SUPPLEMENT TO "ADMISSIBILITY IN GAMES": APPENDICES

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Section S.1 continues the discussion in the main text of the definition of assumption and of the relationship between assumption and the concepts of (i) infinitely more likely than (Blume, Brandenburger, and Dekel (1991), henceforth BBD) and (ii) strong belief (Battigalli and Siniscalchi (2002), henceforth B-S). The focus is on the behavior of the assumption concept in the game, theoretic setting. Section S.2 examines our negative result (Theorem 10.1). It is impossible (under certain conditions) for rationality and common assumption of rationality (RCAR) to hold in a complete structure based on lexicographic probability systems (our Theorem 10.1). But it is possible for rationality and common strong belief of rationality (RCSBR) to hold in a complete structure based on conditional probability systems (Proposition 6 in B-S). What accounts for the difference? Finally, Section S.3 expands on the coverage of the literature in the main text.

S.1. ASSUMPTION

IN THE CONTEXT of a finite space and a full-support LPS σ, BBD showed that E is infinitely more likely than not-E if and only if E satisfies conditions (i) and (ii) of Proposition 5.1 in the main text. Here, we show that, for a general space, conditions (i) and (ii) do not suffice for this result. Specifically, we can have a full-support lexicographic probability system (LPS) σ and an event E so that conditions (i) and (ii) are satisfied, but not-E is infinitely more likely than part of E. We will also argue that if we ask only for conditions (i) and (ii) of assumption, we can end up with a conceptually unsatisfactory game analysis.

Our discussion will focus on the game in Figure 1. Fix type spaces $T^a = \{t^a\}$ and $T^b = [0, 1]$. Let $\lambda^a(t^a) = (\mu_0, \mu_1)$, where μ_0 is uniform on $\{L\} \times [0, 1]$ and μ_1 is uniform on $\{C, R\} \times [0, 1]$. Let $\lambda^b(0)$ be a one-level LPS that assigns probability $\frac{1}{2}$ to each of (U, t^a) and (D, t^a). For each $t^b \in (0, 1]$, let $\lambda^b(t^b)$ be a one-level LPS that assigns probability $\frac{1}{3}$ to (U, t^a) and probability $\frac{2}{3}$ to (D, t^a). Then

$$R_1^a = \{U, D\} \times T^a,$$
$$R_1^b = (\{L\} \times T^b) \cup \{(C, 0)\}.$$

Provisionally, take assumption to mean only conditions (i) and (ii). Note that $\mu_0(R_1^b) = 1$ and $\mu_1(R_1^b) = 0$. Therefore, under our provisional definition of assumption, we get $R_2^a = R_1^a$ and $R_2^b = R_1^b$, and so, by induction, $R_m^a = R_1^a$ and $R_m^b = R_1^b$ for all m. We conclude that if the analyst's prediction is RCAR, then she predicts $\{U, D\} \times \{L, C\}$—that is, the projection of $\bigcap_{m=1}^{\infty} R_m^a \times \bigcap_{m=1}^{\infty} R_m^b$ into $S^a \times S^b$.

But now ask: What if Ann steps into the analyst's shoes? Specifically, let us now focus on the strategies that can be played (i.e., on the space S^b) and

		Bob		
		L	C	R
Ann	U	2, 1	2, 2	0, 0
	D	2, 1	0, 0	2, 0

FIGURE 1.

imagine that the analyst's prediction is available to Ann. What happens if the measures in an initial segment of her sequence of hypotheses each assign probability 1 to $\{L, C\}$? Then Ann cannot rationally play D. Yet the prediction is that she can play D.

So Ann's reasoning fails a natural consistency check, namely, that if she steps into the analyst's shoes (in the above sense), then her answer should not change.

To see what is missing, go back to the epistemic analysis. Type t^a is confident that Bob is rational, in the sense that the LPS associated with t^a satisfies conditions (i) and (ii) of assumption for the event R_1^b. But while the choice C is consistent with the event that Bob is rational, type t^a never considers the possibility that Bob is rational and plays C. (Formally, the event $R_1^b \cap [\{C\} \times T^b] = \{(C, 0)\}$ receives probability 0 under both μ_0 and μ_1.) Indeed, Ann is more confident that Bob is irrational than that Bob is rational and plays C. This is different from what the consistency check gave: There, because Ann is confident of her prediction $\{L, C\}$, she must be more confident that "Bob plays in accordance with the prediction and, in particular, plays C" than that "Bob violates his prediction and plays R."

The missing requirement is this: If Ann is to assume an event E, and U is a "significant" event with $U \cap E \neq \varnothing$, then we should require that Ann considers $U \cap E$ possible. What are the significant events U? If Ann has a full-support LPS, the natural answer is that these are the (nonempty) open events. So the consistency check corresponds exactly to condition (iii) of assumption.

Without this consistency check—that is, without condition (iii)—the event that Bob is irrational is considered infinitely more likely than part of the R_1^b (namely $\{(C, 0)\}$). This is different from the idea that R_1^b be considered infinitely more likely than not-R_1^b. (In particular, then, it does not accord with BBD's original idea for what the term "infinitely more likely than" means.)

Also, without this consistency check, Ann does not strongly believe (in the sense of B-S) the event that Bob is rational. To see this, first use the fact that the LPS $\lambda^a(t^a)$ induces a conditional probability system (CPS) where the conditioning events are the nonempty open sets U. (See Brandenburger, Friedenberg, and Keisler (2006) for details.) Next, consider the conditioning event $\{C\} \times T^b$. Under the construction, we would have that $R_1^b \cap [\{C\} \times T^b] \neq \varnothing$

even though the associated CPS assigns probability 0 to R_1^b given the conditioning event $\{C\} \times T^b$. This says that the associated CPS does not strongly believe R_1^b.

Finally, we note that this example also indicates the need for condition (a) in Definition 5.1. Without (a), $\{(C, 0)\}$ is infinitely more likely than not-R_1^b under $\lambda^a(t^a)$, and not-R_1^b is infinitely more likely than $\{(C, 0)\}$ under $\lambda^a(t^a)$. This does not make sense.

S.2. THE NEGATIVE RESULT

Theorem 10.1 says that if Ann is not indifferent (Definition 10.1), and the type structure is continuous (Definition 7.8) and complete, then RCAR is impossible. By contrast, Proposition 6 in B-S says that RCSBR is possible in a complete structure. We now explore the source of the difference between the two cases.

Here is an overview of the comparison. We will cast both cases in abstract CPS-based probability-theoretic terms. From this we will see that the key distinction is in the choice of conditioning events on which the CPS is built.

First, the B-S analysis. If RCSBR is to hold, Ann must strongly believe each of a decreasing sequence of compact events E_m. That is, there must be a CPS under which each E_m is strongly believed. The conditioning events correspond to the information sets in the tree—and are therefore closed (we will use this below).

Next, for our analysis of admissibility, the natural family of conditioning events to consider is the family of all nonempty open sets. Call a CPS where these are the conditioning events an *open* CPS. In Brandenburger, Friedenberg, and Keisler (2006), we showed that for every full-support LPS σ there is a corresponding open CPS p such that every event which is assumed under σ is strongly believed under p. If RCAR is to hold, Ann must assume each of a (different) decreasing sequence of events E_m. That is, there must be an LPS under which each E_m is assumed. It follows that for RCAR to hold there must be an open CPS which strongly believes each E_m.

Below, we will give a simple property $(*)$ which holds for a family of conditioning events if and only if there exists a CPS which strongly believes each event E_m.

Property $(*)$ will be satisfied when each conditioning event is closed and each event E_m is compact. This explains the positive result of B-S.

But property $(*)$ will fail if each open set is a conditioning event and the closures of the events E_m are strictly decreasing. So in this case, there cannot be an open CPS which strongly believes each E_m. This explains the negative result of our Theorem 10.1.

To begin the formal treatment, let Ω be a Polish space and let $\mathcal{B}(\Omega)$ be the Borel σ-algebra on Ω.

DEFINITION S.1: Fix a family $\mathcal{F}$ of nonempty events in $\mathcal{B}(\Omega)$ (the family of conditioning events). A *conditional probability system* (*CPS*) on $(\Omega, \mathcal{B}(\Omega), \mathcal{F})$ is a map $p : \mathcal{B}(\Omega) \times \mathcal{F} \to [0,1]$ so that:

(i) for all $F \in \mathcal{F}$, $p(F|F) = 1$;

(ii) for all $F \in \mathcal{F}$, $p(\cdot|F)$ is a probability measure on $(\Omega, B(\Omega))$;

(iii) for all $E \in \mathcal{B}(\Omega)$ and $F, G \in \mathcal{F}$, if $E \subseteq F \subseteq G$, then $p(E|G) = p(E|F)p(F|G)$.

DEFINITION S.2—B-S: Fix a CPS p on $(\Omega, \mathcal{B}(\Omega), \mathcal{F})$ and an event E in $\mathcal{B}(\Omega)$. Say *p strongly believes E* if, for all $F \in \mathcal{F}$, $F \cap E \neq \varnothing$ implies $p(E|F) = 1$.

Fix a decreasing sequence of nonempty events E_m in $\mathcal{B}(\Omega)$. We will add the event $E_0 = \Omega$ to the beginning of the sequence $E_1, E_2, \ldots$. (The event E_0 may or may not be distinct from E_1.) We now give our criterion for the existence of a CPS which strongly believes each event E_m.

THEOREM S.1: *Let $\mathcal{F}$ be a family of nonempty events in $\mathcal{B}(\Omega)$. Then the following statements are equivalent*:

($*$) *For each $F \in \mathcal{F}$, either F meets the intersection $\bigcap_n E_n$ or there is a greatest integer m such that F meets E_m.*

($**$) *There exists a CPS p on $(\Omega, \mathcal{B}(\Omega), \mathcal{F})$ which strongly believes each event E_m.*

PROOF: Suppose first that ($**$) holds and let p be a CPS on $(\Omega, \mathcal{B}(\Omega), \mathcal{F})$ which strongly believes each event E_m. Fix $F \in \mathcal{F}$. Suppose there is no greatest m such that F meets E_m. Since $E_0 = \Omega$, F meets E_0. Therefore, F meets each E_m. By strong belief, $p(E_m|F) = 1$ for each E_m. Therefore, $p(\bigcap_n E_n|F) = 1$ and hence F meets $\bigcap_n E_n$. This proves ($*$).

Now assume ($*$). Let $D_m = E_m \backslash E_{m+1}$ and $D_\infty = \bigcap_n E_n$. The sets D_m, $m \leq \infty$, form a partition of Ω. Therefore, each $F \in \mathcal{F}$ meets some D_m. By ($*$), if F meets D_m for infinitely many m, then F meets D_∞. Therefore, there is a greatest $m_F \leq \infty$ such that F meets m_F. Well order each of the sets $D_m, m \leq \infty$ and let x_F be the first element of $F \cap D_{m_F}$ under this ordering. Define $p : \mathcal{B}(\Omega) \times \mathcal{F} \to [0,1]$ by

$$p(E|F) = \begin{cases} 1, & \text{if } x_F \in E, \\ 0, & \text{otherwise.} \end{cases}$$

Since $x_F \in F$ for each $F \in \mathcal{F}$, condition (i) of Definition S.1 holds. Condition (ii) is easily checked, since $p(\cdot|F)$ is a point mass at x_F. For condition (iii), let $E \subseteq F \subseteq G$ with $E \in \mathcal{B}(\Omega)$ and $F, G \in \mathcal{F}$. Suppose first that $x_F = x_G$. Then $p(E|F) = p(E|G)$. Moreover, $x_G \in F$, so $p(F|G) = 1$. Condition (iii) now follows.

Next suppose that $x_F \neq x_G$. Since $x_F \in F \subseteq G$, either $m_F = m_G$ and x_G is earlier than x_F in the well ordering of D_{m_F} or $m_F < m_G$. In either case, x_G cannot belong to F. Therefore, $p(F|G) = 0$. Since $E \subseteq F$, we also have $p(E|G) = 0$. Thus condition (iii) holds with both sides equal to 0.

This shows that p is a CPS on $(\Omega, \mathcal{B}(\Omega), \mathcal{F})$. It remains to prove that p strongly believes E_m for each finite m. Suppose that $F \in \mathcal{F}, m < \infty$, and F meets E_m. Then F meets D_n for some $n \geq m$, so $m_F \geq m$ and $x_F \in D_{m_F} \subseteq E_m$. Therefore, $p(E_m|F) = 1$. This shows that p strongly believes E_m. *Q.E.D.*

We first apply this theorem to the positive result in B-S.

COROLLARY S.1: *If each $F \in \mathcal{F}$ is closed and each event E_m is compact, then there exists a CPS p on $(\Omega, \mathcal{B}(\Omega), \mathcal{F})$ which strongly believes each event E_m.*

PROOF: Let $F \in \mathcal{F}$ and suppose there is no greatest m such that F meets E_m. But F meets $E_0 = \Omega$, so $F \cap E_m$ is nonempty for each m. Since F is closed, $F \cap E_m$ is compact for each m. Therefore $\bigcap_n (F \cap E_n) = F \cap \bigcap_n E_n$ is nonempty, so $(*)$ holds. By Theorem S.1, $(**)$ holds. *Q.E.D.*

B-S (p. 373) showed (in a complete continuous structure) that, for each m, the set of strategy–type pairs for Ann (resp. Bob) that satisfy rationality and mth-order strong belief of rationality is nonempty closed. Our E_m events correspond to these events. In fact, the type spaces in B-S are compact, so each E_m is compact. The conditioning events correspond to the information sets in the tree. They are clopen (therefore closed). We can now apply Corollary S.1 to get a CPS which strongly believes "rationality and mth-order strong belief of rationality" for all m. This is a way to see how B-S get a positive result on RCSBR in a complete structure (their Proposition 6).

Note that Corollary S.1 fails if the hypothesis that each E_m is compact is replaced by the weaker hypothesis that each E_m is closed. We illustrate this with an extreme example.

EXAMPLE S.1: Suppose that $\mathcal{F}$ contains at least the maximal event Ω and that E_m is a decreasing sequence of nonempty closed sets such that $\bigcap_n E_n$ is empty. Then no CPS p on $(\Omega, \mathcal{B}(\Omega), \mathcal{F})$ strongly believes each event E_m. To see this, note that if a CPS p strongly believed each E_m, we would have $p(E_m|\Omega) = 1$ for each m and hence $p(\bigcap_n E_n|\Omega) = 1$, which is impossible if $\bigcap_n E_n$ is empty.

We now turn to our negative result, Theorem 10.1.

COROLLARY S.2: *Suppose $E_m \backslash \overline{E_{m+1}} \neq \varnothing$, for each m (where the bar denotes closure). Let $\mathcal{F}$ be the family of nonempty open events in Ω. Then there does not exist a CPS p on $(\Omega, \mathcal{B}(\Omega), \mathcal{F})$ which strongly believes each event E_m.*

PROOF: Let $F = \Omega \setminus \bigcap_n \overline{E_n}$. Then $F \in \mathcal{F}$. By hypothesis, for each m, there is a point $x_m \in E_m \setminus \overline{E_{m+1}}$. Since $\bigcap_n \overline{E_n} \subseteq \overline{E_{m+1}}$ we have $x_m \in F$, and F meets E_m. But $\bigcap_n E_n \subseteq \bigcap_n \overline{E_n}$, so F does not meet $\bigcap_n E_n$. Thus property $(*)$ fails, so $(**)$ fails. *Q.E.D.*

In the main text, the E_m events are the R_m^b events (the RmAR sets for Bob). The hypothesis of Corollary S.2 is established via Lemmas F.1 and F.2. In Brandenburger, Friedenberg, and Keisler (2006), we showed that if there is no open CPS which strongly believes an event E_m, then there is no LPS which assumes E_m. So Corollary S.2 is an abstract probability-theoretic result which implies our negative result (Theorem 10.1).

S.3. RELATED LITERATURE

Finally, we discuss some other related papers.

A. *Consistent Pairs*: We noted in the main text that Samuelson (1992) pointed out the fundamental inclusion-exclusion challenge in this area. To understand Samuelson's solution (different from ours), go back to the game of Figure 2.1. Suppose an analysis yields the answer that Ann plays (only) U. Then Samuelson requires Ann to include all of Bob's strategies that are optimal with respect to U, so Ann must include both L and R. Turning to Bob, by the same principle he should then include D for Ann, since D could be optimal with respect to $\{L, R\}$. Contradiction. What if the answer includes Ann's playing D? But then Bob must play L (admissibility). From this, Ann will play U, so D is excluded, not included. Another contradiction.

More generally, Samuelson formalized a condition of "common knowledge of admissibility" (CKA) and showed that a "consistent pair" (Börgers and Samuelson (1992))—another weak dominance analog to a Pearce best-response set—is always consistent with CKA.[1] Consistent pairs may or may not exist. In particular, Figure 2.1 is a game where no consistent pair exists. By contrast, self-admissible sets (SAS's) always exist: the iteratively admissible (IA) set is an SAS. In Figure 2.1, $\{(U, L)\}$ is the (unique) SAS. The reason for the difference is that while we also require Ann to include R, she can consider R infinitely less likely than L, in which case only U (and not D) is optimal. A consistent pair (when it exists) may contain inadmissible strategies and so may not be an SAS. If it contains only admissible strategies, it is an SAS.

B. *Other Routes to IA*: Other papers provide foundations for IA. Stahl (1995) used LPS's and supposed that Ann considers Bob's strategy s^b infinitely less likely than his strategy r^b, if s^b is eliminated on an earlier round of IA than

[1] He gave a mixed analog to consistent pairs as defined in Börgers and Samuelson (1992). We are stating his result for the case of pure strategies.

r^b. We want this condition as an output, not an input, of our analysis. For us, the crucial ingredient, to get IA, is completeness. We saw that without this we get an SAS, not the IA set.

Ewerhart (2002) also provided epistemic conditions for IA. His conditions use provability (in the sense of mathematical logic). In his model, Ann assigns probability 0 to a strategy of Bob's if and only if it is not provable that it is possible that Bob plays that strategy. In effect, Ann eliminates a strategy of Bob's unless it is provable that it should not be eliminated. The philosophy in Ewerhart seems almost opposite to ours. Ewerhart's players are "aggressive" (his terminology) in eliminating strategies of the other player. Our players are, in a sense, cautious in eliminating strategies of the other player, since no strategy is ever entirely ruled out. Ewerhart showed that if the players follow his rule and if they work in a self-referential system such as Peano Arithmetic, then they will choose IA strategies. Completeness does not appear to play a role in Ewerhart's analysis.

C. *IA vs. Properness*: Kohlberg and Mertens (1986, p. 1009) explained that forward induction and properness are distinct principles. Note that the former finds its modern epistemic expression in the form of complete type structures. So, arguably, IA captures forward-induction reasoning. In this sense, our analysis is quite different from the properness route.

An example of the properness route is Asheim (2001), who provided epistemic conditions for proper rationalizability (Schuhmacher (1999))—a non-equilibrium analog to proper equilibrium and a refinement of $S^{\infty}W$. Asheim supposed that Ann has a full-support LPS and believes the relevant events. He further asked that Ann's LPS "respect preferences"—that is, if Bob strictly (lexicographically) prefers (s^b, t^b) to (r^b, t^b), then Ann should consider (s^b, t^b) infinitely more likely than (r^b, t^b). Go back to the game of Figure 2.7 and the type structure of Figure 2.8. Bob's LPS there does not respect preferences. Given her type t^a, Ann strictly prefers D to M, but Bob considers (M, t^a) infinitely more likely than (D, t^a). As we saw, Bob does assume Ann is rational. Both (M, t^a) and (D, t^a) are irrational, and considered infinitely less likely than the rational pair (U, t^a). Our route to overturning the (U, L) answer was different—we did so by rationalizing D (but not M) by adding more types (à la completeness). To repeat, the understanding that there are these two different routes goes back to Kohlberg and Mertens.

Asheim (2001, Proposition 2) said that, in a two-player game, if (σ^a, σ^b) is a proper equilibrium, then $\mathrm{Supp}\,\sigma^a \times \mathrm{Supp}\,\sigma^b$ is proper rationalizable. Thus, the proper rationalizable profiles need not be contained in the IA set. Asheim also gave an example where the IA profiles are strictly contained in the proper rationalizable profiles. We do not know if this relationship must hold more generally.

D. *Other Solution Concepts*: Asheim and Dufwenberg (2003) defined another refinement of $S^{\infty}W$, called a fully permissible set. An SAS (in fact, the

IA set) may be disjoint from every fully permissible set. Likewise, a fully permissible set need not be an SAS (so, in particular, need not be the IA set). Naturally, since the fully permissible set concept is distinct from SAS/IA, so is its epistemic characterization. For more on the epistemic analysis of fully permissible sets, see Asheim and Dufwenberg (2003).

Two final solution concepts: Adapting Ewerhart's (1998) definition of a modified consistent pair to pure vs. mixed strategies, we have that a modified consistent pair is an SAS, but not conversely. We can have an SAS which is not a (tight) CURB (closed under rational behavior) set (Basu and Weibull (1991)) and a (tight) CURB set which is not an SAS.

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