*Econometrica Supplementary Material*

# SUPPLEMENT TO "SEMIPARAMETRIC POWER ENVELOPES FOR TESTS OF THE UNIT ROOT HYPOTHESIS"

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BY MICHAEL JANSSON

PROOF OF LEMMA 2: Suppose $f$ satisfies Assumption DQM.

The result $\ell_f \in \mathcal{L}_f$ follows from standard arguments. Specifically, $E[\ell_f(\varepsilon)] = 0$ and $E[\ell_f(\varepsilon)^2] < \infty$ by van der Vaart (2002, Lemma 1.8). Furthermore, using van der Vaart (2002, Example 1.15), the property $E[\varepsilon \ell_f(\varepsilon)] = 1$ can be deduced from the fact that the functional $\int_{-\infty}^{\infty} f(\varepsilon - \theta)\, d\varepsilon = \theta$ is differentiable in the ordinary sense and the sense of van der Vaart (2002, Definition 1.14). Finally, by the Cauchy–Schwarz inequality, $E[\ell_f(\varepsilon)^2] \geq E[\varepsilon^2]/E[\varepsilon \ell_f(\varepsilon)]^2 = 1$.

To establish the locally asymptotically quadratic (LAQ) property, let $c_T$ be a bounded sequence. The log likelihood ratio $L_T^f(c_T)$ admits the expansion

$$L_T^f(c_T) = \frac{c_T}{T} \sum_{t=2}^{T} y_{t-1} \ell_f(\Delta y_t) + \sum_{t=2}^{T} R_{Tt} - \frac{1}{4} \sum_{t=2}^{T} \left[ \frac{c_T}{T} y_{t-1} \ell_f(\Delta y_t) + R_{Tt} \right]^2 (1 + \beta_{Tt}),$$

where $R_{Tt} := R_f(\Delta y_t, c_T y_{t-1}/T)$, $\beta_{Tt} := \beta[c_T y_{t-1} \ell_f(\Delta y_t)/T + R_{Tt}]$, and the defining properties of $R_f(\cdot)$ and $\beta(\cdot)$ are

$$\sqrt{\frac{f(\varepsilon - \theta)}{f(\varepsilon)}} = 1 + \frac{1}{2} \theta \ell_f(\varepsilon) + \frac{1}{2} R_f(\varepsilon, \theta),$$

$$\log(1 + r) = r - \frac{1}{2} r^2 [1 + \beta(2r)].$$

The proof of Lemma 2 will be completed by showing that

$$\text{(S1)} \qquad \sum_{t=2}^{T} R_{Tt} = -\frac{1}{4} c_T^2 \frac{\mathcal{I}_{ff}}{T^2} \sum_{t=2}^{T} y_{t-1}^2 + o_{p_{0,f}}(1),$$

$$\text{(S2)} \qquad \sum_{t=2}^{T} \left[ \frac{c_T}{T} y_{t-1} \ell_f(\Delta y_t) + R_{Tt} \right]^2 (1 + \beta_{Tt}) = c_T^2 \frac{\mathcal{I}_{ff}}{T^2} \sum_{t=2}^{T} y_{t-1}^2 + o_{p_{0,f}}(1).$$

In the rest of the proof, suppose $H_0$ holds and let $\vartheta_T$ be any positive sequence for which $\vartheta_T \to 0$ and $\sqrt{T}\vartheta_T \to \infty$ (as $T \to \infty$).

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*Equation* (S1). Let $\tilde{R}_{Tt} := 1(|c_T y_{t-1}/T| \leq \vartheta_T) R_{Tt}$ denote a truncated version of $R_{Tt}$. Because $\max_{2\leq t\leq T} |c_T y_{t-1}/\sqrt{T}| = O_p(1)$ and $\sqrt{T}\vartheta_T \to \infty$, the sequences $\tilde{R}_{Tt}$ and $R_{Tt}$ are asymptotically equivalent in the sense that $\sum_{t=2}^T R_{Tt} = \sum_{t=2}^T \tilde{R}_{Tt} + o_p(1)$.

Now

$$E_{t-1}(\tilde{R}_{Tt}^2) = 1\left(\left|\frac{c_T}{T} y_{t-1}\right| \leq \vartheta_T\right) E_{t-1}\left[R_f\left(\varepsilon_t, \frac{c_T}{T} y_{t-1}\right)^2\right]$$
$$\leq V_f(\vartheta_T) \frac{c_T^2}{T^2} y_{t-1}^2,$$

where $V_f(\vartheta) := \sup_{|\theta|\leq\vartheta, \theta\neq 0} \theta^{-2} E[R_f(\varepsilon, \theta)^2]$ and $E_{t-1}[\cdot]$ denotes conditional expectation given $\{\varepsilon_1, \ldots, \varepsilon_{t-1}\}$. By Assumption DQM, $\lim_{\vartheta\downarrow 0} V_f(\vartheta) = 0$. As a consequence, using $\vartheta_T = o(1)$ and $E(y_{t-1}^2) = t-1$,

$$\sum_{t=2}^T E_{t-1}(\tilde{R}_{Tt}^2) \leq V_f(\vartheta_T) E\left(\frac{c_T^2}{T^2} \sum_{t=2}^T y_{t-1}^2\right) = V_f(\vartheta_T) O(1) = o(1),$$

implying that $\sum_{t=2}^T \tilde{R}_{Tt} = \sum_{t=2}^T E_{t-1}(\tilde{R}_{Tt}) + o_p(1)$. Moreover,

$$\sum_{t=2}^T E_{t-1}(\tilde{R}_{Tt}) = -\frac{1}{4}\mathcal{I}_{ff} \frac{c_T^2}{T^2} \sum_{t=2}^T 1\left(\left|\frac{c_T}{T} y_{t-1}\right| \leq \vartheta_T\right) y_{t-1}^2$$
$$+ \sum_{t=2}^T 1\left(\left|\frac{c_T}{T} y_{t-1}\right| \leq \vartheta_T\right) r_f\left(\frac{c_T}{T} y_{t-1}\right),$$

where $r_f(\theta) := \frac{1}{4}\mathcal{I}_{ff}\theta^2 + E[R_f(\varepsilon, \theta)]$ and

$$\frac{1}{T^2} \sum_{t=2}^T 1\left(\left|\frac{c_T}{T} y_{t-1}\right| \leq \vartheta_T\right) y_{t-1}^2 = \frac{1}{T^2} \sum_{t=2}^T y_{t-1}^2 + o_p(1)$$

because $\max_{2\leq t\leq T} |c_T y_{t-1}/\sqrt{T}| = O_p(1)$ and $\sqrt{T}\vartheta_T \to \infty$. The proof of (S1) can therefore be completed by showing that

$$\sum_{t=2}^T 1\left(\left|\frac{c_T}{T} y_{t-1}\right| \leq \vartheta_T\right) r_f\left(\frac{c_T}{T} y_{t-1}\right) = o_p(1).$$

The relationship in the preceding display follows from $\vartheta_T = o(1)$ and the fact that

$$\left|\sum_{t=2}^T 1\left(\left|\frac{c_T}{T} y_{t-1}\right| \leq \vartheta_T\right) r^f\left(\frac{c_T}{T} y_{t-1}\right)\right| \leq v_f(\vartheta_T) \frac{c_T^2}{T^2} \sum_{t=2}^T y_{t-1}^2$$

$$= v_f(\vartheta_T)O_p(1),$$

where $v_f(\vartheta) := \sup_{|\theta|\le\vartheta,\theta\neq 0}\theta^{-2}|r_f(\theta)| = o(1)$ as $\vartheta \downarrow 0$ (Pollard (1997, Lemma 1)).

*Equation* (S2). To prove (S2), it suffices to show that

$$\sum_{t=2}^{T}\left[\frac{c_T}{T}y_{t-1}\ell_f(\varepsilon_t)+R_{Tt}\right]^2 = c_T^2\frac{\mathcal{I}_{ff}}{T^2}\sum_{t=2}^{T}y_{t-1}^2 + o_p(1)$$

and

$$\max_{2\le t\le T}\left|\beta[c_T T^{-1}y_{t-1}\ell_f(\varepsilon_t)+R_{Tt}]\right| = o_p(1).$$

By Taylor's theorem, $\beta(r)\to 0$ as $|r|\to 0$. Moreover,

$$\max_{2\le t\le T}\left|\frac{y_{t-1}}{T}\ell_f(\varepsilon_t)\right| \le \max_{2\le t\le T}\left|\frac{y_{t-1}}{\sqrt{T}}\right|\max_{2\le t\le T}\left|\frac{\ell_f(\varepsilon_t)}{\sqrt{T}}\right| = O_p(1)o_p(1) = o_p(1)$$

and $\max_{2\le t\le T}|R_{Tt}| \le \sqrt{\sum_{t=2}^{T}R_{Tt}^2}$. Therefore, the desired result will follow from

$$\text{(S3)} \qquad \frac{1}{T^2}\sum_{t=2}^{T}y_{t-1}^2\ell_f(\varepsilon_t)^2 = \frac{\mathcal{I}_{ff}}{T^2}\sum_{t=2}^{T}y_{t-1}^2 + o_p(1)$$

and

$$\text{(S4)} \qquad \sum_{t=2}^{T}R_{Tt}^2 = o_p(1).$$

As noted by Jeganathan (1995, Lemma 24), (S3) can be deduced with the help of the proof of Hall and Heyde (1980, Theorem 2.23) if it can be shown that

$$\frac{1}{T^2}\sum_{t=2}^{T}E_{t-1}\left[y_{t-1}^2\ell_f(\varepsilon_t)^2 1\left(\left|\frac{y_{t-1}}{T}\ell_f(\varepsilon_t)\right| > \varrho\right)\right] = o_p(1) \quad \forall \varrho > 0.$$

To do so, let $\varrho > 0$ be given and define $Q_f(r) := E[\ell_f(\varepsilon)^2 1(|\ell_f(\varepsilon)| > r)]$. Because $Q_f$ is nonincreasing and $\lim_{r\to\infty} Q_f(r) = 0$,

$$\frac{1}{T^2}\sum_{t=2}^{T}E_{t-1}\left[y_{t-1}^2\ell_f(\varepsilon_t)^2 1\left(\left|\frac{y_{t-1}}{T}\ell_f(\varepsilon_t)\right| > \varrho\right)\right]$$

$$= \frac{1}{T^2}\sum_{t=2}^{T}y_{t-1}^2 Q_f\left(\frac{\sqrt{T}\varrho}{|y_{t-1}/\sqrt{T}|}\right)$$

$$\leq \left(\frac{1}{T^2}\sum_{t=2}^{T} y_{t-1}^2\right) \max_{2\leq t\leq T} Q_f\left(\frac{\sqrt{T}\varrho}{|y_{t-1}/\sqrt{T}|}\right)$$
$$= O_p(1)o_p(1) = o_p(1),$$

where the penultimate equality uses $\max_{2\leq t\leq T} |y_{t-1}/\sqrt{T}| = O_p(1)$.

It can be shown that $\sum_{t=2}^{T} R_{Tt}^2 = \sum_{t=2}^{T} \tilde{R}_{Tt}^2 + o_p(1)$. Moreover,

$$\sum_{t=2}^{T} E_{t-1}\big[\tilde{R}_{Tt}^2 1(|\tilde{R}_{Tt}| > \varrho)\big] \leq \sum_{t=2}^{T} E_{t-1}(\tilde{R}_{Tt}^2) = o_p(1) \quad \forall \varrho > 0,$$

where the equality was established in the proof of (S1). A second application of the proof of Hall and Heyde (1980, Theorem 2.23) therefore establishes (S4). *Q.E.D.*

PROOF OF LEMMA 7: For any $b$, any $c < 0$, any $\alpha \in (0,1)$, and any symmetric $2\times 2$ matrix $\mathcal{I}_F$ for which

$$\operatorname{Var}\begin{pmatrix} W(1) \\ B_F(1) \end{pmatrix} = \begin{pmatrix} 1 & e_1' \\ e_1 & \mathcal{I}_F \end{pmatrix}$$

is positive semidefinite, let $K_\alpha^S(b, c; \mathcal{I}_F)$ be the $1-\alpha$ quantile of

$$G(W, Z, b, c; \mathcal{I}_F)$$
$$:= c\left[\int_0^1 W(r)\,dW(r) + \frac{\mathcal{H}_{f\eta}}{\mathcal{H}_{\eta\eta}} b + \sqrt{\mathcal{H}_{ff.\eta} - \int_0^1 W(r)^2\,dr}\, Z\right]$$
$$- \frac{1}{2}c^2\mathcal{H}_{ff},$$

where $Z \sim \mathcal{N}(0,1)$ is independent of $W$ and $\mathcal{H}_{f\eta}$, $\mathcal{H}_{\eta\eta}$, etc. are as in Section 4.

The function $K_\alpha^S$ satisfies $E[\psi_F^S(\mathcal{S}_F, \mathcal{H}_F|c,\alpha)|\mathcal{S}_\eta] = \alpha$ because it follows from elementary facts about Brownian motions that

$$\frac{\mathcal{S}_{f.\eta} - \int_0^1 W(r)\,dW(r)}{\sqrt{\mathcal{H}_{ff.\eta} - \int_0^1 W(r)^2\,dr}} \sim \mathcal{N}(0,1)$$

independent of $W$ and $\mathcal{S}_\eta$, where $\mathcal{S}_{f.\eta}$ and $\mathcal{S}_\eta$ are as in Section 4.

Continuity of $K_\alpha^S$ follows from the fact that $G(W, Z, b_n, c_n; \mathcal{I}_{F,n})$ converges in distribution to a continuous random variable whenever the sequence $(b_n, c_n, \mathcal{I}_{F,n})$ is convergent (and $G(W, Z, b_n, c_n; \mathcal{I}_{F,n})$ is well defined for each $n$). *Q.E.D.*

PROOF OF (27): Let $f \in \mathcal{F}_{\mathrm{DQM}}$ and $c < 0$ be given, suppose $F$ satisfies Assumption DQM*, and let $(S_T^F, H_T^F)$, $(W, B_f, B_\eta)$, etc. be as in Section 4. Because $K_\alpha^S$ is continuous (Lemma 7) and

$$(S_T^f, H_T^{ff}, S_T^{f,S}, S_T^\eta) \to_{d_{0,f}} (\mathcal{S}_f, \mathcal{H}_{ff}, \mathcal{S}_f^S, \mathcal{S}_\eta),$$

the sequence $\phi_{f,T}^S(\cdot|c,\alpha)$ satisfies

$$\phi_{f,T}^S(Y_T|c,\alpha) \to_{d_{0,f}} \psi_f^S(\mathcal{S}_f, \mathcal{H}_{ff}, \mathcal{S}_f^S|c,\alpha).$$

It follows from these convergence results, Le Cam's third lemma, and the result

$$L_T^F(c,h) \to_{d_{0,f}} \Lambda_F(c,h) := (c,h)\mathcal{S}_F - \frac{1}{2}(c,h)\mathcal{H}_F(c,h)' \quad \forall (c,h)$$

that

$$\begin{aligned}&\lim_{T\to\infty} E_{\rho_T(c'),\eta_T(h)} \phi_T^S(Y_T|c,\alpha;f)\\ &\quad = E\big[\psi_f^S(\mathcal{S}_f, \mathcal{H}_{ff}, \mathcal{S}_f^S|c,\alpha)\exp(\Lambda_F(c',h))\big]\end{aligned}$$

for every $(c',h)$. In particular, $\lim_{T\to\infty} E_{\rho_T(c),\eta_T(0)} \phi_T^S(Y_T|c,\alpha;f) = \Psi_f^S(c,\alpha)$, implying that the proof of (27) can be completed by showing that $\phi_{f,T}^S(\cdot|c,\alpha)$ is locally asymptotically $\alpha$-similar in $F$.

To do so, it suffices to show that $E[\psi_f^S(\mathcal{S}_f, \mathcal{H}_{ff}, \mathcal{S}_f^S|c,\alpha)|\mathcal{S}_\eta] = \alpha$. Let

$$\mathcal{S}_\eta^\perp := \mathcal{S}_\eta - \frac{\mathcal{I}_{f\eta}}{\mathcal{I}_{ff}-1}\mathcal{S}_f^S.$$

Because $B_\eta - \mathcal{I}_{f\eta}(\mathcal{I}_{ff}-1)^{-1}(B_f - W)$ and $(W, B_f)$ are independent, $\mathcal{S}_\eta^\perp$ is independent of $(\mathcal{S}_f, \mathcal{H}_{ff}, \mathcal{S}_f^S)$ and

$$E[\psi_f^S(\mathcal{S}_f, \mathcal{H}_{ff}, \mathcal{S}_f^S|c,\alpha)|\mathcal{S}_f^S, \mathcal{S}_\eta^\perp] = E[\psi_f^S(\mathcal{S}_f, \mathcal{H}_{ff}, \mathcal{S}_f^S|c,\alpha)|\mathcal{S}_f^S] = \alpha,$$

where the second equality is the defining property of $K_\alpha^S$. Because $\mathcal{S}_\eta$ is a function of $(\mathcal{S}_f^S, \mathcal{S}_\eta^\perp)$, it therefore follows from the law of iterated expectations that

$$\begin{aligned}E[\psi_f^S(\mathcal{S}_f, \mathcal{H}_{ff}, \mathcal{S}_f^S|c,\alpha)|\mathcal{S}_\eta] &= E\big(E[\psi_f^S(\mathcal{S}_f, \mathcal{H}_{ff}, \mathcal{S}_f^S|c,\alpha)|\mathcal{S}_f^S, \mathcal{S}_\eta^\perp]\big|\mathcal{S}_\eta\big)\\ &= \alpha,\end{aligned}$$

as was to be shown. *Q.E.D.*

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*Dept. of Economics, University of California at Berkeley, 508-1 Evans Hall, 3880, Berkeley, CA 94720-3880, U.S.A. and Center for Research in Econometric Analysis of Time Series (CREATES), School of Economics and Management, University of Aarhus, DK-8000 Aarhus C, Denmark; mjansson@econ.berkeley.edu.*

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