SUPPLEMENT TO "FOUNDATIONS OF INTRINSIC HABIT FORMATION": APPENDIX C

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This supplement contains additional results that combine with Lemmas 1–22 in Appendix A to prove Theorem 2, giving monotonicity properties of the period utility under Axiom GM.[1]

C.1. ADDITIONAL RESULTS FOR SUFFICIENCY

LEMMA 23: $\forall T \in \mathbb{N}$, $\bar{x} = (x_0, x_1, \dots, x_T) \in \mathbb{R}^{T+1}$, $\exists h_{\bar{x},T} \in H$ *such that*

$$(x_0, x_1, \dots, x_T, 0, 0, \dots) \in C^*_{h_{\bar{x},T+1}}.$$

PROOF: For arbitrary h, define c^h by $c_0^h = x_0 + \varphi(h)$, $c_t^h = x_t + \varphi(hc_0^h c_1^h \cdots c_{t-1}^h)$ for all $1 \le t \le T$, and $c_t^h = \varphi(hc_0^h c_1^h \cdots c_{t-1}^h)$ for $t > T$. φ is strictly increasing, so we may choose $h_{\bar{x},T} \in H$ sufficiently large so that $(c_0^{h_{\bar{x},T}}, c_1^{h_{\bar{x},T}}, \dots, c_T^{h_{\bar{x},T}})$ is nonnegative. But if $(c_0^{h_{\bar{x},T}}, c_1^{h_{\bar{x},T}}, \dots, c_T^{h_{\bar{x},T}})$ is nonnegative, then so is ${}^{T+1}c^{h_{\bar{x},T}}$. Moreover, the stream is ultimately weakly decreasing. Therefore, $c^{h_{\bar{x},T}} \in C$. *Q.E.D.*

LEMMA 24: *Under Axiom* GM, *the period utility u is an increasing function*.

PROOF: Suppose u is not increasing. Because it is continuous, there exist some $x \in \mathbb{R}$ and $\alpha > 0$ such that $\forall \alpha' \in (0, \alpha]$, $u(x + \alpha') < u(x)$.

Let T be arbitrary for the moment. Note that by Lemma 23 there is h' such that $(x, x, \dots, x, 0, 0, \dots) \in C^*_{h'}$ (where x is repeated $T + 1$ times). Again by Lemma 23, there is h'' such that $(x + \alpha, x, x, \dots, x, 0, 0, \dots) \in C^*_{h''}$ (where x by itself is repeated T times). Let $h \ge h', h''$ and recall that the $C^*_{\hat{h}}$ are nested. Using the representation for $\succeq^*$ and the fact that $u(x + \alpha) < u(x)$,

$$u(x) + \sum_{t=1}^{T} \delta^t u(x) + \sum_{t=T+1}^{\infty} \delta^t u(0) > u(x+\alpha) + \sum_{t=1}^{T} \delta^t u(x) + \sum_{t=T+1}^{\infty} \delta^t u(0). \tag{27}$$

Since $(x, x, \dots, x, 0, 0, \dots) \in C^*_h$, there is $c \in C$ with $g(h, c) = (x, x, \dots, x, 0, 0, \dots)$. Clearly $c + \alpha \in C$ and, by GM, we know $c + \alpha \succ_h c$. Moreover, $g(h,$

[1]To ease cross-referencing, we continue the enumeration of results and equations from the main paper.

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$c+\alpha$) is

$$\left(x+\alpha, x+\alpha(1-\lambda_1), \dots, \right. \\ \left. x+\alpha\left(1-\sum_{k=1}^{T}\lambda_k\right), \alpha\left(1-\sum_{k=1}^{T+1}\lambda_k\right), \alpha\left(1-\sum_{k=1}^{T+2}\lambda_k\right), \dots\right), \tag{28}$$

where x appears $T+1$ times. Therefore, by the representation theorem for $\succeq^*$,

$$\begin{aligned} & u(x+\alpha)+\sum_{t=1}^{T}\delta^t u\left(x+\alpha\left(1-\sum_{k=1}^{t}\lambda_k\right)\right) \\ & \quad + \sum_{t=T+1}^{\infty}\delta^t u\left(\alpha\left(1-\sum_{k=1}^{t}\lambda_k\right)\right) \\ & > \sum_{t=0}^{T}\delta^t u(x)+\sum_{t=T+1}^{\infty}\delta^t u(0). \end{aligned} \tag{29}$$

Combine the right-hand side of (27) and the left-hand side of (29), and rearrange by subtracting the right-hand side of (27). This obtains

$$\begin{aligned} & \sum_{t=1}^{T}\delta^t\left[u\left(x+\alpha\left(1-\sum_{k=1}^{t}\lambda_k\right)\right)-u(x)\right] \\ & \quad + \sum_{t=T+1}^{\infty}\delta^t\left[u\left(\alpha\left(1-\sum_{k=1}^{t}\lambda_k\right)\right)-u(0)\right] \\ & > u(x)-u(x+\alpha), \end{aligned} \tag{30}$$

which is strictly positive. Since each $\lambda_k > 0$ and $\sum_{k=1}^{\infty}\lambda_k \le 1$, we know that the value $\alpha(1-\sum_{k=1}^{t}\lambda_k) \in [0,\alpha)$ for every t and is, in fact, strictly positive as $t<\infty$. The assumption that u dips below $u(x)$ just to the right of x implies that

$$\sum_{t=1}^{T}\delta^t\left[u\left(x+\alpha\left(1-\sum_{k=1}^{t}\lambda_k\right)\right)-u(x)\right]<0.$$

This sum decreases in T. By continuity, u is bounded on $[0,\alpha]$. Choose T large enough so that $\sum_{t=T+1}^{\infty}\delta^t[u(\alpha(1-\sum_{k=1}^{t}\lambda_k))-u(0)]$ is small enough to bring about the contradiction $0>0$ from (30). This is possible because Lemma 23 permits us to find h large enough so that the constructed streams are in C^*. *Q.E.D.*

LEMMA 25: *Assume Axiom* GM. *If* $\sum_{k=1}^{\infty}\lambda_k < 1$, *then* $u(\cdot)$ *is strictly increasing on* $(0,\infty)$, *and if* $\sum_{k=1}^{\infty}\lambda_k = 1$, *then there is* a *with* $0 < a \le \infty$ *such that* $u(\cdot)$ *is strictly increasing either on* $(-a,\infty)$ *or on* $(-\infty,a)$.

PROOF: By Lemma 24 we know that $u(\cdot)$, is an increasing function. To prove it is strictly increasing on the relevant ranges, we will consider the two cases separately.

Case (i)—$\sum_{k=1}^{\infty}\lambda_k = 1$. First we will show that $u(\cdot)$ is strictly increasing in some interval around 0. To complete the proof, we will show that there cannot exist $x > 0 > y$ such that $u(\cdot)$ does not increase strictly at both x and y. To see the first point, take any $q > 0$ and let $h = (\dots, q, q)$ and $c = (q, q, \dots)$. Then $g(h, c) = (0, 0, \dots)$ and for small α, both $c + \alpha \succ_h c$ and $c \succ_h c - \alpha$ by Axiom GM. Using the representation for $\succeq^*$, we get

$$\sum_{t=0}^{\infty}\delta^t u\left(\alpha\left(1-\sum_{k=1}^{t}\lambda_k\right)\right) > \sum_{t=0}^{\infty}\delta^t u(0) > \sum_{t=0}^{\infty}\delta^t u\left(-\alpha\left(1-\sum_{k=1}^{t}\lambda_k\right)\right).$$

By monotonicity of $u(\cdot)$, it must be that $u(\cdot)$, increases strictly in a neighborhood of 0. For the second point, suppose by contradiction that there exist $x > 0 > y$ such that $u(\cdot)$ does not increase strictly at both x and y. By continuity and monotonicity of $u(\cdot)$ there is $\alpha > 0$ such that $u(\cdot)$ is constant on $(x, x+\alpha)$ and on $(y, y+\alpha)$. Without loss of generality suppose that x, y are rational (else take some rational x, y inside the interval). Since x, y are rational there exist m, n such that $mx = -ny$. Let $c^* = (x^m, y^n, x^m, y^n, \dots)$ (i.e., x is repeated m times, then y is repeated n times, etc.). Because the compensating streams are constant, we may use the characterization (25) in Lemma 21 to find $h \in H$ large enough so that there is $c \in C$ satisfying $g(h, c) = c^*$. Observe by GM that $c + \alpha/2 \succ_h c$, a contradiction to the assumption that $u(\cdot)$ is constant on $(x, x+\alpha)$ and $(y, y+\alpha)$.

Case (ii)—$\sum_{k=1}^{\infty}\lambda_k < 1$. In this case, for any $q \in Q$, if we set $h = (\dots, q, q)$ and $c = (q, q, \dots)$, then

$$g(h, q) = \left(q\left[1-\sum_{k=1}^{\infty}\lambda_k\right], q\left[1-\sum_{k=1}^{\infty}\lambda_k\right], \dots\right).$$

As q is arbitrary, for any $x \ge 0$, $(x, x, x, \dots) \in C^*$. Suppose to the contrary that $u(\cdot)$ is not increasing from the right at x. Since $u(\cdot)$ is continuous and weakly increasing, this implies that there exists some $\beta^+ > 0$ such that for every $0 < \beta \le \beta^+$, $u(x+\beta) = u(x)$. Take h, c such that $g(h, c) = (x, x, x, \dots)$. By GM, $c + \beta \succ_h c$. Then the representation says that

$$\sum_{t=0}^{\infty}\delta^t u\left(x+\beta\left(1-\sum_{k=1}^{t}\lambda_k\right)\right) > \sum_{t=0}^{\infty}\delta^t u(x).$$

Since $0 < \beta \le \beta^+$ and $\sum_{k=1}^{t} \lambda_k < 1$, $u(x + \beta(1 - \sum_{k=1}^{t} \lambda_k)) = u(x)$ for every $t \ge 0$, we have a contradiction. *Q.E.D.*

C.2. ADDITIONAL RESULTS FOR NECESSITY

LEMMA 26: *Suppose that* $\sum_{k=1}^{\infty} \lambda_k < 1$. *Then, the following situations exist*:

(i) *For any* $\gamma > 0$, *there are no* $c \in C$, $h \in H$ *such that* $c \ge (\gamma, \gamma, \dots)$ *and* $g(h, c) \le (0, 0, \dots)$.

(ii) *For any* $\gamma < 0$, *there are no* $c \in C$, $h \in H$ *such that* $g(h, c) \le (\gamma, \gamma, \dots)$.

PROOF: To see (i), we first note that if $g(h, c) \le (0, 0, \dots)$, then $c_0 \le \varphi(h)$, $c_1 \le \varphi(hc_0)$, $c_2 \le \varphi(hc_0c_1)$, and so forth. Using the monotonicity of φ and recursive substitution, we see that $c_1 \le \varphi(h\varphi(h))$, $c_2 \le \varphi(h\varphi(h)\varphi(h\varphi(h)))$, and so forth. But by Lemma 8, the compensating streams $(\varphi(h), \varphi(h\varphi(h)), \varphi(h\varphi(h)\varphi(h\varphi(h))), \dots)$ tend to zero.

Similarly, to see (ii), note that if $g(h, c) \le (\gamma, \gamma, \dots)$, then $c_0 \le \varphi(h) + \gamma$, $c_1 \le \varphi(hc_0) + \gamma \le \varphi(h\varphi(h)) + \lambda_1\gamma + \gamma$. But since $\gamma < 0$, we may drop the term $\lambda_1\gamma$ to obtain $c_1 \le \varphi(h\varphi(h)) + \gamma$. In this manner, $c_2 \le \varphi(h\varphi(h)\varphi(h\varphi(h))) + \gamma$ and so on. The stream $(\varphi(h), \varphi(h\varphi(h)), \varphi(h\varphi(h)\varphi(h\varphi(h))), \dots)$ tends to zero asymptotically and $\gamma < 0$ is fixed, implying c is eventually negative, a contradiction. *Q.E.D.*

When $\sum_{k=0}^{\infty} \lambda_k < 1$, part (i) in Lemma 26 means the argument of u cannot always be strictly negative when the consumption stream is bounded from zero (we cannot shift down a stream using GM to conclude u is increasing in the negative range). Part (ii) means the argument of u cannot be bounded below zero (we cannot shift up a stream using GM to conclude u is increasing in the negative range). It suffices that u is sensitive on the nonnegative domain to satisfy GM. To see why it suffices that for some $0 < a \le \infty$, u is only strictly increasing either on $(-\infty, a)$ or $(-a, \infty)$ when $\sum_{k=0}^{\infty} \lambda_k = 1$, use Lemma 20. By (25), there cannot exist h and c such that $g(h, c)$ is always positive and bounded from zero (c would be unbounded) or always negative and bounded from zero (c would violate nonnegativity).

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