Econometrica Supplementary Material

SUPPLEMENT TO "MARKOV PERFECT INDUSTRY DYNAMICS WITH MANY FIRMS"—TECHNICAL APPENDIX

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A. PROOFS AND MATHEMATICAL ARGUMENTS FOR SECTION 4: LONG-RUN BEHAVIOR AND THE INVARIANT INDUSTRY DISTRIBUTION

LEMMA A.3: *Let Assumptions* 3.2 *and* 3.3 *hold. Assume that firms follow a common oblivious strategy* $\mu \in \tilde{\mathcal{M}}$, *the expected entry rate is* $\lambda \in \tilde{\Lambda}$, *and the expected time that each firm spends in the industry is finite. Let* $\{Z_x : x \in \mathbb{N}\}$ *be a sequence of independent Poisson random variables with means* $\{\tilde{s}_{\mu,\lambda}(x) : x \in \mathbb{N}\}$, *and let* Z *be a Poisson random variable with mean* $\sum_{x\in\mathbb{N}} \tilde{s}_{\mu,\lambda}(x)$. *Then*:

(a) $\{s_t : t \geq 0\}$ *is an irreducible, aperiodic, and positive recurrent Markov process*;

(b) *the invariant distribution of* s_t *is a product form of Poisson random variables*;

(c) *for all* x, $s_t(x) \Rightarrow Z_x$;

(d) $n_t \Rightarrow Z$.

PROOF: If every firm uses a strategy $\mu \in \tilde{\mathcal{M}}$ and entry is according to an entry rate function $\lambda \in \tilde{\Lambda}$, then $A = \{s_t : t \geq 0\}$ is clearly an irreducible Markov process. All states reach the state $\varnothing = \{0, 0, \ldots\}$ with positive probability and all states can be reached from $\varnothing$ as well. Moreover, state $\varnothing$ is aperiodic; hence, A is aperiodic. Finally, A is positive recurrent because the expected time to come back from state $\varnothing$ to itself is finite (Kleinrock (1975)).

Now, let us write

$$s_t(x) = \sum_{\tau=0}^{t} \sum_{i=1}^{W_\tau} \mathbf{1}_{\{X_{i,t-\tau}=x\}}, \tag{S.1}$$

where W_τ are i.i.d. Poisson random variables with mean λ, the first sum is taken over all periods previous to (and including) t, the second sum is taken over the firms that entered the industry in each period, and for each τ, $X_{i,t-\tau}$ are random variables that represent the state of firm i after $t - \tau$ periods inside the industry when using strategy μ. Since firms use oblivious strategy $\mu \in \tilde{\mathcal{M}}$ and shocks are idiosyncratic, their state evolutions are independent, so $\mathbf{1}_{\{X_{i,t-\tau}=x\}}$ are i.i.d. across i. It follows that $\sum_{i=1}^{W_\tau} \mathbf{1}_{\{X_{i,t-\tau}=x\}}$ is a filtered Poisson random variable, so it is a Poisson random variable. Thus $s_t(x)$, as a sum of independent Poisson random variables, is also Poisson. Given that the expected time a firm spends inside the industry is finite, using characteristic functions it is straightforward to show that $s_t(x) \Rightarrow Z_x$ $\forall x \in \mathbb{N}$. To show that $\{Z_x : x \in \mathbb{N}\}$ is a

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sequence of independent random variables, note that by the filtering property of Poisson random variables, for all t, $\{s_t(x) : x \in \mathbb{N}\}$ is a sequence of independent random variables (Durrett (1996)). By summing over $x \in \mathbb{N}$, we can show that $n_t \Rightarrow Z$. *Q.E.D.*

LEMMA A.4: *Let Assumptions* 3.2 *and* 3.3 *hold. Assume that firms follow a common oblivious strategy* $\mu \in \tilde{\mathcal{M}}$, *the expected entry rate is* $\lambda \in \tilde{\Lambda}$, *and the expected time that each firm spends in the industry is finite. Let* $\{Y_n : n \in \mathbb{N}\}$ *be a sequence of integer-valued i.i.d. random variables, each distributed according to* $\tilde{s}_{\mu,\lambda}(\cdot)/\sum_{x\in\mathbb{N}} \tilde{s}_{\mu,\lambda}(x)$. *Then, for all* $n \in \mathbb{N}$,

$$(x_{(1)t}, \dots, x_{(n_t)t} | n_t = n) \Rightarrow (Y_1, \dots, Y_n).$$

PROOF: The proof relies on a well known result for Poisson processes; conditional on n arrivals on an interval $[0, T]$, the unordered arrival times have the same distribution as n i.i.d. uniform random variables in $[0, T]$.

Let us condition on $n_t = n$. $\{x_{(j)t} : j = 1, \dots, n\}$ are the random variables that represent the state of each of the n firms in the industry when they are sampled randomly. The expected time a firm spends inside the industry is finite, so the time a firm spends inside the industry is finite with probability 1. A firm can increase its quality level by at most $\overline{w}$ states each period. Therefore, for all $\varepsilon > 0$, there exists a state z, such that, for all $j \in \{1, \dots, n\}$ and for all t, $\mathcal{P}[x_{(j)t} > z] < \frac{\varepsilon}{n}$. Hence, $\mathcal{P}[\bigcup_{j=1}^{n}\{x_{(j)t} > z\}|n_t = n] < \varepsilon$, for all t, so the sequence of random vectors $\{(x_{(1)t}, \dots, x_{(n_t)t}|n_t = n) : t \geq 0\}$ is tight. By Theorem 9.1 in Durrett (1996) and tightness, to prove the lemma it is enough to show that for all n, for all $(z_1, \dots, z_n)$,

$$\lim_{t\to\infty} \mathcal{P}[x_{(j)t} = z_j, j = 1, \dots, n|n_t = n] = \prod_{j=1}^{n} p(z_j),$$

where $p(\cdot)$ is the probability mass function (pmf) $\tilde{s}_{\mu,\lambda}(\cdot)/\sum_{x\in\mathbb{N}} \tilde{s}_{\mu,\lambda}(x)$. Let $\tilde{T}_j$ be the entry time period for firm (j) and let $T_j = t - \tilde{T}_j$ be its age. Then we can write

$$\text{(S.2)} \quad \begin{aligned} &\mathcal{P}[x_{(j)t} = z_j,\ j = 1, \dots, n|n_t = n] \\ &\quad = \sum_{\substack{0\leq t_1<\infty,\dots,\\ 0\leq t_n<\infty}} \mathcal{P}[x_{(j)t} = z_j,\ j = 1, \dots, n|T_1 = t_1, \dots, T_n = t_n, n_t = n] \\ &\qquad \times \mathcal{P}[T_1 = t_1, \dots, T_n = t_n|n_t = n] \\ &\quad = \sum_{\substack{0\leq t_1<\infty,\dots,\\ 0\leq t_n<\infty}} \prod_{j=1}^{n} \mathcal{P}[x_{(j)t} = z_j|T_j = t_j] \end{aligned}$$

$$\times \mathcal{P}[T_1 = t_1, \ldots, T_n = t_n | n_t = n].$$

The last equation follows because the evolution of firms is independent across firms. Note that if any t_j has a value greater than t, then $\mathcal{P}[T_1 = t_1, \ldots, T_n = t_n | n_t = n] = 0$. We can write

$$\text{(S.3)} \qquad \begin{aligned} \mathcal{P}[x_{(j)t} = z_j | T_j = t_j] &= \frac{\mathcal{P}[x_{(j)t} = z_j,\ T_j = t_j]}{\mathcal{P}[T_j = t_j]} \\ &= \frac{\mathcal{P}[T_j = t_j,\ X_{j,t_j} = z_j]}{\mathcal{P}[T_j = t_j]} \\ &= \frac{\mathcal{P}[T_j = t_j]\mathcal{P}[X_{j,t_j} = z_j]}{\mathcal{P}[T_j = t_j]} \\ &= \mathcal{P}[X_{j,t_j} = z_j], \end{aligned}$$

where X_{j,t_j} is a random variable that represents a firm's state after t_j periods, conditional on having stayed in the industry. Note that for all k, $\{X_{j,k} : j \geq 1\}$ are i.i.d. The second to last equation follows because the evolution of a firm is independent of its entry time.

Now we show that

$$\lim_{t\to\infty} \mathcal{P}[T_1 = t_1, \ldots, T_n = t_n | n_t = n] = \prod_{j=1}^{n} u[t_j]$$

for some pmf u. We derive this equation by invoking the relationship between n_t and a Poisson process.

Similarly to equation (S.1), we can write

$$n_t = \sum_{\tau=0}^{t} \sum_{i=1}^{W_\tau} A_{i,t-\tau},$$

where $A_{i,t-\tau}$ are i.i.d. Bernoulli random variables that equal one if the firm is still in the industry after $t - \tau$ periods when using strategy μ and zero otherwise. Since $A_{i,t-\tau}$ are i.i.d., $n_{t,\tau} = \sum_{i=1}^{W_\tau} A_{i,t-\tau}$ is a filtered Poisson random variable and is therefore Poisson. Let us denote its mean by $\alpha_{t,\tau}$. It follows that n_t is a sum of independent Poisson random variables, so it is Poisson with mean $\sum_{\tau=0}^{t} \alpha_{t,\tau}$.

Consider $\{N(t) : t \geq 0\}$, a homogeneous Poisson process on the real line with rate 1. Note that $N(t)$ and n_t are equivalent in the sense that we can construct n_t using the process $\{N(s) : 0 \leq s \leq \sum_{\tau=0}^{t} \alpha_{t,\tau}\}$. For each $0 \leq \tau \leq t$, with some abuse of notation, let $N(\alpha_{t,\tau-1},\ \alpha_{t,\tau-1} + \alpha_{t,\tau})$ be the total number of events of the Poisson process in the interval $[\alpha_{t,\tau-1},\ \alpha_{t,\tau-1} + \alpha_{t,\tau}]$, where $\alpha_{t,-1} = 0$. Then we can construct $n_t = \sum_{\tau=0}^{t} n_{t,\tau}$ by defining $n_{t,\tau} = N(\alpha_{t,\tau-1},\ \alpha_{t,\tau-1} + \alpha_{t,\tau})$ for all τ.

Now, conditional on the event $N(\sum_{\tau=0}^{t} \alpha_{t,\tau}) = n$, the unordered arrival times of $N(t)$ have the same distribution as n i.i.d. uniform random variables in $[0, \sum_{\tau=0}^{t} \alpha_{t,\tau}]$ (Durrett (1996)). By the equivalence argument described above, conditional on $n_t = n$, the unordered arrival times of the n firms are i.i.d. discrete random variables with pmf:

$$v_t(\tau) = \frac{\alpha_{t,\tau}}{\sum_{j=0}^{t} \alpha_{t,j}}, \quad 0 \le \tau \le t.$$

Recall that $\alpha_{t,\tau}$ is the expected number of firms that entered at time τ and are still inside the industry at time t. Since the entry rate is oblivious, all firms use the same oblivious strategy and shocks are idiosyncratic, $\alpha_{t,\tau} = \tilde{\alpha}_{t-\tau}$, where $\tilde{\alpha}_{t-\tau}$ is the expected number of firms that entered the industry at time s, for any s, and are still inside the industry at time $s + t - \tau$. This suggests making a change of variable and defining

$$u_t(k) = \frac{\tilde{\alpha}_k}{\sum_{j=0}^{t} \tilde{\alpha}_j}, \quad 0 \le k \le t.$$

$u_t(k)$ is the probability a random sampled firm from the industry at time t entered k periods ago, conditional on $n_t = n$. Taking the limit as t tends to infinity, we get that

$$\lim_{t\to\infty} u_t(k) = u(k) = \frac{\tilde{\alpha}_k}{\sum_{j=0}^{\infty} \tilde{\alpha}_j}, \quad 0 \le k < \infty,$$

provided that $\lim_{t\to\infty} E[n_t] = \sum_{j=0}^{\infty} \tilde{\alpha}_j < \infty$, which is true because the expected time that each firm spends in the industry is finite. $u(k)$ is the probability a random sampled firm, while the industry state is distributed according to its invariant distribution, entered k periods before the sampling period. Therefore,

$$\lim_{t\to\infty} \mathcal{P}[T_1 = t_1, \ldots, T_n = t_n | n_t = n] = \prod_{j=1}^{n} u[t_j].$$

Replacing the previous equation together with equation (S.3) into equation (S.2) we obtain

$$\lim_{t\to\infty} \mathcal{P}[x_{(j)t} = z_j, j = 1, \ldots, n | n_t = n] = \prod_{j=1}^{n} \sum_{0 \le t < \infty} \mathcal{P}[X_{j,t} = z_j] u(t),$$

where the interchange between the infinite sum and the limit follows by the dominated convergence theorem. The sum yields the pmf $p(\cdot)$. The previous equation proves that, for all $n \in \mathbb{N}$, $(x_{(1)t}, \ldots, x_{(n_t)t} | n_t = n) \Rightarrow (Y_1, \ldots, Y_n)$, where $Y_1, \ldots, Y_n$ are i.i.d. random variables with pmf $p(\cdot)$ which does not depend on n.

To finish, consider a very large time period. Formally, suppose that s_0 is sampled from the invariant distribution of $\{s_t : t \geq 0\}$ (which is well defined by Lemma A.3). In this case, s_t is a stationary process; s_t is distributed according to the invariant distribution for all $t \geq 0$:

$$\tilde{s}_{\mu,\lambda}(x) = E[s_t(x)] = E\left[\sum_{j=1}^{n_t} \mathbf{1}_{\{x_{(j)t}=x\}}\right].$$

Conditioning on n_t and considering that we already proved that $\{x_{(j)t} : j = 1, \ldots, n\}$ are i.i.d. with pmf $p(\cdot)$, we conclude that $p(\cdot) = \tilde{s}_{\mu,\lambda}(\cdot)/\sum_{x\in\mathbb{N}} \tilde{s}_{\mu,\lambda}(x)$. *Q.E.D.*

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