamma'(q)_i^n=\sum_{k=1}^5\zeta(q,k)_i^n$ and $\Gamma''(q)_i^n=\sum_{k=6}^9\zeta(q,k)_i^n$; when $p\geq 4$, we define $\Gamma'(q)_i^n=\sum_{k=1}^6\zeta(q,k)_i^n$ and $\Gamma''(q)_i^n=\sum_{k=7}^9\zeta(q,k)_i^n$. In either case, $\Gamma(q)_i^n=\Gamma'(q)_i^n+\Gamma''(q)_i^n$ because of (S.A.60). It is obvious that $\Gamma'(q)_i^n$ and $\Gamma''(q)_i^n$ are $\mathcal{F}_{(i+k_n)\Delta_n}$-measurable. The claims of the lemma follow from (S.A.65) and (S.A.66) with $r_1=7/6$ and $r_2=2$. *Q.E.D.*

PROOF OF LEMMA 10: Recall the notation in (S.A.55). Observe that

$$\text{(S.A.71)}\quad \left|R(q)_t^n\right|\leq a_n\eta_n^p\left|\sum_{j=1}^{k_n-1}\left(g_j^n\right)^p-\Delta_n^{-1/2}\theta\bar{g}(p)\right|\Sigma_t^q + Ka_n\eta_n^pk_n\Sigma_{k_n\Delta_n}^q+Ka_n\eta_n^pk_n\left(\Sigma_t^q-\Sigma_{t-k_n\Delta_n}^q\right).$$

Following an argument similar to that which led to (S.A.49), we can show that the first term on the majorant side of (S.A.71) is $o_{p,\eta_n}(1)$. Moreover,

$$\begin{aligned}\mathbb{E}\left[\Sigma_{k_n\Delta_n}^q\right]&=\mathbb{E}\left[\int_0^{k_n\Delta_n}\int\left|\delta(s,z)\right|^p1_{\{\gamma(z)\leq 1/q\}}\mu(ds,dz)\right]\\&\leq\int_0^{k_n\Delta_n}\int\gamma(z)^p1_{\{\gamma(z)\leq 1/q\}}\nu(ds,dz)\\&\leq K\Delta_n^{1/2}.\end{aligned}$$

Hence, $\Sigma_{k_n\Delta_n}^q=O_p(\Delta_n^{1/2})$, which implies $a_n\eta_n^pk_n\Sigma_{k_n\Delta_n}^q=o_{p,\eta_n}(1)$. By the same argument, we can show that the third term on the majorant side of (S.A.71) is also $o_{p,\eta_n}(1)$. Therefore, for each $q\geq 1$,

$$\text{(S.A.72)}\quad R(q)_t^n=o_{p,\eta_n}(1).$$

With the same notation as in Lemma 12, we have $\Gamma(q)_i^n=\Gamma'(q)_i^n+\Gamma''(q)_i^n$ and

$$\text{(S.A.73)}\quad \mathbb{E}\left|\sum_{i=0}^{\lfloor t/\Delta_n\rfloor-k_n}\Gamma'(q)_i^n\right|\leq K\Delta_n^{r_1-1}+K\varphi(q).$$

Observe that $\Gamma''(q)_i^n$ and $\Gamma''(q)_{i+j}^n$ are uncorrelated whenever $|j|\geq k_n$, because $\Gamma''(q)_i^n\in\mathcal{F}_{(i+k_n)\Delta_n}$ and $\mathbb{E}[\Gamma''(q)_i^n|\mathcal{F}_{i\Delta_n}]=0$. By the Cauchy–Schwarz inequality and Lemma 12, we derive

$$\text{(S.A.74)}\quad \mathbb{E}\left[\left(\sum_{i=0}^{\lfloor t/\Delta_n\rfloor-k_n}\Gamma''(q)_i^n\right)^2\right]\leq K\Delta_n^{r_2-3/2}+K\varphi(q).$$

By (S.A.55), the triangle inequality, and Markov's inequality, we have for each $q \geq 1$,

$$\begin{aligned}
&\limsup_{n\to\infty} \mathbb{P}\big(\big|VJ'(g,p,q)_t^n\big| > \varepsilon\big) \\
&\quad \leq \limsup_{n\to\infty} \mathbb{P}\big(\big|R(q)_t^n\big| > \varepsilon/3\big) + K\varepsilon^{-1} \limsup_{n\to\infty} \mathbb{E}\left|\sum_{i=0}^{\lfloor t/\Delta_n\rfloor - k_n} \Gamma'(q)_i^n\right| \\
&\qquad + K\varepsilon^{-2}\mathbb{E}\left[\left(\sum_{i=0}^{\lfloor t/\Delta_n\rfloor - k_n} \Gamma''(q)_i^n\right)^2\right] \\
&\quad \leq K\big(\varepsilon^{-1} + \varepsilon^{-2}\big)\varphi(q),
\end{aligned}$$

where the second inequality follows from (S.A.72), (S.A.73), and (S.A.74), recalling from Lemma 12 that $r_1 > 1$, $r_2 > 3/2$. Since $\lim_{q\to\infty}\varphi(q) = 0$, the claim of Lemma 10 readily follows. *Q.E.D.*

PROOF OF LEMMA 11: *Step 1.* In this step, we show that for each $l \in \{1, \dots, p/2\}$,

$$a_n \sum_{i=0}^{\lfloor t/\Delta_n\rfloor - k_n} \mathbb{E}\big[\big|\big|\bar{Z}_i^n\big|^{p-2l} - \big|\bar{Z}_i^{*n}\big|^{p-2l}\big|\big(\hat{\chi}_i^n\big)^l\big] \to 0. \tag{S.A.75}$$

When $l = p/2$, the claim holds trivially, because the left-hand side of (S.A.75) is identically zero. It remains to consider $p \geq 4$ and $l \in \{1, \dots, p/2 - 1\}$.

For any $m > 1$, we have

$$\begin{aligned}
&\mathbb{E}\big[\big|\big|\bar{Z}_i^n\big|^{p-2l} - \big|\bar{Z}_i^{*n}\big|^{p-2l}\big|\big(\hat{\chi}_i^n\big)^l\big] \\
&\quad \leq K\mathbb{E}\big[\big|\bar{Z}_i^n - \bar{Z}_i^{*n}\big|^{p-2l}\big(\hat{\chi}_i^n\big)^l\big] \\
&\qquad + K\mathbb{E}\big[\big|\bar{Z}_i^n - \bar{Z}_i^{*n}\big|\big|\bar{Z}_i^{*n}\big|^{p-2l-1}\big(\hat{\chi}_i^n\big)^l\big] \\
&\quad \leq K\Delta_n^{l/2}\big\{\mathbb{E}\big[\big|\bar{Z}_i^n - \bar{Z}_i^{*n}\big|^{(p-2l)m}\big]\big\}^{1/m} \\
&\qquad + K\Delta_n^{(p-2l-1)/4}\Delta_n^{l/2}\big\{\mathbb{E}\big[\big|\bar{Z}_i^n - \bar{Z}_i^{*n}\big|^m\big]\big\}^{1/m} \\
&\quad \leq K\Delta_n^{l/2}\eta_n^{p-2l}\Delta_n^{1/(2m)} + K\Delta_n^{(p-1)/4}\eta_n\Delta_n^{1/(2m)},
\end{aligned} \tag{S.A.76}$$

where the first inequality holds because for any $v \geq 1$, $||x+y|^v - |x|^v| \leq K|x|^{v-1}|y| + K|y|^v$; the second inequality is obtained by using Hölder's inequality, and then using (S.A.4) and (S.A.5); the third inequality follows from (S.A.16).

Hence,

$$
\begin{aligned}
a_n & \sum_{i=0}^{\lfloor t/\Delta_n \rfloor - k_n} \mathbb{E}\big[\big| |\bar{Z}_i^n|^{p-2l} - |\bar{Z}_i^{*n}|^{p-2l} \big| (\hat{\chi}_i^n)^l\big] \\
& \le K a_n \Delta_n^{-1} \big(\Delta_n^{l/2} \eta_n^{p-2l} + \Delta_n^{(p-1)/4} \eta_n\big) \Delta_n^{1/(2m)} \\
& \le K \big(\Delta_n^{-(2p-3)/(4p-4)} + \Delta_n^{-p/(4p-4)}\big) \Delta_n^{1/(2m)} \\
& \le K \Delta_n^{1/(2m)-(2p-3)/(4p-4)},
\end{aligned}
$$

where the first inequality follows from (S.A.76); the second inequality is due to (S.A.22); the last inequality is due to $p \ge 4$. This estimate holds for any $m > 1$. If we take $m < (2p-2)/(2p-3)$, the bound in the above inequality goes to zero as $\Delta_n \to 0$. We hence have (S.A.75) as desired.

Step 2. In this step, we show the claim of the lemma. For each $l \in \{1, \dots, p/2\}$, by the triangle inequality,

$$
\begin{aligned}
\text{(S.A.77)} \quad \big|D(g, p, l)_t^n\big| \le a_n & \sum_{i=0}^{\lfloor t/\Delta_n \rfloor - k_n} |\bar{Z}_i^n|^{p-2l} \big| (\hat{Z}_i^n)^l - (\hat{\chi}_i^n)^l \big| \\
& + a_n \sum_{i=0}^{\lfloor t/\Delta_n \rfloor - k_n} |\bar{Z}_i^{*n}|^{p-2l} \big| (\hat{Z}_i^{*n})^l - (\hat{\chi}_i^n)^l \big| \\
& + a_n \sum_{i=0}^{\lfloor t/\Delta_n \rfloor - k_n} \big| |\bar{Z}_i^n|^{p-2l} - |\bar{Z}_i^{*n}|^{p-2l} \big| (\hat{\chi}_i^n)^l.
\end{aligned}
$$

By Lemma 8(a), the first term on the right-hand side of (S.A.77) is $o_{p,\eta_n}(1)$. Taking η_n in that lemma to be 0, we use it to show that the second term on the right-hand side of (S.A.77) is $o_p(1)$. In view of (S.A.75), we conclude $D(g, p, l)_t^n = o_{p,\eta_n}(1)$ as desired. *Q.E.D.*

S.A.5. *Proof of Theorem 3*

To simplify notation, we denote

$$
x_{n,\eta} = \mathbb{P}_\eta\bigg(\sum_{s \le t} |\Delta J_s|^p \in \mathrm{CS}_{1-c}^n\bigg).
$$

By Corollary 2 and Assumption V, it is easy to see that

$$
\text{(S.A.78)} \quad (\eta_n)_{n \in \mathbb{N}} \subset [0, 1] \text{ and } \Delta_n^{-\bar{r}} \eta_n \to h \in [0, \infty] \quad \Rightarrow \quad x_{n,\eta_n} \to 1 - c.
$$

Taking any η_n that satisfies (S.A.78), we have

$$\limsup_{n\to\infty} \inf_{\eta\in[0,1]} x_{n,\eta} \le \limsup_{n\to\infty} x_{n,\eta_n} = 1-c. \tag{S.A.79}$$

We now show that

$$\liminf_{n\to\infty} \inf_{\eta\in[0,1]} x_{n,\eta} \ge 1-c. \tag{S.A.80}$$

If (S.A.80) were false, then there would exist $\varepsilon > 0$, a subsequence $\mathbb{N}_1 \subseteq \mathbb{N}$, and a sequence $(\tilde{\eta}_n)_{n\in\mathbb{N}_1} \subset [0,1]$, such that for every $n \in \mathbb{N}_1$, $x_{n,\tilde{\eta}_n} < 1-c-\varepsilon$. Then we could extract a further subsequence $\mathbb{N}_2 \subseteq \mathbb{N}_1$ such that $\Delta_n^{-\bar{r}}\tilde{\eta}_n \to h$ along $\mathbb{N}_2$ for some $h \in [0,\infty]$. Given $(\tilde{\eta}_n)_{n\in\mathbb{N}_2}$ and h, we construct a sequence $(\eta_n^*)_{n\in\mathbb{N}}$ as follows: if $n \in \mathbb{N}_2$, we set $\eta_n^* = \tilde{\eta}_n$; if $n \notin \mathbb{N}_2$, we set

$$\eta_n^* = \begin{cases} 1 & \text{if } h = \infty, \\ \min\{\Delta_n^{\bar{r}} h, 1\} & \text{if } h \in [0,\infty). \end{cases}$$

By construction, we have $(\eta_n^*)_{n\in\mathbb{N}} \subset [0,1]$. Moreover, since $\bar{r} > 0$ whenever $p > 2$, we have $\Delta_n^{-\bar{r}}\eta_n^* \to h$. Applying (S.A.78) to η_n^*, we get $x_{n,\eta_n^*} \to 1-c$ along $\mathbb{N}$ and, thus, also along $\mathbb{N}_2$. But this contradicts the fact that for every $n \in \mathbb{N}_2$, $x_{n,\eta_n^*} = x_{n,\tilde{\eta}_n} < 1-c-\varepsilon$. We hence derive (S.A.80). By (S.A.79) and (S.A.80), we readily have

$$\lim_{n\to\infty} \inf_{\eta\in[0,1]} \mathbb{P}_\eta\left(\sum_{s\le t} |\Delta J_s|^p \in \mathrm{CS}_{1-c}^n\right) = 1-c; \tag{S.A.81}$$

this finishes the proof of the first claim.

Next, replacing CS_{1-c}^n in (S.A.81) with $\mathrm{CS}_c^n \equiv \mathbb{R}\backslash\mathrm{CS}_{1-c}^n$ and noting that CS_c^n is a confidence set associated with the nonrandom set $\mathbb{R} \setminus S_{1-c}$, we derive

$$\lim_{n\to\infty} \inf_{\eta\in[0,1]} \mathbb{P}_\eta\left(\sum_{s\le t} |\Delta J_s|^p \notin \mathrm{CS}_{1-c}^n\right) = c.$$

This is equivalent to

$$\lim_{n\to\infty} \sup_{\eta\in[0,1]} \mathbb{P}_\eta\left(\sum_{s\le t} |\Delta J_s|^p \in \mathrm{CS}_{1-c}^n\right) = 1-c,$$

which is the second claim of the theorem. *Q.E.D.*

S.A.6. *Proof of Theorem 4*

LEMMA 13: *Suppose that Assumptions* S.H-1 *and* S.N *hold. Let* $p \geq 4$ *and* η_n *be a sequence in* $[0,1]$ *such that* $\Delta_n^{-\bar{r}}\eta_n \to h$ *for some* $h \in [0,\infty]$. *Let* $\varpi \in ((p-1)/(4p-2), 1/4)$. *Then for* $l \in \{0,\dots,p\}$,

$$a_n^2 \Delta_n^{-1/2} V^*\big(Z^{\eta_n}, g, 2p-2l, l\big)_t^n \xrightarrow{\mathbb{P},\eta_n} \frac{1}{(1+h^{p-1})^2} V(g,2p,l)_t. \tag{S.A.82}$$

PROOF: *Step 1*. In this step, we prove the following elementary (but probably not obvious) result: for any $k \geq 1$, there exists $K > 0$ such that for any $\varepsilon > 0$ and $x, y \in \mathbb{R}$,

$$\big||x+y|^k 1_{\{|x+y|\leq\varepsilon\}} - |x|^k\big| \leq K\big(|x|^k 1_{\{|x|>\varepsilon/2\}} + \varepsilon^{k-1}|y|\big). \tag{S.A.83}$$

Fix $k \geq 1$. We first suppose that $|x+y| \leq \varepsilon$, so the left-hand side of (S.A.83) is $||x+y|^k - |x|^k|$. By Taylor's expansion and the c_r-inequality, there exists $K > 0$ such that for any $x, y \in \mathbb{R}$,

$$\big||x+y|^k - |x|^k\big| \leq K|y|^k + K|x|^{k-1}|y|. \tag{S.A.84}$$

Note that $|y| \leq |x+y| + |x| \leq |x| + \varepsilon$. Hence, if $|x| > \varepsilon/2$, then $|y| \leq K|x|$ and the right-hand side of (S.A.84) is bounded by $K|x|^k$; if $|x| \leq \varepsilon/2$, then $|y| \leq K\varepsilon$ and the right-hand side of (S.A.84) is bounded by $K\varepsilon^{k-1}|y|$. In both cases, we have (S.A.83).

Now suppose $|x+y| > \varepsilon$, so the left-hand side of (S.A.83) is $|x|^k$. If $|x| > \varepsilon/2$, then (S.A.83) is obvious. If $|x| \leq \varepsilon/2$, then $|x+y| > \varepsilon$ implies $|y| > \varepsilon/2$, and thus $|x|^k \leq K\varepsilon^{k-1}|y|$. Again, we have (S.A.83).

Step 2. By using (S.A.26) with p replaced by $2p$, we have

$$\Delta_n^{1-p/2} V\big(Z^*, g, 2p-2l, l\big)_t^n \xrightarrow{\mathbb{P}} V(g,2p,l)_t. \tag{S.A.85}$$

Therefore,

$$a_n^2 \Delta_n^{-1/2} V\big(Z^*, g, 2p-2l, l\big)_t^n \xrightarrow{\mathbb{P},\eta_n} \frac{1}{(1+h^{p-1})^2} V(g,2p,l)_t.$$

It remains to show that

$$a_n^2 \Delta_n^{-1/2}\big(V^*\big(Z^{\eta_n}, g, 2p-2l, l\big)_t^n - V\big(Z^*, g, 2p-2l, l\big)_t^n\big) = o_{p,\eta_n}(1).$$

By the triangle inequality, the left-hand side of the above expression is bounded by $\sum_{j=1}^3 \zeta(l,j)_t^n$, where

$$\zeta(l,1)_t^n = a_n^2 \Delta_n^{-1/2} \sum_{i=1}^{\lfloor t/\Delta_n\rfloor - k_n} \big|\bar{Z}_i^{*n}\big|^{2p-2l} \big|\big(\hat{Z}_i^{*n}\big)^l - \big(\hat{\chi}_i^n\big)^l\big|,$$

$$\zeta(l,2)_t^n = a_n^2 \Delta_n^{-1/2} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor - k_n} \left|\bar{Z}_i^n\right|^{2p-2l} 1_{\{|\bar{Z}_i^n| \le u_n\}} \left|\left(\hat{Z}_i^n\right)^l - \left(\hat{\chi}_i^n\right)^l\right|,$$

$$\zeta(l,3)_t^n = a_n^2 \Delta_n^{-1/2} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor - k_n} \left|\left|\bar{Z}_i^n\right|^{2p-2l} 1_{\{|\bar{Z}_i^n| \le u_n\}} - \left|\bar{Z}_i^{*n}\right|^{2p-2l}\right|\left(\hat{\chi}_i^n\right)^l.$$

It remains to show that $\zeta(l,j)_t^n = o_{p,\eta_n}(1)$ for each $l \in \{0,\dots,p\}$ and $j = 1,2,3$; this is the task below.

Step 3. We show $\zeta(l,1)_t^n = o_{p,\eta_n}(1)$ in this step. Clearly, $\zeta(0,1)_t^n = 0$. When $l \ge 1$, Lemma 8(a) implies

$$\Delta_n^{3/4-p/2} \sum_{i=0}^{\lfloor t/\Delta_n \rfloor - k_n} \mathbb{E}\left[\left|\bar{Z}_i^{*n}\right|^{2p-2l}\left|\left(\hat{Z}_i^{*n}\right)^l - \left(\hat{\chi}_i^n\right)^l\right|\right] \to 0.$$

Note that $a_n^2 \Delta_n^{-1/2} = O(\Delta_n^{1-p/2})$. The above convergence then implies that $\zeta(l,1)_t^n = o_{p,\eta_n}(1)$ holds for $l \ge 1$.

Step 4. We now show $\zeta(l,2)_t^n = o_{p,\eta_n}(1)$. When $l = 0$, $\zeta(l,2)_t^n = 0$. When $l = 1$, we have

$$\begin{aligned}\mathbb{E}\left[\zeta(1,2)_t^n\right] &\le a_n^2 \Delta_n^{-1/2} u_n^{2p-2} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor - k_n} \mathbb{E}\left|\hat{Z}_i^n - \hat{\chi}_i^n\right| \\ &\le K a_n^2 \Delta_n^{-1/2} u_n^{2p-2} \to 0,\end{aligned}$$

where the second inequality follows (S.A.14) and the convergence follows our choice of u_n. When $l \ge 2$, we have

$$\begin{aligned}\mathbb{E}\left[\zeta(l,2)_t^n\right] &\le a_n^2 \Delta_n^{-1/2} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor - k_n} \mathbb{E}\left[\left|\bar{Z}_i^n\right|^{2p-2l}\left|\left(\hat{Z}_i^n\right)^l - \left(\hat{\chi}_i^n\right)^l\right|\right] \\ &\le K a_n^2 \Delta_n^{-3/2}\big(\Delta_n^{p/2+1/2} + \eta_n \Delta_n^{p/2+1/(2p)} \\ &\quad + \eta_n^{2p-2l} \Delta_n^{1+l/2-l/(2p)} + \eta_n^{2p-2l+1} \Delta_n^{1/2-l/(2p)+l/2+1/(2p)}\big) \\ &\le K\Delta_n^{1/2} + K\Delta_n^{(p^2-2)/(4p(p-1))} \\ &\quad + K\Delta_n^{(p^2-2p+2)/(2p(p-1))} + K\Delta_n^{(p^2-2p+2)/(4p(p-1))} \\ &\to 0,\end{aligned}$$

where the second inequality follows a calculation similar to (S.A.36); the third inequality follows (S.A.22). Hence, $\zeta(l,2)_t^n = o_{p,\eta_n}(1)$ for all $l \in \{0,\dots,p\}$.

Step 5. In this step, we show $\zeta(l,3)_t^n = o_{p,\eta_n}(1)$. First suppose that $l \in \{0,\ldots,p-1\}$. Applying (S.A.83) with $k = 2p-2l$, we have

$$\zeta(l,3)_t^n \le K\zeta'(l,3)_t^n + K\zeta''(l,3)_t^n,$$

where

$$\zeta'(l,3)_t^n = a_n^2 \Delta_n^{-1/2} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor - k_n} \left|\bar{Z}_i^{*n}\right|^{2p-2l} 1_{\{|\bar{Z}_i^{*n}|>u_n/2\}} \left(\hat{\chi}_i^n\right)^l,$$

$$\zeta''(l,3)_t^n = a_n^2 \Delta_n^{-1/2} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor - k_n} u_n^{2p-2l-1} \left|\bar{Z}_i^n - \bar{Z}_i^{*n}\right| \left(\hat{\chi}_i^n\right)^l.$$

For $\zeta'(l,3)_t^n$, we note that

$$\begin{aligned}
\mathbb{E}\left[\left|\bar{Z}_i^{*n}\right|^{2p-2l} 1_{\{|\bar{Z}_i^{*n}|>u_n/2\}} \left(\hat{\chi}_i^n\right)^l\right] &\le K u_n^{-1} \mathbb{E}\left[\left|\bar{Z}_i^{*n}\right|^{2p-2l+1} \left(\hat{\chi}_i^n\right)^l\right] \\
&\le K u_n^{-1} \Delta_n^{(2p-2l+1)/4+l/2} \\
&\le K \Delta_n^{1/4-\varpi} \Delta_n^{p/2},
\end{aligned}$$

where the first inequality follows $1_{\{|\bar{Z}_i^{*n}|>u_n/2\}} \le K u_n^{-1} |\bar{Z}_i^{*n}|$; the second inequality is obtained by applying Hölder's inequality and then (S.A.4) and (S.A.5); the last inequality is due to the definition of u_n. Since $a_n = O(\Delta_n^{3/4-p/4})$,

$$\mathbb{E}\left[\zeta'(l,3)_t^n\right] \le K\Delta_n^{1/4-\varpi} \to 0.$$

Hence, $\zeta'(l,3)_t^n = o_{p,\eta_n}(1)$ for $l \in \{0,\ldots,p-1\}$.

The term $\zeta''(l,3)_t^n$ satisfies the following: for every $m > 1$,

$$\begin{aligned}
\mathbb{E}\left[\zeta''(l,3)_t^n\right] &= a_n^2 \Delta_n^{-1/2} u_n^{2p-2l-1} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor - k_n} \mathbb{E}\left[\left|\bar{Z}_i^n - \bar{Z}_i^{*n}\right| \left(\hat{\chi}_i^n\right)^l\right] \\
&\le K a_n^2 \Delta_n^{-1/2} u_n^{2p-2l-1} \Delta_n^{l/2} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor - k_n} \left\{\mathbb{E}\left[\left|\bar{Z}_i^n - \bar{Z}_i^{*n}\right|^m\right]\right\}^{1/m} \\
&\le K a_n^2 \Delta_n^{-3/2} u_n^{2p-2l-1} \Delta_n^{l/2} \eta_n \Delta_n^{1/(2m)} \\
&\le K \Delta_n^{-p/2+l/2+1/(2m)} \Delta_n^{(2p-2l-1)\varpi} \\
&\le K \Delta_n^{-p/2+1/(2m)} \Delta_n^{(2p-1)\varpi},
\end{aligned}$$

where the first inequality follows Hölder's inequality and (S.A.5); the second inequality follows (S.A.16); the third inequality holds because $a_n = O(\Delta_n^{3/4-p/4})$ and η_n is bounded; the fourth inequality is due to $l \ge 0$ and $\varpi < 1/4$. By assumption, $(2p-1)\varpi > (p-1)/2$. Therefore, by choosing $m > 1$ sufficiently close to 1, we have $(2p-1)\varpi - p/2 + 1/(2m) > 0$. Hence, $\mathbb{E}[\zeta''(l,3)_t^n] \to 0$ and $\zeta''(l,3)_t^n = o_{p,\eta_n}(1)$ for $l \in \{0, \ldots, p-1\}$. Consequently, $\zeta(l,3)_t^n = o_{p,\eta_n}(1)$ for $l \in \{0, \ldots, p-1\}$.

Finally, we consider $\zeta(p,3)_t^n$. By definition,

$$\zeta(p,3)_t^n = a_n^2 \Delta_n^{-1/2} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor - k_n} 1_{\{|\bar{Z}_i^n| > u_n\}} \big(\hat{\chi}_i^n\big)^p.$$

By Chebyshev's inequality and (S.A.17) with $v=2$, we derive $\mathbb{P}(|\bar{Z}_i^n| > u_n) \leq K\Delta_n^{1/2-2\varpi}$. By the Cauchy–Schwarz inequality and (S.A.5), $\mathbb{E}[1_{\{|\bar{Z}_i^n|>u_n\}}(\hat{\chi}_i^n)^p] \leq K\mathbb{P}(|\bar{Z}_i^n| > u_n)^{1/2}\Delta_n^{p/2}$. Since $a_n = O(\Delta_n^{3/4-p/4})$, we have

$$\mathbb{E}\big[\zeta(p,3)_t^n\big] \leq K a_n^2 \Delta_n^{-3/2} \Delta_n^{1/4-\varpi} \Delta_n^{p/2} \leq K\Delta_n^{1/4-\varpi} \to 0.$$

Hence, $\zeta(p,3)_t^n = o_{p,\eta_n}(1)$. This finishes the proof of this lemma. *Q.E.D.*

PROOF OF THEOREM 4: By localization, we can and will suppose that Assumptions S.H-1 and S.N hold without loss of generality. We first recall from (C.3) of AJL the following elementary result: for any weight function ϕ and $x, y \in \mathbb{R}$, we have for any integer w,

$$\text{(S.A.86)} \quad \sum_{l=0}^{w} \rho(2w)_l \big(2y^2 \bar{\phi}'(2)\big)^l m_{2w-2l}(\phi; x, y) = m_{2w} x^{2w} \big(\bar{\phi}(2)\big)^w.$$

By (S.A.23) and (S.A.86),

$$\sum_{l=0}^{w} \rho(2w)_l V(\phi, 2p, p+l-w)_t$$
$$= \theta^{-p} m_{2w} 2^{p-w} \bar{\phi}(2)^w \bar{\phi}'(2)^{p-w} \int_0^t (\theta\sigma_s)^{2w} \alpha_s^{2(p-w)}\, ds.$$

Then by Lemma 13,

$$\text{(S.A.87)} \quad \frac{a_n^2 \Delta_n^{-1/2}}{m_{2w} 2^{p-w} \bar{\phi}(2)^w \bar{\phi}'(2)^{p-w}} \sum_{l=0}^{w} \rho(2w)_l V^*\big(Z^{\eta_n}, \phi, 2w-2l, p+l-w\big)_t^n$$
$$\xrightarrow{\mathbb{P},\eta_n} \frac{1}{(1+h^{p-1})^2} \theta^{-p} \int_0^t (\theta\sigma_s)^{2w} \alpha_s^{2(p-w)}\, ds.$$

As shown in Lemma 4 of AJL, for weight functions g_i, g_j and $x, y \in \mathbb{R}$,

$$\mu(g_i, g_j; x, y) = \sum_{w=0}^{p} x^{2w} y^{2p-2w} A'(g_i, g_j; w).$$

Hence,

$$\Sigma_C^{ij} = \theta^{1-p} \sum_{w=0}^{p} A'(g_i, g_j; w) \int_0^t (\theta\sigma_s)^{2w} \alpha_s^{2p-2w}\, ds. \tag{S.A.88}$$

Combining (S.A.87) with (S.A.88), we have $a_n^2 \hat{\Sigma}_C^{n,ij} \xrightarrow{\mathbb{P},\eta_n} \frac{1}{(1+h^{p-1})^2} \Sigma_C^{ij}$ as claimed. *Q.E.D.*

S.A.7. *Proof of Theorem 5*

The proof follows the same scheme as Theorem 1, but is slightly more complicated. We only emphasize the key modifications here. For brevity, we only consider $N(Z^{\eta_n}, \phi, \psi, 0, -)_t^n$, as the other three cases follow essentially the same argument. By localization, we assume that Assumptions S.H-1 and S.N hold without loss of generality.

Step 1. In this step, we introduce some notation and outline the scheme of the proof. To simplify notation, we denote $\xi_i^n = \xi(Z^{\eta_n}, \phi, 0)_i^n$. In view of Lemma 3, we have

$$\mathbb{E}\left|\xi_i^n\right| \le K, \qquad \xi_{i-k_n-k_n'}^n \in \mathcal{F}_{i\Delta_n}. \tag{S.A.89}$$

We set, for $l \in \{0, \dots, p-1\}$ and any process Y,

$$\begin{aligned} v(Y, \psi, l)_i^n &= \left|\bar{Y}(\psi)_i^n\right|^{2p-2-2l} \left(\hat{Y}(\psi)_i^n\right)^l, \\ \tilde{N}(Y, \phi, \psi, l)_t^n &= \sum_{i=k_n+k_n'}^{\lfloor t/\Delta_n \rfloor - k_n} v(Y, \psi, l)_i^n \xi_{i-k_n-k_n'}^n, \\ Q(\psi, 0, l)_t &= \theta^{-(p-1)} \int_0^t \left(2\alpha_s^2 \bar{\psi}'(2)\right)^l m_{2p-2-2l}(\psi; \theta\sigma_s, \alpha_s) \sigma_s^2\, ds. \end{aligned} \tag{S.A.90}$$

Then we have

$$N(Y, \phi, \psi, 0, -)_t^n = \sum_{l=0}^{p-1} \rho(2p-2)_l \tilde{N}(Y, \phi, \psi, l)_t^n \tag{S.A.91}$$

and, by (S.A.86),

$$\bar{\psi}(2)^{p-1} Q(0)_t = \sum_{l=0}^{p-1} \rho(2p-2)_l Q(\psi, 0, l)_t. \tag{S.A.92}$$

In Step 3 below, we show

$$\text{(S.A.93)}\quad a_n^2\tilde{N}\big(Z^{\eta_n},\phi,\psi,0\big)_t^n \xrightarrow{\mathbb{P},\eta_n} \frac{1}{(1+h^{p-1})^2}Q(\psi,0,0)_t + \left(\frac{h^{p-1}}{1+h^{p-1}}\right)^2\bar{\psi}(2p-2)N(0,-)_t,$$

and in Step 4, we show, for $l\in\{1,\dots,p-1\}$,

$$\text{(S.A.94)}\quad a_n^2\tilde{N}\big(Z^{\eta_n},\phi,\psi,l\big)_t^n \xrightarrow{\mathbb{P},\eta_n} \frac{1}{(1+h^{p-1})^2}Q(\psi,0,l)_t.$$

Combining (S.A.91)–(S.A.94), we readily derive the claim of this theorem.

Step 2. In this step, we consider the limiting behavior of $\tilde{N}(Z^*,\phi,\psi,l)_t^n$ for $l\in\{0,\dots,p-1\}$. We decompose

$$\Delta_n^{3/2-p/2}\tilde{N}\big(Z^*,\phi,\psi,l\big)_t^n = \zeta(l)_t^n + \zeta'(l)_t^n,$$

where

$$\zeta(l)_t^n = \Delta_n^{3/2-p/2}\sum_{i=k_n+k_n'}^{\lfloor t/\Delta_n\rfloor-k_n} v\big(Z^*,\psi,l\big)_i^n\big(\xi_{i-k_n-k_n'}^n-\sigma_{i\Delta_n}^2\big),$$

$$\zeta'(l)_t^n = \Delta_n^{3/2-p/2}\sum_{i=k_n+k_n'}^{\lfloor t/\Delta_n\rfloor-k_n} v\big(Z^*,\psi,l\big)_i^n\sigma_{i\Delta_n}^2.$$

A straightforward adaptation of Theorem 3.3 in JPV and Lemma 2 in AJL complements (S.A.26) with

$$\text{(S.A.95)}\quad \zeta'(l)_t^n = Q(\psi,0,l)_t + o_p(1).$$

By Lemma 2 and Hölder's inequality, $\mathbb{E}[|v(Z^*,\psi,l)_i^n|\mathcal{F}_{i\Delta_n}]\le K\Delta_n^{p/2-1/2}$. Observing that $\xi_{i-k_n-k_n'}^n-\sigma_{i\Delta_n}^2$ is $\mathcal{F}_{i\Delta_n}$-measurable, we use the triangle inequality and repeated conditioning to get

$$\text{(S.A.96)}\quad \mathbb{E}\big|\zeta(l)_t^n\big| \le K\mathbb{E}\left(\Delta_n\sum_{i=k_n+k_n'}^{\lfloor t/\Delta_n\rfloor-k_n}\gamma_i^n\right),\quad \text{where}\quad \gamma_i^n=\big|\xi_{i-k_n-k_n'}^n-\sigma_{i\Delta_n}^2\big|.$$

On the product space $(\Omega\times[0,t],\mathcal{F}\otimes\mathcal{B}[0,t],\mathbb{P}\otimes\text{Leb})$, we define a sequence of measurable functions as

$$f_n(\omega,s)=\gamma_{\lfloor s/\Delta_n\rfloor}^n(\omega)1_{[(k_n+k_n')\Delta_n,(\lfloor t/\Delta_n\rfloor-k_n+1)\Delta_n)}(s).$$

By construction,

$$\text{(S.A.97)}\quad \Delta_n \sum_{i=k_n+k_n'}^{\lfloor t/\Delta_n\rfloor - k_n} \gamma_i^n(\omega) = \int_0^t f_n(\omega, s)\, ds.$$

Consider $s \in [0, t]$ such that $s \neq i\Delta_n$ for any $i \in \mathbb{N}$ and $n \in \mathbb{N}$. This condition is satisfied for Lebesgue almost every (a.e.) $s \in [0, t]$ and implies $\sigma^2_{\lfloor s/\Delta_n\rfloor \Delta_n} \to \sigma^2_{s-}$. Moreover, noting that η_n is bounded, we can apply Lemma 3 in AJL with $i_n = \lfloor s/\Delta_n\rfloor - k_n - k_n'$ to get $\xi^n_{\lfloor s/\Delta_n\rfloor - k_n - k_n'} = \sigma^2_{s-} + o_{p,\eta_n}(1)$. Therefore, for Lebesgue a.e. $s \in [0, t]$, $f_n(\cdot, s) = o_{p,\eta_n}(1)$. With an appeal to Fubini's theorem and the bounded convergence theorem, we see that $f_n(\cdot,\cdot)$ converges to zero in measure on the product space. Moreover, under Assumptions S.H-1 and S.N, it is easily seen that the sequence $(f_n(\cdot,\cdot))_{n\geq 1}$ is uniformly integrable. Hence,

$$\text{(S.A.98)}\quad \mathbb{E}\left(\int_0^t f_n(\cdot, s)\, ds\right) \to 0.$$

Combining (S.A.96), (S.A.97), and (S.A.98), we derive $\zeta(l)_t^n = o_{p,\eta_n}(1)$. By (S.A.95), we obtain $\Delta_n^{3/2-p/2}\tilde{N}(Z^*, \phi, \psi, l)_t^n = Q(\psi, 0, l)_t + o_{p,\eta_n}(1)$. Recalling the definition of a_n, we readily have

$$\text{(S.A.99)}\quad a_n^2 \tilde{N}\big(Z^*, \phi, \psi, l\big)_t^n \xrightarrow{\mathbb{P},\eta_n} \frac{1}{(1+h^{p-1})^2} Q(\psi, 0, l)_t.$$

Step 3. In this step, we show (S.A.93). For each $q \geq 1$, let $\Omega_n(t, q)$ be the collection of paths on which $|T_m - T_{m'}| > 2k_n\Delta_n + (k_n'+1)\Delta_n$ and $2k_n\Delta_n < T_m < t - 2k_n\Delta_n$ whenever $T_m, T_{m'} \leq t$ for some m, m' in P_q. We have $\Omega_n(t, q) \to \Omega$ almost surely as $n \to \infty$. Denote $I_m^n = \lfloor T_m/\Delta_n\rfloor$. On $\Omega_n(t, q)$, we decompose

$$\begin{aligned}\text{(S.A.100)}\quad &\tilde{N}\big(Z^* + \eta_n J^q, \phi, \psi, 0\big)_t^n \\ &\quad = \tilde{N}\big(Z^*, \phi, \psi, 0\big)_t^n + \tilde{N}\big(\eta_n J^q, \phi, \psi, 0\big)_t^n + R(q)_t^n,\end{aligned}$$

where

$$R(q)_t^n = \sum_{m\in P_q: T_m \leq t} \sum_{i=I_m^n - k_n + 2}^{I_m^n} \tilde{G}\big(\bar{Z}_i^{*n}, \eta_n \bar{J}_i^{q,n}\big)\xi^n_{i-k_n-k_n'},$$

$$\tilde{G}(x, y) = |x+y|^{2p-2} - |x|^{2p-2} - |y|^{2p-2}, \quad x, y \in \mathbb{R}.$$

We also denote for each $q \geq 1$,

$$\text{(S.A.101)}\quad N(0, -, q)_t = \theta \sum_{m\in P_q: T_m \leq t} |\Delta X_{T_m}|^{2p-2} \sigma^2_{T_m -}.$$

By Lemma 6 in AJL (taking Z there to be J^q), we have $\Delta_n^{1/2}\tilde{N}(J^q,\phi,\psi,0)_t^n = \bar{\psi}(2p-2)N(0,-,q)_t + o_p(1)$. Consequently,

$$\text{(S.A.102)}\quad a_n^2\tilde{N}\big(\eta_n J^q,\phi,\psi,0\big)_t^n \xrightarrow{\mathbb{P},\eta_n} \left(\frac{h^{p-1}}{1+h^{p-1}}\right)^2\bar{\psi}(2p-2)N(0,-,q)_t.$$

Let $\beta>0$ be arbitrary. There exists $K_\beta>0$ such that $||x+y|^{2p-2}-|x|^{2p-2}|\le K_\beta|y|^{2p-2}+\beta|x|^{2p-2}$. For such β and K_β, we have, on $\Omega_n(t,q)$,

$$\text{(S.A.103)}\quad \big|R(q)_t^n\big|\le (K_\beta+1)\bar{R}(q)_t^n+\beta\tilde{N}\big(\eta_n J^q,\phi,\psi,0\big)_t^n,$$

where

$$\text{(S.A.104)}\quad \bar{R}(q)_t^n=\sum_{m\in P_q:T_m\le t}\ \sum_{i=I_m^n-k_n+2}^{I_m^n}\big|\bar{Z}_i^{*n}\big|^{2p-2}\big|\xi\big(Z^*+\eta_n J'^q,\phi,0\big)_{i-k_n-k_n'}^n\big|.$$

Recall the definition of $\mathcal{H}_t$ from the proof of Lemma 6. Following a similar argument as in that lemma, we derive $\mathbb{E}[|\bar{Z}_i^{*n}|^{2p-2}|\mathcal{H}_{i\Delta_n}]\le K\Delta_n^{p/2-1/2}$ and $\mathbb{E}[|\xi(Z^*+\eta_n J'^q,\phi,0)_{i-k_n-k_n'}^n||\mathcal{H}_0]\le K$. By applying repeated conditioning to (S.A.104), we have $\mathbb{E}[\bar{R}(q)_t^n 1_{\Omega_n(t,q)}]\le K\Delta_n^{p/2-1}$. Since $a_n=O(\Delta_n^{3/4-p/4})$,

$$\text{(S.A.105)}\quad a_n^2\bar{R}(q)_t^n=o_{p,\eta_n}(1).$$

Combining (S.A.102), (S.A.103), and (S.A.105), and noting that β is arbitrary, we derive for each $q\ge 1$,

$$\text{(S.A.106)}\quad a_n^2 R(q)_t^n=o_{p,\eta_n}(1).$$

By (S.A.99), (S.A.100), (S.A.102), and (S.A.106), we have for each $q\in\mathbb{N}$,

$$\text{(S.A.107)}\quad a_n^2\tilde{N}\big(Z^*+\eta_n J^q,\phi,\psi,0\big)_t^n \xrightarrow{\mathbb{P},\eta_n} \frac{1}{(1+h^{p-1})^2}Q(\psi,0,0)_t + \left(\frac{h^{p-1}}{1-h^{p-1}}\right)^2\bar{\psi}(2p-2)N(0,-,q)_t.$$

By repeated conditioning and (S.A.89), we can easily adapt the proof of Lemma 5 to show that for any $\varepsilon>0$,

$$\text{(S.A.108)}\quad \lim_{q\to\infty}\limsup_{n\to\infty}\mathbb{P}\big(\big|a_n^2\tilde{N}\big(\eta_n J'^q,\phi,\psi,0\big)_t^n\big|>\varepsilon\big)=0.$$

Finally, observe that $N(0,-,q)_t$ converges in probability to $N(0,-)_t$ as $q\to\infty$. Following the same steps that led to (S.A.40), we combine (S.A.107) and (S.A.108) to derive (S.A.93).

Step 4. We prove (S.A.94) in this step for $l \in \{1, \dots, p-1\}$. By repeated conditioning and (S.A.89), we can easily adapt the proofs of Lemmas 7 and 8(b) to derive

$$\begin{cases} a_n^2 \sum_{i=k_n+k_n'}^{[t/\Delta_n]-k_n} \left|\bar{Z}_i^n\right|^{2p-2-2l} \left(\left(\hat{Z}_i^n\right)^l - \left(\hat{\chi}_i^n\right)^l\right) \xi_{i-k_n-k_n'}^n = o_{p,\eta_n}(1), \\ a_n^2 \sum_{i=k_n+k_n'}^{[t/\Delta_n]-k_n} \left|\bar{Z}_i^{*n}\right|^{2p-2-2l} \left(\left(\hat{Z}_i^{*n}\right)^l - \left(\hat{\chi}_i^n\right)^l\right) \xi_{i-k_n-k_n'}^n = o_{p,\eta_n}(1), \\ a_n^2 \sum_{i=k_n+k_n'}^{[t/\Delta_n]-k_n} \left(\left|\bar{Z}_i^n\right|^{2p-2-2l} - \left|\bar{Z}_i^{*n}\right|^{2p-2-2l}\right) \left(\hat{\chi}_i^n\right)^l \xi_{i-k_n-k_n'}^n = o_{p,\eta_n}(1), \end{cases}$$

which imply

(S.A.109) $$a_n^2\left(\tilde{N}\left(Z^{\eta_n}, \phi, \psi, l\right)_t^n - \tilde{N}\left(Z^*, \phi, \psi, l\right)_t^n\right) = o_{p,\eta_n}(1).$$

Combining (S.A.99) and (S.A.109), we have (S.A.94). *Q.E.D.*

APPENDIX S.B: SIMULATION RESULTS

S.B.1. *The Baseline Setting*

In this supplemental appendix, we examine the validity of the asymptotic theory in the main text in a simulation setting designed to approximate the constraints faced in a typical real life application.[1] We adopt a similar simulation setting as in AJL. The log price Z_t is generated according to the model

$$\begin{aligned} &Z_t = X_t^* + J_t + \chi_t, \\ &X_t^* = X_0 + \int_0^t \sigma_s \, dW_s, \quad \sigma_t = v_t^{1/2}, \\ &dv_t = \kappa(\beta - v_t)\, dt + \gamma v_t^{1/2}\, dB_t, \quad \mathbb{E}[dW_t\, dB_t] = \rho\, dt, \quad \text{where} \\ &\qquad \beta^{1/2} = 0.4, \quad \gamma = 0.5, \quad \kappa = 5, \quad \rho = -0.5, \quad X_0 = \log(100). \end{aligned}$$

Here, X_t^* is the continuous part with instantaneous volatility σ_t, J_t is a pure jump process, and χ_t is the additive noise. The drift in X_t^* is excluded because it plays little role in the high frequency setting. Parameters that govern the stochastic volatility process are calibrated according to the estimates in Aït-Sahalia and Kimmel (2007). We use an observation length of $T = 5$ days, with

[1] I wish to thank an anonymous referee whose suggestions significantly improved the scope of this simulation study.

each day consisting of 6.5 hours, and sample the continuous-time process every 5 seconds. There are 2,000 simulations in each experiment.

The additive noise χ_t is generated according to

$$\text{(S.B.1)}\quad \begin{aligned} &\chi_t = 3\sigma_t \Delta_n^{1/2}\big(\chi_t^A + \chi_t^B\big),\\ &\chi_t^A \sim \mathcal{N}(0,1),\\ &\chi_t^B = \frac{f_{25}(\tilde{\chi}_t^B)}{\mathrm{SD}(f_{25}(\tilde{\chi}_t^B))}, \quad f_{25}(x) = \min\big\{\max\{x, -25\}, 25\big\},\\ &\tilde{\chi}_t^B \sim t\text{-distribution with degrees of freedom 2.5,}\end{aligned}$$

where $\mathrm{SD}(\cdot)$ is the standard deviation operator, and χ_t^A and χ_t^B are i.i.d. draws and mutually independent. The instantaneous standard deviation of either the Gaussian noise or the (truncated) t-distributed noise is three times that of the diffusive increment, that is, $\sigma_t \Delta_n^{1/2}$. This experimental design allows temporal heteroskedasticity and dependence in χ_t. The t-distributed noise is introduced to capture the large bouncebacks commonly observed in transaction data. In this setting, the microstructure noise clearly dominates the diffusive increment. Moreover, with the t-distributed noise present, one could observe many large returns even in the absence of jumps. The task of estimating and making inference on jump characteristics is thus fairly challenging.

We simulate the jump process J_t from a centered symmetric α-stable process with activity index 0.5, 1, 1.5, or 1.75. To compare results across activity levels, we scale J_t so that in each realization, the realized quadratic variation of J_t is fixed at $\beta T/9$ (resp. $\beta T/4$). In other words, the jump quadratic variation (JV) is 10% (resp. 20%) of the total quadratic variation (QV) on average; this configuration is motivated by the empirical findings in Aït-Sahalia and Jacod (2012).[2] Our design allows for a wide spectrum of jump behaviors. When the jump activity level is low, the jump process is dominated by a few big jumps, featuring the situation with "infrequent big jumps"; when the activity is high, jumps have relatively similar sizes, featuring the situation with "many small jumps." As the activity index approaches 2, that is, the index of the Brownian motion, it becomes more difficult to disentangle the jump part from the continuous part.

Throughout the simulations, we fix $p = 4$ and consider weight functions $g_1(x) = \max\{1 - |2x - 1|^2, 0\}$ and $g_k(x) = g_1(kx)$ for $k \geq 1$. We use weight functions g_1 and g_2 in the computation of the bias-corrected estimator $\hat{H}_T^n$, which we denote by $\hat{H}$ below for notational simplicity. We also compute the standard uncorrected pre-averaging estimators for $\sum_{s\leq T} |\Delta J_s|^p$: we denote

[2]In an application to transaction prices of 30 component stocks of the Dow Jones Industrial Average (DJIA), Aït-Sahalia and Jacod (2012) found that the JV/QV ratio is about 25% for individual stocks and ranges from 5% to 15% for the DJIA index.

$\tilde{H}_k = \Delta_n^{1/2}(\theta\bar{g}_k(p))^{-1}\bar{V}(Z, g_k, p)_T^n$ for $k \geq 1$ (recall Corollary 1 in the main text). Note that $\hat{H}$ is a linear combination of $\tilde{H}_1$ and $\tilde{H}_2$, so these uncorrected estimators serve as natural benchmarks for comparison. This comparison is made in Section S.B.2. In Section S.B.3, we further discuss the finite-sample behavior of $\tilde{H}_k$ and the associated nonrobust confidence intervals (CI) for $1 \leq k \leq 15$. Section S.B.4 reports a length comparison between the robust and the nonrobust CI's. We set the pre-averaging window $k_n = 80$, 100, or 120. In the computation of the variance estimator $\hat{\Sigma}_n$, we set $\phi = \psi_1 = g_1$, $\psi_2 = g_2$, and $k'_n = 3k_n$, and set the truncation level $u_n = 5(\bar{V}(Z, g_1, 2)_T^n/T)^{1/2}\Delta_n^{0.49}$. Robustness checks for the choice of k'_n and u_n are presented in Section S.B.5, where we also report simulation results for cases with rounding effect.

S.B.2. *Baseline Results*

Figure S.1 plots the distributions of estimation errors of the bias-corrected estimator $\hat{H}$ and the uncorrected estimators $\tilde{H}_1$ and $\tilde{H}_2$. We start with the case with low-activity jumps (top panel). When the jump signal is weak (top left), the standard estimators are clearly upward biased, while the estimation error of the bias-corrected estimator is properly centered around 0. These findings are consistent with our asymptotic theory. As the jump signal becomes stronger (top right), the standard estimators still appear to be upward biased, but only mildly. However, the performance of the standard estimators deteriorates substantially when jumps become more active and smaller (bottom panel). Even in the case with $\mathrm{JV}/\mathrm{QV} = 0.2$ (bottom right), the standard estimators are still quite biased; indeed, the distributions of their estimation errors put almost all mass on the positive real line. In contrast, the estimation error of the bias-corrected estimator remains properly centered around zero, regardless of the strength of the jump signal.

Table S.I summarizes the relative estimation error, defined as the ratio of the estimation error to the estimand $\sum_{s\leq T}|\Delta J_s|^4$ and expressed in percentage terms. We report the sample median of the relative estimation error, henceforth the median relative bias (MRB).[3] As clearly shown in the table, the uncorrected estimators $\tilde{H}_1$ and $\tilde{H}_2$ are always upward biased, and the bias can be quite substantial when the jump signal is weak. On the other hand, the MRB of the bias-corrected estimator $\hat{H}$ is much smaller, insensitive to the strength of the jump signal, and fairly close to zero. We remind the reader that $\hat{H}$ is a linear combination of $\tilde{H}_1$ and $\tilde{H}_2$, so the comparison here is natural and clearly demonstrates the effectiveness of the bias correction. These findings are robust with respect to the choice of k_n.

[3] We report the median instead of the sample average for the following reason. As $\sum_{s\leq T}|\Delta J_s|^4$ is a random variable, it takes very small values in some realizations, leading to very large relative

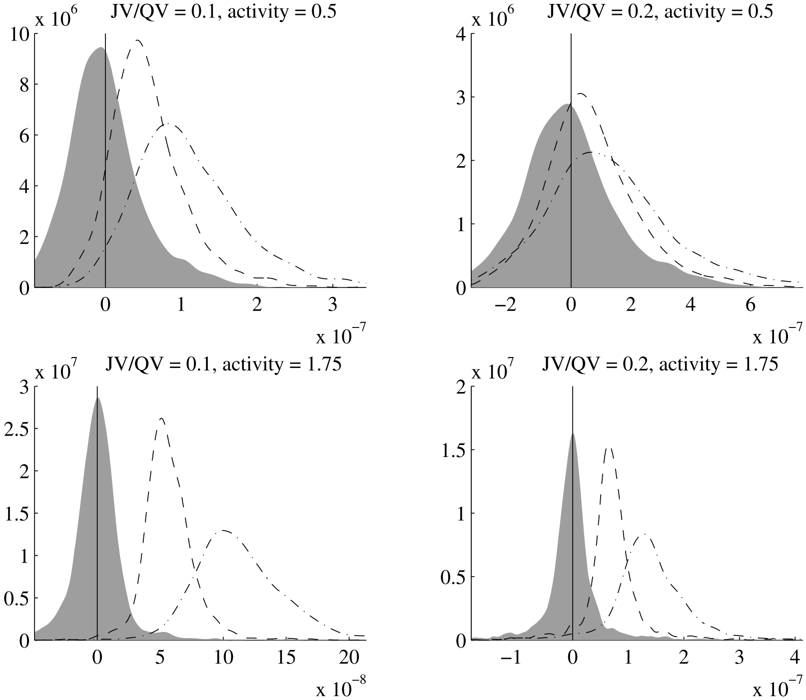

FIGURE S.1.—Comparison of the distributions of estimation errors of bias-corrected and uncorrected estimators. The estimation error is defined as the estimate less the estimand $\sum_{s\leq T} |\Delta J_s|^4$. We consider three estimators: the bias-corrected estimator $\hat{H}$ (shaded area), as well as standard uncorrected estimators $\tilde{H}_1$ (dot–dash) and $\tilde{H}_2$ (dash).

Next we examine the finite-sample coverage rate of the robust confidence set. We set the nominal level to be 95% and consider the symmetric two-sided CI given by

$$\text{(S.B.2)} \quad \mathrm{CI}^{\mathrm{Rbst}} = \big[\hat{H} - z_{0.975}\Delta_n^{1/2}\sqrt{\boldsymbol{\kappa}^\top \hat{\Sigma}_n \boldsymbol{\kappa}}, \hat{H} + z_{0.975}\Delta_n^{1/2}\sqrt{\boldsymbol{\kappa}^\top \hat{\Sigma}_n \boldsymbol{\kappa}}\big],$$

where for any $c \in (0,1)$, z_c is the c-quantile of $\mathcal{N}(0,1)$, that is. $\mathbb{P}(\xi \leq z_c) = c$ for $\xi \sim \mathcal{N}(0,1)$. This CI is a special case of the robust confidence set CS^n_{1-c} introduced in Section 3.3 of the main text.

estimation errors. In face of these “outliers,” we consider the sample median as a better measure of the center of the distribution.

TABLE S.I

MEDIAN RELATIVE BIAS (%) OF BIAS-CORRECTED AND UNCORRECTED ESTIMATORS

	JV/QV = 0.1			JV/QV = 0.2		
Activity	Rbst	Std 1	Std 2	Rbst	Std 1	Std 2
			Panel A: $k_n = 80$			
0.50	−5.4	93	46	−3.7	20	9
1.00	−5.3	163	84	−2.7	41	21
1.50	−7.3	475	241	−4.7	119	61
1.75	−3.7	1601	813	−6.6	394	202
			Panel B: $k_n = 100$			
0.50	−5.4	114	58	−3.3	26	12
1.00	−5.4	201	104	−3.1	49	27
1.50	−4.4	594	302	−4.7	149	76
1.75	−0.9	2015	1007	−4.2	498	251
			Panel C: $k_n = 120$			
0.50	−7.4	137	69	−4.3	31	15
1.00	−5.0	239	124	−3.2	58	32
1.50	−3.9	706	361	−4.1	176	89
1.75	6.2	2402	1191	−4.2	591	300

Note: We report the sample median of the relative bias (i.e., the ratio of the estimation error to the estimand $\sum_{s\le T}|\Delta J_s|^4$) for the robust estimator $\hat{H}$ (Rbst), as well as the standard uncorrected estimators $\tilde{H}_1$ (Std 1) and $\tilde{H}_2$ (Std 2).

For comparison, we also consider CI's that would be justified by the standard asymptotics. We denote for $k = 1, 2$,

$$\widetilde{\mathrm{CI}}_k^{\mathrm{Std}} = \left[\tilde{H}_k - \frac{z_{0.975}\Delta_n^{1/4}\sqrt{\Sigma_{J,k}}}{\theta\bar{g}_k(p)}, \tilde{H}_k + \frac{z_{0.975}\Delta_n^{1/4}\sqrt{\Sigma_{J,k}}}{\theta\bar{g}_k(p)}\right],$$

where $\Sigma_{J,k}$ is the kth diagonal element of Σ_J, that is, the asymptotic variance of $\bar{V}(Z, g_k, p)_T^n$ in the presence of jumps under the standard asymptotics (recall comment (iii) of Theorem 2 in the main text). We note that $\widetilde{\mathrm{CI}}_k^{\mathrm{Std}}$ is infeasible because it depends on the unknown variable $\Sigma_{J,k}$. A feasible version demands an estimator for $\Sigma_{J,k}$ that is consistent under the standard asymptotics. We construct such an estimator as follows. Recall the notation in Appendix B of the main text. We set

$$\hat{N}_k^{\mathrm{Std}}(m, \pm)_T^n = \frac{\Delta_n^{1/2} N(Z, \phi, \psi, m, \pm)_T^n}{\bar{\psi}(2p-2)}, \quad m = 0, 1,$$

$$\hat{\Sigma}_{J,k}^{\mathrm{Std}} = \theta^2 p^2\big(\Psi_{k-}\hat{N}_k^{\mathrm{Std}}(0,-)_T^n + \Psi_{k+}\hat{N}_k^{\mathrm{Std}}(0,+)_T^n + \Psi'_{k-}\hat{N}_k^{\mathrm{Std}}(1,-)_T^n + \Psi'_{k+}\hat{N}_k^{\mathrm{Std}}(1,+)_T^n\big),$$

where ϕ and ψ are weight functions (we set $\phi = \psi = g_k$ in the simulation), and the scalars $\Psi_{k\pm}$ and $\Psi'_{k\pm}$ are defined in the same way as $\Psi_\pm$ and $\Psi'_\pm$, but only for each weight function g_k. By specializing Theorem 5 of the main text to the case with $\eta_n \equiv 1$, it is easy to see that $\hat{N}_k^{\mathrm{Std}}(m, \pm)_T^n \xrightarrow{\mathbb{P}} N(m, \pm)_T^n$, $m = 0, 1$, and thus $\hat{\Sigma}_{J,k}^{\mathrm{Std}} \xrightarrow{\mathbb{P}} \Sigma_{J,k}$. A feasible CI under the standard asymptotics can then be constructed as

$$\mathrm{CI}_k^{\mathrm{Std}} = \left[\tilde{H}_k - \frac{z_{0.975}\Delta_n^{1/4}\sqrt{\hat{\Sigma}_{J,k}^{\mathrm{Std}}}}{\theta \bar{g}_k(p)}, \tilde{H}_k + \frac{z_{0.975}\Delta_n^{1/4}\sqrt{\hat{\Sigma}_{J,k}^{\mathrm{Std}}}}{\theta \bar{g}_k(p)} \right].$$

Table S.II compares the Monte Carlo coverage rate of $\mathrm{CI}^{\mathrm{Rbst}}$ with that of $\mathrm{CI}_1^{\mathrm{Std}}$ and $\mathrm{CI}_2^{\mathrm{Std}}$. In all cases, the standard CI's exhibit undercoverage. Their best performance occurs in the case when the jump activity is 0.5 and $\mathrm{JV}/\mathrm{QV} = 0.2$, so jumps are relatively large. In this case, the standard CI's undercover only by 3–4 percentage points and slightly outperform the robust CI. This finding supports the standard asymptotic theory, as well as the intuition that the standard asymptotics should perform well when jumps are “big.” However, as the jump signal becomes weaker, the undercoverage problem is quite severe for the standard CI's. In contrast, the coverage rate of the robust CI is fairly close to the nominal level in all cases. These findings are robust to the choice of k_n, at least within the range considered here.

TABLE S.II

FINITE-SAMPLE COVERAGE RATES (%) OF 95% NOMINAL LEVEL CI'S

	JV/QV = 0.1			JV/QV = 0.2		
Activity	$\mathrm{CI}^{\mathrm{Rbst}}$	$\mathrm{CI}_1^{\mathrm{Std}}$	$\mathrm{CI}_2^{\mathrm{Std}}$	$\mathrm{CI}^{\mathrm{Rbst}}$	$\mathrm{CI}_1^{\mathrm{Std}}$	$\mathrm{CI}_2^{\mathrm{Std}}$
Panel A: $k_n = 80$						
0.50	91.5	70.6	80.4	90.0	91.4	92.1
1.00	93.5	42.8	54.8	91.3	83.0	86.7
1.50	95.2	17.2	24.6	93.3	51.0	58.6
1.75	95.6	7.9	10.9	94.4	25.4	31.0
Panel B: $k_n = 100$						
0.50	93.2	66.8	77.5	90.7	91.4	91.9
1.00	95.1	39.0	50.7	93.4	82.3	86.1
1.50	95.9	15.6	21.6	94.6	47.6	56.4
1.75	96.5	6.9	9.7	95.0	23.5	29.0
Panel C: $k_n = 120$						
0.50	94.8	64.0	75.0	90.8	91.2	92.4
1.00	95.7	36.8	47.6	92.7	80.5	85.7
1.50	96.5	14.7	20.3	94.8	45.8	54.2
1.75	96.9	6.2	8.8	95.9	22.8	26.9

TABLE S.III

NUMBER OF NEGATIVE REALIZATIONS OF BIAS-CORRECTED ESTIMATORS AND ROBUST CI'S

	JV/QV = 0.1		JV/QV = 0.2	
Activity	$\hat{H} < 0$	$\mathrm{CI}^{\mathrm{Rbst}} \cap \mathbb{R}_+ = \emptyset$	$\hat{H} < 0$	$\mathrm{CI}^{\mathrm{Rbst}} \cap \mathbb{R}_+ = \emptyset$
		Panel A: $k_n = 80$		
0.50	21	1	1	0
1.00	62	0	1	1
1.50	246	3	42	0
1.75	499	8	218	2
		Panel B: $k_n = 100$		
0.50	35	1	2	1
1.00	94	1	6	1
1.50	303	4	80	0
1.75	556	10	282	2
		Panel C: $k_n = 120$		
0.50	64	1	4	1
1.00	143	1	14	2
1.50	341	4	114	0
1.75	582	7	331	2

Note: The total number of Monte Carlo trials is 2,000. The nominal level of each CI is 95%.

The good performance of the robust CI in cases with active jumps may be surprising. Indeed, when the jump activity is greater than 1, Assumption H-1 is not satisfied. Somewhat more surprisingly, the robust CI actually performs slightly better when the jump activity is higher. Our conjecture is that one might be able to prove Theorems 2 and 3 in the main text under weaker conditions. The pursuit of such generality is beyond the scope of the current paper.

We further discuss some seemingly "irregular" behaviors of $\hat{H}$ and $\mathrm{CI}^{\mathrm{Rbst}}$. While the jump power variation is nonnegative, the bias-corrected estimator $\hat{H}$ may be negative in finite samples due to sampling errors. The robust CI around a negative estimate of $\hat{H}$ should be wide enough to cover zero, that is, $\mathrm{CI}^{\mathrm{Rbst}} \cap \mathbb{R}_+ \neq \emptyset$. However, for a given nominal level, $\mathrm{CI}^{\mathrm{Rbst}} \cap \mathbb{R}_+$ may still be empty in some realizations, but the probability of such an event should be small. To be concrete, when the nominal level is 95%, the probability of the event $\{\mathrm{CI}^{\mathrm{Rbst}} \cap \mathbb{R}_+ = \emptyset\}$ is bounded above by 5% asymptotically, in general, and bounded above by 2.5% for the symmetric two-sided CI considered here. Table S.III reports the number of Monte Carlo realizations for $\hat{H} < 0$ and $\mathrm{CI}^{\mathrm{Rbst}} \cap \mathbb{R}_+ = \emptyset$ out of 2,000 Monte Carlo trials. The results are quite intuitive: the number of such realizations is small when the jump signal is strong and vice versa. In particular, the number of realizations with $\mathrm{CI}^{\mathrm{Rbst}} \cap \mathbb{R}_+ = \emptyset$ is always well below the theoretical bound $2{,}000 \times 2.5\% = 50$. In this regard, the performance of the robust CI is quite satisfactory.

In applications, one may adopt post-estimation regularization to incorporate the prior knowledge that the jump power variation is nonnegative. Perhaps the simplest option is to report $\max\{\hat{H}, 0\}$ as the estimate and $\mathrm{CI}^{\mathrm{Rbst}} \cap \mathbb{R}_+$ as the CI ($\{0\}$ in the case with empty intersection). In general, the necessity and the specific choice of regularization method should depend on the problem at hand and likely involve decision-theoretic arguments. A discussion in this direction is clearly beyond our scope here. We stress that, throughout the paper, we only report the "raw" estimates for all estimators and CI's, so as to maintain the consistency between the simulation setting and the theoretical results in the main text.

S.B.3. *Extended Results for Nonrobust Estimators and CI's*

Tables S.I and S.II reveal an interesting pattern for the standard estimators: $\tilde{H}_2$ always has a smaller MRB than $\tilde{H}_1$ and $\mathrm{CI}_2^{\mathrm{Std}}$ always has less size distortion than $\mathrm{CI}_1^{\mathrm{Std}}$. Hence, for fixed k_n, using g_2 as the weight function appears to be strictly better than using g_1. Observe that pre-averaging under g_2 results in less smoothing than under g_1. Indeed, since g_k is supported on $[0, 1/k]$, pre-averaging under g_k only involves $\lfloor k_n/k \rfloor$ raw returns within each averaging window. In other words, the "effective" averaging window of g_k is k_n/k. The jump signal is thus better preserved under the weight function g_2 than is g_1. It is then not surprising to find g_2 outperforming g_1. This intuition suggests that using the weight function g_k with a larger k may further improve the performance of the standard estimators. This being said, we note that choosing k too large will introduce other finite-sample complications. In the standard pre-averaging theory, as well as in the current paper, we need the effective averaging window to go to infinity sufficiently fast, so that the noise can be sufficiently smoothed and higher-order effects vanish sufficiently fast. Hence, when k is large (so the effective averaging window is small), the finite-sample behavior of the standard estimators tends to be confounded by higher-order effects on which the existing theory is silent. In the absence of theoretical guidance, we examine the choice of the weighting scale k via simulation for a broad range of k.

In the same simulation setting as above, we compute $\tilde{H}_k$ and $\mathrm{CI}_k^{\mathrm{Std}}$ for all $1 \leq k \leq 15$. For brevity, we fix $k_n = 100$. Table S.IV shows the MRB of uncorrected estimators. We have two findings. First, the MRB of $\tilde{H}_k$ decreases as we increase k up to $k \leq 5$. In the case with $\mathrm{JV}/\mathrm{QV} = 0.2$ and jump activity $= 0.5$, the MRB of $\tilde{H}_5$ is only 1.7%, which is smaller in magnitude than that of the robust estimator (−3.3%). This being said, we note that when $k \leq 5$, the uncorrected estimators carry evidently large MRB, in general, and, consistent with our theory, the MRB's are positive. Second, as we further increase k above 5, the MRB decreases in numerical value but eventually increases in magnitude. Indeed, in each simulation setting, the MRB becomes negative when k is large

TABLE S.IV

MEDIAN RELATIVE BIAS OF UNCORRECTED ESTIMATORS

Activity	Std 3	Std 4	Std 5	Std 7	Std 9	Std 11	Std 13	Std 15
			Panel A: JV/QV = 0.1					
0.50	37	26	19	9.3	−1.9	−13	−22	−34
1.00	71	52	39	23	6.9	−11	−24	−40
1.50	202	150	116	72	34	−8.0	−32	−63
1.75	681	506	395	251	126	−2.2	−49	−131
			Panel B: JV/QV = 0.2					
0.50	7.0	4.1	1.7	−2.1	−6.5	−12	−17	−24
1.00	17	12	8.2	2.4	−3.8	−11	−17	−25
1.50	52	38	29	16	4.9	−7.4	−18	−31
1.75	169	127	98	61	32	−1.4	−21	−43

Note: Std k, $1 \leq k \leq 15$, stands for the uncorrected estimator $\tilde{H}_k$ associated with the weight function $g_k(x) = g_1(kx) = \max\{1 - |2kx - 1|^2, 0\}$.

enough and becomes "more negative" when k is larger. The negative sign of the MRB contradicts our theoretical prediction. As hinted in the previous paragraph, some higher-order confounding effect brought on by large values of k is likely in force, which leads to a negative bias. This confounding effect appears to take effect when $k > 5$ and becomes dominant in all settings when $k \geq 11$. Hence, the seemingly good performance of $\tilde{H}_k$ for large k ($k \approx 11$) is likely due to a cancellation of biases from two sources with opposite signs and, hence, can only be taken with a grain of salt.

We further examine the coverage property of $\mathrm{CI}_k^{\mathrm{Std}}$ and its infeasible counterpart $\widetilde{\mathrm{CI}}_k^{\mathrm{Std}}$ for $1 \leq k \leq 15$. The purpose of considering the infeasible CI's is to directly examine the approximation quality of the nonlocal asymptotic distribution. Indeed, the coverage property of the feasible CI inevitably depends on our choice of the asymptotic variance estimator $\hat{\Sigma}_{J,k}^{\mathrm{Std}}$, which some readers may find arbitrary. The infeasible CI's are immune to such choice.

Figure S.2 plots the Monte Carlo coverage rates of $\widetilde{\mathrm{CI}}_k^{\mathrm{Std}}$ and $\mathrm{CI}_k^{\mathrm{Std}}$ versus the weighting scale k. In light of the discussion above, we discuss cases with $k \leq 5$ and $k > 5$ separately, as the latter are likely to be confounded by higher-order effects that are not captured by the existing theory (including the current paper). We have the following findings for cases with $k \leq 5$. First, both the feasible and infeasible nonrobust CI's often have evident undercoverage, except for cases when jumps are big. This finding is consistent with those in Table S.II. Second, by moderately increasing k, say from 1 to 5, hence effectively reducing the degree of smoothing, the size distortion of nonrobust CI's is mitigated in most cases. Third, infeasible nonrobust CI's often suffer from even larger size distortion than their feasible counterparts. The reason is simple: the variance estimator $\hat{\Sigma}_{J,k}^{\mathrm{Std}}$ tends to overestimate $\Sigma_{J,k}^{\mathrm{Std}}$, but the overestimation

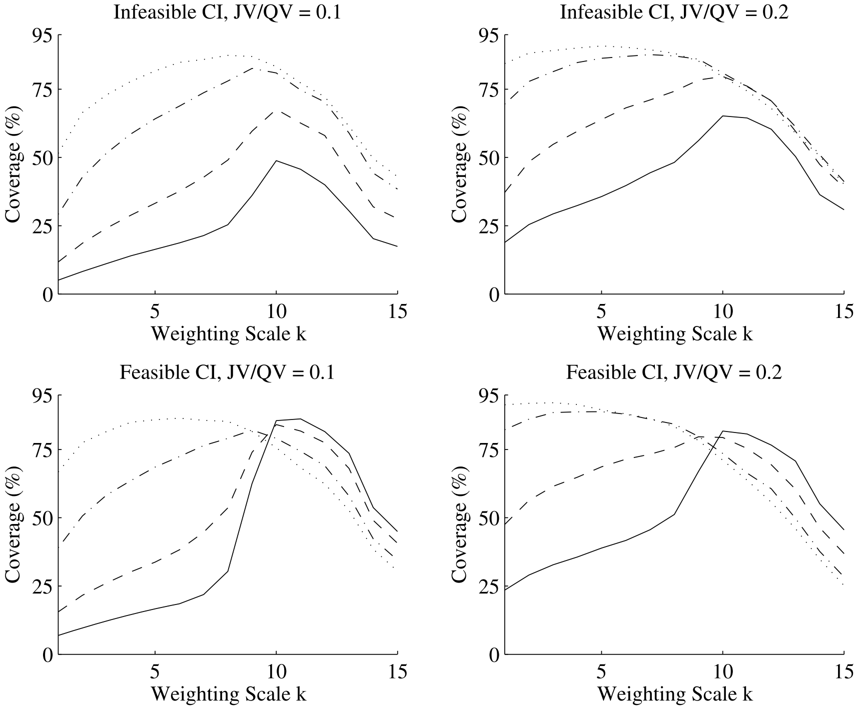

FIGURE S.2.—Finite-sample coverage rates of nominal level 95% nonrobust CI's. We plot the coverage rate for $\widetilde{\mathrm{CI}}_k^{\mathrm{Std}}$ (top) and $\mathrm{CI}_k^{\mathrm{Std}}$ (bottom) versus the scaling factor k in the weight function, $1 \leq k \leq 15$. The quadratic variation of jumps is set to be 10% (left) or 20% (right) of the total quadratic variation, on average. The jump process is α-stable with activity index 0.5 (dot), 1 (dot–dash), 1.5 (dash), and 1.75 (solid).

helps reduce the undercoverage problem; the overestimation can be explained by Theorem 5 in the main text. Therefore, the better coverage of the feasible nonrobust CI's relative to the infeasible ones should be taken with caution.

We now turn to cases with $k > 5$ and summarize our findings as follows. First, the coverage rate no longer increases in k monotonically, as evidenced by the hump shapes in the plots. The "optimal" choice of k clearly depends on the simulation setting and appears to increase with the jump activity. Second, the coverage rates of the infeasible CI's exhibit an intuitive ordering: the undercoverage is almost always more severe when jumps are small (high activity). This pattern is not preserved for feasible CI's. Indeed, when k is large enough, this ordering is reversed, likely due to additional confounding effects associated with large k. Third, we note that even in the ideal case in which $\mathrm{CI}_k^{\mathrm{Std}}$ and $\widetilde{\mathrm{CI}}_k^{\mathrm{Std}}$ are implemented at the "optimal" k that maximizes the coverage rate ex post, the overall performance of these nonrobust CI's is still worse than that of the robust CI's (cf. panel B of Table S.II). In practice, the optimal choice of a tuning parameter like k is difficult, as it involves higher-order asymptotic expansions and other unknown functionals of the underlying process. Nevertheless, based on the aforementioned evidence, the nonrobust CI's are unlikely to outperform the robust CI's, even if the optimal tuning is feasible.

In summary, we find that moderately increasing k in the weight function g_k tends to improve the performance of the uncorrected estimators and the nonrobust CI's. However, further increasing k brings in additional confounding effects, which may improve or worsen the finite-sample performance of the standard methods. Overall, the bias-corrected estimators and robust CI's tend to outperform the standard methods, especially when jumps are relatively small.

S.B.4. *Length Comparison for CI's*

In this section, we compare the average length of the robust and the nonrobust CI's to better understand the relative advantage/disadvantage of these methods. Table S.V reports the relative average length of the nonrobust CI's with respect to that of the robust CI. The robust CI's are computed in the same way as in Table S.II. For the nonrobust CI's, we consider both infeasible and feasible versions associated with weight functions g_k, $1 \leq k \leq 5$. Results for $6 \leq k \leq 15$ are omitted here for brevity, but are available on request.

In view of the coverage results in Table S.II and Figure S.2, Table S.V shows the trade-off between coverage and length in the comparison of robust versus nonrobust CI's: the nonrobust CI's in general are tighter than their robust counterparts, which partially explains their undercoverage. We also observe

TABLE S.V

RELATIVE AVERAGE LENGTH OF THE NONROBUST 95% NOMINAL LEVEL CI'S

	JV/QV = 0.1					JV/QV = 0.2				
Activity	$\widetilde{\mathrm{CI}}_1^{\mathrm{Std}}$	$\widetilde{\mathrm{CI}}_2^{\mathrm{Std}}$	$\widetilde{\mathrm{CI}}_3^{\mathrm{Std}}$	$\widetilde{\mathrm{CI}}_4^{\mathrm{Std}}$	$\widetilde{\mathrm{CI}}_5^{\mathrm{Std}}$	$\widetilde{\mathrm{CI}}_1^{\mathrm{Std}}$	$\widetilde{\mathrm{CI}}_2^{\mathrm{Std}}$	$\widetilde{\mathrm{CI}}_3^{\mathrm{Std}}$	$\widetilde{\mathrm{CI}}_4^{\mathrm{Std}}$	$\widetilde{\mathrm{CI}}_5^{\mathrm{Std}}$
Panel A: Relative average length of infeasible CI's										
0.50	1.14	0.81	0.66	0.57	0.51	1.35	0.95	0.78	0.68	0.60
1.00	1.02	0.72	0.59	0.51	0.45	1.25	0.88	0.72	0.62	0.56
1.50	0.85	0.60	0.49	0.42	0.38	1.11	0.79	0.64	0.56	0.50
1.75	0.65	0.46	0.38	0.33	0.29	0.97	0.68	0.56	0.48	0.43
Panel B: Relative average length of feasible CI's										
0.50	1.36	0.86	0.68	0.58	0.51	1.41	0.97	0.78	0.67	0.59
1.00	1.38	0.82	0.63	0.53	0.46	1.38	0.93	0.74	0.63	0.56
1.50	1.55	0.81	0.59	0.48	0.41	1.40	0.88	0.69	0.59	0.51
1.75	1.79	0.82	0.55	0.43	0.36	1.52	0.86	0.65	0.54	0.47

Note: In each simulation setting, we compute the average length of each nonrobust CI and report the length as its ratio with respect to the average length of the robust CI ($\mathrm{CI}^{\mathrm{Rbst}}$) in the same setting. We fix $k_n = 100$.

that the robust CI's are not much wider than their nonrobust counterparts. Given the size distortion of the nonrobust CI's, we consider the extra length of the robust CI's to be reasonable and likely necessary for achieving good coverage.

S.B.5. *Additional Robustness Checks*

We now examine the robustness of the performance of the bias-corrected estimator and the robust CI against (i) the choice of k'_n, (ii) the choice of the truncation level u_n, and (iii) the case with rounding effect. We remind the reader that k'_n and u_n are tuning parameters in the computation of the asymptotic variance estimator $\hat{\Sigma}_n$; hence they are only relevant for the CI's. For brevity, we fix $k_n = 100$. Previous findings in Section S.B.2 appear to be robust to these changes. The details are given below.

The Choice of k'_n

We set $k'_n = Ck_n$ for $C = 2, 3$, or 4 and keep other settings the same as in the baseline case; the baseline case corresponds to $C = 3$. As shown in Table S.VI, the perturbation on k'_n has only a mild effect on the coverage rate of $\mathrm{CI}^{\mathrm{Rbst}}$.

The Choice of Truncation Threshold u_n

We set $u_n = C(\bar{V}(Z, g_1, 2)^n_T/T)^{1/2}\Delta_n^{0.49}$ for $C = 4, 5$, or 6 and keep other settings the same as in the baseline case; the baseline case corresponds to $C = 5$. As shown in Table S.VII, the coverage rate of $\mathrm{CI}^{\mathrm{Rbst}}$ increases in u_n. In all cases, the perturbation on the truncation threshold affects the coverage rate by less than 4 percentage points.

Rounding

Finally, we consider pure rounding on the price level. That is the situation in which there is no additive noise and we only observe the price level

TABLE S.VI

FINITE-SAMPLE COVERAGE RATES (%) OF THE 95% NOMINAL LEVEL $\mathrm{CI}^{\mathrm{Rbst}}$ FOR VARIOUS VALUES OF k'_n

	JV/QV = 0.1			JV/QV = 0.2		
Activity	$C = 2$	$C = 3$	$C = 4$	$C = 2$	$C = 3$	$C = 4$
0.50	93.2	93.2	93.2	89.5	90.7	90.7
1.00	94.7	95.1	95.0	91.7	93.4	93.2
1.50	95.7	95.9	96.1	94.2	94.6	94.8
1.75	96.2	96.5	96.5	95.1	95.0	95.5

Note: We set $k'_n = Ck_n$ with $k_n = 100$. Other settings are the same as in the baseline case.

TABLE S.VII

FINITE-SAMPLE COVERAGE RATES (%) OF THE 95% NOMINAL LEVEL $\mathrm{CI}^{\mathrm{Rbst}}$ FOR VARIOUS TRUNCATION THRESHOLDS

	JV/QV = 0.1			JV/QV = 0.2		
Activity	$C=4$	$C=5$	$C=6$	$C=4$	$C=5$	$C=6$
0.50	89.5	93.2	96.5	89.3	90.7	93.2
1.00	92.5	95.1	97.2	90.6	93.4	95.3
1.50	94.2	95.9	96.9	92.2	94.6	96.2
1.75	94.9	96.5	96.8	93.7	95.0	96.5

Note: We set $u_n = C(\bar{V}(Z, g_1, 2)^n_T/T)^{1/2}\Delta_n^{0.49}$. Other settings are the same as in the baseline case.

rounded to the nearest multiple of the tick size. The pure rounding case serves as an interesting robustness check because it is not covered by Assumption N. As is typical in stock market applications, we set the tick size to be 1 cent and control the severity of rounding by varying the initial log price level $X_0 \in \{\log(10), \log(100)\}$; the lower the price, the larger the effect of rounding. We keep other settings the same as in the baseline case. Table S.VIII shows the median relative bias of $\hat{H}$ (panel A) and the coverage rate of $\mathrm{CI}^{\mathrm{Rbst}}$ (panel B). In all cases, the results are robust with respect to the price level, suggesting that pure rounding is not very important in our setting. Simulations with both additive noise and rounding on the price level deliver very similar results and, thus, are omitted for brevity.

TABLE S.VIII

MEDIAN RELATIVE BIAS AND FINITE-SAMPLE COVERAGE RATES (%) IN THE CASE WITH PURE ROUNDING

	JV/QV = 0.1		JV/QV = 0.2	
Activity	$X_0 = \log(100)$	$X_0 = \log(10)$	$X_0 = \log(100)$	$X_0 = \log(10)$
	Panel A: Median relative bias of the bias-corrected estimator			
0.50	−5.7	−5.5	−3.3	−3.3
1.00	−5.1	−5.4	−3.2	−3.6
1.50	−3.5	−3.4	−4.6	−4.1
1.75	−0.9	0.2	−4.7	−4.4
	Panel B: Finite-sample coverage of the 95% robust CI's			
0.50	94.5	94.2	91.5	91.4
1.00	95.7	95.6	93.3	93.3
1.50	96.4	96.4	94.8	94.8
1.75	96.8	96.7	95.9	95.6

REFERENCES

AÏT-SAHALIA, Y., AND J. JACOD (2012): "Analyzing the Spectrum of Asset Returns: Jump and Volatility Components in High Frequency Data," *Journal of Economic Literature*, 50 (4), 1007–1050. [44]

AÏT-SAHALIA, Y., J. JACOD, AND J. LI (2012): "Testing for Jumps in Noisy High Frequency Data," *Journal of Econometrics*, 168, 207–222. [1]

AÏT-SAHALIA, Y., AND R. KIMMEL (2007): "Maximum Likelihood Estimation of Stochastic Volatility Models," *Journal of Financial Economics*, 83, 413–452. [43]

JACOD, J. (2008): "Asymptotic Properties of Realized Power Variations and Related Functionals of Semimartingales," *Stochastic Processes and Their Applications*, 118, 517–559. [4,21]

JACOD, J., M. PODOLSKIJ, AND M. VETTER (2010): "Limit Theorems for Moving Averages of Discretized Processes Plus Noise," *The Annals of Statistics*, 38, 1478–1545. [1]

JACOD, J., AND P. PROTTER (2012): *Discretization of Processes*. Berlin: Springer. [3]

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