Comparative statics with linear objectives: normality, complementarity, and ranking multi-prior beliefs

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Abstract

We formulate an order over constraint sets $A \subseteq \mathbb{R}^\ell$, called the *parallelogram order*, which guarantees that argmin $\{p \cdot x : x \in A\}$ increases in the product order as A increases in the parallelogram order, for any vector $p \in \mathbb{R}^\ell$. Using this result, we characterize the utility/production functions that lead to normal demand as well as the closely related class of production functions with marginal costs that increase with factor prices. By generalizing the concept of supermodularity, we also characterize the class of production functions for which factors are complements. In the context of decision-making under uncertainty, our new set order leads to natural generalizations of first order stochastic dominance in multi-prior models.

Keywords: parallelogram order, increasing differences, complementarity, ambiguity, first order stochastic dominance, normal demand, marginal costs

JEL Classification: C61, D21, D24

1 Introduction

This paper studies the monotone comparative statics of optimization problems with linear objectives. We pose the following question: what relation between two subsets A and A'

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of the Euclidean space $\mathbb{R}^\ell$ guarantees that the set $\Phi' = \text{argmin}\{p \cdot x : x \in A'\}$ is higher (in an appropriate sense) than $\Phi = \text{argmin}\{p \cdot x : x \in A\}$, for all $p \in \mathbb{R}^\ell$?

When Φ' and Φ are singletons consisting of x' and x, respectively, it is clear that 'higher' means $x' \geq x$. More generally, if Φ' and Φ are nonempty sets, a minimal requirement for Φ' to be higher than Φ, is for the former to dominate the latter in the *weak set order*: for any $x \in \Phi$, $x' \in \Phi'$, there is $y \in \Phi$, $y' \in \Phi'$ such that $x' \geq y$ and $y' \geq x$.

A well-known monotone comparative statics result by Topkis (1978) and Milgrom and Shannon (1994) states that whenever A' dominates A in the *strong set order*, then $\text{argmin}\{F(x) : x \in A'\}$ dominates $\text{argmin}\{F(x) : x \in A\}$ in the same sense, for any submodular objective F.[1]

The strong set order requires that for any $x \in A$ and $x' \in A'$, their greatest upper bound $x \wedge x'$ belongs to A and the lowest upper bound $x \vee x'$ is in A'. Since $x' \geq x \wedge x'$ and $x \vee x' \geq x$, this relation is stronger than the weak set order. Since linear functions are also submodular, this result gives a possible solution to the problem we pose. However, in many economic applications the strong set order is too restrictive. Since we are considering a class of objective functions that is narrower than the class of submodular functions, we are able to propose a more general relation between sets that implies monotone comparative statics.

We say that set A' dominates A by the *parallelogram order*, if for any $x \in A$, $x' \in A'$, there is $y \in A$, $y' \in A'$ such that $x' \geq y$, $y' \geq x$, and the four points form a parallelogram, i.e., $x + x' = y + y'$. Clearly, the parallelogram order is stronger than the weak set order, since the weak set order does not require the final condition. On the other hand, it is weaker than the strong set order since, if the latter property holds, one can always choose $y = x \wedge x'$ and $y' = x \vee x'$.

In our main result, we show that if A' dominates A in the parallelogram order, then $\Phi' = \text{argmin}\{p \cdot x : x \in A'\}$ also dominates $\Phi = \text{argmin}\{p \cdot x : x \in A\}$ in the parallelogram order, for any vector $p \in \mathbb{R}^\ell$ (and the converse holds under certain ancillary conditions). Furthermore, one obtains the useful property that the value function $f(\tilde{p}, \tilde{A}) := \min\{\tilde{p} \cdot x : x \in \tilde{A}\}$ satisfies *increasing differences* in $(\tilde{p}, \tilde{A})$, i.e., for any $p' \geq p$,

$$f(p', A') - f(p, A') \;\geq\; f(p', A) - f(p, A).$$

[1] The function F is *submodular* if $F(x) + F(x') \geq F(x \vee x') + F(x \wedge x')$ for all $x, x' \in \mathbb{R}^\ell$. The function is *supermodular* if, and only if, $-F$ is submodular.

We employ our results to investigate three broad economic applications. The first problem we study is *normality* in a firm's conditional factor demand or (what is formally similar) a consumer's Marshallian demand. Suppose a firm transforms ℓ factors into a single good using the production function $F : \mathbb{R}^\ell_+ \to \mathbb{R}_+$. Given a price vector $p \in \mathbb{R}^\ell_{++}$, the firm's factor demand at output q is $H(p, q) = \text{argmin} \{p \cdot x : F(x) \geq q\}$. It is natural to ask what conditions on F will guarantee that factor demand is *normal*, in the sense that $H(p, q)$ rises with q, at least with respect to the weak set order, for any price p; among other things, normality guarantees that a firm's marginal cost is increasing with factor prices and (thus) a rise in factor prices leads to a lower profit-maximizing output. Given our main result, it is clear that a sufficient condition for normality is for the upper-contour correspondence $U(q) := \{x : F(x) \geq q\}$ to be increasing in the parallelogram order, i.e., if $q' > q$, then $U(q')$ dominates $U(q)$ in the parallelogram order.

The toolkit provided by Topkis (1978) and Milgrom and Shannon (1994) is too stringent to address this problem, as the correspondence U fails to increase in the strong set order whenever F is strictly increasing.[2] On the other hand, our approach allows us to extend the existing work of Quah (2007) and characterize the class of production functions that induce the normality of factor demand.

Our second application pertains to complementarity in a quasilinear setting. Consider a firm with technology F that chooses a bundle of factors $x \in \mathbb{R}^\ell_+$ in order to maximize profit $F(x) - p \cdot x$, given factor prices $p \in \mathbb{R}^\ell_{++}$. The factors are *gross complements* if lowering the price of a factor i raises the (unconditional) demand for *all* factors. It is well-known that supermodularity of the production function is sufficient for complementarity (see Topkis, 1978), but this property is not necessary and we present natural situations in which it fails.

We approach this problem by exploiting the fact that the profit maximization problem can be thought of as a linear optimization problem over the firm's production possibility set and, furthermore, the change in price of a factor can be modelled as a change in the firm's production possibility set. It follows that establishing the appropriate comparative statics can be reduced to finding a property on F under which this change in the production possibility set can be ranked according to the parallelogram order. We dub this

[2] Let $x \in U(q)$ with $F(x) = q$ and $x' \in U(q')$, where $x' \ngeq x$. In such a case, we obtain $x > x \wedge x'$ which implies $F(x) = q > F(x \wedge x')$ and, thus, $x \wedge x' \notin U(q)$.

condition *super*modularity*.[3] It strictly generalises supermodularity and, whenever F is concave, super*modularity is also necessary for factors to be complements. Our approach also allows us to develop conditions under which just a *subset* of factors are complements of each other.

Our final application concerns the *comparison of multi-prior beliefs* in models of choice under uncertainty. Imagine an agent who has to take an action under uncertainty. Let $g(x, s)$ be the agent's payoff when action $x \in X \subseteq \mathbb{R}$ is taken and state $s \in S \subseteq \mathbb{R}$ occurs. If λ is the cumulative distribution function on S, then the expected utility of action x is $f(x, \lambda) = \int g(x, s) d\lambda(s)$.

It is known that if g has increasing differences, i.e., the marginal payoff of action x is increasing with state s, then the *expected* marginal payoff of a higher action is also greater when higher states are more likely. Formally, whenever g has increasing differences in (x, s) then f has increasing differences in (x, λ). In other words, if, for any $x' > x$, the expression $g(x', s) - g(x, s)$ is increasing in s, then $f(x', \lambda') - f(x, \lambda') \geq f(x', \lambda) - f(x, \lambda)$ whenever the distribution λ' first order stochastically dominates λ. The latter property implies that $\text{argmax}\{f(x, \lambda') : x \in X\}$ dominates $\text{argmax}\{f(x, \lambda) : x \in X\}$ by the strong set order (see Topkis (1978)).[4]

Suppose that instead of maximizing expected utility, the agent is ambiguity averse and has maxmin preferences à la Gilboa and Schmeidler (1989). In this scenario, the ex-ante utility of action x is $f(x, \Lambda) = \min\{\int_S g(x, s) d\lambda(s) : \lambda \in \Lambda\}$, where Λ denotes the set of beliefs (i.e., cumulative distribution functions over S). Assuming that g has increasing differences, what change in beliefs from Λ to Λ' would guarantee that the ex-ante utility f has increasing differences? In other words, when does

$$f(x', \Lambda') - f(x, \Lambda') \geq f(x', \Lambda) - f(x, \Lambda),$$

for any $x' > x$? Since Nature's problem of choosing a distribution in Λ (or Λ') that minimizes expected utility is essentially an optimization problem with a linear objective, our main result is applicable and tells us that the beliefs have to shift with respect to the parallelogram order. In this context, it means that for any cumulative distribution

[3] We suggest reading 'super*modular' as 'super-star-modular.'

[4] For a simple example, let x be the agent's current consumption in a two-date model where s is the uncertain income of tomorrow. Assuming that tomorrow's consumption has diminishing marginal utility, a first order shift in tomorrow's income distribution will increase today's consumption (see Section 5.1).

functions $\lambda \in \Lambda$ and $\lambda' \in \Lambda'$, there is $\mu \in \Lambda$ and $\mu' \in \Lambda'$ such that λ' first order stochastically dominates μ, μ' first order stochastically dominates λ, and $\lambda + \lambda' = \mu + \mu'$.

As an example, suppose an investor uses different models of the return on a risky asset and these models give an interval of distributions $\Lambda' = [\underline{\nu}', \bar{\nu}']$, i.e., the set of all distributions lying between $\bar{\nu}'$ and $\underline{\nu}'$ in the sense of first order stochastic dominance, where $\bar{\nu}'$ is the most optimistic distribution and $\underline{\nu}'$ is the least. Suppose that the government decides to impose a proportional tax on this return. This will shift each distribution in $[\underline{\nu}', \bar{\nu}']$ downwards with respect to first order stochastic dominance, leading to a new interval of distributions $\Lambda = [\underline{\nu}, \bar{\nu}]$ where $\underline{\nu}$ is the distribution of after-tax returns corresponding to $\underline{\nu}'$ and, similarly, $\bar{\nu}$ is the after-tax return distribution corresponding to $\bar{\nu}'$. One could show that Λ' dominates Λ with respect to the parallelogram order (see Example 11). In the classical portfolio choice model with one safe and one risky asset, the introduction of such a tax will lower investment in the risky asset, if the investor has maxmin preferences and a sufficiently low level of risk aversion (see Example 15).

Organization of the paper Section 2 is devoted to the basic results and a discussion on the related literature, in particular, Topkis (1978), Milgrom and Shannon (1994), and Quah (2007). The normality of conditional factor demand and the closely related (but weaker) notion of increasing marginal costs are discussed in Section 3; this section also discusses conditions for normal Marshallian demand. Section 4 formulates conditions for factors to be complements and introduces the concept of super*modularity. In Section 5 we formulate first order stochastic dominance for multi-prior models; it covers the maxmin model as well as the variational and multiplier preference models. There is an Appendix containing the more elaborate proofs and an Online Supplement. In particular, the latter contains results on monotone decision rules for ambiguity averse agents in a dynamic setting, generalizing known results for agents maximizing discounted expected utility (see Hopenhayn and Prescott, 1992).

2 The parallelogram order

A *partial order* $\geq_X$ over a set X is a reflexive, transitive, and antisymmetric binary relation. A partially ordered set, or a *poset*, is a pair $(X, \geq_X)$ consisting of a set X and a partial order $\geq_X$. Whenever it causes no confusion, we denote $(X, \geq_X)$ with X. A poset is a *lattice* if, for any x and x' in X, their meet (the greatest lower bound) $x \wedge x'$ and their join (the least upper bound) $x \vee x'$ both belong to X. A subset Y of a lattice X is a *sublattice* if for any x, $x' \in Y$, $x \vee x'$ and $x \wedge x'$ are also in Y.

Most of our analysis is carried out in the Euclidean space $\mathbb{R}^\ell$. For any vector $x \in \mathbb{R}^\ell$, we denote its i'th entry by x_i; for any set $K \subseteq \{1, 2, \ldots, \ell\}$, let $x_K := (x_i)_{i \in K}$ be the sub-vector of entries in x that belong to K. Thus, we can write x as (x_K, x_{-K}), where $x_{-K} := (x_i)_{i \notin K}$. The product order $\geq$ on $\mathbb{R}^\ell$ is defined as follows: for any x, $x' \in \mathbb{R}^\ell$, $x' \geq x$ if $x'_i \geq x_i$ for all $i = 1, 2, \ldots, \ell$. The relation is said to be *strict*, and denoted by $x' > x$, whenever $x' \geq x$ and $x' \neq x$. It is straightforward to check that $(\mathbb{R}^\ell, \geq)$ constitutes a lattice, with $(x \wedge x')_i = \min\{x_i, x'_i\}$ and $(x \vee x')_i = \max\{x_i, x'_i\}$, for $i = 1, 2, \ldots, \ell$.

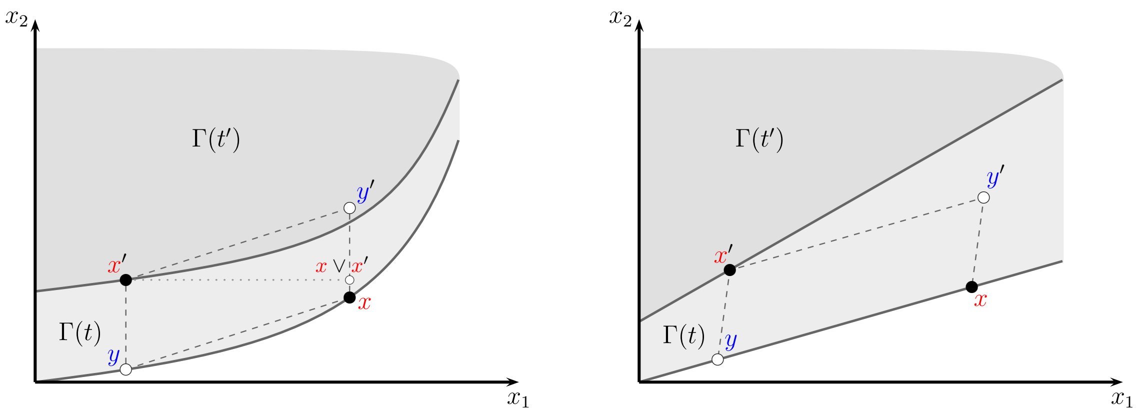
Definition 1 (Parallelogram order)**.** Let A, $A' \subseteq \mathbb{R}^\ell$ and $K \subseteq \{1, 2, \ldots, \ell\}$. The set A' *dominates* A *in* K *by the parallelogram order* if for any $x \in A$, $x' \in A'$, there is $y \in A$, $y' \in A'$ such that $x + x' = y + y'$ and $x'_K \geq y_K$, $y'_K \geq x_K$. Whenever we refer to the parallelogram order without mentioning K, our default is $K = \{1, 2, \ldots, \ell\}$.

Given two nonempty sets $A, A' \subseteq \mathbb{R}^\ell$, we say that A' dominates A in K by the *weak order* if, for any $x \in A$ there $y' \in A'$ such that $x_K \leq y'_K$, and for any $x' \in A'$ there is $y \in A$ such that $y_K \leq x'_K$. Clearly, whenever A' dominates A in K by the parallelogram order, then A' also dominates A in K by the weak order.

A widely-used property in monotone comparative statics is the *strong set order* (see Topkis, 1978). Given a lattice X and subsets A, A' of X, the set A' dominates A by the strong set order if, for any $x \in A$, $x' \in A'$, we have $x \wedge x' \in A$ and $x \vee x' \in A'$. If X is a sublattice of $\mathbb{R}^\ell$, then when A' dominates A by the strong set order, A' also dominates A by the parallelogram order, since we can choose $y = x \wedge x'$ and $y' = x \vee x'$.

Our comparative statics results are typically formulated in a setting where there is a collection of sets related by the parallelogram order. This is formally captured through a correspondence Γ from a poset $(T, \geq_T)$ to $\mathbb{R}^\ell$; we say that Γ is $\mathcal{P}$*-increasing in* K if, for any $t' \geq_T t$, $\Gamma(t')$ dominates $\Gamma(t)$ in K by the parallelogram order. Clearly, if Γ is nonempty-valued and $\mathcal{P}$-increasing in K then it is also $\mathcal{W}$*-increasing in* K, in the sense that, if $t' \geq_T t$, then the set $\Gamma(t')$ dominates $\Gamma(t)$ in K by the weak order.[5]

[5] Similarly, we could speak of Γ being $\mathcal{P}$-decreasing or $\mathcal{W}$-decreasing in K.



(a) $\Gamma(t')$ dominates $\Gamma(t)$ by parallelogram order for $K = \{1, 2\}$.

(b) $\Gamma(t')$ fails to dominate $\Gamma(t)$ for $K = \{1\}$ and $K = \{1, 2\}$.

Figure 1: The parallelogram order

Example 1. Figure 1a depicts values of a correspondence Γ for $t' >_T t$. The mapping is $\mathcal{P}$-increasing for $K = \{1, 2\}$ since, given $x \in \Gamma(t)$ and $x' \in \Gamma(t')$, we can find $y \in \Gamma(t)$ and $y' \in \Gamma(t')$ such that $x' \geq y$ and $y' \geq x$, and the four points form a parallelogram. This holds because the boundary of the set $\Gamma(\tilde{t})$ becomes flatter as $\tilde{t}$ increases. Formally, if $x_2 = \bar{x}_2(x_1, \tilde{t})$ is the equation of the boundary of $\Gamma(\tilde{t})$ (for x_1 in an interval X_1), then Γ is $\mathcal{P}$-increasing if $\bar{x}_2$ is increasing in $(x_1, \tilde{t})$ and $d\bar{x}_2/dx_1$ is decreasing in $\tilde{t}$. (See the Appendix for a proof of this claim and the converse.) Note that Γ is *not* increasing in the strong set order. Indeed, in the figure, vector $x \vee x'$ is not in $\Gamma(t')$.

In contrast, in Figure 1b, the boundary of $\Gamma(t')$ is steeper rather than flatter than that of $\Gamma(t)$. The figure depicts $x \in \Gamma(t)$ and $x' \in \Gamma(t')$ with $x'_1 < x_1$ but it is impossible to find $y \in \Gamma(t)$ and $y' \in \Gamma(t')$ such that $x'_1 \geq y_1$, $y'_1 \geq x_1$, and $x + x' = y + y'$. For the 'unsuccessful' choice of y and y' shown in the figure, $x'_1 \geq y_1$, $y \in \Gamma(t)$, and $x + x' = y + y'$, but $y' \notin \Gamma(t')$. Thus Γ is not $\mathcal{P}$-increasing in $\{1\}$ or in $\{1, 2\}$. We leave the reader to check that Γ *is* $\mathcal{P}$-increasing in $\{2\}$.

The parallelogram order is closed under scalar multiplication and addition; these features turn out to be important in certain applications (see Examples 6 and 13). In contrast, the strong set order is closed under scalar multiplication, but not under addition. For example, although the set $A' = \{1\}$ dominates $A = \{0\}$, and $B = B' = \{0, 2\}$ are (trivially) ranked in the strong set order, the set $A + B = \{0, 2\}$ is not dominated by $A' + B' = \{1, 3\}$ in this sense. We omit the obvious proof of the next result.

Proposition 1. *Let the correspondences* Γ, $\Gamma' : T \to \mathbb{R}^\ell$ *be nonempty-valued and* $\mathcal{P}$*-increasing in* $K \subseteq \{1, \ldots, \ell\}$. *Then the correspondence* $\alpha\Gamma + \Gamma'$ *is* $\mathcal{P}$*-increasing in* K*, for any* $\alpha \geq 0$.

2.1 Increasing optimal solutions

Our first main result justifies the attention we give to the parallelogram order. It states that when a family of constraint sets are ordered in this sense, then so are the solutions to an optimization problem with a linear objective.

Theorem 1. *Suppose the correspondence* $\Gamma : T \to \mathbb{R}^\ell$ *is* $\mathcal{P}$*-increasing in* K. *Then for any* $p \in \mathbb{R}^\ell$*, the correspondence* $\Phi : T \to \mathbb{R}^\ell$*, given by*

$$\Phi(t) \;:=\; \text{argmin}\Big\{p \cdot y : y \in \Gamma(t)\Big\}, \tag{1}$$

is $\mathcal{P}$*-increasing in* K. *Furthermore, if the values of* Γ *are nonempty, compact, and convex, then* Γ *is* $\mathcal{P}$*-increasing in* K *if* Φ *is* $\mathcal{W}$*-increasing in* K *for every* $p \in \mathbb{R}^\ell$.

We postpone discussion of the second claim in Theorem 1, which is an immediate consequence of Theorem 2 (see Remark 2.5). The proof of the first claim is straightforward and useful for building intuition.

Proof of the first part of Theorem 1. Take any $p \in \mathbb{R}^\ell$, $t' \geq_T t$, and $x \in \Phi(t)$, $x' \in \Phi(t')$. Since $x \in \Gamma(t)$, $x' \in \Gamma(t')$, and Γ is $\mathcal{P}$-increasing, there is $y \in \Gamma(t)$, $y' \in \Gamma(t')$ such that $x + x' = y + y'$ and $x'_K \geq y_K$, $y'_K \geq x_K$. We claim that $y \in \Phi(t)$ and $y' \in \Phi(t')$. Since $y \in \Gamma(t)$ and $x \in \Phi(t)$, it must be that $p \cdot y \geq p \cdot x$. Similarly, $p \cdot y' \geq p \cdot x'$. Thus,

$$p \cdot (y + y') \;\geq\; p \cdot (x + x') \;=\; p \cdot (y + y'),$$

which holds only if $p \cdot y = p \cdot x$ and $p \cdot y' = p \cdot x'$, and so $y \in \Phi(t)$, $y' \in \Phi(t')$. □

Remark 2.1. The first part of this result makes no ancillary assumptions on Γ and, thus, $\Phi(t)$ may be empty for some values of t. When $\Phi(t)$ is nonempty (as it will be if Γ is nonempty and compact) then Φ will also be $\mathcal{W}$-increasing in K if it is $\mathcal{P}$-increasing in K. The second part of this result states that, under ancillary assumptions on Γ which (in particular) guarantee that Φ has nonempty, compact, and convex values, then Γ must be $\mathcal{P}$-increasing in K if we require Φ to be $\mathcal{W}$-increasing in K.

Remark 2.2. While Theorem 1 focuses on minimization problems, it is clear that, for any $p \in \mathbb{R}^\ell$, the correspondence $\Psi(t) := \text{argmax}\,\{p \cdot y : y \in \Gamma(t)\}$ inherits the parallelogram order from Γ, since $y \in \Gamma(t)$ maximizes $p \cdot y$ if, and only if, it minimizes $(-p) \cdot y$.

Remark 2.3. (Transitivity of the parallelogram order) Consider three nonempty compact and convex sets where A^3 dominates A^2, and A^2 dominates A^1 in K by the parallelogram order. Define Γ by $\Gamma(t) = A^t$. By Theorem 1, for every $p \in \mathbb{R}^\ell$, $\Phi(3)$ dominates $\Phi(2)$ in K by the parallelogram order and thus also by the weak order. Similarly, $\Phi(2)$ dominates $\Phi(1)$ in K by the weak order. Clearly, this implies that $\Phi(3)$ dominates $\Phi(1)$ in K by the weak order. The converse part of Theorem 1 then allows us to conclude that $\Gamma(3) = A^3$ dominates $\Gamma(1) = A^1$ in K by the parallelogram order. We conclude that domination in K by the parallelogram order is a transitive relation on the family of nonempty, compact, and convex subsets of $\mathbb{R}^\ell$. Obviously, this relation is also reflexive, and thus it constitutes a preorder. We show in Section S.1 of the Online Supplement that this relation is also anti-symmetric if $K = \{1, 2, \ldots, \ell\}$.

Remark 2.4. The conclusion in the first part of Theorem 1 that Φ is $\mathcal{P}$-increasing, rather than just $\mathcal{W}$-increasing is important. It guarantees that an increasing selection exists (see Proposition 2). It also guarantees comparative statics in models where the agent applies a two- (or multi-)stage optimization procedure, with a linear objective applied to obtain a set of preliminary choices, before another linear objective is applied to this set to obtain the final set of choices. For example, Pareto optimal or efficient outcomes can be characterized by applying a sequence of linear objectives (see Che et al. (2020)), and thus our result can be used to study the comparative statics of Pareto optimal points. The Online Supplement (Section S.3) discusses such an application in detail.

The following example illustrates the use of Theorem 1.

Example 2. A firm hires an employee with a utility u that depends on the effort level $e \geq 0$ the employee exerts, and the payment $c \geq 0$ to the employee. The employee has an outside opportunity that yields utility t. The firm transforms e into revenue re, where $r > 0$. Thus, it chooses (e, c) to maximize $re - c$ subject to $\Gamma(t) = \{(e, c) \in X : u(e, c) \geq t\}$, where X is the domain of u. Assuming that $u(e, c)$ is strictly decreasing in e and strictly increasing in c, the indifference curves are upward sloping, as depicted in

Figure 1(a) (with effort on the horizontal axis). If Γ is $\mathcal{P}$-increasing, then an improvement in the outside opportunity t will lead to the firm paying more and requiring higher effort. This holds if the indifference curves $c = \bar{c}(e, t)$ become flatter with higher t, i.e., for each e, the derivative $d\bar{c}/de$ increases with t (see Example 1). A quick check with implicit differentiation will confirm that this occurs if u is supermodular in (e, c) and convex in c.

Theorem 1 guarantees the pairwise comparability of the optimal solutions, in the sense that $\Phi(t)$ and $\Phi(t')$ are ordered whenever t and t' are ordered. The next result (proved in the Appendix) states that there is an increasing selection from Φ. We note that existence of an increasing selection relies on Φ being $\mathcal{P}$-increasing in K; if Φ is merely $\mathcal{W}$-increasing in K then there is no guarantee that an increasing selection exists (see Example 3.4 in Kukushkin (2013)).

Proposition 2. *Suppose that the correspondence $\Phi : T \to \mathbb{R}^\ell$ is $\mathcal{P}$-increasing in K and has nonempty and compact values. Then there is a function $\phi : T \to \mathbb{R}^\ell$ such that $\phi(t) \in \Phi(t)$ for all $t \in T$, and $\phi_K(t') \geq \phi_K(t)$ whenever $t' \geq_T t$.*

2.2 Value functions

In this subsection we investigate the properties of value functions corresponding to optimisation problems with linear objectives.

Given the constraint sets $\Gamma(t)$ and $p \in \mathbb{R}^\ell$, we define the *value function* $f : \mathbb{R}^\ell \times T \to \mathbb{R}$ by $f(p, t) := \min \{p \cdot y : y \in \Gamma(t)\}$. This function has *increasing differences in* (p_K, t) if, for any $p'_K \geq p_K$ and $t' \geq_T t$, and p_{-K},

$$f\big((p'_K, p_{-K}), t'\big) - f\big((p_K, p_{-K}), t'\big) \;\geq\; f\big((p'_K, p_{-K}), t\big) - f\big((p_K, p_{-K}), t\big). \qquad (2)$$

For certain comparative statics applications (see Propositions 3 and 8), this property on f plays a crucial role and the next theorem gives a characterization of this property.

The condition on Γ needed to guarantee that f satisfies increasing differences is close to, but not identical with, what is needed to guarantee that Φ is $\mathcal{P}$-increasing in K. The required property is that the convex hull of $\Gamma(t')$, which we denote by $\operatorname{co}\Gamma(t')$, dominates $\operatorname{co}\Gamma(t)$ in K by the parallelogram order whenever $t' \geq_T t$ (see Theorem 2 below); in other words, the correspondence $\operatorname{co}\Gamma$ is $\mathcal{P}$-increasing in K. This property is weaker than requiring Γ to be $\mathcal{P}$-increasing in K. Indeed, one could check that $\operatorname{co}\Gamma(t')$ dominates $\operatorname{co}\Gamma(t)$ in K by the parallelogram order if (and, obviously, only if) for any $x \in \Gamma(t)$, $x' \in \Gamma(t')$, there is $y \in \operatorname{co}\Gamma(t)$, $y' \in \operatorname{co}\Gamma(t')$ such that $x + x' = y + y'$ and $x'_K \geq y_K$, $y'_K \geq x_K$.[6] It immediately follows from this observation that Γ is $\mathcal{P}$-increasing in K only if the correspondence $\operatorname{co}\Gamma$ is $\mathcal{P}$-increasing in K.

Theorem 2. *Let T be a poset and $\Gamma : T \to \mathbb{R}^\ell$ be a correspondence with nonempty and compact values. For any $K \subseteq \{1, 2, \ldots, \ell\}$, the following statements are equivalent.*

(i) $\operatorname{co}\Gamma$ *is $\mathcal{P}$-increasing in K.*

(ii) *For any $p \in \mathbb{R}^\ell$, $\operatorname{co}\Phi$ is $\mathcal{P}$-increasing in K, where Φ is defined as in (1).*

(iii) *For any $p \in \mathbb{R}^\ell$, Φ is $\mathcal{W}$-increasing in $\{i\}$, for each $i \in K$.*

(iv) *The value function $f : \mathbb{R}^\ell \times T \to \mathbb{R}$ has increasing differences in (p_K, t).*

Proof. By Theorem 1, if (i) holds then $\Psi(t) := \operatorname{argmin} \{p \cdot y : y \in \operatorname{co}\Gamma(t)\}$ is $\mathcal{P}$-increasing in K; (ii) follows immediately since $\Psi(t) = \operatorname{co}\Phi(t)$.

To show that (ii) implies (iii), let $\operatorname{co}\Phi$ be $\mathcal{P}$-increasing for K. Then, $t' \geq_T t$ and $x \in \Phi(t)$ imply $y'_K \geq x_K$, for some $y' \in \operatorname{co}\Phi(t')$. Thus, there are vectors $z^j \in \Phi(t')$ and numbers $\alpha^j \geq 0$, for $j = 1, \ldots, m$, such that $y' = \sum_{j=1}^m \alpha^j z^j$ and $\sum_{j=1}^m \alpha^j = 1$. Since $y'_K \geq x_K$, there is j with $z_i^j \geq x_i$. Analogously, for any $x' \in \Phi(t')$ and $i \in K$, there is some $z \in \Phi(t)$ satisfying $x'_i \geq z_i$.

We prove that (iii) implies (iv). It is well-known that f is a concave function. In particular, the map from $z \in [p_i, p'_i]$ to $f(z, p_{-i}, t)$ is concave and continuous over the interval $[p_i, p'_i]$. Hence, it is absolutely continuous and, thus, almost everywhere differentiable (see Theorem 25.5 in Rockafellar, 1970), with

$$f\big((p'_i, p_{-i}), t\big) - f\big((p_i, p_{-i}), t\big) \;=\; \int_{p_i}^{p'_i} \frac{\partial f}{\partial p_i}\big((z, p_{-i}), t\big) dz.$$

By Theorem 25.1 in Rockafellar (1970) (Shephard's Lemma), if $\partial f/\partial p_i\big((z, p_{-i}), t\big)$ exists then $y_i = y'_i = \partial f/\partial p_i\big((z, p_{-i}), t\big)$, for any $y, y' \in \Phi(t)$, where $p = (z, p_{-i})$. Since Φ is $\mathcal{W}$-increasing $\{i\}$ (for $i \in K$), for any $t' \geq_T t$, we have $\partial f/\partial p_i(z, p_{-i}, t) \leq \partial f/\partial p_i(z, p_{-i}, t')$,

[6] Indeed, for any $x \in \operatorname{co}\Gamma(t)$ and $x' \in \operatorname{co}\Gamma(t')$, we can find $\alpha^j \geq 0$, x^j and x'^j such that $\sum_{j=1}^J \alpha^j x^j = x$ and $\sum_{j=1}^J \alpha^j x'^j = x'$. For each pair, x^j and x'^j, there is $y^j \in \operatorname{co}\Gamma(t)$ and $y'^j \in \operatorname{co}\Gamma(t')$ such that $y^j \leq x'^j$, $x^j \leq y'^j$, and $x^j + x'^j = y^j + y'^j$. The required condition holds with $y = \sum_{j=1}^J \alpha^j y^j$ and $y' = \sum_{j=1}^J \alpha^j y'^j$.

for almost all $z \in [p_i, p_i']$. This leads to $f\big((p_i', p_{-i}), t\big) - f\big((p_i, p_{-i}), t\big) \ \leq \ f\big((p_i', p_{-i}), t'\big) - f\big((p_i, p_{-i}), t'\big)$. Thus, we have shown that the function f has increasing differences in (p_i, t), for any $i \in K$, which implies that f has increasing differences in (p_K, t).

The claim that (iv) implies (i) is harder to prove and we leave that to the Appendix. Our argument employs the separating hyperplane theorem to show that if $\operatorname{co}\Gamma$ fails to be $\mathcal{P}$-increasing, then f violates increasing differences. □

Remark 2.5. We claim in Theorem 1 that, when Γ has nonempty, compact, and convex values then it is $\mathcal{P}$-increasing for K if Φ is $\mathcal{W}$-increasing in K (for all $p \in \mathbb{R}^\ell$). This follows from the equivalence of (i) and (iii) in Theorem 2 and the following observations: firstly, if Γ is convex-valued, then $\Gamma = \operatorname{co}\Gamma$ and, secondly, when Φ is $\mathcal{W}$-increasing in K then it is $\mathcal{W}$-increasing in $\{i\}$ for each $i \in K$.

Remark 2.6. Comparing this result with Theorem 1, we see that the $\mathcal{P}$-increasing property on $\operatorname{co}\Gamma$ guarantees that Φ is increasing in $\{i\}$ for each $i \in K$, but not necessarily in K *jointly*. This phenomenon is illustrated in Example 4 in the following section.

Theorem 2 determines the comparative statics of minimization problems with an arbitrary linear objective. However, in some applications we require comparative statics over the narrower class of *strictly increasing* linear objectives. The next theorem (which we prove in the Appendix) covers that case. Instead of requiring Γ to be compact-valued, we impose the following, more permissive, regularity requirement on the correspondence.

Definition 2 (Regularity)**.** Correspondence Γ is *regular* if the set $\Gamma(t)$ is closed, upward comprehensive, and its asymptotic cone, $\mathbf{A}\Gamma(t)$, is equal to $\mathbb{R}^\ell_+$, for all $t \in T$.[7]

The regularity condition on Γ guarantees that $\operatorname{co}\Gamma(t)$ is a closed set and that $\Phi(t)$ is nonempty for all $p \in \mathbb{R}^\ell_{++}$.[8] A sufficient (but by no means necessary) condition for $\mathbf{A}\Gamma(t) = \mathbb{R}^\ell_+$ is for $\Gamma(t)$ to be *bounded from below*, i.e., there is $\underline{x}^t$ such that $y \geq \underline{x}^t$, for all $y \in \Gamma(t)$.

[7] A set $S \subseteq \mathbb{R}^\ell$ is *upward comprehensive* if $x \in S$ and $x' \geq x$ implies $x' \in S$. The *asymptotic cone* of S contains limits of sequences $\{\lambda_n x_n\}$, where $x_n \in S$ and $\{\lambda_n\}$ is a positive sequence converging to 0. Since $\Gamma(t)$ is upward comprehensive, it is straightforward to check that $\mathbf{A}\Gamma(t)$ contains $\mathbb{R}^\ell_+$; regularity requires that the two sets are equal.

[8] This claim is related to fairly standard results in convex analysis; for completeness we provide a proof in Section S.2 (Proposition S.2) of the Online Supplement.

Theorem 3. *Let T be a poset and $\Gamma : T \to \mathbb{R}^\ell$ be a regular correspondence. For any $K \subseteq \{1, 2, \dots, \ell\}$, the following statements are equivalent.*

(i) $\operatorname{co}\Gamma$ *is $\mathcal{P}$-increasing in K.*

(ii) *For any $p \in \mathbb{R}^\ell_{++}$, $\operatorname{co}\Phi$ is $\mathcal{P}$-increasing in K, where Φ is given in (1).*

(iii) *For any $p \in \mathbb{R}^\ell_{++}$, Φ is $\mathcal{W}$-increasing in $\{i\}$, for each $i \in K$.*

(iv) *The value function $f : \mathbb{R}^\ell_{++} \times T \to \mathbb{R}$ has increasing differences in (p_K, t).*

Remark 2.7. Suppose that, in addition to the properties listed in this theorem, Γ is convex-valued. Then, the equivalence of (i) and (iii) means that Γ is $\mathcal{P}$-increasing in K if, for every $p \in \mathbb{R}^\ell_{++}$, Φ is $\mathcal{W}$-increasing in $\{i\}$, for each $i \in K$.

Remark 2.8. There is an analogous version of Theorem 3 for maximization problems. If Γ has closed, downward comprehensive, and bounded from above values, the following are equivalent: (i) $\operatorname{co}\Gamma$ is $\mathcal{P}$-increasing in K; (ii) $\operatorname{co}\tilde{\Phi}$ is $\mathcal{P}$-increasing in K, where $\tilde{\Phi}(t) := \operatorname{argmax}\{p \cdot y : y \in \Gamma(t)\}$, for any $p \in \mathbb{R}^\ell_{++}$; (iii) $\tilde{\Phi}$ is $\mathcal{W}$-increasing in $\{i\}$, for each $i \in K$ and for any $p \in \mathbb{R}^\ell_{++}$; (iv) the value $\tilde{f}(p, t) := \max\{p \cdot y : y \in \Gamma(t)\}$ has increasing differences in (p_K, t). This result can be obtained by applying Theorem 3 to the correspondence $\Gamma^* : T \to \mathbb{R}^\ell$, where $\Gamma^*(t) = -\Gamma(t)$; note that $\operatorname{co}\Gamma^*$ is $\mathcal{P}$-decreasing in K if, and only if, $\operatorname{co}\Gamma$ is $\mathcal{P}$-increasing in K.

We now discuss the relationship between our results and the monotone comparative statics results by Topkis (1978), Milgrom and Shannon (1994), and Quah (2007).

Let $(X, \geq_X)$ be a lattice and $\Gamma : T \to X$ be a correspondence that is increasing in the strong set order. Topkis (1978) showed that the correspondence of optimal points $\Phi(t) := \operatorname{argmin}\{\phi(y) : y \in \Gamma(t)\}$ is also increasing in the strong set order if the objective function $\phi : X \to \mathbb{R}$ is *submodular*, i.e., satisfies $\phi(x) + \phi(x') \geq \phi(x \wedge x') + \phi(x \vee x')$, for any $x, x' \in X$. Observing that any comparative statics result on Φ must be independent of strictly increasing transformations of the objective function, Milgrom and Shannon (1994) generalize Topkis' result by showing that it suffices for ϕ to satisfy the ordinal counterpart of submodularity, called *quasisubmodularity*; this property requires that $\phi(x \wedge x') \geq (>)\, \phi(x)$ implies $\phi(x') \geq (>)\, \phi(x \vee x')$, for any $x, x' \in X$.

Quah (2007) observes that for certain economic problems, the strong set order on Γ is an overly strong assumption. He develops a comparative statics result that requires an ordinal condition on the objective function ϕ that is stronger than quasisubmodularity (called $\mathcal{C}$-quasisubmodularity[9]), while relaxing the strong set order requirement on Γ. Specifically, the correspondence Γ is required to be increasing in the *$\mathcal{C}$-flexible set order for $K \subseteq \{1, 2, \ldots, \ell\}$*, which means that, for any $t' \geq_T t$, $x \in \Gamma(t)$, and $x' \in \Gamma(t')$ with $x'_K \not\geq x_K$, there is some $\lambda \in [0, 1]$ such that $\left[\lambda x' + (1 - \lambda)(x \wedge x')\right] \in \Gamma(t)$ and $\left[\lambda x + (1 - \lambda)(x \vee x')\right] \in \Gamma(t')$. Obviously, this order is weaker than the strong set order, which corresponds to the case where $\lambda = 0$. It is shown that if Γ increases in the $\mathcal{C}$-flexible set order for K, then so does Φ, for any $\mathcal{C}$-quasisubmodular function ϕ.

In this paper, we push the approach in Quah (2007) even further, by requiring the objective ϕ to be *linear*, in order to obtain the most permissive conditions on Γ needed for monotone comparative statics. Notice that the $\mathcal{C}$-flexible set order is a special case of the parallelogram order since, if we let $y = \lambda x' + (1 - \lambda)(x \wedge x')$ and $y' = \lambda x + (1 - \lambda)(x \vee x')$, then $y' \geq x$, $x' \geq y$, and $x + x' = y + y'$. In some applications, the parallelogram order holds but the $\mathcal{C}$-flexible set order does not. For example, we know that $\Gamma(t')$ dominates $\Gamma(t)$ by the parallelogram order in Figure 1a, yet the $\mathcal{C}$-flexible set order *cannot* hold between these sets. This is because the boundaries of $\Gamma(t)$ and $\Gamma(t')$ are strictly upward sloping and we could always choose x and x' on the boundary of $\Gamma(t)$ and $\Gamma(t')$, respectively, as depicted in Figures 1(a), with the entire line connecting x' and $x \vee x'$ outside of $\Gamma(t')$.

3 Normal demand and monotone marginal cost

The first part of this section applies our main results to obtain comparative statics properties on conditional factor demand, marginal cost, and firm output. The second part applies our results to analyze Marshallian demand. The section ends with a comparison of our findings with the existing literature.

3.1 Factor demand, marginal cost, and optimal output

It is commonplace to hear that a firm has raised the price of its output because the prices of raw materials have gone up. While this link may seem very natural it is not always correct and, in this subsection, we spell out the precise conditions under which it is.[10]

[9] A sufficient condition for this property is that ϕ is submodular and convex.

[10] We would like to thank the anonymous referee who suggested this motivation.

This involves examining the relationships linking factor demand, marginal cost, and the firm's profit-maximizing output.

We assume that the firm has a continuous and increasing production function $F : \mathbb{R}^\ell_+ \to \mathbb{R}$. We denote the range of F by $Q := \{F(x) : x \in \mathbb{R}^\ell_+\}$.[11] For any $q \in Q$, let

$$U(q) \;:=\; \Big\{x \in \mathbb{R}^\ell : F(x) \geq q\Big\}.$$

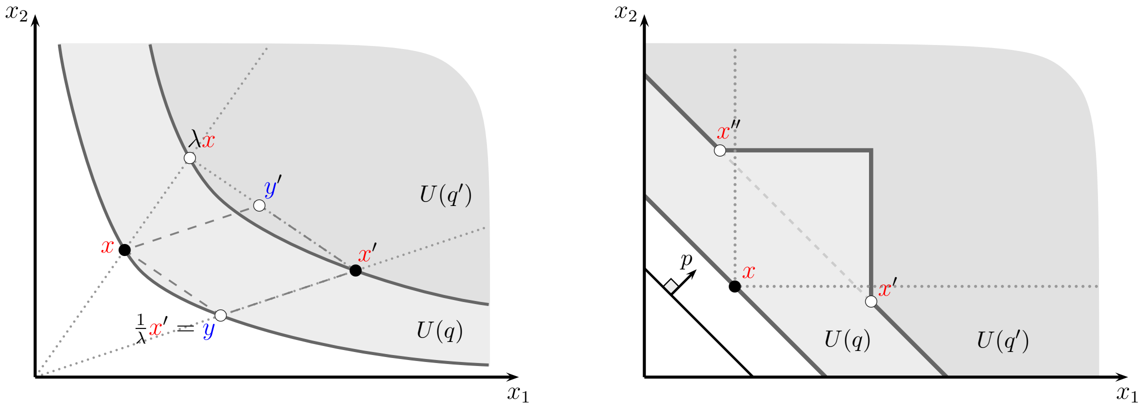
We refer to U as the *upper contour correspondence of* F. The set $U(q)$ is upward comprehensive, bounded from below by 0, and closed (since F is continuous).

Definition 3. The production function $F : \mathbb{R}^\ell \to \mathbb{R}$ is $\mathcal{P}$*-increasing* in K if its upper contour correspondence $U : Q \to \mathbb{R}$ is $\mathcal{P}$-increasing in K (where Q is the range of F). The function is *quasi-*$\mathcal{P}$*-increasing* in K whenever the correspondence $q \to \operatorname{co} U(q)$ is $\mathcal{P}$-increasing in K.

We assume that the firm is a price-taker in the market for factors/inputs and faces *strictly* positive input prices $p = (p_1, p_2, \ldots, p_\ell)$. The *conditional factor/input demand* at p and output $q \in Q$ refers to those bundles that achieve output of at least q with the least cost. Formally, input demand is the correspondence $H : \mathbb{R}^\ell_{++} \times Q \to \mathbb{R}^\ell_+$, where $H(p, q) = \operatorname{argmin}\{p \cdot x : x \in U(q)\}$. Our assumptions on F guarantee that $H(p, q)$ is nonempty and compact, for all (p, q) in $\mathbb{R}^\ell_{++} \times Q$. The associated cost function is $C(p, q) = \min\{p \cdot x : x \in U(q)\}$.

Conditional Factor Demand Theorem 1 tells us that if U is $\mathcal{P}$-increasing in K then the conditional factor demand $H(p, \cdot)$ is also $\mathcal{P}$-increasing in K. In particular, the factor demand is $\mathcal{W}$-increasing in K (as target output q increases); the conventional usage in this context would say that demand is *normal* (more precisely, *jointly normal*) *in* K. We may also conclude that, for each p, $H(p, \cdot)$ admits a selection $h(p, q) \in H(p, q)$, for all $q \in Q$, such that $h_K(p, q') \geq h_K(p, q)$ whenever $q' \geq q$ (see Proposition 2). If F is quasiconcave (so that $U(q)$ is convex) we obtain the converse result that U is $\mathcal{P}$-increasing in K if, at every factor price $p \in \mathbb{R}^\ell_{++}$, the factor demand for each $i \in K$ is normal. This follows from Theorem 3 (Remark 2.7). The following is a basic example of a $\mathcal{P}$-increasing production function; more examples are presented at the end of this subsection.

[11] Our analysis could be performed for functions F defined over a general domain $X \subseteq \mathbb{R}^\ell_+$. However, to keep the exposition simple, we restrict our attention to the special case of $X = \mathbb{R}^\ell_+$.



(a) A homothetic function is $\mathcal{P}$-increasing.

(b) The set of cost-minimisers is not $\mathcal{P}$-increasing in $\{1,2\}$, but its convex hull is.

Figure 2: Illustrations for Examples 3 (left) and 4 (right).

Example 3. A production function $F : \mathbb{R}^\ell_+ \to \mathbb{R}$ is homothetic if $F(x') \geq F(x)$ implies $F(\lambda x') \geq F(\lambda x)$, for any scalar $\lambda > 0$. It is well-known that if F is homothetic then its factor demand is jointly normal; in fact, the following (much stronger) property holds: if $x \in H(p,q)$, then $tx \in H(p,q')$, where $t > 1$ if $q' > q$ and $t < 1$ if $q' < q$. Thus a production function that is homothetic and quasiconcave must be $\mathcal{P}$-increasing.

This can be directly confirmed. Suppose $x \in U(q)$ and $x' \in U(q')$, with $q' > q$. We focus on the case where $F(x) < q' \leq F(x')$ and x' and x are not ordered (the other cases being straightforward). Since F is continuous and increasing, there is a scalar $a > 1$ such that $F(x) = F(x'/a)$ and, by the homotheticity of F, we obtain $F(ax) = F(x')$ (see Figure 2a). Set $y = x'/a$ and $y' = x' + (x - (x'/a))$. The bundle y' is on the line segment joining ax and x' and, since F is quasiconcave, $y' \in U(q')$. We also have $y \leq x'$, $x \leq y'$ and $x + x' = y + y'$, as required by the parallelogram property.

Marginal cost The cost function C has increasing differences in (p_K, q) if, for any input prices $p'_K \geq p_K$, p_{-K}, and output levels $q' \geq q$, we have

$$C\big((p'_K, p_{-K}), q'\big) - C\big((p'_K, p_{-K}), q\big) \;\geq\; C\big((p_K, p_{-K}), q'\big) - C\big((p_K, p_{-K}), q\big). \tag{3}$$

Thus, the increase in cost when output is raised from q to q' is greater at the input prices (p'_K, p_{-K}) compared to the prices (p_K, p_{-K}), when $p'_K \geq p_K$. When C is differentiable in output, this is equivalent to $\partial C/\partial q\big((p_K, p_{-K}), q\big)$ being increasing in p_K, i.e., an increase in the price of a factor in the set K leads to a higher marginal cost.

By Theorem 3, the cost function C has increasing differences in (p_K, q) if, and only if, $\operatorname{co} U$ is $\mathcal{P}$-increasing in K (equivalently, the function F is quasi-$\mathcal{P}$-increasing). This is enough to guarantee that demand is normal in $\{i\}$ for each $i \in K$, but not enough to guarantee that demand is *jointly* normal in K (which requires U to be $\mathcal{P}$-increasing in K). In other words, joint normality is a strictly stronger property. The next example illustrates this distinction.

Example 4. The production function $F : \mathbb{R}^2_+ \to \mathbb{R}$ has isoquants depicted in Figure 2b. It is quasi-$\mathcal{P}$-increasing, since $\operatorname{co} U(q) = \big\{(x_1, x_2) : x_1 + x_2 \geq q\big\}$, for all $q \in Q$. In line with statement (iii) in Theorem 3, factor demand is normal in factor 1 and in factor 2 separately. For example, as shown in the figure, when $p_1 = p_2$ and given $x \in H(p, q)$, there is $x' \in H(p, q')$ such that $x'_1 \geq x_1$, and $x'' \in H(p, q')$ such that $x''_2 \geq x_2$. However, U is not $\mathcal{P}$-increasing for $K = \{1, 2\}$ and joint normality in both goods does not hold: there is no bundle in $H(p, q')$ that is higher than x in both goods. Nonetheless, due to the quasi-$\mathcal{P}$-increasing property, the cost function has increasing differences in $\big((p_1, p_2), q\big)$.

Optimal output Our ability to sign the impact of higher factor prices on marginal cost allows us to predict how the firm's profit-maximizing output will change. Let $R : Q \to \mathbb{R}_+$ be the revenue that the firm earns when it produces $q \in Q$. The firm chooses $q \in Q$ to maximize profit $\Pi(p, q) = R(q) - C(p, q)$. When C has increasing differences in (p_K, q), a rise in the price of any factor in K raises marginal cost and reduces the profit-maximizing output. Conversely, if the optimal output decreases with p_K for any increasing revenue function R, then C must have increasing differences in (p_K, q). The following result (which we prove in the Appendix) summarizes our claims.

Proposition 3. *For any continuous and increasing production function $F : \mathbb{R}^\ell_+ \to \mathbb{R}$, the following statements are equivalent.*

(i) *F is quasi-$\mathcal{P}$-increasing in K.*

(ii) *For all $p \in \mathbb{R}^\ell_{++}$, $H(p, \cdot)$ is normal in $\{i\}$, for each $i \in K$.*

(iii) *C has increasing differences in (p_K, q).*

(iv) *The set $\operatorname{argmax}\big\{R(q) - C(p, q) : q \in Q\big\}$ is decreasing in $p_K \gg 0$ in the strong set order, for any function $R : Q \to \mathbb{R}$.*

We are now in a position to address the issue we raised at the beginning of this subsection. Proposition 3 states that the quasi-$\mathcal{P}$-increasing property in K on the production function F is sufficient and (in a certain specific sense) necessary for the profit-maximizing output to fall when p_K increases. Provided the price of output is determined by a downward sloping demand curve, this also implies that the firm charges a higher price for its output when p_K increases.

Further examples We end this subsection with three more examples of $\mathcal{P}$-increasing production functions (besides the homothetic case already considered in Example 3).

Example 5. A function $F : \mathbb{R}^\ell_+ \to \mathbb{R}$ is *increasing in* $K \subseteq \{1, 2, \ldots, \ell\}$ *by the* $\mathcal{C}$*-flexible set order* if the correspondence $q \to U(q)$ is increasing in K by the $\mathcal{C}$-flexible set order; this property implies that F is $\mathcal{P}$-increasing in K. (Recall the discussion on the $\mathcal{C}$-flexible set order at the end of Section 2.) It is known that F is increasing in K by the $\mathcal{C}$-flexible set order if it is continuous, increasing, supermodular, and concave in x_{-i}, for $i \in K$; for a proof see Quah (2007) or Section S.4 of the Online Supplement. For example, $F(x_1, x_2, x_3) := \sqrt{x_1 x_2} + x_3$ for $(x_1, x_2, x_3) \in \mathbb{R}^3_+$ is increasing, supermodular, and concave and thus is increasing (in $K = \{1, 2, 3\}$) by the $\mathcal{C}$-flexible set order.

The next two examples give economically interpretable ways of constructing new $\mathcal{P}$-increasing production functions from other production functions with that property.

Example 6. Suppose that one unit of output can be produced from one unit each of n intermediate goods. Each intermediate good is produced from ℓ factors, with $f^j : \mathbb{R}^\ell_+ \to \mathbb{R}$ being the production function for the jth intermediate good. Assuming that any bundle x of factors is efficiently assigned, the firm's production function $F : \mathbb{R}^\ell_+ \to \mathbb{R}$ is

$$F(x) := \max \left\{ \min_{j=1,\ldots,n} \{f^j(y^j)\} : x \geq \sum_{j=1}^n y^j \right\}.$$

It is straightforward to check that its upper contour correspondence U satisfies $U(q) = \sum_{j=1}^n U^j(q)$, where U^j is the upper contour correspondence associated with f^j. By Proposition 1, U is $\mathcal{P}$-increasing in K if each U^j is $\mathcal{P}$-increasing in K. For example, $F(x_1, x_2, x_3) := \min \{x_1^2, x_2 + x_3\}$ is $\mathcal{P}$-increasing since $f^1(x) = x_1^2$ and $f^2(x) = x_2 + x_3$ are both homothetic and thus $\mathcal{P}$-increasing. (In fact, both functions are also increasing in the $\mathcal{C}$-flexible set order.) While F is $\mathcal{P}$-increasing, it is clearly not homothetic and neither is it increasing in the $\mathcal{C}$-flexible set order.[12]

Example 7. Another aggregation procedure preserving the parallelogram order is as follows. Given production functions $f^j : \mathbb{R}^\ell_+ \to \mathbb{R}$ $(j = 1, 2 \ldots, n)$, define $F : \mathbb{R}^\ell_+ \to \mathbb{R}$ by

$$F(x) \;:=\; \max\left\{G\Big(f^1(y^1), f^2(y^2), \ldots, f^n(y^n)\Big) : x \geq \sum\nolimits_{j=1}^{n} y^j\right\}, \tag{4}$$

where $G : \times_{j=1}^n Q^j \to \mathbb{R}$ is an increasing function that aggregates the values of f^j, and Q^j contains the range of f^j. We show in the Appendix that F is $\mathcal{P}$-increasing in K provided that f^j is continuous, concave, and is $\mathcal{P}$-increasing in K (for each j), and the aggregating function G is increasing in the $\mathcal{C}$-flexible set order.[13]

For example, if $G(q^1, q^2, \ldots, q^n) = \sum_{k=1}^n q^k$, then G is increasing in the $\mathcal{C}$-flexible set order. In this case F can be interpreted as the production function of a firm that allocates its output efficiently across n plants, with plant j having the production function f^j. Thus, the following functions are $\mathcal{P}$-increasing.

(i) $F(x_1, x_2, x_3, x_4) := \sqrt{x_1\, x_2} + \sqrt{x_3 + x_4}$;

(ii) $F(x_1, x_2, x_3) := \max\left\{\sqrt{x_1\, y} + \sqrt{x_3 + z} : x_2 \geq y + z\right\}$.

In (i) the two plants produce output with completely distinct factors (the first plant with factors 1 and 2 and the second with factors 3 and 4) while in case (ii), one factor (factor 2) is used by both plants. Notice that both plants have homothetic production functions, but since one plant has constant returns to scale while the other has diminishing returns, F is not homothetic in both (i) and (ii).

3.2 Consumer demand

Our results on conditional factor demand can be straightforwardly re-formulated to guarantee that Marshallian demand is normal. Apart from being an intrinsically appealing

[12] If $x = (\sqrt{5}, 0, 5)$ and $x' = (\sqrt{6}, 6, 0)$, then $F(x) = 5$, $F(x') = 6$, $F(x \wedge x') = F\big((\sqrt{5}, 0, 0)\big) = 0$, and $F(x \vee x') = F\big((\sqrt{6}, 6, 5)\big) = 6$. For F to be increasing in the $\mathcal{C}$-flexible set order, we must find $\lambda \in [0, 1]$ such that $F\big(\lambda x' + (1-\lambda)(x \wedge x')\big) \geq F(x) = 5$ and $F\big(\lambda x + (1-\lambda)(x \vee x')\big) \geq F(x') = 6$. But this is impossible since $F(x \wedge x') < 5$ and if $\lambda > 0$, $F\big(\lambda x + (1-\lambda)(x \vee x')\big) < 6$.

[13] This example does not generalize Example 6: while the 'min' function is increasing in the $\mathcal{C}$-flexible set order, Example 6 does not require f^j to be concave, which is required in this example.

property for most product categories, normality plays an important role in many model settings. For example, normality is used in Bergstrom et al. (1986) to guarantee the uniqueness of Nash equilibria in a public goods game; the results on general equilibrium comparative statics in Nachbar (2002) and Quah (2003) hinge on their assumption that demand is normal; in Blundell et al. (2005), normality helps to determine how provision of a public good varies with intra-household bargaining power; and normality simplifies the non-parametric estimation of demand functions in Blundell et al. (2003). Thus it is important to have a thorough understanding of the foundations of this property.

Suppose a consumer has a utility function $u : \mathbb{R}^\ell_+ \to \mathbb{R}$, defined over bundles of ℓ commodities. At prices $p \in \mathbb{R}^\ell_{++}$ and income $m \geq 0$, the consumer chooses a consumption bundle $x \in \mathbb{R}^\ell_+$ that is affordable and maximizes her utility; the solution to this problem is captured by the *Marshallian demand correspondence* $D : \mathbb{R}^\ell_{++} \times \mathbb{R}_+ \to \mathbb{R}^\ell_+$, where $D(p, m) := \operatorname{argmax}\big\{u(x) : p \cdot x \leq m\big\}$. The indirect utility function $v : \mathbb{R}^\ell_{++} \times \mathbb{R}_+ \to \mathbb{R}$ is given by $v(p, m) := \max\big\{u(x) : p \cdot x \leq m\big\}$.

Given the range V of u, the *Hicksian demand* $H : \mathbb{R}^\ell_{++} \times V \to \mathbb{R}^\ell_+$ maps prices p and utility levels v to those bundles that minimize the expenditure $p \cdot x$ over all alternatives satisfying $u(x) \geq v$. Obviously, the Hicksian demand is formally identical to the input demand in the production context, while the analog to the cost function is the expenditure function $e : \mathbb{R}^\ell_{++} \times V \to \mathbb{R}_+$, where $e(p, v) := \min\big\{p \cdot x : u(x) \geq v\big\}$.

Suppose that utility u is continuous and locally non-satiated.[14] In such a case, correspondences D and H are well-defined. Moreover, we have $p \cdot x = m$, for all $x \in D(p, m)$, while the two demands are related by the identity $D(p, m) = H\big(p, v(p, m)\big)$, for any prices p and income m (see Proposition 3.E.1 in Mas-Colell et al., 1995).

Let $K \subseteq \{1, 2, \ldots, \ell\}$. We say that D is *normal* in K if, for any prices p, income levels m, m', and $x \in D(p, m)$, there is $x' \in D(p, m')$ such that $m' \geq m$ implies $x'_K \geq x_K$ and $m' \leq m$ implies $x'_K \leq x_K$. The demand D is normal in input i if it is normal in $K = \{i\}$. Finally, if $K = \{1, 2, \ldots, \ell\}$, we simply say that D is normal.

The equivalence of demands D and H allows us to translate normality results on Hicksian demand into results on Marshallian demand. In particular, the following result on utility functions that are $\mathcal{P}$-increasing (in the sense defined in Section 3.1) follows

[14] Utility function u is *locally non-satiated* if, for any bundle $x \in \mathbb{R}^\ell_+$, there is another bundle y arbitrarily close to x such that $u(y) > u(x)$.

immediately from Theorem 3 and Remark 2.7.

Proposition 4. *Let $u : \mathbb{R}^\ell_+ \to \mathbb{R}$ be a continuous and increasing utility function. If u is $\mathcal{P}$-increasing in $K \subseteq \{1, 2, \ldots, \ell\}$ then, for any prices $p \in \mathbb{R}^\ell_{++}$, the correspondence $m \to D(p, m)$ is $\mathcal{P}$-increasing in K. In particular, D is normal in K. Furthermore, if u is quasiconcave and D is normal in K, then u is $\mathcal{P}$-increasing in K.*

It is well-known that if a Marshallian demand function d is normal for good i, then the demand for i obeys the *law of demand*, i.e., function $d_i\big((p_i, p_{-i}), m\big)$ is decreasing in p_i, for all p_{-i} and m. We know that if, for all $p \in \mathbb{R}^\ell_{++}$, the Marshallian demand $D(p, \cdot)$ is $\mathcal{P}$-increasing in K, then D admits a selection d that satisfies normality (see Proposition 2). Therefore, an immediate consequence of Proposition 4 is the following: *for any continuous and locally non-satiated utility $u : \mathbb{R}^\ell_+ \to \mathbb{R}$ that is $\mathcal{P}$-increasing in K, there is $d(p, m) \in D(p, m)$ that obeys the law of demand for every good $i \in K$.*

Example 8. Consider an agent who lives for ℓ periods and has a preference over consumption streams $x = (x_1, x_2, \ldots, x_\ell) \in \mathbb{R}^\ell_+$. In this context, it is natural to assume that this agent's utility has a recursive form, where

$$u(x) \; := \; h_1\Bigg(x_1, h_2\bigg(x_2, h_3\Big(x_3, \ldots h_{\ell-1}\big(x_{\ell-1}, x_\ell\big)\Big)\bigg)\Bigg)$$

Let $h_i : \mathbb{R}^2_+ \to \mathbb{R}_+$ be continuous, increasing, concave, and supermodular. Then h_i is increasing in the $\mathcal{C}$-flexible set order (see Example 5); consequently, the map from $(x_{\ell-2}, x_{\ell-1}, x_\ell)$ to $h_{\ell-2}\big(x_{\ell-2}, h_{\ell-1}(x_{\ell-1}, x_\ell)\big)$ is $\mathcal{P}$-increasing[15] and concave, because both $h_{\ell-2}$ and $h_{\ell-1}$ are concave functions. Repeating this argument, we eventually conclude that u is concave and $\mathcal{P}$-increasing, and hence it has a normal Marshallian demand.

3.3 Related results on normal demand

Some version of the equivalence between normality of demand and monotonicity of marginal costs is known at least since Fisher (1990). Fisher's original argument assumes that F is differentiable and generates a unique demand; we dispense with these assumptions. Our argument that normality implies increasing marginal costs (which is

[15] We obtain this by applying the result in Example 7 with $G = h_{\ell-2}$, $f^1(x_{\ell-2}, x_{\ell-1}, x_\ell) = x_{\ell-2}$, and $f^2(x_{\ell-2}, x_{\ell-1}, x_\ell) = h_{\ell-1}(x_{\ell-1}, x_\ell)$.

the proof that statement (iii) implies (iv) in Theorem 2) does not significantly break new ground, but our converse result is stronger, because it does not assume that demand is unique or that F is quasiconcave. Our proof goes through statement (i) and, thus, hinges on our characterization of normality using the parallelogram order.[16]

Alarie et al. (1990) and Bilancini and Boncinelli (2010) also characterize normal demand, under the condition that the objective function $F : \mathbb{R}^{\ell}_{++} \to \mathbb{R}$ is strictly increasing, strictly quasiconcave, and twice-differentiable. Note that these conditions on F are sufficient for demand to be strictly positive for all goods and a smooth *function* of prices. Let $\mathbf{G}(x)$ and $\mathbf{J}(x)$ denote the gradient and the Hessian of F, respectively, at some bundle x. The corresponding bordered Hessian $\tilde{\mathbf{J}}(x)$ is given by

$$\tilde{\mathbf{J}}(x) \;:=\; \begin{bmatrix} \mathbf{J}(x) & \mathbf{G}(x) \\ \mathbf{G}'(x) & 0 \end{bmatrix},$$

where $\mathbf{G}'(x)$ is the transposition of the column vector $\mathbf{G}(x)$. Let $\tilde{\mathbf{J}}_{i,j}(x)$ denote the (i,j)-th minor of $\tilde{\mathbf{J}}(x)$ and $\left|\tilde{\mathbf{J}}_{i,j}(x)\right|$ be its determinant. Bilancini and Boncinelli (2010) show that the demand function is normal in $K \subseteq \{1, 2, \ldots, \ell\}$ if, and only if,

$$(-1)^{i-1}\left|\tilde{\mathbf{J}}_{i,(\ell+1)}(x)\right| \;\geq\; 0, \tag{5}$$

for all $i \in K$ and each $x \in \mathbb{R}^{\ell}_{++}$. By combining Proposition 4 with this result, we conclude that the condition (5) is equivalent to F being $\mathcal{P}$-increasing in K, when F satisfies the ancillary smoothness assumptions in their setup.

4 Factor complementarity

In this section, we apply our main results to the standard quasilinear optimizing model. This model appears in many contexts in economic modeling, but to keep the exposition focused, we shall refer to the problem as one of a firm choosing a bundle of factors/inputs

[16] One could obtain a weaker version of (ii) from (iii) through a direct argument. Given any $\hat{p}_{-i}$, $p''_i > p'_i$, and $q'' > q'$, $H_i(p_i, \hat{p}_{-i}, q)$ is unique at $q = q', q''$, for almost every $p_i \in [p'_i, p''_i]$. We claim that, if (iv) holds, then there cannot be a generic violation of normality, in the sense of having $H_i(p_i, \hat{p}_{-i}, q') > H_i(p_i, \hat{p}_{-i}, q'')$ for almost every $p_i \in [p'_i, p''_i]$. Indeed, since $C(p''_i, \hat{p}_{-i}, q) - C(p'_i, \hat{p}_{-i}, q) = \int_{p'_i}^{p''_i} H_i(p_i, \hat{p}_{-i}, q)\, dp_i$, we would obtain $C(p''_i, \hat{p}_{-i}, q') - C(p'_i, \hat{p}_{-i}, q') > C(p''_i, \hat{p}_{-i}, q'') - C(p'_i, \hat{p}_{-i}, q'')$, which violates (iii). Obviously, this conclusion is weaker than (ii) (as stated in Proposition 3), where normality holds at every price for each good $i \in K$, whether or not demand is unique.

to maximize profit. We characterize production functions that lead to complementary demand, in the sense that a fall in the price of one factor leads to an increase in demand for *all* factors. We also analyze the conditions under which some (but not all) factors are complements to each other.

4.1 The parallelogram order on production possibility sets

We assume the firm chooses ℓ factors from a set X, a nonempty and closed subset of $\mathbb{R}^\ell_+$, to maximize profit. We denote the firm's production function by $F : X \to \mathbb{R}$. In some models of quasilinear demand (see, for example, Ausubel and Milgrom (2002)) the domain X is finite or discrete and we allow that here; note also that F need not be concave or quasiconcave.

Definition 4 (Regularity)**.** Let X be a closed subset of $\mathbb{R}^\ell_+$. The function $F : X \to \mathbb{R}$ is *regular* if it is continuous and $P = \Big\{(-y, v) : y \geq x \text{ and } v \leq F(x), \text{ for } x \in X\Big\}$ (its associated production possibility set) has an asymptotic cone that equals $\mathbb{R}^{\ell+1}_-$.

Since the production possibility set P is downward comprehensive, its asymptotic cone must contain $\mathbb{R}^{\ell+1}_-$. The condition that it *equals* $\mathbb{R}^{\ell+1}_-$ requires (loosely speaking) that $F(x)$ does not grow too quickly as x increases; a sufficient but not necessary condition is that F is uniformly bounded above.[17]

At factor prices $p \in \mathbb{R}^\ell_{++}$, the firm's (unconditional) *input/factor demand* consists of those bundles that maximize profit, i.e.,

$$\mathcal{H}(p) \ := \ \text{argmax}\Big\{F(x) - p \cdot x : x \in X\Big\}.$$

Our assumptions on X and F guarantee that $\mathcal{H}(p)$ is nonempty, for all $p \in \mathbb{R}^\ell_{++}$.[18] The firm's profit is $\pi(p) = F(x^*) - p \cdot x^*$, for $x^* \in \mathcal{H}(p)$.

We say that a good i is a complement of good j if the demand for i increases with the price of j. Formally, the property could be stated as follows.

Definition 5 (Complements)**.** For any j and $K \subseteq \{1, 2, \ldots, \ell\}$, the set K is a *complement* of j if, for any $p, p' \in \mathbb{R}^\ell_{++}$ satisfying $p_{-j} = p'_{-j}$, and any $x \in \mathcal{H}(p)$, there is $x' \in \mathcal{H}(p')$

[17] For example, $F(x) = \sqrt{x}$ is not bounded on $X = \mathbb{R}_+$, but is regular in our sense.

[18] In formal terms, this claim is similar to the claim that Φ is nonempty for any $p \in \mathbb{R}^\ell_{++}$ under the assumptions of Theorem 3 (see footnote 8). We prove this claim in Section S.2 of the Online Supplement. For general results linking the domain of $\mathcal{H}$ and asymptotic cones, see, e.g., Neuefeind (1980).

such that $x'_K \geq x_K$ if $p'_j \leq p_j$, and $x'_K \leq x_K$ if $p'_j \geq p_j$. The factors in K are *joint complements* if K is a complement of j, for any $j \in K$.

It is straightforward to check that the factors in K are joint complements if, and only if, for any $x \in \mathcal{H}(p_K, p_{-K})$, there is $x' \in \mathcal{H}(p'_K, p_{-K})$ such that $x'_K \geq x_K$ if $p'_K \leq p_K$, and $x'_K \leq x_K$ if $p'_K \geq p_K$.[19]

We claim that our main results could be used to characterize complementary demand. This may be surprising since those results concern how optimal solutions vary with constraint sets for a fixed linear objective, whereas complementarity deals with how optimal solutions vary as the linear objective changes (in other words, how demand varies with prices). However, a change in factor prices can be equivalently reformulated as a change in the production possibility set and thus our main results are indeed applicable.

To be precise, let Γ^j be the correspondence defined on $T = \mathbb{R}_-$ such that

$$\Gamma^j(t_j) \;:=\; \Big\{(y, v) \in \mathbb{R}^{\ell+1} : y \geq x \text{ and } v \geq -F(x) - t_j x_j, \text{ for } x \in X\Big\}. \tag{6}$$

It is easy to verify that, for any $p \in \mathbb{R}^{\ell}_{++}$, we have $x \in \mathcal{H}(p_j - t_j, p_{-j})$ if and only if $(x, -F(x) - t_j x_j) \in \Phi^j(t_j)$, where

$$\Phi^j(t_j) = \operatorname{argmin}\big\{(p, 1) \cdot y : y \in \Gamma^j(t_j)\big\} \tag{7}$$

(and thus $\pi(p_j - t_j, p_{-j}) = -\min\{(p, 1) \cdot y : y \in \Gamma^j(t_j)\}$). Notice that we have converted the complementarity property into a format suitable for the application of our main theorems, since K is a complement of j if and only if, for any $p \in \mathbb{R}^{\ell}_{++}$, the correspondence Φ^j is $\mathcal{W}$-increasing in K. We exploit this observation in the remainder of this section.

4.2 Complementarity is symmetric

We first address a fundamental question: *if factor i is a complement of factor j, then is j a complement of i?* A well-known and widely-applied result in monotone comparative statics states that the set of *all* factors are joint complements if F is a supermodular function (see Topkis, 1978). Thus the question we pose does not arise if F is supermodular, but when complementarity patterns are more complicated, is it still true that

[19] This is essentially because a change in factor prices from (p_K, p_{-K}) to (p'_K, p_{-K}) (with factor prices outside the set K being fixed) can be broken into $|K|$ steps, with the price of one good in K changing at each step.

complementarity is a symmetric property? The next result (which allows for nonconvex domains and nonconcave production functions) states that it is.

Proposition 5. *Let $F : X \to \mathbb{R}_+$ be a regular function. For any $i, j \in \{1, 2, \ldots, \ell\}$, factor i is a complement of j if, and only if, j is a complement of i. Moreover, any two factors in $K \subseteq \{1, 2, \ldots, \ell\}$ are complements if, and only if, the function $p_K \to \pi(p_K, p_{-K})$ is supermodular, for any fixed p_{-K}.*[20]

Proof. It is easy to check that the regularity of F guarantees the regularity of Γ^j. As we have observed, i is a complement of j if and only if, for any $p \in \mathbb{R}^\ell_{++}$, the correspondence Φ^j (as defined by (7)) is $\mathcal{W}$-increasing in $K = \{i\}$. Thus Theorem 3 guarantees the equivalence of the following statements: (i) $\operatorname{co}\Gamma^{\{j\}}$ is $\mathcal{P}$-increasing in $\{i\}$; (ii) i is a complement of j; and (iii) $-\pi(p_j - t_j, p_{-j})$ has increasing differences in (t_j, p_i) or, equivalently, π is supermodular in (p_i, p_j). Since supermodularity is a symmetric property, i is a complement of j if, and only if, j is a complement of i. □

4.3 Super*modularity

While the supermodularity of the production function is sufficient for all factors to be joint complements, it is certainly *not* a necessary condition for complementarity and there is an instructive way to notice this.

Suppose there are N firms (or N divisions within a firm), with firm n having the regular production function $F^n : X^n \to \mathbb{R}$. We denote its factor demand at prices $p \in \mathbb{R}^\ell_{++}$ by $\mathcal{H}^n(p)$. Whenever F^n is supermodular, then $\mathcal{H}^n$ has the property that all factors are joint complements, and hence aggregate demand $\mathcal{H}(p) = \sum_{i=1}^N \mathcal{H}^n(p)$ will also exhibit the joint complementarity property. On the other hand, it is well-known that the aggregate demand of multiple agents with quasilinear objectives admits a *representative agent*, in the sense that $\mathcal{H}(p) = \mathcal{H}^*(p)$, where $\mathcal{H}^*$ is the factor demand generated by $F^* : X^* \to \mathbb{R}$, such that $X^* = \sum_{n=1}^N X^n$ and $F^*(x) := \max\left\{ \sum_{n=1}^N F^n(y^n) : \sum_{n=1}^N y^n = x \right\}$. This raises

[20] Theorem 10 in Ausubel and Milgrom (2002) states that all inputs are substitutes (rather than complements) if, and only if, the profit function π is submodular with respect to their prices. However, their definition of substitutes applies only to those prices at which the demand is a singleton, whereas our definition of complementarity (and its obvious modification for substitutability) applies at *all* prices. Modifying our proof of Proposition 5 in the obvious way will allow us to conclude that (a) for $i \neq j$, i is a substitute of j if, and only if, j is a substitute of i, and (b) π is submodular if, and only if, i and j are substitutes for any $i \neq j$. See Section S.5 of the Online Supplement for a fuller discussion.

the question of whether the representative agent's utility function F^* is also supermodular if every F^n is supermodular. This is *not* generally true.[21] In this respect, supermodularity is more fragile than a property such as concavity (which *is* preserved by aggregation[22]) and one may wish to be more careful when imposing it as a modeling assumption.

We now introduce a condition on production functions that leads to complementarities in factor demand *and* is preserved by aggregation.

Definition 6 (Super*modularity)**.** Let X be a closed subset of $\mathbb{R}^\ell_+$. A function $F : X \to \mathbb{R}$ is *super*modular* in $K \subseteq \{1, \dots, \ell\}$ if, for any $x, x' \in X$ there is $y, y' \in X$ such that $(x \wedge x')_K \geq y_K$, $y'_K \geq (x \vee x')_K$, $x + x' = y + y'$, and $F(x) + F(x') \leq F(y) + F(y')$. When F is super*modular in $K = \{1, 2, \dots, \ell\}$, we simply refer to it as super*modular.

Implicit in our definition of super*modularity is that the domain X is *lattice-like in* K in the sense that, for any $x, x' \in X$ there is $y, y' \in X$ such that $(x \wedge x')_K \geq y_K$, $y'_K \geq (x \vee x')_K$, and $x+x' = y+y'$. This property is strictly weaker than requiring X to be a sublattice of $\mathbb{R}^\ell$ (see Example 9). Clearly, any supermodular function is super*modular, since the condition required by the latter holds folds for $y = x \wedge x'$ and $y' = x \vee x'$.[23] The next result states that super*modularity is preserved by aggregation.

Proposition 6. *Let $F^n : X^n \to \mathbb{R}$ be a super*modular function in $K \subseteq \{1, \dots, \ell\}$, for $n = 1, \dots, N$. Then the function $F^* : X^* \to \mathbb{R}$, where $X^* = \sum_{n=1}^N X^n$ and $F^*(x) := \max \left\{ \sum_{n=1}^N F^n(y^n) : \sum_{n=1}^N y^n = x \right\}$, is super*modular in K.*

Proof. For any $x, x' \in X$, there is $\tilde{x}^n, \tilde{x}^{n\prime} \in X^n$, for $n = 1, \dots N$, such that $x = \sum_n \tilde{x}^n$, $x' = \sum_n \tilde{x}^{n\prime}$ and $F^*(x) = \sum_n F^n(\tilde{x}^n)$, and $F^*(x') = \sum_n F^n(\tilde{x}^{n\prime})$. By super*modularity of F^n, for $n = 1, \dots, N$, there is $\tilde{y}^n, \tilde{y}^{n\prime} \in X^n$ where $\tilde{x}^n_K, \tilde{x}^{n\prime}_K \geq \tilde{y}^n_K$, $\tilde{x}^n + \tilde{x}^{n\prime} = \tilde{y}^n + \tilde{y}^{n\prime}$, and $F^n(\tilde{x}^n) + F^n(\tilde{x}^{n\prime}) \geq F^n(\tilde{y}^{n\prime}) + F^n(\tilde{y}^{n\prime})$. Denote $y = \sum_n \tilde{y}^n$, $y' = \sum_n \tilde{y}^{n\prime}$. Summing across n, guarantees that $x_K, x'_K \geq y_K$, $x + x' = y + y'$, and

$$F^*(x) + F^*(x') = \sum_n F^n(\tilde{x}^n) + \sum_n F^n(\tilde{x}^{n\prime}) \leq \sum_n F^n(\tilde{y}^{n\prime}) + \sum_n F^n(\tilde{y}^{n\prime}) \leq F^*(y) + F^*(y').$$

[21] Suppose $F^1(x_1, x_2, x_3) = x_1^{1/2} x_2^{1/4}$ and $F^2(x_1, x_2, x_3) = x_1^{1/2} x_3^{1/4}$ for $(x_1, x_2, x_3) \in \mathbb{R}^3_+$. Then one could check that $F^*(x_1, x_2, x_3) = \sqrt{x_1}\sqrt{\sqrt{x_2} + \sqrt{x_3}}$. While F^1 and F^2 are supermodular functions, F^* is not, since the cross derivative of x_2 and x_3 is negative.

[22] To be precise, F^* is concave if F^n is concave for all n.

[23] For example of a super*modular function that is not supermodular, suppose $X = \{0, 1, 2, 3\} \times \{0, 1\}$, and define $F : X \to \mathbb{R}$ by $F(x_1, 0) = x_1$ and $F(0, 1) = 1$, $F(1, 1) = F(2, 1) = 2$, $F(3, 1) = 4$. Since $3 = F(1, 0) + F(2, 1) < F(1, 1) + F(2, 0) = 4$, the function is *not* supermodular. However, one could check that F is super*modular; in particular, $F(3, 1) + F(0, 0) = F(1, 1) + F(2, 0)$.

Thus, the function F^* is super*modular in K. $\square$

Super*modularity is a sufficient and (under certain ancillary conditions) necessary condition for joint complementarity.

Proposition 7. *Let $F : X \to \mathbb{R}$ be a regular function. If F a super*modular function in $K \subseteq \{1, \ldots, \ell\}$, then the factors in K are joint complements; indeed, the map $p_K \to \mathcal{H}(p_K, p_{-K})$ is $\mathcal{P}$-decreasing in K, for any p_{-K}.*[24] *Moreover, if $X = \mathbb{R}^\ell_+$ and F is increasing and concave, then the factors in K are joint complements only if F is super*modular in K.*

The proof of this result (in the Appendix) relies on our earlier observation that the factors in K are joint complements if and only if, for each $j \in K$ and for any $p \in \mathbb{R}^\ell_{++}$, the correspondence Φ^j (defined by (7)) is $\mathcal{W}$-increasing in K. Theorem 3 tells us that this holds if and (when Γ^j is convex-valued) only if Γ^j is $\mathcal{P}$-increasing in K. The super*modularity of F in K is precisely what is needed to guarantee that Γ^j has this property.

We conclude this section with two other examples of super*modular functions.

Example 9. Let $X = \{B \cdot z : z \in \mathbb{R}^k_+\}$, where B be an $\ell \times k$ matrix B with positive entries. Given a function $f : \mathbb{R}^k_+ \to \mathbb{R}$, define $F : X \to \mathbb{R}$ by $F(x) := \max\{f(z) : B \cdot z = x\}$. We can interpret F as the production function of a two-stage production process: the final output is produced from k intermediate goods and governed by the production function f; in turn, each intermediate good is produced from ℓ factors, with the jth column of B being the vector of factors needed to produce one unit of the jth intermediate good.

We claim that F is super*modular if f is super*modular. Indeed, take any $x, x' \in X$ and $z, z' \in Z$ such that $x = B \cdot z$, $x' = B \cdot z'$, $f(z) = F(x)$ and $f(z') = F(x')$. Since all entries in B are positive, $B \cdot (z \wedge z') \leq B \cdot z$ and $B \cdot (z \wedge z') \leq B \cdot z'$, implying $B \cdot (z \wedge z') \leq (B \cdot z) \wedge (B \cdot z') = x \wedge x'$. Since f is super*modular, there is $\tilde{z}, \tilde{z}' \in Z$ such that $(z \wedge z') \geq \tilde{z}'$, $z + z' = \tilde{z} + \tilde{z}'$, and $f(\tilde{z}) + f(\tilde{z}') \geq f(z) + f(z')$. Note that $B \cdot \tilde{z}' \leq B \cdot (z \wedge z') \leq x \wedge x'$. We also have $(B \cdot \tilde{z}) + (B \cdot \tilde{z}') = x + x'$. Let $y = B \cdot \tilde{z}$ and $y' = B \cdot \tilde{z}'$. Then $x \wedge x' \geq y'$, $x + x' = y + y'$, and

$$F(x) + F(x') \;=\; f(z) + f(z') \;\leq\; f(\tilde{z}) + f(\tilde{z}') \;\leq\; F(y) + F(y').$$

[24] This means that $\mathcal{H}(p_K, p_{-K})$ dominates $\mathcal{H}(p'_K, p_{-K})$ in K by the parallelogram order for $p'_K \geq p_K$.

Note that while F is a super*modular function, it need not be supermodular, even when f is supermodular. In fact, the set X need not even be a sublattice of $\mathbb{R}^\ell$.[25]

Unlike supermodularity, the notion of super*modularity can be applied to a subset of inputs (see Definition 6), which in turn allows us to guarantee complementarity on that subset but not necessarily across all inputs. The next example discusses such a case.

Example 10. Let $I_1, I_2, \ldots, I_n$ be a partition of $\{1, \ldots, \ell\}$ and $X_j \subseteq \mathbb{R}_+^{|I_j|}$, for all $j = 1, \ldots, n$. Suppose that $g_j : X_j \to Y_j$ (where $Y_j \subseteq \mathbb{R}$) is $\mathcal{P}$-increasing in $\{i_j\} \in I_j$. Then for any increasing and supermodular function $A : \times_j Y_j \to \mathbb{R}$, the function $F : \times_j X_j \to \mathbb{R}$,

$$F(x) \;:=\; A\Big(g_1(x_{I_1}), g_2(x_{I_2}), \ldots, g_n(x_{I_n})\Big),$$

is super*modular in $K = \{i_1, \ldots, i_n\}$. We prove this claim in the Appendix. Notice that the factors within each group I_j are not necessarily complements, even if g_j is $\mathcal{P}$-increasing in I_j (rather than just $\{i_j\}$). For example, suppose $F(x) = (x_1 + x_2)(x_3 + x_4)$. Then $A = y_1 y_2$ is supermodular, and $h_1 = x_1 + x_2$ and $h_2 = x_3 + x_4$ are both supermodular and concave functions (and hence $\mathcal{P}$-increasing in $\{1, 2\}$ and $\{3, 4\}$ respectively). Thus factors *across* groups – such as 1 and 3 or 1 and 4 – are joint complements. However, factors within a group are clearly not complements. Indeed suppose there is a drop in the price of factor 1, so it goes from being above to below the price of 2. The firm would use only factor 2 initially since it is cheaper than 1, and it switches completely to factor 1 at the new price. Thus the demand for 1 increases but that of 2 drops to zero.

5 First order stochastic dominance under ambiguity

We consider an agent making decisions in an uncertain environment. Suppose that the possible states of the world are represented by a set $S \subseteq \mathbb{R}$; to keep our exposition focused on the essentials we assume that the set $S = \{s_1, s_2, \ldots, s_\ell, s_{\ell+1}\}$ is finite, where $s_1 < s_2 < \ldots < s_\ell < s_{\ell+1}$. We denote the set of cumulative distribution functions on S by $\triangle_S$. Let λ, $\mu : S \to \mathbb{R}$ be two cumulative distribution functions. The distribution λ *first order stochastically dominates* μ if $\lambda(s) \leq \mu(s)$ for all $s \in S$; we denote this by $\lambda \succeq \mu$. An

[25] For example, suppose three factors are used to produce two intermediate goods and B is the matrix with two columns $(1, 1, 0)$ and $(0, 1, 1)$. Let $x = B \cdot (1, 0)^T = (1, 1, 0)^T$ and $x' = B \cdot (0, 1)^T = (0, 1, 1)^T$. Note that there is no $z \in Z$ for which $B \cdot z = (0, 1, 0)^T = x \wedge x'$. Hence, X is not a sublattice of $\mathbb{R}^2$.

important feature of $(\triangle_S, \succeq)$ is that it is a lattice. For distributions λ and λ' their meet and join are defined by $(\lambda \wedge \lambda')(s) = \max\{\lambda(s), \lambda'(s)\}$ and $(\lambda \vee \lambda')(s) = \min\{\lambda(s), \lambda'(s)\}$, respectively.

The concept of first order stochastic dominance (FSD) allows us to compare distributions by expected utility; indeed, $\lambda \succeq \mu$ if, and only if, $\int_S u(s)d\lambda(s) \geq \int_S u(s)d\mu(s)$ for all increasing functions $u : S \to \mathbb{R}$. This basic result also has a simple and widely-used corollary that allows us to compare the actions of an agent maximizing expected utility. Suppose this agent chooses an action from a set $X \subseteq \mathbb{R}$ and her utility from action x is $g(x, s)$ when state s is realized. Let $\lambda(\cdot, t)$ be a distribution over S (parameterized by t in a poset T) which captures the agent's belief about the likelihood of different states. Then the expected utility of taking action x is $f(x, t) = \int_S g(x, s)d\lambda(s, t)$. Suppose that $g(x, s)$ has increasing differences in (x, s) (equivalently, is supermodular in (x, s)) and λ is ordered by first order stochastic dominance in the sense that $\lambda(\cdot, t') \succeq \lambda(\cdot, t)$ whenever $t' \geq_T t$. In such a case, $x' \geq x$ implies that

$$f(x', t) - f(x, t) \;=\; \int_S \big[g(x', \tilde{s}) - g(x, \tilde{s})\big] d\lambda(\tilde{s}, t)$$

is increasing in t, since $s \to \big[g(x', s) - g(x, s)\big]$ is increasing in s. In other words, f has increasing differences in (x, t), which guarantees that argmax $\{f(x, t) : x \in X\}$ increases with t in the strong set order (see Topkis (1978) or Milgrom and Shannon (1994)).

Our objective in this section is to extend this simple result on comparative statics to some widely-used multi-prior models of decision-making under uncertainty.

5.1 FSD in the maxmin model

In the *maxmin* model of Gilboa and Schmeidler (1989), the agent evaluates an uncertain environment with a convex *set* of distributions over $S \subseteq \mathbb{R}$. If $g(x, s)$ is the utility from action $x \in X \subseteq \mathbb{R}$ when $s \in S$ is realized, then the agent's utility (ex ante) is

$$f(x, t) \;:=\; \min\left\{\int_S g(x, s)d\lambda(s) : \lambda \in \Lambda(t)\right\}, \tag{8}$$

where $\Lambda(t)$ is a convex set of distributions parameterized by $t \in T$. This leads naturally to the following question: when g is a supermodular function, what shift in the set $\Lambda(t)$ would guarantee that the agent chooses a higher action? The next definition gives the set generalization of first order stochastic dominance that is appropriate for this purpose.

Definition 7. Let T be a poset. The correspondence $\Lambda : T \to \Delta_S$ is *FSD-increasing by the parallelogram order* (or $\mathcal{P}_{FSD}$*-increasing*, for short) if for any $t' \geq_T t$ and distributions $\lambda \in \Lambda(t)$, $\lambda' \in \Lambda(t')$, there is some $\mu \in \Lambda(t)$, $\mu' \in \Lambda(t')$ such that

$$\lambda' \succeq \mu, \quad \mu' \succeq \lambda, \quad \text{and} \quad \tfrac{1}{2}\lambda + \tfrac{1}{2}\lambda' = \tfrac{1}{2}\mu + \tfrac{1}{2}\mu'.$$

Our set-generalization of first order stochastic dominance is clearly just a version of the parallelogram order. To be precise, the correspondence $\Gamma : T \to \mathbb{R}^\ell$ given by

$$\Gamma(t) \; := \; \Big\{ y \in \mathbb{R}^\ell : y_i = -\lambda(s_i), \text{ for all } i = 1, 2, \ldots, \ell \text{ and } \lambda \in \Lambda(t) \Big\} \tag{9}$$

is $\mathcal{P}$-increasing if, and only if, Λ is $\mathcal{P}_{\text{FSD}}$-increasing. Thus our main theorems can be applied to obtain monotone comparative statics in the maxmin model.[26]

Proposition 8. *Suppose that $\Lambda : T \to \triangle_S$ has compact and convex values. Then $f : X \times T \to \mathbb{R}$, as defined by (8), has increasing differences in (x, t) for any supermodular function $g : X \times S \to \mathbb{R}$ if, and only if, Λ is $\mathcal{P}_{\text{FSD}}$-increasing.*[27]

Proof. We show here that the $\mathcal{P}_{FSD}$-increasing property is sufficient. The converse is found in the Appendix. Define $\Gamma : T \to \mathbb{R}^\ell$ by (9). If Λ is $\mathcal{P}_{\text{FSD}}$-increasing, then Γ is $\mathcal{P}$-increasing. For any function $g : X \times S \to \mathbb{R}$ and distribution λ,

$$\begin{aligned} \int_S g(x,s) d\lambda(s) \; &= \; g(x, s_1)\lambda(s_1) + \sum_{i=1}^{\ell} g(x, s_{i+1}) \big[\lambda(s_{i+1}) - \lambda(s_i)\big] \\ &= \; g(x, s_{\ell+1}) + \sum_{i=1}^{\ell} \big[g(x, s_{i+1}) - g(x, s_i)\big]\big[-\lambda(s_i)\big]. \end{aligned} \tag{10}$$

Given $x' \geq x$, we define $p, p' \in \mathbb{R}^\ell$ by $p_i = g(x, s_{i+1}) - g(x, s_i)$ and $p'_i = g(x, s_{i+1}) - g(x, s_i)$, for $i = 1, 2, \ldots, \ell$. Then inequality (10) gives

$$f(x, t') - f(x, t) \; = \; \min \Big\{ p \cdot y : y \in \Gamma(t') \Big\} - \min \Big\{ p \cdot y : y \in \Gamma(t) \Big\}, \tag{11}$$

[26] Definition 7 is designed to guarantee monotone comparative statics. Quite naturally we can also ask what set-generalization of first order stochastic dominance will guarantee that the map from t to $\min\big\{ \int u(s) d\lambda(s) : \lambda \in \Lambda(t) \big\}$ is increasing, for any increasing utility function u We show in Section S.6 of the Online Supplement that this leads to a set-generalization that is strictly weaker than the $\mathcal{P}_{\text{FSD}}$-increasing property. This notion of stochastic dominance is not sufficient to guarantee comparative statics except in special cases. One such special case is considered in Ui (2015) which studies global games with ambiguity. The Online Supplement explains Ui's formulation in greater detail.

[27] Milgrom and Shannon (1994) show that a weaker condition on f than supermodularity, called *single crossing differences* in (x, t), is sufficient for $\operatorname{argmax}\big\{ f(x, t) : x \in X \big\}$ to be increasing in the strong set order. We show in Section S.7 of the Online Supplement that $\mathcal{P}_{\text{FSD}}$ monotonicity of Λ is also necessary for f to have single crossing differences in (x, t), for any supermodular function g.

with a similar formula for $f(x', t') - f(x', t)$. If g is supermodular, then $p' \geq p$ and Theorem 2 guarantees that $\min\{p \cdot y : y \in \Gamma(t')\} - \min\{p \cdot y : y \in \Gamma(t)\}$ is less than $\min\{p' \cdot y : y \in \Gamma(t')\} - \min\{p' \cdot y : y \in \Gamma(t)\}$. Thus, $f(x, t') - f(x, t) \leq f(x', t') - f(x', t)$, and so f has increasing differences. □

Remark 5.1. We show in the Appendix that Proposition 8 remains true if S is a compact interval of $\mathbb{R}$ and function $g(x, \cdot)$ is Riemann-Stieltjes integrable with respect to each $\lambda \in \Lambda(t)$, for all $x \in X$ and $t \in T$. This holds if any of the following conditions are satisfied: (a) function $g(x, s)$ is continuous in $s \in S$; (b) $g(x, s)$ is bounded on S and has only finitely many discontinuities in s, and all distributions in $\Lambda(t)$ are atomless; or (c) $g(x, s)$ is bounded and monotone on S, and all distributions in $\Lambda(t)$ are atomless.

Remark 5.2. Whenever $g(x, s)$ is increasing in s, we can assume, without loss of generality, that Λ is upper comprehensive, i.e., if $\lambda \in \Lambda(t)$ and $\lambda' \succeq \lambda$, then $\lambda' \in \Lambda(t)$.[28] In Section S.8 of the Online Supplement, we show that when Λ is upper comprehensive, the $\mathcal{P}_{\text{FSD}}$-increasing property on Λ remains *necessary* for f to have increasing differences for all $g(x, s)$ that are supermodular in (x, s) and increasing in s.

The following are examples of $\mathcal{P}_{\text{FSD}}$-increasing correspondences.

Example 11 (Strong set order)**.** Suppose Λ is increasing in the strong set order, i.e., for any $t' \geq t$, $\lambda \in \Lambda(t)$, and $\lambda' \in \Lambda(t')$, we have $\lambda \wedge \lambda' \in \Lambda(t)$ and $\lambda \vee \lambda' \in \Lambda(t')$. By setting $\mu = \lambda \wedge \lambda'$ and $\mu' = \lambda \vee \lambda'$, we conclude that Λ is $\mathcal{P}_{\text{FSD}}$-increasing. For example, let $\bar{\nu}^t$ and $\underline{\nu}^t$ be distributions on $S \subset \mathbb{R}$ that increase with respect to first order stochastic dominance in t and satisfy $\bar{\nu}^t \succeq \underline{\nu}^t$. for all t. Then $\Lambda(t) = [\bar{\nu}^t, \underline{\nu}^t]$, which consists of all distributions ordered between $\bar{\nu}^t$ and $\underline{\nu}^t$, is convex-valued, and the correspondence Λ increases with t in the strong set order.

For another simple example, let $\Lambda(t)$ be a set of normal distributions with a fixed variance and means drawn from a set $M^{\Lambda}(t) \subset \mathbb{R}$. In this case, the family of normal distributions is totally ordered by the mean, i.e, one distribution first order stochastically dominates another distribution if and only if the former has a higher mean than the latter.

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[28] Given a correspondence Λ, let $\bar{\Lambda}(t) = \{\lambda \in \triangle_S : \lambda \succeq \lambda', \text{ for } \lambda' \in \Lambda(t)\}$. It is clear that $\bar{\Lambda}$ is upper comprehensive and that $\min\{\int_S g(x, s) d\lambda(s) : \lambda \in \Lambda(t)\} = \min\{\int_S g(x, s) d\lambda(s) : \lambda \in \bar{\Lambda}(t)\}$.

Then it is clear that $\Lambda(t)$ increases with t in the strong set order if and only if $M^{\Lambda}(t)$ (as sets in $\mathbb{R}$) increases with t in the strong set order.[29]

Example 12 (Increasing mean)**.** Take an increasing function $h : S \to \mathbb{R}$ and suppose that $\Lambda(t)$ consists of all distributions over S for which the expected value of h is equal to t, i.e., $\Lambda(t) = \left\{\lambda \in \triangle_S : \int_S h(s)d\lambda(s) = t\right\}$. It is clear that Λ is not increasing in the strong set order since the supremum or infimum of two distributions μ and μ' will not generally have the same mean as μ or μ'. However, we show in the Appendix that Λ is $\mathcal{P}_{\text{FSD}}$-increasing because Γ (as defined by (9)) is increasing in the $\mathcal{C}$-flexible set order.

Example 13 (Convex combinations)**.** Since the $\mathcal{P}_{\text{FSD}}$-increasing property is just a version of the parallelogram order, it is also preserved by convex combinations, i.e., if Λ_1 and Λ_2 are $\mathcal{P}_{\text{FSD}}$-increasing, then so is $\Lambda := \alpha\Lambda_1 + (1-\alpha)\Lambda_2$, for $\alpha \in (0, 1)$ (see Proposition 1). In particular, instances of the $\mathcal{P}_{\text{FSD}}$-increasing property given in the two previous examples could be combined to generate more examples. We give two instances where such combinations occur naturally.

Firstly, in ϵ-contamination models of ambiguity aversion (see, for example, Epstein and Wang (1994) and Nishimura and Ozaki (2004)), the agent has a set of priors which is the convex combination of the set of all distributions on S and a single distribution interpreted as the agent's belief, held with incomplete confidence. The weight on the former is ϵ so, in our notation, $\Lambda_1(t)$ is the set of all priors, $\Lambda_2(t)$ is a singleton set, and $\alpha = \epsilon$. Clearly, if the distribution $\Lambda_2(t)$ is FSD-increasing in t, then $\Lambda(t)$ has the $\mathcal{P}_{\text{FSD}}$-increasing property.

Secondly, prior sets which are convex combinations of other sets of distributions could arise because of set predictions. For example, suppose a firm uses a model to forecast future demand for its product. This model gives a set prediction of demand levels conditional on the prevailing state of the economy $\omega \in \Omega$ and some other parameter $t \in T$ (such as the firm's advertising expenditure in the current period). We denote by $A(\omega, t)$ the finite set of demand forecasts at (ω, t). Suppose that, for any ω, $A(\omega, t)$ increases with t in the strong set order and let $\Lambda^{\omega}(t)$ be the set of degenerate probability distri-

[29] It is common in applications to model ambiguity with a parametric family of distributions having different means, while keeping other parameters unchanged (see Bianchi et al. (2018) and Ilut and Schneider (2022)). In these cases, the distributions are often totally ordered by first order stochastic dominance, so our example applies beyond the family of normal distributions.

butions corresponding to $A(\omega, t)$.[30] Assuming that the firm knows that ω occurs with probability $\pi(\omega)$, the set of possible demand distributions (for a given t and before the realization of ω) is $\Lambda(t) = \sum_{\omega \in \Omega} \pi(\omega) \Lambda^\omega(t)$. In other words, a typical element of $\Lambda(t)$ is a distribution where some $s^\omega \in A(\omega, t)$ occurs with probability $\pi(\omega)$. Since Λ^ω is increasing in the strong set order, Λ, and thus also $\operatorname{co} \Lambda$, is $\mathcal{P}_{\text{FSD}}$-increasing.[31] However, Λ need not increase in the strong set order, nor in the $\mathcal{C}$-flexible sense.[32]

We conclude this subsection with two economic applications. More applications are found in Section S.11 of the Online Supplement where, among other things, we formulate conditions under which ambiguity averse agents in a dynamic model have monotone decision rules; this generalizes known results on monotone decision rules (see Hopenhayn and Prescott, 1992) for agents maximizing discounted expected utility.

Example 14 (Optimal savings)**.** An agent lives for two periods, with income m in period 1 and uncertain income s in period 2.[33] With saving $x \in [0, m]$ in period 1, the agent's utility conditional on s is $g(x, s) = u(m - x) + \beta u\big(x(1 + r) + s\big)$, where u is the per-period utility, β is the discount rate, and r is the interest. The function g is increasing in s if u is increasing, and it is submodular in (x, s) (equivalently, $g_{xs} \leq 0$) if u is concave. Suppose the agent has maxmin preferences of the form (8). Since g is submodular, f has decreasing differences in (x, t) if Λ is $\mathcal{P}_{\text{FSD}}$-increasing. It follows that the agent saves less with higher t; formally, $\operatorname{argmax}_{x \in [0,m]} f(x, t)$ falls with t in the strong set order.

In particular, suppose that through the news and other channels, this agent is confident that the *mean* income in period 2 is t, but is not confident of the precise distribution that s takes. In this case, he may behave as though $\Lambda(t)$ consists of all distributions with mean t (as in Example 12); if so, any news that raises the agent's belief about the mean income in period 2 will cause him to save less in period 1.

[30] For example, suppose the firm models the possible demand outcomes (given (ω, t)) as the optimal choices of a representative agent with the quasilinear utility function $Q(s, \omega, t) = \phi(s, \omega, t) - s$, where ϕ is supermodular in (s, t). Then $A(\omega, t) = \operatorname{argmax}_{s \in S} Q(s, \omega, t)$ increases with t in the strong set order.

[31] Elements of $\operatorname{co} \Lambda(t)$ have a natural interpretation: each element is a distribution over demand that arises from choosing a distribution over $A(\omega, t)$ (for each ω), with ω occurring with probability $\pi(\omega)$. The maxmin model requires the set of priors to be convex, but it makes no difference here whether the set of priors is $\Lambda(t)$ or $\operatorname{co} \Lambda(t)$, since the value of $f(x, t)$ (as defined by (8)) is the same in either case.

[32] Section S.9 of the Online Supplement provides a specific example.

[33] For other discussions of the two-period savings problem with ambiguity aversion, see Miao (2004) and Ilut and Schneider (2022). The latter also contains a review of infinite horizon consumption-saving problems with ambiguity aversion and of the evidence of ambiguity aversion in household survey data.

Example 15 (Portfolio problem)**.** An investor divides her wealth $m > 0$ between a *safe asset*, that pays out $r > 0$ for sure, and a *risky asset* with an uncertain return of s in $S \subseteq \mathbb{R}_+$. The investor's beliefs over the risky return is captured by the correspondence Λ, which we assume is $\mathcal{P}_{\text{FSD}}$-increasing. Recall that, in the Introduction, we provided a natural example of such a correspondence. Suppose the investor uses different models of the return on the risky asset and these models lead to an interval of distributions $[\underline{\nu}, \bar{\nu}]$, with $\bar{\nu}$ being the most optimistic and $\underline{\nu}$ the least. This return attracts tax and we denote by t the proportion of the return that is retained after tax. Then $\Lambda(t) = [\underline{\nu}^t, \bar{\nu}^t]$ is the set of distributions after tax, where $\underline{\nu}^t$ and $\bar{\nu}^t$ are the after-tax return distributions corresponding to $\underline{\nu}$ and $\bar{\nu}$ respectively. Both $\underline{\nu}^t$ and $\bar{\nu}^t$ are FSD-increasing with t and thus Λ is $\mathcal{P}_{\text{FSD}}$-increasing (see Example 11).[34]

The investor chooses to invest $x \in X \subseteq \mathbb{R}$ in the risky asset, with the rest of his wealth invested in the safe security. We allow the investor to go short on either asset but require her to be solvent, i.e., it must be that $xs + (m - x)r \geq 0$, for all $s \in S$ and $x \in X$. Assuming that her Bernoulli index is $u : \mathbb{R}_+ \to \mathbb{R}$ and the investor is ambiguity averse, the investor's utility at $x \in X$ is

$$f(x,t) \ := \ \min \left\{ \int_S u\big(xs + (m-x)r\big) d\lambda(s) : \lambda \in \Lambda(t) \right\}. \tag{12}$$

We know that f has increasing differences in (x,t) if Λ is $\mathcal{P}_{\text{FSD}}$-increasing and $g(x,s) := u\big(xs + (m-x)r\big)$ is supermodular. Assuming that u is strictly increasing, concave, and twice continuously differentiable, it is straightforward to check that g is supermodular if the coefficient of relative risk aversion of u is less than 1.[35] With this condition on u, f has increasing differences in (x,t) and (consequently) the investor's holding in the risky asset increases with t.[36]

In both Examples 14 and 15, the choice variable is x while t is just a parameter, but

[34] Assuming that only positive returns are taxed proportionately, we obtain $\underline{\nu}^t(s) = \underline{\nu}(s)$, for $s \leq 0$, and $\underline{\nu}^t(s) = \underline{\nu}(s/t)$ otherwise.

[35] Note that, since x can take negative values, function g does not increase in s.

[36] There are other discussions of the portfolio choice model under ambiguity. For example, Gollier (2011) examines how the demand for the risky asset changes with the level of ambiguity aversion, in the context of the smooth ambiguity model. Cherbonnier and Gollier (2015) study both the smooth ambiguity model and the α-maxmin model; the authors provide conditions under which the demand for the risky asset increases with respect to initial wealth. See also the survey of Ilut and Schneider (2022). Our result here also extends to other models besides the maxmin model; see Section 5.2.

there are also applications where both x and t are actions taken by the agent. Section S.10 of the Online Supplement contains as example of that type.

5.2 α-maxmin, variational, and multiplier preferences

Proposition 8 can be extended to obtain comparative statics results for other multi-prior models of preferences.

α-maxmin preferences. The α-maxmin model of Ghirardato et al. (2004) allows for both ambiguity averse and ambiguity loving behavior, with the agent's utility function having the form

$$\alpha \ \min\left\{\int_S g(x,s)d\lambda(s) : \lambda \in \Lambda(t)\right\} \ + \ (1-\alpha) \ \max\left\{\int_S g(x,s)d\lambda(s) : \lambda \in \Lambda(t)\right\},$$

for some $\alpha \in [0,1]$. We claim that this function has increasing differences in (x,t) if Λ is $\mathcal{P}_{\text{FSD}}$-increasing (which in turn guarantees that the set of optimal actions increases with t in the strong set order). This is true because Proposition 8 can be equivalently formulated as saying that the map from (x,t) to $\max\left\{\int_S g(x,s)d\lambda(s) : \lambda \in \Lambda(t)\right\}$ has increasing differences in (x,t), for any supermodular g if, and only if, Λ is $\mathcal{P}_{\text{FSD}}$-increasing.[37]

Variational preferences. This model is introduced in Maccheroni et al. (2006) and also generalizes the maxmin model. In this model, the utility of action $x \in X \subset \mathbb{R}$ is

$$f(x,t) \ = \ \min\left\{\int_S g(x,s)d\lambda(s) + c(\lambda,t) : \lambda \in \triangle_S\right\}, \tag{13}$$

where $c(\cdot,t)$ is a convex function parameterized by $t \in T$. Loosely speaking, the agent's utility from action x is obtained by minimizing her expected utility over the set of all probability distributions; unlike the maxmin model where the agent is restricted to a subset of $\triangle_S$, any distribution in $\triangle_S$ could be 'picked' in the variational preferences model, though each distribution λ is associated with a different cost $c(\lambda,t)$. The next result identifies those shifts in the cost function c which guarantee that the agent's utility has increasing differences in (x,t).

Proposition 9. *Let $c : \triangle_S \times T \to \mathbb{R}_+$ be a continuous and convex function on $\triangle_S$, for all $t \in T$. The following statements are equivalent.*

[37] Indeed, Proposition 8 guarantees that $\min\left\{\int_S -g(x,s)d\lambda(s) : \lambda \in \Lambda(t)\right\}$ has decreasing differences in (x,t) since $-g(x,s)$ is submodular; therefore $\max\left\{\int_S g(x,s)d\lambda(s) : \lambda \in \Lambda(t)\right\} = -\min\left\{\int_S -g(x,s)d\lambda(s) : \lambda \in \Lambda(t)\right\}$ has increasing differences in (x,t).

(i) *The function c satisfies the following property:*

(C) *for any $t' \geq_T t$ in T and λ, λ' in $\triangle_S$ there is μ, μ' in $\triangle_S$ such that*

$$\lambda' \succeq \mu,\ \mu' \succeq \lambda,\ \tfrac{1}{2}\lambda + \tfrac{1}{2}\lambda' = \tfrac{1}{2}\mu + \tfrac{1}{2}\mu',\ \textit{and}\ c(\lambda, t) + c(\lambda', t') \geq c(\mu, t) + c(\mu', t').$$

(ii) *The function $f : X \times T \to \mathbb{R}$ defined in (13) is supermodular, for any supermodular function $g : X \times S \to \mathbb{R}$.*[38]

To better understand condition (C), which may seem opaque initially, notice that it captures the change in the function c that leads to an upward revision in the agent's belief about the state. To be specific, suppose that λ_* and λ'_* are distributions that minimize $\int_S g(x,s)d\lambda(s) + c(\lambda, t)$ and $\int_S g(x,s)d\lambda(s) + c(\lambda, t')$, respectively, with $t' \geq_T t$. (C) guarantees that there are distributions μ_* and μ'_* such that $\lambda'_* \succeq \mu_*$, $\mu'_* \succeq \lambda_*$, and

$$\int_S g(x,s)d\lambda_*(s) + \int_S g(x,s)d\lambda'_*(s) + c(\lambda_*, t) + c(\lambda'_*, t') \ \geq$$
$$\int_S g(x,s)d\mu_*(s) + \int_S g(x,s)d\mu'_*(s) + c(\mu_*, t) + c(\mu'_*, t').$$

Thus, μ_* also minimizes $\int_S g(x,s)d\lambda(s) + c(\lambda, t)$ and μ'_* minimizes $\int_S g(x,s)d\lambda(s) + c(\lambda, t')$. In other words, as t increases the distribution the agent uses to evaluate the utility of an action x shifts up in the sense of first order stochastic dominance (from λ_* to μ'_*).

The proof of Proposition 9 is in the Appendix. Note that (C) can be thought of as a generalization of the $\mathcal{P}_{\text{FSD}}$-increasing property imposed on $\Lambda : T \to \triangle_S$. Indeed, given Λ, let $c(\lambda, t) = 0$ if $\lambda \in \Lambda(t)$, and ∞ otherwise. Then c obeys (C) if, and only if, Λ is $\mathcal{P}_{\text{FSD}}$-increasing (while (13) reduces to the maxmin form (8) in this case). Below are two more examples of cost functions that satisfy property (C).[39]

Example 16 (Submodular cost and decreasing differences)**.** Let $c : \triangle_S \times T \to \mathbb{R}_+$ be a submodular function of λ that has decreasing differences in (λ, t). Then, for all λ, $\lambda' \in \triangle_S$ and t, $t' \in T$ with $t' \geq_T t$, we have

$$c(\lambda', t) - c(\lambda' \wedge \lambda, t) \ \geq \ c(\lambda' \vee \lambda, t) - c(\lambda, t) \ \geq \ c(\lambda' \vee \lambda, t') - c(\lambda, t')$$

and condition (C) holds, if we choose $\mu = \lambda \wedge \lambda'$ and $\mu' = \lambda \vee \lambda'$.

[38] As in the case of Proposition 8, statement (ii) in Proposition 9 is equivalent to $f(x,t)$ having single crossing differences in (x,t), for all supermodular functions g that are increasing in s.

[39] Note that (C) restricts how $c(\lambda, t)$ varies jointly with λ and t; for a *fixed* t, it has no content.

An important sub-class of variational preferences where the cost function c has the submodular property is the class of **multiplier preferences**; these preferences are used in Sargent and Hansen (2001) and axiomatized in Strzalecki (2011a). In this case, $c(\lambda, t) = \theta R\big(\lambda \| \lambda^*(\cdot, t)\big)$, for $\theta \geq 0$ and $\lambda^*(\cdot, t) \in \triangle_S$, where

$$R\big(\lambda \| \lambda^*(\cdot, t)\big) \;:=\; \int_S \ln\left(\frac{d\lambda(s)}{d\lambda^*(s,t)}\right) d\lambda(s)$$

is the *relative entropy*.[40] Note that $d\lambda(s)$, $d\lambda^*(s,t)$ denote the probability of state s in the distribution λ, $\lambda^*(\cdot, t)$, respectively. This representation can be interpreted in the following manner. The decision maker has a belief over the states of the world given by a *reference* or *benchmark* distribution $\lambda^*(\cdot, t)$, but she is not completely confident that she is exactly correct. To accommodate this concern, the decision maker takes all distributions in $\triangle_S$ into account when evaluating her utility from a given action, though distributions further away from $\lambda^*(\cdot, t)$ cost more and are thus less likely to be the distribution that solves the minimization problem in (13).

We show in the Appendix that *for multiplier preferences, c is a submodular function of λ. Furthermore, c has decreasing differences in* (λ, t) *if $\lambda^*(\cdot, t)$ is increasing in t with respect to the monotone likelihood ratio* (MLR).[41] For many commonly used distributions (such as the normal, lognormal, or exponential distributions) the MLR condition is satisfied if t is the mean of the distribution. In other words, for reference distributions drawn from one of these classes, an increase in its mean is sufficient to guarantee an increase in the optimal choice of the action x.

Example 17. Suppose that $\tilde{c} : \mathbb{R} \times T \to \mathbb{R}$ has decreasing differences in (m, t) and the cost function $c : \triangle_S \times T \to \mathbb{R}$ is evaluated by $c(\lambda, t) := \tilde{c}\left(\int_S h(s) d\lambda(s), t\right)$ for some increasing function $h : S \to \mathbb{R}$. In other words, the cost function depends only on the mean of the random variable h with respect to the distribution λ, and the parameter t. We claim that c satisfies (C). Let $t' \geq_T t$; take any λ, λ' in $\triangle_S$ and denote the mean of function h corresponding to each distribution by m, m', respectively. Suppose that $m' \geq m$; then there are distributions μ, μ' with means m, m', respectively, such that

[40] See Strzalecki (2011b) for a detailed discussion on the relation between variational preferences, multiplier preference, and subjective expected utility.

[41] This requires that, for any $t' \geq t$, the ratio $d\lambda^*(s, t')/d\lambda^*(s, t)$ be increasing with s. This property implies $\lambda^*(\cdot, t') \succeq \lambda^*(\cdot, t)$, hence, it is stronger than the first order stochastic dominance.

$\lambda' \succeq \mu$, $\mu' \succeq \lambda$, and $(1/2)\lambda + (1/2)\lambda' = (1/2)\mu + (1/2)\mu'$.[42] Since $c(\lambda, t) = c(\mu, t)$ and $c(\lambda', t') = c(\mu', t')$, we obtain (as required) $c(\lambda, t) + c(\lambda', t') = c(\mu, t) + c(\mu', t')$. If $m' < m$, then choose $\mu = \lambda'$ and $\mu' = \lambda$; since $\tilde{c}$ has decreasing differences in (m, t) we obtain

$$c(\lambda, t) + c(\lambda', t') \;=\; \tilde{c}(m, t) + \tilde{c}(m', t') \;\geq\; \tilde{c}(m', t) + \tilde{c}(m, t') \;=\; c(\mu, t) + c(\mu', t').$$

In Examples 14 and 15, we gave economic applications of Proposition 8, which assumes that the agent has maxmin utility. It is clear that, by appealing to Proposition 9, the conclusions in those examples will continue to hold, mutatis mutandi, if the agent has maxmin, variational, or multiplier preferences.

Appendix

Proof of the claim in Example 1 First, we show that the correspondence in $\mathcal{P}$-increasing in $K = \{1, 2\}$ whenever $d\bar{x}_2/dx_1$ is decreasing in $\tilde{t}$. Take $t' \geq_T t$ and $x \in \Gamma(t), x' \in \Gamma(t')$. Let $\bar{x}_2(x_1, t) = x_2$ and $\bar{x}_2(x_1', t') = x_2'$. Γ is $\mathcal{P}$-increasing if we can find $y \in \Gamma(t)$, $y' \in \Gamma(t')$ such that $y' \geq x$, $x' \geq y$, and $x + x' = y + y'$. If $x_1' \geq x_1$, then $x_2' = \bar{x}_2(x_1', t') \geq \bar{x}_2(x_1, t) = x_2$, and choose $y' = x'$ and $y = x$. If $x_1' < x_1$, let y be given by $y_1 = x_1'$ and $y_2 = \bar{x}_2(x_1', t) \leq \bar{x}_2(x_1', t') = x_2'$. Therefore, $x' \geq y$ and $y \in \Gamma(t)$. Set $y' = x + x' - y$. Since $d\bar{x}_2/dx_1$ decreases in $\tilde{t}$, we obtain $\bar{x}_2(x_1, t) - \bar{x}_2(x_1', t) \geq \bar{x}_2(x_1, t') - \bar{x}_2(x_1', t')$, which implies that $y_2' \geq \bar{x}_2(x_1, t') = \bar{x}_2(y_1', t')$, and so $y' \in \Gamma(t')$.[43]

If the function $x_1 \to \bar{x}_2(x_1, \tilde{t})$ is C^1 and convex (in x_1), then the converse is also true. Otherwise, there is $t' \geq_T t$ and z_1 such that $d\bar{x}_2/dx_1(z_1, t) < d\bar{x}_2/dx_1(z_1, t')$; then, since $\bar{x}_2$ is C^1, there is $z_1' < z_1$ such that $d\bar{x}_2/dx_1(z_1, t) < d\bar{x}_2/dx_1(z_1', t')$. By convexity of $\bar{x}_2$, $d\bar{x}_2/dx_1(v_1, t) < d\bar{z}_2/dx_1(v_1', t')$ for any $v_1' \geq x_1'$ and $v_1 \leq z_1$. Thus $\bar{x}_2(z_1, t) - \bar{x}_2(y_1, t) < \bar{x}_2(y_1', t') - \bar{x}_2(z_1', t')$, for any y_1, y_1' such that $y_1' \geq z_1$, $z_1' \geq y_1$, and $z_1 + z_1' = y_1 + y_1'$. This guarantees that there is no $y \in \Gamma(t), y' \in \Gamma(t')$ such that $y_1' \geq z_1$, $z_1' \geq y_1$, and $z + z' = y + y'$ and thus Γ is not $\mathcal{P}$-increasing in $K = \{1\}$. $\square$

Proof of Proposition 2 Without loss of generality, suppose that $K = \{1, 2, \ldots, n\}$, for some $n \leq \ell$. Let $\phi(t) := \big\{x \in \Phi(t) : x >_{lex} y, \text{ for all } y \in \Phi(t)\big\}$, where $>_{lex}$ denotes

[42] For a proof of this claim, see the proof of Example 12 in the Appendix.

[43] It is clear from this proof that so long as $d\bar{x}_2/dx_1$ is decreasing in $\tilde{t}$, then Γ is $\mathcal{P}$-increasing in $\{1\}$. The further assumption that $\bar{x}_2$ is increasing in (x_1, t) guarantees that Γ is $\mathcal{P}$-increasing in $\{1, 2\}$.

the lexicographic order.[44] Take any $p \in \mathbb{R}^\ell$. Since Φ is compact-valued, ϕ is well-defined and $\phi(t) \in \Phi(t)$, for all $t \in T$. We claim that $\phi_K(t') \geq \phi_K(t)$, for any $t' \geq_T t$. Since Φ is $\mathcal{P}$-increasing in K, there is $y \in \Phi(t)$, $y' \in \Phi(t')$ such that $\phi(t) - y = y' - \phi(t')$ and $y'_K \geq \phi_K(t)$, $\phi_K(t') \geq y_K$. If $\phi_K(t') \ngeq \phi_K(t)$, then $\phi_K(t) \neq y_K$, and so $\phi_K(t) >_{lex} y$. Thus, there is $j \leq n$ such that $\phi_i(t) = y_i$, for all $i \leq j$, and $\phi_j(t) > y_j$. However, $\phi_K(t) - y = y' - \phi_K(t')$, and so $y'_i = \phi_i(t')$, for all $i \leq j$, and $y'_j > \phi_j(t')$. Hence, $y' >_{lex} \phi(t')$, which contradicts the definition of $\phi(t')$. $\square$

Continuation of the proof of Theorem 2 We show that statement (iii) implies (i) by contradiction. Suppose $\operatorname{co}\Gamma$ is not $\mathcal{P}$-increasing. There is $t' \geq_T t$ and $x \in \operatorname{co}\Gamma(t)$, $x' \in \operatorname{co}\Gamma(t')$ for which there is no $y \in \operatorname{co}\Gamma(t)$, $y' \in \operatorname{co}\Gamma(t')$ satisfying $x + x' = y + y'$ and $x'_K \geq y_K$, $y'_K \geq x_K$. Take any such x, x' and define

$$\begin{aligned} C &:= \Big\{(x - y', x' - y) \in \mathbb{R}^\ell \times \mathbb{R}^\ell : y \in \operatorname{co}\Gamma(t) \text{ and } y' \in \operatorname{co}\Gamma(t')\Big\} \text{ and} \\ D &:= \Big\{(d, d') \in \mathbb{R}^\ell \times \mathbb{R}^\ell : d + d' = 0 \text{ and } d'_K \geq 0\Big\}. \end{aligned}$$

Clearly, both sets are closed, convex, and $C \cap D = \emptyset$. Moreover, since C is compact, one can show that the difference $D - C$ is closed.[45] By the strong separating hyperplane theorem, there are non-zero vectors $p, p' \in \mathbb{R}^\ell$ and a number b that satisfy

$$\sup\Big\{p \cdot c + p' \cdot c' : (c, c') \in C\Big\} \ < \ b \ < \ \inf\Big\{p \cdot d + p' \cdot d' : (d, d') \in D\Big\}. \tag{A1}$$

Since $(0, 0) \in D$, we have $b < 0$. Let $\epsilon_i \in \mathbb{R}^\ell_+$ be the vector with the i'th entry equal to 1 and zeros elsewhere, for $i = 1, 2, \ldots, \ell$. Given that $\alpha(-\epsilon_i, \epsilon_i) \in D$, for all numbers $\alpha \geq 0$ and $i \in K$, we have $p'_K \geq p_K$. Since $\alpha(-\epsilon_i, \epsilon_i)$ belongs to D, for all α and $i \notin K$, it must be that $p'_{-K} = p_{-K}$. The first inequality in (A1) gives $p \cdot x + p' \cdot x' < b + p \cdot y' + p' \cdot y$, for all $y' \in \Gamma(t')$ and $y \in \Gamma(t)$. Therefore, we obtain

$$\begin{aligned} f(p, t) + f(p', t') \ &\leq \ p \cdot x + p' \cdot x' \\ &< \ \min\Big\{p \cdot z : z \in \Gamma(t')\Big\} + \min\Big\{p' \cdot z : z \in \Gamma(t)\Big\} \ = \ f(p, t') + f(p', t), \end{aligned}$$

which contradicts our assumption that $f(p, t)$ has increasing differences in (p_K, t). $\square$

[44] By definition, $x >_{lex} y$ if $x_i = y_i$, for all $i \leq j$, and $x_j > x_j$, for some $j \leq \ell$.
[45] We denote $(D - C) := \big\{d - c : d \in D \text{ and } c \in C\big\}$.

Proof of Theorem 3 Implication (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) follows directly from Theorem 2. It suffices to show that (iv) $\Rightarrow$ (i). Suppose that this property is violated for some $x \in \operatorname{co}\Gamma(t)$, $x' \in \operatorname{co}\Gamma(t')$. Define the nonempty closed sets C and D as in the the proof of Theorem 2. Since $\mathbf{A}\Gamma(t) = \mathbb{R}^\ell_+$ and $\Gamma(t)$ is upward comprehensive, we have $\mathbf{A}\big(\operatorname{co}\Gamma(t)\big) = \mathbb{R}^\ell_+$.[46] Thus, we have $\mathbf{A}C = \mathbb{R}^{2\ell}_-$. Since $\mathbf{A}D = D$ and any nonzero element of D must have entries with strictly different signs, it must be that $\mathbf{A}C \cap \mathbf{A}D = \{0\}$; this suffices for $D - C$ to be closed (see Border, 1985, Proposition 2.38). By the strong separating hyperplane theorem, there are non-zero vectors $p, p' \in \mathbb{R}^\ell$ and a number $b > 0$ that satisfy (A1), and $p'_K \geq p_K$, $p_K = p'_K$, as in the proof of Theorem 2. Finally, since C is downward comprehensive, $p, p' \geq 0$.

Let $\tilde{p} := (p + \delta\mathbf{1})$ and $\tilde{p}' := (p' + \delta\mathbf{1})$, where $\delta > 0$ and $\mathbf{1} \in \mathbb{R}^\ell$ is the unit vector. Clearly, $\tilde{p}, \tilde{p}' \in \mathbb{R}^\ell_{++}$, $\tilde{p}'_K \geq \tilde{p}_K$, and $\tilde{p}'_{-K} = \tilde{p}_{-K}$. Since $d + d' = 0$, we obtain $\tilde{p} \cdot d + \tilde{p}' \cdot d' = p \cdot d + p' \cdot d'$ for any $(d, d') \in D$ and $\delta > 0$. Thus, $\inf\big\{p \cdot d + p' \cdot d' : (d, d') \in D\big\} = \inf\big\{\tilde{p} \cdot d + \tilde{p}' \cdot d' : (d, d') \in D\big\}$. Note that $\mathbf{1} \cdot y$ is uniformly bounded below over $\Gamma(t)$ and $\Gamma(t')$, since both $\Phi(t)$ and $\Phi(t')$ are nonempty for $p = \mathbf{1}$. Thus, $\sup\big\{\tilde{p} \cdot c + \tilde{p}' \cdot c' : (c, c') \in C\big\}$ is arbitrarily close to $\sup\big\{p \cdot c + p' \cdot c' : (c, c') \in C\big\}$ for an arbitrarily small $\delta > 0$, and we can guarantee that the former term (like the latter) is strictly lower than b. We conclude that (A1) still holds, even with $\tilde{p}, \tilde{p}' \in \mathbb{R}^\ell_{++}$ taking the place of p and p'. Re-tracing the proof that (iv) $\Rightarrow$ (i) in Theorem 2, we obtain $f(\tilde{p}, t) + f(\tilde{p}', t') < f(\tilde{p}, t') + f(\tilde{p}', t)$, contradicting the assumption that f has increasing differences in (p_K, t). □

Proof of Proposition 3 The equivalence of statements (i), (ii), and (iii) follows from Theorem 3. That (iii) implies (iv) follows from Topkis (1978). It remains to show that (iv) implies (iii). Suppose (iii) fails and there is $p''_i \geq p'_i$ and $q'' \geq q'$ such that $C\big((p''_i, p_{-i}), q''\big) - C\big((p'_i, p_{-i}), q''\big) < C\big((p''_i, p_{-i}), q'\big) - C\big((p'_i, p_{-i}), q'\big)$, for some p_{-i}. Let $R(q) := C\big((p''_i, p_{-i}), q\big)$, for all $q < q''$, and $R(q) := C\big((p''_i, p_{-i}), q''\big)$, for all $q \geq q''$. Since $C\big((p''_i, p_{-i}), q\big)$ is increasing in q, at price (p''_i, p_{-i}) the firm is maximizing profit (which equals zero) at $q = q''$. However, the profit is *not* maximized at any $q \geq q''$ when (p'_i, p_{-i}), since $R(q') - C\big((p'_i, p_{-i}), q'\big) > R(q'') - C\big((p'_i, p_{-i}), q''\big) \geq R(q) - C(p, q)$, for any $q \geq q'$, since R is constant for $q \geq q''$ and C is increasing in q. □

[46] For the proof of this claim, see Proposition S.1 in Section S.2 of the Online Supplement.

Proof of Example 7 Take any $q' \geq q$ and $x, x' \in X$ satisfying $F(x) \geq q$, $F(x') \geq q'$. If $F(x) \geq F(x')$, set $y := x'$, $y' := x$, which trivially satisfy the required condition. Suppose that $F(x') > F(x)$. Take any vectors $\tilde{x}^n, \tilde{x}^{n\prime} \in \mathbb{R}^\ell_+$, $n = 1, \ldots, N$, such that $x \geq \sum_{n=1}^N \tilde{x}^n$, $x' \geq \sum_{n=1}^N \tilde{x}^{n\prime}$ and $F(x) = G\big(f^1(\tilde{x}^1), \ldots, f^N(\tilde{x}^N)\big)$, $F(x') = G\big(f^1(\tilde{x}^{1\prime}), \ldots, f^N(\tilde{x}^{N\prime})\big)$.

Denote $v := \big(f^j(\tilde{x}^j)\big)_{n=1}^N$ and $v' := \big(f^n(\tilde{x}^{j\prime})\big)_{n=1}^N$. Since the function G is increasing in the $\mathcal{C}$-flexible sense, there is $\lambda \in [0,1]$ such that $G\big(\lambda v' + (1-\lambda)(v \wedge v')\big) \geq q$ and $G\big(\lambda v + (1-\lambda)(v \vee v')\big) \geq q'$.[47] Let $\tilde{v} := \lambda v' + (1-\lambda)(v \wedge v')$ and $\tilde{v}' := \lambda v + (1-\lambda)(v \vee v')$.

We claim that, for all $n = 1, \ldots, N$, there is $\tilde{y}^n, \tilde{y}^{n\prime} \in \mathbb{R}^\ell_+$ such that $f^n(\tilde{y}^n) \geq \tilde{v}_n$, $f^n(\tilde{y}^{n\prime}) \geq \tilde{v}'_n$ and $\tilde{y}^{n\prime} \geq \tilde{x}^n$, $\tilde{x}^{n\prime} \geq \tilde{y}^n$, $\tilde{x}^n + \tilde{x}^{n\prime} = \tilde{y}^n + \tilde{y}^{n\prime}$. Let $L := \{n : f^n(\tilde{x}^{n\prime}) < f^n(\tilde{x}^n)\}$ and M be its complement. Given that G is increasing and $F(x') > F(x)$, M is non-empty. For each $n \in L$, set $\tilde{y}^n = \tilde{x}^{n\prime}$ and $\tilde{y}^{n\prime} = \tilde{x}^n$. Since $f^n(x^n) = v_n = \tilde{v}'_n$ and $f^n(x^{n\prime}) = v'_n = \tilde{v}_n$, for all $n \in L$, vectors $\tilde{y}^n$, $\tilde{y}^{n\prime}$ satisfy the required condition.

Since f^n is $\mathcal{P}$-increasing in K, for all $n \in M$ there is $\tilde{z}^n, \tilde{z}^{n\prime}$ satisfying $f^n(\tilde{z}^n) \geq v_n$, $f^n(\tilde{z}^{n\prime}) \geq v'_n$, and $\tilde{x}^{n\prime}_K \geq \tilde{z}^n_K$, $\tilde{z}^{n\prime}_K \geq \tilde{x}^n_K$, $\tilde{z}^n + \tilde{z}^{n\prime} = \tilde{x}^n + \tilde{x}^{n\prime}$. Set $\tilde{y}^n := \big[\lambda\tilde{x}^{n\prime} + (1-\lambda)\tilde{z}^n\big]$ and $\tilde{y}^{n\prime} := \big[\lambda\tilde{x}^n + (1-\lambda)\tilde{z}^{n\prime}\big]$. Since f^n is concave, we have $f^n\big(\tilde{y}^n\big) \geq \tilde{v}_n$ and $f^n\big(\tilde{y}^{n\prime}\big) \geq \tilde{v}'_n$. Moreover, $\tilde{x}^{n\prime}_K \geq \tilde{y}^n_K$, $\tilde{y}^{n\prime}_K \geq \tilde{x}^n_K$, and $\tilde{x}^n + \tilde{x}^{n\prime} = \tilde{y}^n + \tilde{y}^{n\prime}$.

Define $y := \sum_{n=1}^N \tilde{y}^n$, and note that $x'_K \geq \sum_{n=1}^N \tilde{x}^{n\prime}_K \geq \sum_{n=1}^N \tilde{y}^n_K = y_K$. Since G is increasing, $G\big(f^1(\tilde{y}^1), \ldots, f^N(\tilde{y}^N)\big) \geq G(\tilde{v}) \geq q$, and so $y \in U(q)$. We define $y' \in U(q')$ analogously for $\tilde{y}^{n\prime}$, $n = 1, \ldots, N$. Clearly, we have $y'_K \geq x_K$ and $x + x' = y + y'$. □

Proof of Proposition 7 Note that the correspondence Φ^j (given by (7)) is $\mathcal{W}$-increasing in K for all $j \in K$ if and only if the correspondence mapping $t \in \mathbb{R}^K_-$ to $\Phi^K(t) = \operatorname{argmin}\big\{(p,1)\cdot y : y \in \Gamma^K(t)\big\}$ is $\mathcal{W}$-increasing in K, where

$$\Gamma^K(t) \;:=\; \Big\{(y,v) \in \mathbb{R}^\ell \times \mathbb{R} : (y,v) \geq \big(x, -F(x) - t\cdot x_K\big), \text{ for some } x \in X\Big\}.$$

It is straightforward to check that Γ^K is regular if F is regular. By Theorem 3, if Γ^K is $\mathcal{P}$-increasing in K, then Φ^K is $\mathcal{P}$-increasing in K and the latter is equivalent to $\mathcal{H}(p_K, p_{-k})$ being $\mathcal{P}$-decreasing in K. Furthermore, if Γ^K is convex-valued (which holds if X is convex and F is concave) then the converse holds: Γ^K is $\mathcal{P}$-increasing in K if Φ^K is $\mathcal{W}$-increasing in K for all $p \in \mathbb{R}^\ell_{++}$ (see Remark 2.7).

[47] If $v' \geq v$, simply set $\lambda = 0$.

Given these observations, the sufficiency of super*modularity holds if it implies that Γ^K is $\mathcal{P}$-increasing in K. Let $t' > t$, $\big(x, -F(x)-t\cdot x_K\big) \in \Gamma^K(t)$, and $\big(x', -F(x')-t'\cdot x'_K\big) \in \Gamma^K(t')$, for $x, x' \in X$. By super*modularity, there is $y, y' \in X$ such that $(x \wedge x')_K \geq y_K$, $y'_K \geq (x \vee x')_K$, $x + x' = y + y'$, and $F(x) + F(x') \leq F(y) + F(y')$. Then

$$F(x)-F(y)+t\cdot(x_K-y_K) \;\leq\; F(x)-F(y)+t'\cdot(x_K-y_K) \;\leq\; F(y')-F(x')+t'\cdot(y'_K-x'_K)$$

and so $-F(y)-t\cdot y_K-F(y')-t'\cdot y'_K \leq -F(x)-t\cdot x_K-F(x')-t'\cdot x'_K$. Thus we can choose $w \geq -F(y)-t\cdot y_K$, $w' \geq -F(y')-t'\cdot y'_K$ such that $w+w' = -F(x)-t\cdot x_K-F(x')-t'\cdot x'_K$. Since $(y, w) \in \Gamma(t)$ and $(y', w') \in \Gamma^K(t')$, this proves that Γ^K is $\mathcal{P}$-increasing in K.

It remains for us to show that super*modularity is necessary. Suppose $F : \mathbb{R}^\ell_+ \to \mathbb{R}$ is a regular, increasing and concave function and suppose that F fails super*modularity at $x, x' \in \mathbb{R}^\ell_+$. Let

$$Z \;:=\; \Big\{z \in \mathbb{R}^{|K|} \times \mathbb{R} : z_K \leq (x_K - y_K), z_{|K|+1} \leq F(y) + F(y') - F(x) - F(x'),$$
$$\text{for some } y, y' \in X \text{ such that } x'_K \geq y_K \text{ and } y + y' \leq x + x'\Big\}.$$

Z is nonempty (since $(x_K - x'_K, 0) \in Z$), convex (because F is concave), closed (because F is continuous and X is closed and bounded from below), and downward comprehensive (by construction). Furthermore, we have $0 \notin Z$. Otherwise, there is y and y' such that $y_K \leq (x \wedge x')_K$, $y + y' \leq x + x'$ and $F(x) + F(x') \leq F(y) + F(y')$. Since F is increasing we can always find $y'' \geq y'$ such that $x + x' = y + y''$ and $F(x) + F(x') \leq F(y) + F(y'')$, contradicting our assumption about x and x'. By the strong separating hyperplane theorem, there is a vector $p > 0$ and a number b such that $p \cdot z < b < 0$, for all $z \in Z$, where $p_{|K|+1} > 0$.[48] Without loss of generality, let $p_{|K|+1} = 1$ and $t = -p_K$, $t' = 0$. Thus, there is $t' \geq t$ such that, for any $y, y' \in X$ satisfying $x'_K \geq y_K$ and $y + y' \leq x + x'$,

$$\big[-F(x) - t\cdot x_K\big] + \big[-F(x') - t'\cdot x'_K\big] \;<\; \big[-F(y) - t\cdot y_K\big] + \big[-F(y') - t'\cdot y'_K\big],$$

which is incompatible with Γ^K being $\mathcal{P}$-increasing in K. □

Continuation of Example 10 Take any $x, x' \in X$ and denote $t_j = g_j(x_{I_j})$, $t'_j = g_j(x'_{I_j})$, for all $j = 1, \ldots, n$. Since g_j is $\mathcal{P}$-increasing in $\{i_j\}$, there is some $y_{I_j}, y'_{I_j} \in X_j$,

[48] Suppose that $p_{|K|+1} = 0$. Since X is lattice-like in K, there is y, y' such that $(x \wedge x')_K \geq y_K$ and $x + x' = y + y'$. Thus, there is $\tilde{z} \in Z$ such that $\tilde{z}_K \geq 0$, which leads to $p \cdot \tilde{z} \geq 0$, yielding a contradiction.

such that $g(y'_{I_j}) \geq t_j \vee t'_j$, $g(y_{I_j}) \geq t_j \wedge t'_j$, $(x \wedge x')_{i_j} \geq y_{i_j}$ and $x_{I_j} + x'_{I_j} = y_{I_j} + y'_{I_j}$. Let $y = (y_{I_j})_{j=1}^n$ and $y' = (y'_{I_j})_{j=1}^n$. Clearly, $(x \wedge x')_{i_j} \geq y_{i_j}$, for all j, and $x + x' = y + y'$. Since A is supermodular and increasing,

$$\begin{aligned} F(x) + F(x') &= A(t) + A(t') \leq A(t \wedge t') + A(t \vee t') \\ &\leq A\big(g_1(y_{I_1}), \ldots, g_n(y_{I_n})\big) + A\big(g_1(y'_{I_1}), \ldots, g_n(y'_{I_n})\big) = F(y) + F(y'). \end{aligned}$$

Thus F is super*modular in $K = \{i_1, \ldots, i_n\}$. □

Continuation of proof of Proposition 8 It remain for us to show the converse. Suppose Λ is not $\mathcal{P}_{\text{FSD}}$-increasing and thus Γ is not $\mathcal{P}$-increasing. By Theorem 2, there are vectors $p' \geq p$ in $\mathbb{R}^\ell$ and $t' \geq_T t$ such that

$$\begin{aligned} &\min\Big\{p \cdot y : y \in \Gamma(t')\Big\} - \min\Big\{p \cdot y : y \in \Gamma(t)\Big\} \\ &\qquad > \min\Big\{p' \cdot y : y \in \Gamma(t')\Big\} - \min\Big\{p' \cdot y : y \in \Gamma(t)\Big\}. \end{aligned} \tag{A2}$$

Take any $x, x' \in X$ satisfying $x' > x$.[49] Define a supermodular function $g : X \times S \to \mathbb{R}$ as follows: $g(y, s_1) = 0$ for all $y \in X$ and for $i > 2$, $g(y, s_i) := \sum_{j=1}^{i-1} p_i$ if $y \leq x$ and $g(y, s_i) := \sum_{j=1}^{i-1} p'_i$ otherwise. The formula (11), together with (A2), gives $f(x, t') - f(x, t) > f(x', t') - f(x', t)$, so f violates increasing differences. □

Proof of Remark 5.1 Suppose $S = [a, b]$. Let $\{s_i^n\}_{i=0}^n$ be a sequence with $n+1$ terms such that $a = s_0^n < s_1^n < \ldots < s_{n-1}^n < s_n^n = b$. Since at each (x, t), function $g(x, \cdot)$ is the Riemann-Stieltjes integrable with respect to $\lambda \in \Lambda(t)$, we can choose $\{s_i^n\}_{i=0}^n$ so that

$$\int_S g(x, s) d\lambda(s) = \lim_{n \to \infty} \sum_{i=0}^{n-1} g(x, s_{i+1}) \big[\lambda(s_{i+1}) - \lambda(s_i)\big]$$

for all $\lambda \in \Lambda(t)$. This guarantees that $\lim_{n \to \infty} f_n(x, t) = f(x, t)$ for all (x, t), where

$$f_n(x, t) := \min\left\{\sum_{i=0}^{n-1} g(x, s_{i+1}^n) \big[\lambda(s_{i+1}^n) - \lambda(s_i^n)\big] : \lambda \in \Lambda(t)\right\}.$$

We know, from the case where S is finite, that $f_n : X \times T \to \mathbb{R}$ is a supermodular function. Since supermodularity is preserved by pointwise convergence, f is supermodular. □

[49] Clearly, the proof requires that X is nonempty and contains at least two elements.

Continuation of Example 12 Take any $t' \geq t$ and $\lambda \in \Lambda(t)$, $\lambda' \in \Lambda(t')$. Since $\int_S h(s)d(\lambda \wedge \lambda')(s) \leq \int_S h(s)d\lambda'(s) = t$ and $\int_S h(s)d\lambda'(s) = t'$, there is $\alpha \in [0,1]$ such that

$$\alpha \int_S h(s)d\lambda'(s) + (1-\alpha) \int_S h(s)d(\lambda \wedge \lambda')(s) \;=\; t.$$

Let $\mu = \alpha\lambda' + (1-\alpha)(\lambda \wedge \lambda')$ and $\mu' = \alpha\lambda + (1-\alpha)(\lambda \vee \lambda')$. Clearly, $\mu \in \Lambda(t)$, $\lambda' \succeq \mu$, and $\lambda \succeq \mu'$. Since $\lambda + \lambda' = (\lambda \vee \lambda') + (\lambda \wedge \lambda')$, we also obtain $\lambda + \lambda' = \mu + \mu'$. Hence,

$$\int_S h(s)d\mu'(s) \;=\; \int_S h(s)d\lambda(s) + \int_S h(s)d\lambda'(s) - \int_S h(s)d\mu(s) \;=\; t + t' - t \;=\; t'.$$

Thus $\mu' \in \Lambda(t')$. We conclude that $\Lambda(t')$ dominates $\Lambda(t)$ in the $\mathcal{C}$-flexible set order. □

Proof of Proposition 9 Let $t'' >_T t'$ and let M satisfy $M > \max\big\{c(\lambda,t'), c(\lambda,t'')\big\}$, for all $\lambda \in \triangle_S$. The correspondence $\Gamma : T' \to \mathbb{R}^{\ell+1}$, where $T' = \{t', t''\}$, is defined by

$$\Gamma(t) \;:=\; \Big\{y \in \mathbb{R}^{\ell+1} : y_i = -\lambda(s_i), \text{ for } i = 1,2,\ldots,\ell, \text{ and } y_{\ell+1} \in \big[c(\lambda,t), M\big] \text{ for } \lambda \in \triangle_S\Big\}.$$

Since c is convex, Γ is convex-valued and it is straightforward to check that Γ is $\mathcal{P}$-increasing if, and only if, c obeys (C). Indeed, if c obeys (C), define $p', p \in \mathbb{R}^\ell$ by $p'_i = g(x', s_{i+1}) - g(x', s_i)$ and $p_i = g(x, s_{i+1}) - g(x, s_i)$, for $i = 1, \ldots, \ell$, for some supermodular function g. By supermodularity of g, we have $p' \geq p$ when $x' \geq x$. By Theorem 2 and the integration formula (10), for any $x' \geq x$ and $t' \geq_T t$, we obtain

$$\begin{aligned} f(x,t') - f(x,t) \;&=\; \min\Big\{(p,1)\cdot y : y \in \Gamma(t')\Big\} - \min\Big\{(p,1)\cdot y : y \in \Gamma(t)\Big\} \\ &\leq\; \min\Big\{(p',1)\cdot y : y \in \Gamma(t')\Big\} - \min\Big\{(p',1)\cdot y : y \in \Gamma(t)\Big\} \;=\; f(x',t') - f(x',t). \end{aligned}$$

We prove the converse by contradiction. Suppose c violates (C) and so Γ is not $\mathcal{P}$-increasing. By Theorem 2, the function $\tilde{f} : \mathbb{R}^{\ell+1} \times T' \to \mathbb{R}$, where $\tilde{f}\big((\tilde{p},q),t\big) := \min\big\{(\tilde{p},q)\cdot y : y \in \Gamma(t)\big\}$, must violate increasing differences in $(\tilde{p},t)$, i.e., there is $p, p' \in \mathbb{R}^\ell$, $t, t' \in T$, and $q \in \mathbb{R}$ such that $p' \geq p$, $t' >_T t$ and

$$\tilde{f}\big((p',q),t\big) - \tilde{f}\big((p,q),t\big) \;>\; \tilde{f}\big((p',q),t'\big) - \tilde{f}\big((p',q),t'\big). \tag{A3}$$

If $q \leq 0$, then $\tilde{f}\big((p,q),t\big) = \tilde{f}\big((p,q),t'\big)$ and $\tilde{f}\big((p',q),t\big) = \tilde{f}\big((p',q),t'\big)$, so we only need to consider $q > 0$. Given this, we can assume with no loss of generality that $q = 1$, so that $\tilde{f}\big((\tilde{p},1),t\big) \;=\; \min\Big\{\sum_{i=1}^{\ell} \tilde{p}_i\big[-\lambda(s_i)\big] + c(\lambda,t) : \lambda \in \Delta_S\Big\}$. Define the function $g : X \times S \to \mathbb{R}$ as in the proof of Proposition 8. Using (10), we obtain $f(x',t) - f(x,t) = \tilde{f}\big((p',1),t\big) - \tilde{f}\big((p,1),t\big)$ and $\tilde{f}\big((p',1),t'\big) - \tilde{f}\big((p,1),t'\big) \;=\; f(x',t') - f(x,t')$, in which case (A3) implies that f violates increasing differences. □

Continuation of Example 16 We first show that $R(\lambda\|\lambda^*(\cdot,t))$ is submodular in λ. Let λ, $\lambda' \in \triangle_S$ and denote $\mu' = \lambda \vee \lambda'$ and $\mu = \lambda \wedge \lambda'$. To abbreviate the notation, let $p_i^*(t)$, p_i, p_i', q_i, q_i' be the probability of state s_i, for all $i = 1, 2, \dots, (\ell+1)$, corresponding to the cumulative distribution of $\lambda^*(t)$, λ, λ', μ, and μ', respectively. $R(\lambda\|\lambda^*(\cdot,t))$ is submodular in λ if, for all i, $c(p_i) + c(p_i') \geq c(q_i) + c(q_i')$, where $c(x) = x\ln x - x\ln p_i^*(t)$. Clearly, this inequality holds for $i = 1$. Consider $i > 1$. With no loss of generality, let $\mu(s_{i-1}) = \lambda(s_{i-1})$ and $\mu'(s_{i-1}) = \lambda'(s_{i-1})$. Consider two cases. Assume that (i) $p_i' + \lambda'(s_{i-1}) \leq p_i + \lambda(s_{i-1})$, so that $\mu(s_i) = \lambda(s_i)$ and $\mu'(s_i) = \lambda'(s_i)$. Then $q_i = p_i$ and $q_i' = p_i'$ and $c(p_i) + c(p_i') = c(q_i) + c(q_i')$ holds. Suppose, instead, that (ii) $p_i' + \lambda'(s_{i-1}) > p_i + \lambda(s_{i-1})$, which implies $\mu(s_i) = \lambda'(s_i)$ and $\mu'(s_i) = \lambda(s_i)$. Let $\delta = \lambda(s_{i-1}) - \lambda'(s_{i-1})$ and notice that $0 \leq \delta < p_i' - p_i$. Since $q_i = p_i' - \delta$ and $q_i' = p_i + \delta$, and c is convex, $c(q_i) + c(q_i') \leq c(p_i) + c(p_i')$.

To show that $R(\lambda\|\lambda^*(\cdot,t))$ has decreasing differences in (λ, t), take any distribution $\lambda' \succeq \lambda$, $t' \geq t$, and notice that

$$\Big[R\big(\lambda'\|\lambda^*(\cdot,t')\big) - R\big(\lambda\|\lambda^*(\cdot,t')\big)\Big] - \Big[R\big(\lambda'\|\lambda^*(\cdot,t)\big) - R\big(\lambda\|\lambda^*(\cdot,t)\big)\Big]$$
$$= \sum_{i=1}^{\ell} \big[\ln p_i^*(t') - \ln p_i^*(t)\big]\big[p_i - p_i'\big].$$

This is nonpositive since $\ln p_i^*(t') - \ln p_i^*(t)$ is increasing in i (because $\lambda^*(t)$ is increasing in t with respect to the monotone likelihood ratio order) and $\lambda' \succeq \lambda$. □

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