

SUPPLEMENT TO “THE REVEALED PREFERENCE THEORY OF STABLE AND EXTREMAL STABLE MATCHINGS”
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BY FEDERICO ECHENIQUE, SANGMOK LEE,
MATTHEW SHUM, AND M. BUMIN YENMEZ

PROOF OF THEOREM 1—SUFFICIENCY: To prove sufficiency, we explicitly construct an orientation d that satisfies for all (m_1, w_1) and (m_2, w_2) , if $x_{1,1} > 0$ and $x_{2,2} > 0$, then

$$(S1) \quad d_{m_1, w_2, w_1} d_{w_2, m_1, m_2} = 0 \quad \text{and} \quad d_{m_2, w_1, w_2} d_{w_1, m_2, m_1} = 0.$$

We then show that there is a rationalizing preference profile.

We first deal with the case where all vertices in X are connected and there is at most one minimal cycle. By decomposing an arbitrary X into connected components, we later generalize the argument. If there is no cycle in X , choose a singleton vertex and treat it as the “cycle” in the sequel.

Let C be the submatrix that has the indices in the minimal cycle. If $c = \langle (m, w)_n \rangle$ is the minimal cycle, let $M_1 = \bigcup_n \{m_n\}$ and $W_1 = \bigcup_n \{w_n\}$. Then C is the matrix $(x_{m', w'})_{(m', w') \in M_1 \times W_1}$. Thus C contains the minimal cycle.

We rearrange the indices of X to obtain a matrix of the form

$$\begin{array}{c|ccc} & (W_1) & (W_2) & (W_3) \\ \hline (M_1) & C & X_1 & O \quad \dots \\ (M_2) & Y_1 & O & X_2 \quad \dots \\ (M_3) & O & Y_2 & O \quad \dots \\ & \vdots & \vdots & \vdots \end{array}$$

where O represents submatrices consisting of only zeros.

We define the submatrices X_n and Y_n by induction. For $n \geq 1$, let

$$M_{n+1} = \left\{ m \notin \bigcup_1^n M_k : \exists w \in \bigcup_1^n W_k \text{ such that } (m, w) \in V \right\},$$

$$W_{n+1} = \left\{ w \notin \bigcup_1^n W_k : \exists m \in \bigcup_1^n M_k \text{ such that } (m, w) \in V \right\}.$$

Now, let X_n be the matrix $(x_{m', w'})_{(m', w') \in M_n \times W_{n+1}}$ and let Y_n be the matrix $(x_{m', w'})_{(m', w') \in M_{n+1} \times W_n}$. Finally, relabel the indices such that if $m_i \in M_n$, $m_{i'} \in M_{n'}$, and $n < n'$, then $i < i'$. The numbering of indexes in M_n is otherwise arbitrary. Relabel w 's in a similar fashion.

For every $m \in M_n$, there is a $k < n$ and $w \in W_k$ such that $(m, w) \in V$, and, similarly, for every $w \in W_n$, there is a $k < n$ and $m \in M_k$ such that $(m, w) \in V$. Thus, for $m \in M_n$, there is a sequence

$$(m, w_{k_0}), (m_{k_1}, w_{k_0}), \dots, (m_{k_N}, w_{k_{N'}})$$

with $N = N' + 1$ or $N' = N - 1$, which defines a path connecting (m, w_{k_0}) to the cycle c . Similarly, if $w \in W_n$, there is a path connecting (m_{k_0}, w) to c .

The observation in the previous paragraph has two consequences:

CLAIM S1: *If $m \in M_n$ and $w \in W_n$ ($n > 1$), then $(m, w) \notin V$.*

Claim S1 is true because otherwise there would be two different paths connecting (m, w) to c , one having (m, w_{k_0}) and the other having (m_{k_0}, w) as its second element. Then we would have a distinct second cycle.

CLAIM S2: *Let $m_i \in M_n$ ($n > 1$), and let there be two distinct w_j and $w_{j'}$ ($j' > j$) such that $(m_i, w_j), (m_i, w_{j'}) \in V$. Then $(m_{i'}, w_{j'}) \in V$ implies that $m_{i'} \in M_{n'}$ with $n' > n$.*

Claim S2 is true because otherwise we would again have two different paths connecting $(m_i, w_{j'})$ to c ; one path with (m_i, w_j) and one with $(m_{i'}, w_{j'})$ as its second element.

Define the orientation d as follows. We simply use $d_{i,j,j'}$ for $d_{m_i, w_j, w_{j'}}$, and $d_{j,i,i'}$ for $d_{w_j, m_i, m_{i'}}$.

DEFINITION 1: (i) If $(m_i, w_j) \in c$ and $(m_i, w_{j'}) \in c$, then define $d_{i,j,j'}$ to be 1 if (m_i, w_j) comes immediately after $(m_i, w_{j'})$ in c ; that is, $d_{i,j,j'} = 1$ if there is n such that

$$(m_i, w_{j'}) = (m_i, w_j)_n \quad \text{and} \quad (m_i, w_j) = (m_i, w_{j'})_{n+1}.$$

(ii) If $(m_i, w_j) \in c$ and $(m_{i'}, w_j) \in c$, then define $d_{j,i,i'}$ to be 1 if (m_i, w_j) comes immediately after $(m_{i'}, w_j)$ in c .

(iii) If $(m_i, w_j) \notin c$ and $(m_i, w_{j'}) \in c$, then define $d_{i,j,j'}$ to be 1.

(iv) If $(m_i, w_j) \notin c$ and $(m_{i'}, w_j) \in c$, then define $d_{j,i,i'}$ to be 1.

(v) If $(m_i, w_j) \notin c$ and $(m_i, w_{j'}) \notin c$, then define $d_{i,j,j'}$ to be 1 if and only if $j > j'$.

(vi) If $(m_i, w_j) \notin c$ and $(m_{i'}, w_j) \notin c$, then define $d_{j,i,i'}$ to be 1 if and only if $i > i'$.

Let $d_{i,j',j} = 0$ when (i)–(vi) imply that $d_{i,j,j'} = 1$; similarly, $d_{j,i',i} = 0$ when (i)–(vi) imply that $d_{j,i,i'} = 1$.

LEMMA S1: *If (m_i, w_j) is a vertex in c , then there is at most one $w_{j'}$ such that $j' \neq j$ and $(m_i, w_{j'}) \in c$; in addition, (m_i, w_j) and $(m_i, w_{j'})$ are adjacent in c .*

Similarly, there is at most one $i' \neq i$ such that $(m_{i'}, w_j) \in c$; in addition, (m_i, w_j) and $(m_{i'}, w_j)$ are adjacent in c .

PROOF: We let the index of c range over all the integers by denoting $(m, w)_{n \bmod(N)}$ by $(m, w)_n$.

Let (m, w) be a vertex in c and let $n > 0$ be such that $(m, w) = (m, w)_n$. Suppose there is w' such that $w' \neq w$ and $(m, w') \in c$. If it does not exist, we are done. Since now $N \geq 2$, (m, w) is in the minimal path connecting $(m, w)_{n-1}$ and $(m, w)_{n+1}$. By the second fact stated in the beginning of Appendix A.1, either $m_{n-1} = m$ or $m_{n+1} = m$, and exactly one of these is true. In the first case, we can set $w' = w_{n-1}$ and in the second case, we can set $w' = w_{n+1}$. Suppose, without loss of generality, that $w' = w_{n+1}$.

We show that there is not a $w'' \neq w, w'$ with $(m, w'') \in c$. Suppose that there is such a w'' . Let $(m, w'') = (m, w)_l$. By the second fact stated in the beginning of Appendix A.1, we have either $l < n - 1$ or $l > n + 1$. When $l > n + 1$, the path $\langle (m, w)_{n-1}, \dots, (m, w)_l \rangle$ is not minimal because $\langle (m, w)_{n-1}, (m, w)_n, (m, w)_m \rangle$ is a proper subset connecting $(m, w)_{n-1}$ and $(m, w)_m$. When $l < n - 1$, the path $\langle (m, w)_m, (m, w)_n, (m, w)_{n+1} \rangle$ is not minimal because $(m, w)_m$ and $(m, w)_{n+1}$ are directly connected. Thus c is not a minimal cycle—a contradiction. *Q.E.D.*

LEMMA S2: Let (m, w) be a vertex in c . If $(m, w') \in V$ is not a vertex in c , then for all $m' \neq m$, $(m', w') \notin c$. Similarly, if $(m', w) \in V$ is not a vertex in c , then for all $w' \neq w$, $(m', w') \notin c$.

PROOF: Suppose, for contradiction, that $(m, w) \in c$, $(m', w') \in c$ with $m \neq m'$, $w \neq w'$, and $(m, w') \notin c$. Since $(m, w), (m', w') \in c$, there is a minimal path $\langle (m, w)_n : n = 0, \dots, N \rangle$ connecting (m', w') to (m, w) . Then, since $(m, w') \notin c$, the minimal cycle

$$\langle (m, w)_0, \dots, (m, w)_N, (m, w'), (m', w') \rangle$$

is distinct from c and connected to c .

Q.E.D.

LEMMA S3: (i) If $d_{i,j,j'} = 1$ and $d_{i,j',j''} = 1$, then $d_{i,j,j''} = 1$.
 (ii) If $d_{j,i,i'} = 1$ and $d_{j,i',i''} = 1$, then $d_{j,i,i''} = 1$.

PROOF: We prove only the first statement. The second statement can be proved in similar fashion to the following first three cases.

First, we can rule out that $d_{i,j,j'} = 1$ because $(m_i, w_j) \in c$, $(m_i, w_{j'}) \in c$, and (m_i, w_j) comes immediately after $(m_i, w_{j'})$ in c (case (i), Definition 1). To see this, note that $d_{i,j',j''} = 1$ would imply that $(m_i, w_{j''}) \in c$, which is not possible by Lemma S1.

Second, suppose that $d_{i,j,j'} = 1$ is from the fact that $(m_i, w_j) \notin c$ and $(m_i, w_{j'}) \in c$. Then $d_{i,j',j''} = 1$ implies that $(m_i, w_{j''}) \in c$. Thus $d_{i,j,j''} = 1$ by case (iii) in Definition 1.

Third, suppose that $d_{i,j,j'} = 1$ is from the fact that $(m_i, w_j) \notin c$, $(m_i, w_{j'}) \notin c$, and $j > j'$. If $d_{i,j',j''} = 1$ because $(m_i, w_{j''}) \notin c$ and $j' > j''$, then $d_{i,j,j''} = 1$ by case (v) in Definition 1 and the transitivity of P . On the other hand, if $d_{i,j',j''} = 1$ because $(m_i, w_{j''}) \in c$, then $d_{i,j,j''} = 1$ (case (iii), Definition 1) as well. *Q.E.D.*

LEMMA S4: *The orientation d satisfies (S1).*

PROOF: Let $((m_i, w_j), (m_{i'}, w_{j'}))$ be an antiedge: so $(m_i, w_j), (m_{i'}, w_{j'}) \in V$, $j \neq j'$ and $i \neq i'$. Suppose that $d_{i,j',j} = 1$. We prove that $d_{j',i,i'} = 0$.

Suppose first that $d_{i,j',j} = 1$ because of case (i) in Definition 1. Then $(m_i, w_{j'}) \in c$. So, if $(m_{i'}, w_{j'}) \notin c$, we obtain that $d_{j',i,i'} = 0$ by case (iii) in Definition 1. On the other hand, if $(m_{i'}, w_{j'}) \in c$, then the edges $((m_i, w_j), (m_i, w_{j'}))$ and $((m_i, w_{j'}), (m_{i'}, w_{j'}))$ are in c . In fact, these edges must be consecutive or $(m_i, w_{j'})$ will appear twice in c . Then $d_{i,j',j} = 1$ because case (i) implies that $(m_i, w_{j'})$ comes immediately after (m_i, w_j) in c ; the edge $((m_i, w_{j'}), (m_{i'}, w_{j'}))$ comes after $((m_i, w_j), (m_i, w_{j'}))$ in c , so we obtain that $d_{j',i,i'} = 0$ by case (i) in Definition 1.

Suppose second that $d_{i,j',j} = 1$ because of case (iii) in Definition 1. So $(m_i, w_j) \in c$ and $(m_i, w_{j'}) \notin c$. Then $m_i \in M_1$ because m_i is an index for a vertex in the minimal cycle c . Now, by Lemma S2, there is no \tilde{m}_i with $(\tilde{m}_i, w_{j'}) \in c$. Since $(m_{i'}, w_{j'}) \in V$, we must have $m_{i'} \in M_n$ for $n > 1$. By the labeling we adopted, then, $i < i'$. Hence, $d_{j',i',i} = 1$ by case (vi) in Definition 1.

Third, suppose that $d_{i,j',j} = 1$ because of case (v) in Definition 1. If $m_i \in M_1$, there exists $w_{j''}$ such that $(m_i, w_{j''}) \in c$ and $d_{i,j',j''} = 1$ because of case (iii) in Definition 1, and $d_{j',i',i} = 1$ by the previous result. If $m_i \in M_n$ ($n > 1$), then we have shown in Claim S2 that $(m_{i'}, w_{j'}) \in V$ implies that $m_{i'} \in M_k$ with $k > n$. Hence $d_{j',i',i} = 1$ because of case (v) in Definition 1. *Q.E.D.*

Given the orientation d we have constructed, define two collections of partial orders, $(\tilde{P}_m : m \in M)$ and $(\tilde{P}_w : w \in W)$, where we say that $w\tilde{P}_mw'$ when $d_{m,w,w'} = 1$ and that $m\tilde{P}_mw'$ when $d_{w,m,m'} = 1$. By Lemma S3, these are well defined strict partial orders.

Now define the preference of man type m to be some complete strict extension of \tilde{P}_m to W and similarly for the women. By Lemma S4, these preferences rationalize the matching X .

The previous construction assumes that X has one minimal cycle. If X has more than one minimal cycle, these must not be connected in the graph. Therefore, if we partition the graph into connected components, there will be at most one minimal cycle in each.

In particular, we can partition the set of vertices V of X to be $V = V_1 \cup \dots \cup V_N$ and $V_m \cap V_n = \emptyset$. All vertices in each V_n are connected, but no pair of vertices in different sets is connected. The partition corresponds to the connected components of the graph.

Now relabel the indices of types such that matching X is a diagonal block matrix:

$$X = \begin{pmatrix} X_1 & O & \dots & O \\ O & X_2 & \dots & O \\ \vdots & \vdots & \dots & \vdots \\ O & O & \dots & X_N \end{pmatrix}.$$

All vertices in V_n are positive elements in X_n , and vice versa.

Let \tilde{M}_n (\tilde{W}_n) be the set of types m (w) of men (women) who have a positive elements $x_{m,w}$ in X_n . The previous construction, applied to each X_n separately, yields a rationalizing preference profile of each X_n , which we denote by \tilde{P} . For each $m \in \tilde{M}_n$, we define a partial order P_m on W to agree with \tilde{P}_m by adding relations that any $w \in \tilde{W}_n$ is preferred to every $w \in W \setminus \tilde{W}_n$. We similarly define partial orders for the other types of men and types of women. Subsequently, we define m 's preferences over W to be a complete extension of P_m . Women types' preferences are defined analogously.

The resulting profile of preferences rationalizes X because if (v, v') is an antiedge with $v, v' \in V_n$ for some n , then (S1) is satisfied by the previous construction of preferences, and if v and v' are in different components of the partition of V , then (S1) is satisfied because any agent ranks an index in their component over an index in a separate component. Q.E.D.

Division of the Humanities and Social Sciences, California Institute of Technology, Pasadena, CA 91125, U.S.A.; fede@caltech.edu,

Dept. of Economics, University of Pennsylvania, Philadelphia, PA 19104, U.S.A.; sangmok@sas.upenn.edu,

Division of the Humanities and Social Sciences, California Institute of Technology, Pasadena, CA 91125, U.S.A.; mshum@caltech.edu,

and

Tepper School of Business, Carnegie Mellon University, 5000 Forbes Ave., Pittsburgh, PA 15213, U.S.A.; byenmez@andrew.cmu.edu.

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