Online appendix

C Results on convergence to the limit

C.1 Model with fixed labor

Proposition 8. *Suppose labor is inflexible, so that each sector's production function is still*

$$Y_i = Z_i L_i^{1-\alpha} \left(\sum_j A_{i,j}^{1/\sigma_i} X_{i,j}^{(\sigma_i - 1)/\sigma_i} \right)^{\alpha\sigma_i/(\sigma_i - 1)} \tag{66}$$

but L_i is no longer a choice variable. Then the leading term of the asymptotic expansions for prices and GDP remains unchanged from Lemma 3.1 and Theorem 1:

$$\lim_{t\to\infty} p_i(\theta t)/t = \phi_i(\theta) \tag{67}$$

$$\lim_{t\to\infty} gdp(\theta t)/t = \lambda(\theta) \tag{68}$$

$$\textit{where } \lambda(\theta) = -\beta'\phi(\theta) \tag{69}$$

where ϕ_i is defined as in equation (7).

Proof. In addition to the claims in the proposition itself, we also prove the further results that

$$\lim_{t\to\infty} \frac{y_i}{t} = \lim_{t\to\infty} \frac{c_i}{t} = -\phi_i \tag{70}$$

(now suppressing the θ for convenience).

Normalizing nominal GDP to 1 (which affects only equation (73)), the equilibrium conditions are

$$Y_i = \exp(z_i) L_i^{1-\alpha} \left(\sum_j A_{i,j}^{1/\sigma_i} X_{i,j}^{(\sigma_i - 1)/\sigma_i} \right)^{\alpha\sigma_i/(\sigma_i - 1)} \tag{71}$$

$$Y_j = C_j + \sum_i X_{i,j} \tag{72}$$

$$\beta_j = P_j C_j \tag{73}$$

$$P_j = \alpha P_i \exp(z_i) L_i^{1-\alpha} \left(Y_i / \left(\exp(z_i) L_i^{1-\alpha} \right) \right)^{(\alpha - (\sigma_i - 1)/\sigma_i)/\alpha} A_{i,j}^{1/\sigma_i} X_{i,j}^{-1/\sigma_i} \tag{74}$$

We first prove some small lemmas. Define

$$f_i(\phi) = \begin{cases} \max_{j \in S_i} \phi_j \text{ if } \sigma_i < 1 \\ \sum_j A_{i,j} \phi_j \text{ if } \sigma_i = 1 \\ \min_{j \in S_i} \phi_j \text{ if } \sigma_i > 1 \end{cases} \tag{75}$$

Lemma C1. *$f_i(\phi) + \sigma_i(\phi_j - f_i(\phi)) \geq \phi_j$ for all $j \in S(i)$*

Proof. Trivial algebra. ■

Note that f_i is defined over arbitrary vectors. Consider a vector $\hat{\phi}_i$ with jth element equal to $f_i(\phi) + \sigma_i(\phi_j - f_i(\phi))$.

Lemma C2.

$$f_i\left(\hat{\phi}\right) = f_i(\phi) \tag{76}$$

Proof. This follows from the quasi-linearity of f_i, where for scalars a and b, $f_i(a\phi + b) = af_i(\phi) + b$. In the case of this lemma, $a = \sigma_i$ and $b = (1 - \sigma_i) f_i(\phi)$, so that

$$f_i\left(\hat{\phi}_i\right) = \sigma_i f_i(\phi) + (1 - \sigma_i) f_i(\phi) \tag{77}$$

$$= f_i(\phi) \tag{78}$$

■

To prove the proposition, we also need the use of inputs. We guess that

$$\lim_{t \to \infty} \frac{x_{i,j}}{t} = -f_i(\phi) - \sigma_i[\phi_j - f_i(\phi)] \tag{79}$$

We need to verify that the above, along with the solution in the proposition, satisfies, in the limit, the equilibrium conditions (71)-(74).

We first take limits of the equilibrium conditions. For any variable g_j, define

$$\phi_{g,j} \equiv \lim_{t \to \infty} \frac{g_j}{t} \tag{80}$$

Taking logs of the equilibrium conditions (equations (71)-(74), respectively) and dividing by t and taking limits as $t \to \infty$ yields

$$\phi_{y,i} = \theta_i + \alpha f_i\left([\phi_{x,i,j}]\right) \tag{81}$$

$$\phi_{y,j} = \max\left\{\phi_{c,j}, \max_i \phi_{x,i,j}\right\} \tag{82}$$

$$0 = \phi_{p,j} + \phi_{c,j} \tag{83}$$

$$\phi_{p,j} = \phi_{p,i} + \theta_i + \frac{\alpha - (\sigma_i - 1)/\sigma_i}{\alpha}\left(\phi_{y,i} - \theta_i\right) - \sigma_i^{-1}\phi_{x,i,j} \tag{84}$$

where $[\phi_{x,i,j}]$ is a vector with jth element equal to $\phi_{x,i,j}$.

Equation (81) holds by applying Lemma C2 to $f_i\left([\phi_{x,i,j}]\right)$. Equation (82) holds using the guesses and Lemma C1. Equations (83) and (84) hold trivially after inserting the various guesses.

■

Intuitively, the result here simply says that productivity eventually dominates reallocation of inputs. That idea already underlies the main results, in fact. Reallocation, or lack thereof, affects convergence to the limit (see section 6.2), but it does not affect the value of the limit.

C.2 Quasi-dynamic model with inventories

This section considers an extension of the model in Dew-Becker and Vedolin (2022), which is itself closely related to the model of Jones (2011).

Suppose output in sector i on date τ is

$$Y_{i,\tau} = Z_{i,\tau} X_{i,\tau-1} \tag{85}$$

where $X_{i,\tau-1}$ is the quantity of material inputs purchased by sector i on date $\tau - 1$ (i.e. inventories of materials) and $Z_{i,\tau}$ is productivity. There is a final good produced according to the function

$$Y_\tau = \left(\sum_i a_i^{1/\sigma} Y_{i,\tau}^{(\sigma-1)/\sigma}\right)^{\sigma/(\sigma-1)} \tag{86}$$

(i.e. all of the output of the individual sectors goes to produce the final good) and the resource constraint says that the final good can be allocated to either consumption or inventories of inputs for use on date $\tau + 1$:

$$Y_\tau = C_\tau + \sum_i X_{i,\tau} \tag{87}$$

This can be mapped into the main model by making final good production its own sector, with each sector only using the final good as an input and also consumption only involving the final good (though that is without the dynamics).

Combining the production functions yields

$$Y_\tau = \left(\sum_i a_i^{1/\sigma} \left(Z_{i,\tau} X_{i,\tau-1} \right)^{(\sigma-1)/\sigma} \right)^{\sigma/(\sigma-1)} \tag{88}$$

A fully dynamic version of this model could be studied by specifying processes for the $Z_{i,\tau}$. However, that does not appear to be tractable. I therefore consider a one-time surprise shock. Specifically, I assume that for $\tau < 0$, agents believe that $Z_{i,\tau} = 1$ for all i, and τ. On date $\tau = 0$ a surprise shock occurs, with each sector receives a random $Z_{i,0}$, after which productivity permanently stays at the new level (I discuss the case of a transitory shock, which is less interesting, below).

Specifically, $Z_{i,\tau} = 1$ for all $\tau < 0$, and $Z_{i,\tau} = Z_{i,0}$ for all $\tau > 0$. We proceed by solving the model under the agents' assumption that there are no shocks. If we define $\sum_i X_{i,\tau} = \bar{X}_\tau$, then it is straightforward to show that the optimal choice of $X_{i,\tau}$ each period satisfies

$$X_{i,\tau} = \bar{X}_\tau \frac{a_i Z_i^{\sigma-1}}{\sum_i a_i Z_i^{\sigma-1}} \tag{89}$$

Define effective productivity, output per unit of inputs, to be $Y_\tau / \bar{X}_\tau$. We have

$$Y_\tau / \bar{X}_{\tau-1} = 1 \text{ for all } \tau < 0 \tag{90}$$

$$Y_0 / \bar{X}_{-1} = \left(\sum_i a_i Z_{i,0}^{(\sigma-1)/\sigma} \right)^{\sigma/(\sigma-1)} \tag{91}$$

$$Y_\tau / \bar{X}_{\tau-1} = \left(\sum_i a_i Z_{i,0}^{\sigma-1} \right)^{1/(\sigma-1)} \quad \text{for all } \tau > 0 \tag{92}$$

C.3 Which is the right approximation to use?

The usual Taylor approximation is around $z = 0$, while this paper focuses on $z \to \infty$. As z grows, the tail approximation is eventually superior, so for any statements about limiting probabilities as $gdp \to \pm\infty$, it is the correct representation. But at what point does that transition happen? To shed light on that question, first note that $gdp\,(0) = 0$. So to know the size of the error from using the tail approximation when $z = 0$, we need to know the constants $\mu\,(\theta)$.

The constant in the tail approximation is $-\beta'\mu$ where the vector μ solves the recursion

$$\mu_i = \frac{\alpha}{(1-\sigma_i)} \log \left(\sum_{j \in j^*(i)} A_{i,j} \exp\left((1-\sigma_i)\, \mu_j\right) \right) \tag{93}$$

and

$$j^*(i) \equiv \begin{cases} \{j : \phi_j = \max_{k \in S_i} \phi_k\} & \text{if } \sigma_i < 1 \\ \{j : \phi_j = \min_{k \in S_i} \phi_k\} & \text{if } \sigma_i > 1 \end{cases} \tag{94}$$

When $j^*(i)$ is a singleton,

$$\mu_i = \frac{\alpha}{(1-\sigma_i)} \log A_{i,j^*(i)} + \alpha \mu_{j^*(i)} \tag{95}$$

The constant, $\mu(\theta)$, thus increases when the elasticity of substitution is closer to 1 and when the upstream source of shocks is units that are relatively small (have small $A_{i,j}$). Those factors cause the tail approximation to have a relatively larger error as $t \to 0$.

The concave case

In the case where *gdp* is globally concave in the shocks – $\sigma_i \leq 1 \; \forall \, i$ – a stronger result is available. The error for the tail approximation then is smaller than for the first-order Taylor series when

$$t > \frac{\mu(\theta)}{D_{ss}'\theta - \lambda(\theta)} \tag{96}$$

The tail approximation is superior if t is sufficiently large – larger when the constant $\mu(\theta)$ is larger or the gap between the local and tail approximations, $D_{ss}'\theta - \lambda(\theta)$, is smaller. That immediately implies that when any elasticity gets closer to 1, the cutoff point gets larger, since σ_i has no impact on λ and D_{ss} away from 1. The closer are the various elasticities to 1, the larger the shocks have to be in order for the tail approximation to be superior to a local approximation.

It is less clear what the effects of the $A_{i,j}$ parameters on the cutoff is because they affect both μ and D_{ss}. Note, though, that (in the concave case), when $\lambda(\theta) < 0$ – i.e. when thinking about shocks that reduce GDP – the tail approximation cannot possibly be the better of the two until $\mu(\theta) + \lambda(\theta)\, t < 0$, and the point where that happens necessarily increases as the A parameters for the minimizing units (i.e. the units $j \in j^*(i)$ for some i) decline.

D Extensions and additional results

D.1 Neoclassical growth model

Each sector's output on date τ is

$$Y_{i,\tau} = Z_{i,\tau}\left(K_{i,\tau}^{\eta}L_{i,\tau}^{1-\eta}\right)^{1-\alpha}\bar{X}_{i,\tau}^{\alpha} \tag{97}$$

$$\text{where } \bar{X}_{i,\tau} \equiv \left(\sum_{i} A_{i,j}^{1/\sigma_i}X_{i,j,\tau}^{(\sigma_i-1)/\sigma_i}\right)^{\sigma_i/(\sigma_i-1)} \tag{98}$$

Note that the first-order optimality conditions for each sector's use of capital and labor imply that they all use the same mix of capital and labor. If the aggregate capital stock is $\bar{K}_{\tau}$ and we normalize aggregate labor to 1, $\sum_i L_{i,\tau} = 1$, we have that $K_{i,\tau} = L_{i,\tau}\bar{K}_{\tau}$. Define

$$M_{i,\tau} \equiv K_{i,\tau}^{\eta}L_{i,\tau}^{1-\eta} = L_{i,\tau}\bar{K}_{\tau}^{\eta} \tag{99}$$

Now normalize the price of the labor-capital bundle to 1.[37] Aggregate nominal income is then

$$\sum_{i} M_{i,\tau} = \bar{K}_{\tau}^{\eta} \tag{100}$$

Inserting $M_{i,\tau}$ into the production function yields (trivially)

$$Y_{i,\tau} = Z_{i,\tau}M_{i,\tau}^{1-\alpha}\bar{X}_{i,\tau}^{\alpha} \tag{101}$$

This is exactly the same structure as in section 2, just replacing labor, $L_{i,\tau}$, with the capital-labor bundle, $K_{i,\tau}^{\eta}L_{i,\tau}^{1-\eta}$. Lemma 1 and Theorem 1 then continue to hold, with the only modification that GDP is proportional to $\bar{K}_{\tau}^{\eta}$ (in the baseline case aggregate labor adds up to 1; here, the sum of M_i is instead $\bar{K}_{\tau}^{\eta}$). That is,

$$GDP_{\tau} = \bar{K}_{\tau}^{\eta}/\exp\left(\beta' p_{\tau}\right) \tag{102}$$

where p_{τ} is the log price vector satisfying the recursion in (3) (which depends only on productivity). Note that there is a multiplier effect of α that is absorbed in the solution for p_{τ}.

Now consider a dynamic but nonstochastic version of the model in which households

[37] Again, we can always normalize one price. $M_{i,\tau}$ here plays the same role as labor in the baseline case in the main text, so we normalize its price to 1 analogously to the normalization of the wage to 1 in the baseline case.

maximize lifetime utility. To keep things simple, I assume that capital and final consumption both use the same mix of goods. That is, there is some final good producing sector with the production function in equation (2) that produces interchangeable consumption and capital goods and the household's budget constraint is

$$\bar{K}_{\tau+1} + C_\tau = (1-\delta)\,\bar{K}_\tau + \bar{K}_\tau^\eta \exp\left(-\beta' p_\tau\right) \tag{103}$$

The household's Lagrangian is then

$$\max \sum_{j=0}^{\infty} \beta^j \left[U\left(C_\tau\right) - \lambda_\tau \left(\bar{K}_{\tau+1} + C_\tau - (1-\delta)\,\bar{K}_\tau - \bar{K}_\tau^\eta \exp\left(-\beta' p_\tau\right)\right)\right]$$

Assuming the productivities are fixed at some level $Z_{i,\tau} = Z_i$, the steady-state for GDP is

$$GDP_\tau = \left[\left(\beta^{-1} - 1 + \delta\right)^{-1} \eta\right]^{\eta/(1-\eta)} \exp\left(\frac{-1}{1-\eta}\beta' p_\tau\right) \tag{104}$$

where p_τ solves the recursion from (3) given the productivities Z_i.

D.2 Relaxing the CES assumption

This section extends the baseline result to a broader class of production functions. Consider the same competitive economy as in the main analysis, with the only difference that each sector's production need not be CES. Rather, just assume that it each sector has constant returns to scale. Again, without loss of generality, assume that labor and materials are combined with a unit elasticity of substitution. Those assumptions imply that, in competitive equilibrium, the price of good i is given by

$$P_i = \frac{1}{Z_i} W^{1-\alpha} (C_i(P_1, \ldots, P_n))^\alpha \tag{105}$$

where Z_i is the productivity shock to industry i, C_i is a homogenous function of degree one, and $\alpha < 1$. In addition to the intermediate input producing industries, there is also an industry with cost function C_0 that produces a final good, which is then sold to the representative consumer. Therefore, the final good price, P_0, also satisfies equation (105), with the convention that $\alpha_0 = 1$ and $Z_0 = 1$.

To find cirumstances under which limits of the form in Theorem 1 appear, again normalize $W = 1$, insert the guess that $p_i \to \phi_i t$ and take limits,

$$\phi_i = \lim_{t\to\infty} -\theta_i + \alpha t^{-1} \log C_i \left(\exp\left(\phi_1 t\right), ..., \exp\left(\phi_n t\right)\right) \tag{106}$$

So if it is the case that

$$\lim_{t\to\infty}\frac{1}{t}\log C_i\left(\exp\left(\phi_1 t\right),...,\exp\left(\phi_n t\right)\right)=\tilde{f}_i\left(\phi_l,\phi_1,...,\phi_n\right) \tag{107}$$

for some function $\tilde{f}_i$, then we have a recursion as in the main text. For the CES case in the main text, the function $\tilde{f}$ is the term in braces in (7), which can be seen by just plugging in the CES cost function, $C_i\left(P\right)=\left(\sum_j a_{i,j}P_j^{1-\sigma_i}\right)^{1/(1-\sigma_i)}$ and taking limits.

A sufficient condition for the limit in (107) to exist is that

$$\lim_{t\to\infty}\frac{d}{dt}\log C_i\left(\exp\left(\phi_l t\right),\exp\left(\phi_1 t\right),...,\exp\left(\phi_n t\right)\right) \tag{108}$$

exists. That is, it is sufficient that the gradients of the cost functions have limits, but even that is not strictly necessary. Intuitively, equation (107) requires that the cost function eventually scales approximately linearly. It does not have to be literally linear, though. For example, the function $y\left(t\right)=at+\sin\left(t\right)$ has the limit $\lim_{t\to\infty}t^{-1}y\left(t\right)=a$. The at term dominates for large t.

D.2.1 The heterogeneous CES setup of Chodorow-Reich, Gabaix, and Koijen (2022)

Chodorow-Reich et al. (2022) study an aggregator of the form

$$\sum_i \phi_i\frac{\left(X_i/Y\right)^{(\sigma_i-1)/\sigma_i}-1}{\left(\sigma_i-1\right)/\sigma_i}+\phi_0=0 \tag{109}$$

where the X_i are uses of inputs, The ϕ_i are parameters, and Y is output, which is an implicit function of the inputs (see also Matsuyama and Ushchev (2017) and Baqaee, Farhi, and Sangani (2023)). They show that the unit cost function for this case is solved by

$$C=\mu\sum_i\left(P_i/\mu\right)^{1-\sigma_i} \tag{110}$$

where μ solves

$$\sum_i\frac{\sigma_i}{\sigma_i-1}\left(P_i/\mu\right)^{1-\sigma_i}+\phi_0=0 \tag{111}$$

Now suppose the prices all have limits $\log P_i\to g_i t$ as $t\to\infty$. It is then the case that if all $\sigma_i<1$, $C\to\left(\max_i g_i\right)t$, while if $\sigma_i>1$, $C\to\left(\min g_i\right)t$. That is, in this more general case, the precise value of the elasticity of substitution for each good continues to play no role, as long as all of the elasticities (within a given sector) are above or below 1. In the case where elasticities are mixed within a sector in this model, the analysis, for general g_i, becomes much more difficult and does not yield a simple solution.

E Exponential example

We begin with a general result for Weibull-tailed shocks. The shocks have a Weibull-type tail if, for $t > \bar{t}$,

$$\bar{F}(t) = c \exp\left(-\eta (t-\bar{t})^{\kappa}\right) \tag{112}$$

$$\text{where } c = \Pr\left(t \leq \bar{t}\right) \tag{113}$$

for parameters $\kappa > 0$ and $\eta > 0$. Denote the essential supremum with respect to the measure m over θ of any function $f(\theta)$ by $\|f(\theta)\|_{\infty}$.[38] For example, in the typical case where m has full support, $\|f(\theta)\|_{\infty} = \max_{\theta} f(\theta)$ (note that it is *not* the maximum of $|f(\theta)|$). $\|f(\theta)\|_{\infty;\Theta^*}$ denotes the essential supremum on some subset of the sphere Θ^*.

Proposition 9. *If the shocks have Weibull tails,*

$$\lim_{x\to\infty} \Pr\left[gdp < -x\right]^{1/(x^{\kappa})} = \exp\left(-\eta\left(\frac{1}{\|-s(\theta)\lambda(\theta)\|_{\infty}}\right)^{\kappa}\right) \tag{114}$$

Furthermore, for any set Θ^* *such that* $\|-s(\theta)\lambda(\theta)\|_{\infty;\Theta^*} < \|-s(\theta)\lambda(\theta)\|_{\infty}$,

$$\lim_{x\to\infty} \Pr\left[\theta \in \Theta^* \mid gdp < -x\right] = 0 \tag{115}$$

Analogous results hold for $\Pr\left[gdp > x\right]$.

In the independent exponential case, the probability density in the tail is $\exp\left(-\|z\|_{1;v}/\eta\right)$, where

$$\|z\|_{1,v} \equiv \sum_j |z_j|/v_j \tag{116}$$

denotes an l_1-type norm weighted by a vector v, representing the volatility of each shock. To confirm that $s(\theta) = 1/\|\theta\|_{1,v}$, note that

$$\exp\left(-\left(t/s(\theta)\right)/\eta\right) = \exp\left(-\left(\|z\|\left\|\frac{z}{\|z\|}\right\|_{1,v}\right)/\eta\right) \tag{117}$$

$$= \exp\left(-\|z\|_{1,v}/\eta\right) \tag{118}$$

[38]Formally, $\|f(\theta)\|_{\infty} = \inf\{a \in \mathbb{R} : m(\{\theta : f(\theta) > a\}) = 0\}$.

as required.

The aim is to find $\max_{\tilde{\theta}:\|\tilde{\theta}\|_2=1} \left\| -s\left(\tilde{\theta}\right)\lambda\left(\tilde{\theta}\right)\right\|$. Now note that $b\lambda\left(\tilde{\theta}\right) = \lambda\left(b\tilde{\theta}\right)$, and hence $s\left(\tilde{\theta}\right)\lambda\left(\tilde{\theta}\right) = \lambda\left(\tilde{\theta}s\left(\tilde{\theta}\right)\right)$. We can then apply a change of variables, with $\theta = \tilde{\theta}s\left(\tilde{\theta}\right)$. Note that $\tilde{\theta} = \theta/\|\theta\|$, so we have

$$\max_{\tilde{\theta}:\|\tilde{\theta}\|=1} \left\| -s\left(\tilde{\theta}\right)\lambda\left(\tilde{\theta}\right)\right\| = \max_{\theta:\|\theta/s(\theta/\|\theta\|)\|=1} \|-\lambda(\theta)\| \tag{119}$$

Now in this particular case,

$$\begin{aligned} \|\theta/s(\theta/\|\theta\|)\| &= \left\|\theta\,\|\theta/\|\theta\|\|_{1,v}\right\| &(120)\\ &= \|\theta\|_{1,v} &(121) \end{aligned}$$

The objective is then

$$-\max_{\theta}\max_{n} D_n'\theta = -\max_{n}\max_{\theta} D_n'\theta \tag{122}$$

subject to the constraint $\|\theta\|_{1,v} = 1$. The inner maximization on the right is a problem with a linear objective and a linear constraint, so it is simply solved at the point that maximizes $D_{n,j}v_j$. We then have

$$-\max_{n}\max_{j} D_{n,j}v_j \tag{123}$$

The example in the text is the special case of $v_j = 1\ \forall\ j$.

E.1 Proof of proposition 9

The statement of Theorem 2 is

$$\int_{\theta:\lambda(\theta)<0} \bar{F}\left(\frac{x-\mu(\theta)+\varepsilon(x)}{-s(\theta)\lambda(\theta)}\right)dm(\theta) \le \Pr[gdp < -x] \le \int_{\theta:\lambda(\theta)<0} \bar{F}\left(\frac{x-\mu(\theta)-\varepsilon(x)}{-s(\theta)\lambda(\theta)}\right)dm(\theta) \tag{124}$$

In this case we have

$$\begin{aligned} \bar{F}(s) &= c\exp\left(-\beta\left(t-\bar{t}\right)^{\kappa}\right) &(125)\\ \text{where } c &= \Pr\left(t \le \bar{t}\right) &(126) \end{aligned}$$

If the limits of the two integrals in (124) are the same, then that limit is also the limit for $\Pr[gdp < -x]$. This section gives the derivation for the right-hand side limit, with the arguments holding equivalently on the left with the sign of $\varepsilon(x)$ reversed.

We have

$$\left(\int_{\theta:\lambda(\theta)<0} \bar{F}\left(\frac{x-\mu(\theta)-\varepsilon(x)}{-s(\theta)\lambda(\theta)}\right) dm(\theta)\right)^{1/x^{\kappa}} \tag{127}$$

$$= \left[\int_{\theta\in\Theta} \exp\left(-\left(\frac{1}{-s(\theta)\lambda(\theta)} - \frac{\varepsilon(x)+\mu(\theta)}{x}\frac{1}{s(\theta)\lambda(\theta)} - \frac{\bar{t}}{x}\right)^{\kappa}\right)^{x^{\kappa}} dm(\theta)\right]^{1/x^{\kappa}} \tag{128}$$

Now consider the limit as $x\to\infty$. I show that the limit of the right-hand side is the essential supremum of $\exp\left(-\left(\frac{1}{-s(\theta)\lambda(\theta)}\right)^{\kappa}\right)$ with respect to the measure $m(\theta)$ (i.e. the measure of the set of θ such that $\exp\left(-\left(\frac{1}{s(\theta)\lambda(\theta)}\right)^{\kappa}\right)$ is above the essential supremum is zero). Denote that by $\left\|\exp\left(-\left(\frac{1}{s(\theta)\lambda(\theta)}\right)^{\kappa}\right)\right\|_{\infty}$.

The structure of this proof is from Ash and Doleans-Dade (2000), page 470, with the addition of the convergence of the argument of the integral with respect to x.

Define, for notational convenience,

$$f(\theta) = \exp\left(-\left(\frac{1}{s(\theta)\lambda(\theta)}\right)^{\kappa}\right) \tag{129}$$

$$f(\theta;x) = \exp\left(-\left(\frac{1}{s(\theta)\lambda(\theta)} - \frac{\varepsilon(x)+\mu(\theta)}{x}\frac{1}{s(\theta)\lambda(\theta)} - \frac{\bar{t}}{x}\right)^{\kappa}\right) \tag{130}$$

Lemma E3. $\lim_{x\to\infty}\|f(\theta;x)\|_{\infty} = \|f(\theta)\|_{\infty}$.

Proof. $f(\theta;x)\to f(\theta)$ pointwise trivially. The difference $|f(\theta;x)-f(\theta)|$ is bounded due to the facts that $\varepsilon(x)$ and $\mu(\theta)$ are bounded and that $f(\theta;x)$ is decreasing in $s(\theta)\lambda(\theta)$ (for sufficiently large x), which is bounded from above (and below, by zero). $f(\theta;x)$ then converges uniformly to $f(\theta)$, from which $\|f(\theta;x)\|_{\infty}\to\|f(\theta)\|_{\infty}$ follows, since, using the reverse triangle inequality,

$$\left|\|f(\theta;x)\|_{\infty} - \|f(\theta)\|_{\infty}\right| \leq \|f(\theta)-f(\theta;x)\|_{\infty} \tag{131}$$

■

Lemma E4. $\limsup_{x\to\infty}\left[\int_{\theta\in\Theta} f(\theta;x)^{x^{\kappa}} dm(\theta)\right]^{1/x^{\kappa}} \leq \|f(\theta)\|_{\infty}$

Proof. We have (except possibly on a set of measure zero)

$$\|f(\theta;x)\|_{x^{\kappa}} \leq \left\|\|f(\theta;x)\|_{\infty}\right\|_{x^{\kappa}}$$

Taking limits of both sides

$$\lim_{x\to\infty} \|f(\theta;x)\|_{x^\kappa} \leq \lim_{x\to\infty} \|\|f(\theta;x)\|_\infty\|_{x^\kappa} \quad (132)$$
$$= \lim_{x\to\infty} \|f(\theta;x)\|_\infty \quad (133)$$
$$= \|f(\theta)\|_\infty \quad (134)$$

where the second line follows from the fact that $\|f(\theta;x)\|_\infty$ is constant and the third line uses lemma E3. ■

Lemma E5. $\liminf_{x\to\infty} \left[\int f(\theta;x)^{x^\kappa}\, dm(\theta)\right]^{1/x^\kappa} \geq \|f(\theta)\|_\infty$

Proof. Consider some $\eta > 0$, and set $A = \left\{\theta : \exp\left(-\left(\frac{1}{-s(\theta)\lambda(\theta)}\right)^\kappa\right) \geq \left\|\exp\left(-\left(\frac{1}{-s(\theta)\lambda(\theta)}\right)^\kappa\right)\right\|_\infty - \eta\right\}$. Consider also the set $A' = \left\{\theta : \exp\left(-\left(\frac{1}{s(\theta)\lambda(\theta)} - \frac{\pm\varepsilon(x)+\mu(\theta)}{x}\frac{1}{\lambda(\theta)} - \frac{\bar{t}}{x}\right)^\kappa\right) \geq \left\|\exp\left(-\left(\frac{1}{s(\theta)\lambda(\theta)}\right)^\kappa\right)\right\|_\infty - \eta\right\}$. For any η such that A has positive measure, there exists an $\bar{x}(\eta)$ sufficiently large that A' has positive measure for all $x > \bar{x}(\eta)$ due to the continuity of $\exp\left(-\left(\frac{1}{s(\theta)\lambda(\theta)} - \frac{\pm\varepsilon(x)+\mu(\theta)}{x}\frac{1}{s(\theta)\lambda(\theta)} - \frac{\bar{t}}{x}\right)^\kappa\right)$ and the fact that $\exp\left(-\left(\frac{1}{s(\theta)\lambda(\theta)} - \frac{\pm\varepsilon(x)+\mu(\theta)}{x}\frac{1}{\lambda(\theta)}\right)^\kappa\right) \to \exp\left(-\left(\frac{1}{s(\theta)\lambda(\theta)}\right)^\kappa\right)$ as $x \to \infty$.

It is then the case that for $x > \bar{x}(\eta)$

$$\int \exp\left(-\left(\frac{1}{\lambda(\theta)} - \frac{\pm\varepsilon(x)+\mu(\theta)}{x}\frac{1}{s(\theta)\lambda(\theta)} - \frac{\bar{t}}{x}\right)^\kappa\right)^{x^\kappa} dm(\theta) \quad (135)$$
$$\geq \int_{A'} \exp\left(-\left(\frac{1}{\lambda(\theta)} - \frac{\pm\varepsilon(x)+\mu(\theta)}{x}\frac{1}{s(\theta)\lambda(\theta)} - \frac{\bar{t}}{x}\right)^\kappa\right)^{x^\kappa} dm(\theta) \quad (136)$$
$$\geq \left(\left\|\exp\left(-\left(\frac{1}{\lambda(\theta)}\right)^\kappa\right)\right\|_\infty - \eta\right)^{x^\kappa} \mu(A') \quad (137)$$

Since $\mu(A') > 0$ from the definition of $\left\|\exp\left(-\left(\frac{1}{s(\theta)\lambda(\theta)}\right)^\kappa\right)\right\|_\infty$ (ignoring the trivial case of a constant value for $\exp\left(-\left(\frac{1}{s(\theta)\lambda(\theta)}\right)^\kappa\right)$), and since the above holds for any $\eta > 0$,

$$\liminf_{x\to\infty} \left[\int \exp\left(-\left(\frac{1}{s(\theta)\lambda(\theta)} - \frac{\pm\varepsilon(x)+\mu(\theta)}{x}\frac{1}{\lambda(\theta)} - \frac{\bar{t}}{x}\right)^\kappa\right)^{x^\kappa}\right]^{1/x^\kappa} dm(\theta) \quad (138)$$
$$\geq \left\|\exp\left(-\left(\frac{1}{s(\theta)\lambda(\theta)}\right)^\kappa\right)\right\|_\infty \quad (139)$$

■

Proof of the proposition: Since both the lim inf and lim sup are equal to $\left\|\exp\left(-\left(\frac{1}{s(\theta)\lambda(\theta)}\right)^\kappa\right)\right\|_\infty$, the limit is also.

For the second part, in the set Θ^*, there exists an η such that $|-s(\theta)\lambda(\theta)| < \|-s(\theta)\lambda(\theta)\|_\infty - \eta$. Therefore

$$\frac{\int_{\Theta^*}\exp\left(-\left(\frac{x+\varepsilon(x)-\mu(\theta)}{-s(\theta)\lambda(\theta)}-\bar{t}\right)^\kappa\right)dm(\theta)}{\int\exp\left(-\left(\frac{x-\varepsilon(x)-\mu(\theta)}{-s(\theta)\lambda(\theta)}-\bar{t}\right)^\kappa\right)dm(\theta)} \leq \Pr\left[\begin{array}{c}\theta\in\Theta^* \\ \mid gdp<-x\end{array}\right] \leq \frac{\int_{\Theta^*}\exp\left(-\left(\frac{x-\varepsilon(x)-\mu(\theta)}{-s(\theta)\lambda(\theta)}-\bar{t}\right)^\kappa\right)dm(\theta)}{\int\exp\left(-\left(\frac{x+\varepsilon(x)-\mu(\theta)}{-s(\theta)\lambda(\theta)}-\bar{t}\right)^\kappa\right)dm(\theta)} \tag{140}$$

Again, we show that both sides of the inequality have the same limit. For a sufficiently large x,

$$\frac{\int_{\Theta^*}\exp\left(-\left(\frac{x\pm\varepsilon(x)-\mu(\theta)}{-s(\theta)\lambda(\theta)}-\bar{t}\right)^\kappa\right)dm(\theta)}{\int\exp\left(-\left(\frac{x\pm\varepsilon(x)-\mu(\theta)}{-s(\theta)\lambda(\theta)}-\bar{t}\right)^\kappa\right)dm(\theta)} \leq \frac{\int_{\Theta^*}\exp\left(-\left(\frac{x\pm\varepsilon(x)-\mu(\theta)}{\left(\|-s(\theta)\lambda(\theta)\|_\infty-\eta\right)}-\bar{t}\right)^\kappa\right)dm(\theta)}{\int_{\theta:|\lambda(\theta)|>|\lambda(\theta)|-\eta/2}\exp\left(-\left(\frac{x-\pm\varepsilon(x)-\mu(\theta)}{-s(\theta)\lambda(\theta)}-\bar{t}\right)^\kappa\right)dm(\theta)}$$

$$\begin{aligned}
&\leq \frac{\exp\left(-\left(\frac{x-\pm\varepsilon(x)-\mu(\theta)}{-\left(\|s(\theta)\lambda(\theta)\|_\infty-\eta\right)}-\bar{t}\right)^\kappa\right)}{\exp\left(-\left(\frac{x-\pm\varepsilon(x)-\mu(\theta)}{-\left(\|s(\theta)\lambda(\theta)\|_\infty-\eta/2\right)}-\bar{t}\right)^\kappa\right)}\frac{1}{m\left(\{\theta:|\lambda(\theta)|>\|\lambda(\theta)\|_\infty-\eta/2\}\right)} &(141)\\
&\to 0 &(142)
\end{aligned}$$

■