Generic Indeterminacy of Steady-State Competitive Equilibria in Walras-von Neumann Production Economies*

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Abstract

This paper studies the structure of the set of steady-state equilibria defined in quite generalized von Neumann economic models. First, it shows that in any von Neumann production economy, there is an admissible domain of non-negative interest rates such that for any interest rate within the domain, there exists an associated steady-state equilibrium. Second, for almost all interest rates within the domain, the associated steady-state equilibrium is *indeterminate*. Thus, in summary, for any von Neumann production economy, there is a dense subset of the admissible domain over which the set of steady-state equilibria consists of a finite number of one-dimensional continuums of those equilibria. This feature is observed regardless of whether the underlying economy is *regular* or not, which is a sharp contrast with the finite and discrete features of the other types of Walrasian equilibria in static as well as intertemporal regular economies. These main results suggest, as a new, future research agenda, the necessity of studying an appropriate *equilibrium selection mechanism* which should be applied before market competition.

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1 Introduction

Since the age of Adam Smith, understanding the basic principle of the capitalist economy as a whole has been a central concern for economists, and the general equilibrium theory has played a pivotal role in economics by providing a framework to work on this subject. Indeed, perfect competition and the traditional theory of economic rationale are regarded to describe the enduring tendencies of economic activity, or to be benchmarks against which real-world deviations can be measured.

As such a benchmark theory, the general equilibrium theory has been successful in providing a coherent solution concept for economic resource allocation problems. Moreover, the contemporary general equilibrium theory has extended the classical solution concept of Walrasian competitive equilibrium into various types of more generalized economies, such as the perfectly farsighted equilibrium in intertemporal economies with the finite set of infinitely lived agents as well as with the infinite set of finitely lived agents, and the sequential equilibrium in the multi-stage sequential trading.

The literature on general equilibrium theory also contains works on the *von Neumann equilibrium* (von Neumann, 1945). Note in this paper, that the von Neumann equilibrium means the *balanced-growth* (or *the steady-state*) *competitive equilibrium in the generalized von Neumann production economies*, where the aggregate consumption demands can vary due to the change of prices.[1]

The Neumann equilibrium is a solution concept in intertemporal economies, but it is intrinsically distinctive from the standard perfectly farsighted equilibrium as well as the sequential equilibrium solutions. Indeed, unlike these intertemporal equilibrium solutions, it can be defined by equilibrium conditions for one period's resource allocations, due to its stationary features emerging across all the periods. However, it also has some distinctive aspects, in the existence problem of it, not encountered with the static production economies, as Malinvaud (1972; ch. 10) suggested.

The Neumann equilibrium is usually referred to, in the context of the *turnpike theorem*, as a *golden rule steady state* that attracts the intertemporal competitive equilibrium paths starting from any initial position. Because of this feature, as emphasized by Mandler (2002), it can be recognized as a proper representation of the *long-period equilibrium*, not only by the classical and Marxian schools but also by the neoclassical school, which serves as a 'center of gravitation' for economic activities under the capitalistic market competition, towards which short-run prices would quickly return.

Despite such importance of this solution concept, however, there has been a quite limited number of studies, such as Morishima (1960) and Bidard and Hosoda (1987), on the (*refined*) Neumann equilibrium. In other words, there remain plenty of uncultivated subjects regarding this solution, including even the general existence problem of it. In particular, no full-fledged analysis is devoted to *the equilibrium manifold of the Neumann equilibria*.

[1] That is, we will here discuss a *refined* version of the original definition of von Neumann equilibrium discussed in von Neumann (1945) and Gale (1960). The original definition was often criticized as a model of "slave economies" since it did not properly incorporate individuals' optimal choices of consumption via market exchanges. Given this kind of criticism, Morishima (1960) proposed a refined version with *generalized von Neumann models*, where specific types of Marshallian demand functions of the capitalist and the working classes are introduced as well as an exogenously given rate of saving for the capitalist class. As discussed below, the model of *generalized von Neumann production economies* in this paper is even much more generalized than the Morishima (1960) model.

In this paper, we will develop a full-fledged analysis of the *structure of the set of the Neumann equilibria*. Remember that the general equilibrium theory has developed many full-fledged analyses of the structure of the set of Walrasian competitive equilibria since Debreu (1970), which were motivated to warrant the explanatory power of this theory. Indeed, the theory of *regular economies* initiated by Debreu (1970) shows that for almost all exchange economies, the set of Walrasian competitive equilibria is *finite* and *discrete*. According to Debreu (1976), such structure of the solution set is highly desired as it warrants *generic determinacy* of equilibrium allocations via market competition, which further warrants the predictability and stability of such an economic system.

However, as Mandler (1999a,b, 2002) forcefully argued, the issue of generic (in)determinacy of market equilibria can be also motivated from a different angle. Given the extreme weatlh and income inequality in the present capitalist economies as reported by Oxfam (2024), the task of exploring the mechanisms that generate the expanding inequality would be one of the central concerns. In economics, however, no common view about the mechanism to determine functional income distribution has been established across different schools of economics. For instance, the classical and Marxian schools have often argued that functional income distribution is determined by means of various factors, including not only market competition but also some historical and institutional conditions of the capitalist society. In contrast, the early neoclassical school had originally argued that functional income distribution in competitive market economies is determined due to the principle of marginal productivity, though, as Hahn (1982) emphasized, the contemporary mainstream school would not recognize this principle as indispensable for that determination.

Nowadays, this fundamental debate can be discussed via the angle of the generic (in)determinacy of market equilibria, as Mandler (1999a, b, 2002) emphasized. The classical and Marxian views about the functional income distribution have been formally studied by Sraffa's (1960) system of price equations, which is conceived of as a representation of the *long-period equilibrium under free competition* and is known as underdetermined: in the system, the number of unknown variables is greater than that of equations, which implies that one of the wage and interest rates should be the parameter of the market mechanism, that must be determined outside of market competition in order to close the system of the equations. Such a structure of price determination under free competition is compatible with the classical and Marxian view that the wage rate is determined by historical and institutional factors, rather than the matching mechanism of demands and supplies in labor markets.

In contrast, Debreu's (1970) work on regular economies suggests that there is no room for factors other than market competition in the determination of the functional income distribution, since the Walrasian equilibrium prices and allocations change smoothly as a function of the parameters representing economic environments. Though Debreu (1970) focused only on exchange economies, Mas-Colell (1975) and Kehoe (1980, 1982, 1985) showed that the set of Walrasian competitive equilibria is finite and discrete in (static) production economies with constant returns to scale technologies. Thus, the generic determinacy of the Walrasian competitive equilibria is established even in such economies, which may suggest a criticism against the *one-degree-of-freedom* view of the Sraffian system of price equations.[2]

[2] Indeed, the Sraffian school defines the long-period equilibrium simply by the Sraffian system of price equations, which lacks conditions for demand-supply matching in commodity and factor markets. That

The generic determinacy of Walrasian equilibria is also verified in some intertemporal economic frameworks. As Kehoe and Levine (1985) argued, in the framework of intertemporal economies with a finite number of infinitely lived agents, regular economies are of full measure, each of which has a finite number of isolated perfectly farsighted equilibria. In the framework of intertemporal exchange economies with an *overlapping generation* (OLG, hereafter) structure (the infinite number of finitely lived agents), again the regular economies are generic and each of them has a finite number of steady-state equilibria, as Kehoe and Levine (1984) showed.

However, there is also plenty of literature on the *local indeterminacy* of such steady states in some intertemporal economic frameworks. A steady state is said to be locally indeterminate if there is a continuum of nearby steady-state equilibrium paths, all of which converge to the steady state. Kehoe and Levine (1984, 1985) and Calvo (1978) respectively show that each of the steady states in each intertemporal OLG economy with a pure exchange or simple production technology is locally indeterminate. Even in infinite-horizon intertemporal economies with a finite number of infinitely lived individuals, there is some literature, such as Benhabib and Farmer (1994) and Benhabib and Nishimura (1998), on *local indeterminacy of equilibrium paths* converging to a steady state, which is shown to exist under economies with some degree of *market imperfections*.[3] However, a further discussion will not be devoted to this line of research in this paper, as we focus on Walrasian equilibria under perfectly competitive economies.

There is also literature on *generic sequential indeterminacy* in intertemporal production economies with a finite number of finitely lived agents, like Mandler (1995, 1999a). Using Radner's (1972) method to decompose an intertemporal (Arrow-Debreu) equilibrium into a sequential one, the second-period production activities can be fixed in the second-period continuation equilibrium by the vector of factors endowed and produced in the first period. Therefore, the continuation equilibrium condition consists of the second-period equilibrium price equations and the equations of the second-period excess demand condition for consumption goods, where the only unknown variables are the second-period prices of consumption goods and factors. Then, under some conditions, the second-period continuation equilibrium is indeterminate for almost every induced second-period economy.[4]

There is also recent work by Yoshihara and Kwak (2023, 2024) on generic indeterminacy of steady-state equilibria, which is intrinsically distinctive from local indeterminacy as well as sequential indeterminacy. As shown by Yoshihara and Kwak (2023, 2024), in a simple model of OLG production economies with a Leontief technique, there exists a steady state equilibrium which is regular and indeterminate, in that for a neighborhood of it, there exists a *continuum of other steady-state equilibria* converging to it.

Insert Table 1 aroud here.

The last line of research is the most closely linked to what we will study in this paper. However, the model of simple Leontief production economies in that work is much more

point was criticized by Mandler (1999a).

[3]See also Nishimura and Venditti (2006) for a useful survey of these works.

[4]However, this indeterminacy is limited to economies with activity analysis types of production technology, in that the generic determinacy of both the intertemporal equilibria and the endogenously generated second-period equilibria is verified under differentiable production technology, as argued by Mandler (1997).

specific than our model of von Neumann production economies discussed in this paper. Moreover, the main results of this paper explained below would provide a more full-fledged characterization of the equilibrium manifold of the Neumann equilibria than the results by Yoshihara and Kwak (2023, 2024).

In this paper, we will provide a quite general framework of production economies with von Neumann production technology. Remember that the Morishima (1960) and Bidard and Hosoda (1987) type of generalized von Neumann model implicitly presumes that the aggregate demand functions are derived from homothetic preferences over the consumption space and a saving rate of the capitalist class (resp. of the working class) is exogenously given. None of such stringent conditions is imposed in our economic models.

Precisely speaking, we will introduce a von Neumann production economic model with a simple OLG stucture, which is represented as a profile of one-period data of production economies: a list of von Neumann production technology, a given size of the population as the data of labor endowment, and two Marshallian demand functions. Under this type of economic model, the Neumann equilibrium solution is reduced to a *steady-state* competitive equilibrium.

We will then show that for each economic environment, there is an admissible domain of non-negative interest rates such that for every interest rate within the admissible domain, there exists a Neumann equilibrium associated with this rate. Moreover, in each economic environment, for almost all interest rates within the admissible domain, the corresponding Neumann equilibria are shown to be *indeterminate*, in the sense that within a neighborhood of each of those equilibria, there is a continuum of other Neumann equilibria converging to it.

Summarizing these main theorems in this paper, we can observe that for each economy, there exists a non-empty set of the von Neumann equilibria, which is described as a closed graph of an upper hemi-continuous correspondence between the von Neumann equilibria and their associated interest rates. Moreover, there is a dense subset of the admissible domain over which such a closed graph consists of a finite number of continuous curves, each of which represents a one-dimensional continuum of these equilibria. Interestingly, these significant features can be commonly observed regardless of whether the underlying economy is regular or not.[5] Thus, one-dimensional indeterminacy of Neumann equilibria is quite *generic*.

These main results do *not* depend on the OLG structure of the model *at all*. Indeed, as discussed in Appendix B of this paper, even a more general version of von Neumann production economic model can be defined, in which a population growth rate is exogenously given as $g \geqq 0$, and an economic environment at any given period is specified by a list of von Neumann production technology, size of the population as the labor endowment specific to that period, and a Marshallian aggregate demand function. Correspondingly, a more general solution concept is defined as a *von Neumann balanced growth equilibrium* associated with g as the warranted rate of capital accumulation being equal to the natural rate of population growth. For this solution concept, the above-mentioned two main results are still obtained.

As mentioned above, section 2 provides a von Neumann production economic model with a simple OLG structure. Given this class of economies, section 3 discusses the general

[5] In this paper, as discussed below, an *economy is* defined to be *regular* if all of the von Neumann equilibria in this economy are regular, and a *von Neumann equilibrium is* defined to be *regular* if its corresponding Jacobian has full row rank.

existence of von Neumann steady-state equilibria. Moreover, in section 4, the generic indeterminacy of such equilibria is verified. Section 5 is devoted to some further discussions about the implications of these main results. Section 6 provided some concluding remarks. All of the proofs of the main theorems in sections 3 and 4 are relegated to Appendix A. Appendix B provides another variant of von Neumann production economies and its corresponding solution concept, as mentioned above.

2 A simple overlapping generation economy with von Neumann production technology

A simple overlapping generation model is constructed, in which each generation $t = 1, 2, \ldots,$ is a single individual and lives in two periods, and works only at his young age and is retired and so purchases consumption goods from the wealth due to his past saving at his old age. Let ω_l be the labor endowment of one generation, which is assumed to be fixed throughout the whole periods.

There are $n \geqq 2$ commodities which are produced in this economy and respectively used as consumption goods and/or capital goods. There are m ($\geqq n$) alternative production processes which would be operated in the production activity. Consider a von Neumann production technology (A, B, L), where A is a $n \times m$ nonnegative matrix of material input coefficients, B is a $n \times m$ nonnegative matrix of gross output coefficients, and L is a $1 \times m$ positive vector of direct labor coefficients. For each period t, let $p_t \in \mathbb{R}^n_+$ represent a vector of *prices* of n commodities prevailing at the end of this period; $w_t \in \mathbb{R}_+$ represent a *wage rate* prevailing at the end of this period; and $r_t \in \mathbb{R}$ represent an *interest rate* prevailing at the end of this period.

Let $z_b : \mathbb{R}^n_+ \times \mathbb{R}_+ \times \mathbb{R}^n_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}^n_+$ (*resp.* $z_a : \mathbb{R}^n_+ \times \mathbb{R}_+ \times \mathbb{R}^n_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}^n_+$) be a *Marshallian demand function* of every generation t in his youth (*resp.* in his old age) such that for each commodity price vectors $p_t, p_{t+1} \in \mathbb{R}^n_+$, each wage rates $w_t, w_{t+1} \in \mathbb{R}_+$, and an interest factor $1 + r_{t+1} \in \mathbb{R}_+$, $z_k(p_t, w_t, p_{t+1}, w_{t+1}, 1 + r_{t+1}) \in \mathbb{R}^n_+$ is a consumption vector purchasable for every generation when his age is $k = b, a$. Following the standard literature on the OLG production economies such as Calvo (1978), it is also assumed that each agent t's Marshallian demands satisfy the following budget constraint:

$$p_t z_b^t + \frac{p_{t+1} z_a^t}{1 + r_{t+1}} = w_t \omega_l,$$

where each agent is assumed to supply the whole of her labor endowment ω_l inelastically, and given the wage earning $w_t \omega_l$ received at the end of period t, she expends $p_t z_b^t$ for purchasing her present consumption goods z_b^t after deducting the amount of money $\frac{p_{t+1} z_a^t}{1+r_{t+1}}$ which is saved for the consumption expenditure at her old age. Note that her saved money is devoted to productive investments in the next period when she becomes old. Because of such an underlying OLG structure of each agent's decision making and given that the labor endowment is invariable, we can define each demand function z_k as a function of the price information alone as variables.

The demand function z_k is assumed to be *continuously differentiable* and satisfies *homogeneity*: for $k = b, a$,

$$\begin{aligned} z_k(p_t, w_t, p_{t+1}, w_{t+1}, 1 + r_{t+1}) &= z_k(\lambda p_t, \lambda w_t, \lambda p_{t+1}, \lambda w_{t+1}, 1 + r_{t+1}) \\ &= z_k(p_t, w_t, \lambda p_{t+1}, \lambda w_{t+1}, \lambda(1 + r_{t+1})) \end{aligned}$$

for any $\lambda > 0$ and every $(p_t, w_t, p_{t+1}, w_{t+1}, 1+r_{t+1}) \in \mathbb{R}^n_+ \times \mathbb{R}_+ \times \mathbb{R}^n_+ \times \mathbb{R}_+ \times \mathbb{R}_+$; and *Walras' law*. In addition, assume that each demand function z_k is derived from a strongly monotonic preference on the consumption space at each period. When z_k is evaluated at stationary prices $(p_t, w_t) = (p_{t+1}, w_{t+1}) = (p, w)$ for every t, we will use the notation $z_k(p, w, r)$ for $k = a, b$. Let $z(p, w, r) \equiv z_b(p, w, r) + z_a(p, w, r)$ be the aggregate demand function at every period t when the market prices are stationary.

Thus, an *overlapping generation economy* is given by a profile $E = \langle (A, B, L); \omega_l; z \rangle$. As in the literature of von Neumann production models, we are interested in studying a specific long-period feature of economic resource allocations through market competition where prices are stationary and all of the investment activities are simply of the replacements. Such a long-period feature is given as a steady-state of the Walrasian competitive equilibrium, which is defined, following Mandler (1999a; section 6), as follows:

Definition 1: A *steady-state equilibrium* (in short, $\boldsymbol{SE}$) under the overlapping generation economy $E = \langle (A, B, L); \omega_l; z \rangle$ is a pair of a stationary price vector $(p, w, 1+r) \in \mathbb{R}^n_+ \times \mathbb{R}_+ \times \mathbb{R}_+$ and a production activity vector $y \in \mathbb{R}^m_+$, such that the following conditions hold:

$$pB \leqq (1+r)\, pA + wL, \tag{a}$$

$$By \geqq z(p, w, r) + Ay, \tag{b}$$

where $z(p, w, r) = z_b(p, w, r) + z_a(p, w, r)$; and

$$Ly \leqq \omega_l. \tag{c}$$

In Definition 1, $1 + r \geqq 0$, which implies that non-negativeness of equilibrium interest rates is not requested. However, the negative value of equilibrium interest rates is not reasonable, whenever the underlying microeconomic model of individual optimization program allows each agent to devote a part of her saved money to non-productive investments. Indeed, if each agent at the beginning of her old age can invest a part of her saved money to purchase a bundle of commodities for the sake of speculative investment, as specified in Appendix C,[6] then she would devote all of her saved money to such speculative activities at a stationary price system with a negative interest rate. In this case, if a stationary price system $(p, w, 1+r)$ with $r < 0$ constitutes a $\boldsymbol{SE}$, it would be only a *trivial* one, in that the corresponding equilibrium activity vector is $y = \mathbf{0}$.

Since we are interested in a non-trivial equilibrium, we will introduce a specific case of a $\boldsymbol{SE}$, in which the corresponding equilibrium activity vector is non-zero: $y \geq \mathbf{0}$, which is also a necessary condition for the Walras-von Neumann equilibrium discussed by Morishima (1960). To be compatible with agents' motive of productive investments in such an equilibrium, the corresponding interest rate should be at least non-negative. Thus, a non-trivial steady-state equilibrium is defined as follows.

Definition 2: A *steady-state equilibrium* $((p, w, r), y)$ under the overlapping economy $E = \langle (A, B, L); \omega_l; z \rangle$ is called *Walras-von Neumann* (in short, **W-N** $\boldsymbol{SE}$) if and only if $(p, w, 1+r) > \mathbf{0}$ with $r \geqq 0$ and the condition (c) in Definition 1 holds in equality.

[6] See Appendix C in detail, where a microeconomic model of individual optimization programs is specified. In that program, each agent in her old age can devote her saved money to purchase a bundle of commodities for the sake of speculation, as well as for productive investments necessary in the next production period. Moreover, no speculative investment expenditure is optimal for every agent whenever a price system is stationary and associated with a non-negative interest rate, as explained in Appendix C.

Here, the non-triviality $y \geq \mathbf{0}$ is implied by the condition (c) with equality: $Ly = \omega_l$.

In each economy $E = \langle (A, B, L); \omega_l; z \rangle$, Walras' law is given, as in the standard way, by:

$$w [Ly - \omega_l] + p [z_b + z_a + Ay - By] = 0 \qquad (*).$$

By measn of this condition $(*)$, we can see, as in the standard literature of Walrasian general equilibrium theory, that one of the conditions (a), (b), and (c) of Definition 2 can be redundant. To see this point, let us take any profile $((p, w, r), y) \in \mathbb{R}^n_{++} \times \mathbb{R}_{++} \times \mathbb{R}_+ \times \mathbb{R}^m_{++}$ satisfying conditions (a) and (b). Then, multiplying (b) with p from the left, we have:

$$pBy \geqq pz_b (p, w, r) + pz_a (p, w, r) + pAy. \qquad \text{(b)*}$$

Note that as p is an equilibrium commodity price vector, if $(\boldsymbol{b}_i - \boldsymbol{a}_i) y > z_{bi} (p, w, r) + z_{ai} (p, w, r)$ holds for some commodity $i = 1, \ldots, n$, where $\boldsymbol{b}_i$ (resp. $\boldsymbol{a}_i$) is the i-th row vector of the matrix B (resp. A), then $p_i = 0$ holds. Therefore, the above (b)* is reduced to the following:

$$pBy = pz_b (p, w, r) + pz_a (p, w, r) + pAy. \qquad \text{(b)**}$$

Next, multiplying (a) with y from the right implies:

$$pBy \leqq (1 + r) pAy + wLy. \quad \text{(a)*}$$

Likewise, as y is an equilibrium activity vector which achieves profit maximization, if $pB_j < (1 + r) pA_j + wL_j$ holds for some process $j = 1, \ldots, m$, where B_j (resp. A_j) is the j-th column vector of the matrix B (resp. A), then $y_j = 0$ holds. Therefore, the above (a)* is reduced to the following:

$$pBy = (1 + r) pAy + wLy. \quad \text{(a)**}$$

Thus, by (b)** and (a)**, we have:

$$pz_b (p, w, r) + pz_a (p, w, r) = rpAy + wLy. \quad \text{(b)****}$$

By (a)**, Walras' law $(*)$ also can be reduced to:

$$pz_b (p, w, r) + pz_a (p, w, r) = rpAy + w\omega_l \qquad (**).$$

Thus, we can obtain equation from (b)**** and $(**)$ that:

$$Ly = \omega_l.$$

Thus, if the profile $((p, w, r), y)$ satisfies conditions (a) and (b) of Definition 2, then it also satisfies (c).

Given the above arguments, we will focus on conditions (a) and (b) of Definition 2 in the following discussion, when the existence and the indeterminacy of a **W-N *SE*** $((p, w, r), y)$ are studied.

3 Existence of the Walras-Neumann Steady-State Equilibrium

In this section, we will study the existence problem of **W-N** $\boldsymbol{SE}$. To do it, the following assumptions are introduced.

Assumption 1 (A1): For any non-negative vector $d \in \mathbb{R}^n_+$, there exists $x \in \mathbb{R}^m_+$ such that $[B - A]\, x \geqq d$.

Assumption 2 (A2): Every commodity $i \in \{1, 2, \ldots, n\}$ needs to be produced in some production process: $\boldsymbol{e}_i B \geq \mathbf{0}$ for every commodity $i \in \{1, 2, \ldots, n\}$. Every process $j \in \{1, 2, \ldots, m\}$ needs to use some commodity inputs: $A\boldsymbol{e}_j \geq \mathbf{0}$ for every process $j \in \{1, 2, \ldots, m\}$.

Assumption 3 (A3) (free disposal): For each process producing commodity $i \in \{1, 2, \ldots, n\}$ jointly with commodity $i^* \in \{1, 2, \ldots, n\}$ ($i^* \neq i$), there is another process with the same outputs and the same inputs and labor except that commodity i is not produced.

Assumption 4 (A4): If a sequence of prices $\{(p^q, w^q, r^q)\} \subseteq \mathbb{R}^n_+ \times \mathbb{R}_{++} \times \mathbb{R}_+$ with $p^q \in \mathbb{R}^n_{++}$ for every $q = 1, 2, \ldots$, converges to $(p, w, r) \in \mathbb{R}^n_+ \times \mathbb{R}_{++} \times \mathbb{R}_+$ with $p \in \mathbb{R}^n_+ \backslash \left(\mathbb{R}^n_{++} \cup \{\mathbf{0}\}\right)$, then $\|z\,(p^q, w^q, r^q)\|$ converges to infinity.

Among these assumptions, A4 is standard in the literature of *regular economies*, and it is a natural requirement for economies with strictly monotonic preferences over consumptions. By A4, if a **W-N** $\boldsymbol{SE}$ exists, then its corresponding equilibrium price vector of commodities should be positive. A2 is a standard and a quite natural assumption in the literature of von Neumann production models. A1 looks a little stronger as a condition for productiveness within von Neumann production models. However, it is indeed a standard productiveness condition as it is equivalent to the following: there exists $x \in \mathbb{R}^m_+$ such that $[B - A]\, x > \mathbf{0}$. This condition is also indispensable in this paper, as the aggregate consumption vector is endogenously varied with respect to changes of prices. Finally, A3 is a standard condition of free disposal.

By A2, it follows from the generalized Perron-Frobenius Theorem (Mangasarian, 1971, Theorem 4.1, p. 91) and Fujimoto and Krause (1988, Corollary 1, p. 191) that there exists a semi-positive vector $p^R \in \mathbb{R}^n_+ \backslash \{\mathbf{0}\}$ with a positive $(1 + R) > 0$ such that $p^R B = (1 + R)\, p^R A$ holds. This positive eigenvalue $\frac{1}{1+R}$ is maximal, in that there is no $\lambda > \frac{1}{1+R}$ such that $\lambda pB = pA$ for some $p \in \mathbb{R}^n \backslash \{\mathbf{0}\}$, due to Mangasarian (1971, Theorem 4.1-(i), p. 91). In addition, by A1, $R > 0$ holds from Fujimoto and Krause (1988, Theorem 2, p. 192).

According to Gale (1960), $(1 + R)$ is the solution to the following problem (GP_1):

$$\min_{\beta \in \mathbb{R}_+;\ p \in \mathbb{R}^n_+} \beta \text{ s.t. } pB \leqq \beta pA.$$

Then, by Gale (1960, Lemma 9.4, p. 313), for this $(1 + R)$, there exists $x^R \in \mathbb{R}^m_+ \backslash \{\mathbf{0}\}$ such that $Bx^R \geqq (1 + R)\, Ax^R$, though $(1 + R)$ is not necessarily the optimal solution to the following problem (GP_2):

$$\max_{\alpha \in \mathbb{R}_+;\ x \in \mathbb{R}^m_+} \alpha \text{ s.t. } Bx \geqq \alpha Ax.$$

However, if the von Neumann production technology (A, B, L) is *irreducible* in the sense of Gale (1960, p. 314), then $(1 + R)$ is also the solution to the maximizaiton problem (GP_2), due to Gale (1960, Theorem 9.10, p. 315). In the classical literature of von Neumann production models, where wages are assumed to be paid in advance of production so that the matrix A is also assumed to include a fixed bundle of consumption goods necessary for workers, this unique solution constitutes a von Neumann balanced growth equilibrium. Hence, no issue of equilibrium indeterminacy arises. Because no issue of the freedom of degree one is observed for the distribution between wages and profits, though no explicit examination for workers' optimal choice of consumption goods can be developed.

Unlike such literature on the classical studies of von Neumann models, our model in this paper allows agents' optimal choice of consumption goods by assuming wages to be paid after production. At the same time, our model endogenizes the determination of distribution between wages and profits, which may open the window for the issue of equilibrium indeterminacy. Within our model, the solution $(1 + R)$ to the problems (GP_1) and (GP_2) is not relevant for the determination of equilibrium, but simply to specify the maximal rate of interests $R > 0$ associated with the zero wage rate.

With the above argument, it follows that, for any $r \in [0, R)$, there exists $p^r \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$ such that $p^r [B - (1 + r) A] \leqq L$. Therefore, $[0, R)$ should serve as the domain of interest rates over which the existence of the associated **W-N** $\boldsymbol{SE}$ can be examined. Indeed, the existence of **W-N** $\boldsymbol{SE}$s is warranted for each and every interest rate in $[0, R)$, as the following theorem states:

Theorem 1: Let $E = \langle (A, B, L) ; \omega_l; z \rangle$ be an economy as specified above. Then, for *each interest rate* $r \in [0, R)$, there exists a *Walras-von Neumann steady-state equilibrium* $((p^*, 1, r) , y^*)$ under this economy.

The proof of Theorem 1 is relegated to Appendix A. Here, let us discuss the basic scenario of the proof. Let $((p^*, 1, r) , y^*)$ be a **W-N** $\boldsymbol{SE}$ for the economy $E = \langle (A, B, L) ; \omega_l; z \rangle$. Then, the conditions (a) and (b) of Definition 1 are satisfied by $((p^*, 1, r) , y^*)$.

First, multiplying the inequalities in (a) of Definition 1 by y^* from the right at this equilibrium should hold

$$p^* B y^* = (1 + r) p^* A y^* + L y^*.$$

Likewise, multiplying the inequalities in (b) of Definition 1 by p^* from the left at this equilibrium should hold

$$p^* B y^* = p^* z (p^*, 1, r) + p^* A y^*.$$

Therefore, at this equilibrium, it must hold

$$p^* z (p^*, 1, r) = p^* (B - A) y^* = (r p^* A + L) y^*. \qquad \text{(d)}$$

Note that condition (c) of Definition 1 with equality automatically follows from this condition (d) and Walras' law $(*)$.

Second, consider the following linear programing problems: given r and p^* of the equilibrium $((p^*, 1, r) , y^*)$,

$$(MP_1^*) \quad \max_{p' \in \mathbb{R}^n_+} p' \cdot z (p^*, 1, r) \text{ subject to } p' [B - A] \leqq r p^* A + L.$$

and

$$(MP_2^*) \quad \min_{y \in \mathbb{R}^m_+ \setminus \{\mathbf{0}\}} (r p^* A + L) y \text{ subject to } [B - A] y \geqq z (p^*, 1, r) .$$

Then, as $p^* \in \mathbb{R}^n_+$ is feasible in (MP^*_1) while $y^* \in \mathbb{R}^m_+ \setminus \{\mathbf{0}\}$ is feasible in (MP^*_2), it follows from condition (d) and the duality theorem that p^* is an optimal solution to (MP^*_1) while y^* is an optimal solution to (MP^*_2).

The above linear programing problems indicate that a crucial step for the existence issue of a **W-N** $\boldsymbol{SE}$ would be to find an *appropriate* price vector, like p^*, such that it is an optimal solution to a linear programming problem, like (MP^*_1), which is defined by means of that price vector itself. In other words, denote the set of solutions to a linear programming problem defined by means of each price vector $p \in \mathbb{R}^n_+$ by $\psi^r(p)$. Then, $p \in \psi^r(p)$ holds if and only if this p is an optimal solution to the problem. Thus, the crucial step is formulated by a fixed point problem of such a mapping ψ^r.

One difficulty involved in that crucial step is that the linear programming problem, like (MP^*_1), needs to find an optimal solution from the *universal set* $\mathbb{R}^n_+$, though the fixed point problem of the mapping ψ^r may require that the domain and the range of ψ^r should be identical and become a *proper subset of* $\mathbb{R}^n_+$. In the proof of Theorem 1 developed in Appendix A, we will fulfill this gap by suitably defining the domain and the range of ψ^r, in that the fixed point of such a mapping is indeed optimal over the *universal set* $\mathbb{R}^n_+$.

This existence theorem has a significant feature compared to the standard literature on von Neumann equilibrium. First, we introduce a general model of von Neumann production economy and then define a **W-N** $\boldsymbol{SE}$ with a quite general form of Marshallian demand function and an explicit condition for the labor market equilibrium. In contrast, the standard literature mainly focuses on the analysis of the production side of the economy, and neither explicit argument on the consumption side of the economy nor the labor market equilibrium condition is developed, except for a few cases.[7] Second, under such a general setting of this paper, Theorem 1 explicitly highlights the parametric feature of interest rates in the determination of **W-N** $\boldsymbol{SE}$s: it shows the existence of equilibria corresponding to each and every interest rate, and no condition to determine an equilibrium rate of interest exists even though it is a variable to determine the aggregate demands of all commodities and the optimal production plans.

Remember that in the Sraffian literature, the freedom of degree one feature is often emphasized in the determination of the steady-state equilibrium, in that a steady-state equilibrium can be determined only after either a rate of interest or wage is fixed outside of market competition.[8] In such an argument, however, neither the Marshallian demand function nor the labor market equilibrium condition is seriously discussed in the determination of equilibria.[9] Therefore, it is not obvious whether the Sraffian view of the degree one freedom can be robust even when these factors are introduced into the model and the excess demand conditions for all markets are explicitly examined. Theorem 1 in this paper verifies that the answer to this question is yes in von Neumann production economies with the OLG structure. Indeed, as discussed below, Theorem 1 implies the existence of a

[7] Such a few exceptions include Morishima (1960, 1969), Salvadori (1980), and Bidard and Hosoda (1988), where Marshallian demand functions and the condition for the labor market equilibrium are introduced in a specific way. A more detailed discussion about these works will be given in Appendix B of this paper.

[8] For instance, see Kurz and Salvadori (1995, Chapter 8) and Bidard (2004, Chapter 11).

[9] An exception may be found in Bidard (2004, Chapter 22, Theorem 1, p. 256), in which a Marshallian aggregate demand function is introduced and then the existence of a balanced growth equilibrium is shown for any given profit rate. However, the latter outcome is based on an inappropriate formulation of Walras' law: the form of Walras' law (A2) in Bidard (2004, p. 256) represents an equilibrium condition, rather than the Walras identity.

closed graph relationship between **W-N** $\boldsymbol{SE}$s and interest rates at any given economy.[10]

For the equilibrium $((p^*, 1, r), y^*)$, consider normalization of $(p^*, 1)$ as

$$\left(\frac{p_1^*}{1+\sum_{j=1}^n p_j^*}, \ldots, \frac{p_n^*}{1+\sum_{j=1}^n p_j^*}, \frac{1}{1+\sum_{j=1}^n p_j^*}\right).$$

Then, this normalized price vector belongs to $\triangle \equiv \left\{(p, w) \in \mathbb{R}_+^{n+1} \mid \sum_{i=1}^n p_i + w = 1\right\}$. From now on, let us take $\triangle$ to represent the set of price vectors.

Note that, based on Therorem 1, for each economy $E = \langle (A, B, L); \omega_l; z \rangle$, we can construct a mapping $\Psi_E : [0, R) \twoheadrightarrow \triangle \times [0, R) \times \mathbb{R}_+^m$ as follows: for each $r \in [0, R)$, $\Psi_E(r)$ is the set of **W-N** $\boldsymbol{SE}$s associated with r in the economy E. That is, for each $r \in [0, R)$, every $(p(r), w(r), r; y(r)) \in \Psi_E(r)$ is a **W-N** $\boldsymbol{SE}$ in the economy E. Call this mapping Ψ_E the *equilibrium-graph correspondence of the economy* E. This correspondence Ψ_E is non-empty, according to Theorem 1. Moreover, it can be shown that Ψ_E is *upper hemi-continuous at every* $r \in [0, R)$.

Proposition 1: For each economy $E = \langle (A, B, L); \omega_l; z \rangle$, the equilibrium-graph correspondence Ψ_E of this economy is *upper hemi-continuous.*

Proof. For each economy $E = \langle (A, B, L); \omega_l; z \rangle$, define a correspondence $\gamma : [0, R) \times \triangle^K \twoheadrightarrow \triangle^K$ as: for each $(r, p) \in [0, R) \times \triangle^K$, $\gamma(r, p) = \psi^r(p)$, where ψ^r is defined in the proof of Theorem 1 and it is non-empty, conpact-valued, and upper hemi-continuous. Therefore, γ is closed. Next, define $\Gamma : [0, R) \twoheadrightarrow \triangle^K$ by $\Gamma(r) \equiv \left\{p^* \in \triangle^K \mid p^* \in \gamma(r, p^*)\right\}$ for each $r \in [0, R)$. Then, by Proposition 12.9 of Border (1985, p.65), Γ is upper hemi-continuous. Define $\eta : [0, R) \times \Gamma([0, R)) \twoheadrightarrow \mathbb{R}_+^m$ as: $y^* \in \eta(r, p^*)$ if and only if y^* is a solution to (MP_2) at (r, p^*) with $p^* \in \Gamma(r)$. Then, by Proposition 11.23 of Border (1985, p.60), η is upper hemi-continuous. Finally, define $\Psi_E : [0, R) \twoheadrightarrow \mathbb{R}_+^m \times \Gamma([0, R))$ such that for each $r \in [0, R)$, $(y^*, p^*) \in \Psi_E(r)$ if and only if $p^* \in \Gamma(r)$ and $y^* \in \eta(r, p^*)$ hold. By the proof of Theorem 1, this (y^*, p^*) constitutes a **W-N** $\boldsymbol{SE}$ associated with r. As η is upper hemi-continuous, Ψ_E is also upper hemi-continuous by definition. ■

Thus, according to Proposition 1, the set of **W-N** $\boldsymbol{SE}$s for the economy E is described as a *closed graph in* $[0, R) \times \left(\triangle \times \mathbb{R}_+ \times [0, R) \times \mathbb{R}_+^m\right)$ by the mapping Ψ_E .

However, such a closed graph feature does not necessarily imply indeterminacy of equilibria. Indeed, as Ψ_E is not necessarily lower hemi-continuous, there may exist a **W-N** $\boldsymbol{SE}$ $(p(r'), w(r'), r'; y(r')) \in \Psi^E(r')$ which is *locally unique*, in the sense that for a sufficiently small open neighborhood $\mathcal{O}(r') \subseteq (0, R) \times \Psi_E([0, R))$ of $(p(r'), w(r'), r'; y(r'))$,

$$\mathcal{O}(r') \setminus \{(p(r'), w(r'), r'; y(r'))\} = \varnothing$$

holds. In other words, the upper hemi-continuity *alone* does not generally exclude the possibility of the existence of a *locally isolated equilibrium* within $\Psi_E([0, R))$. Therefore, Theorem 1 *per se* cannot exclude the possibility of the existence of a **W-N** $\boldsymbol{SE}$ which is *determinate.*[11]

[10] This property cannot be highlighted by the existence theorems of the few exceptional works referred in footnote 7. We will discuss this point later in Appendix B.

[11] In this paper, determinacy of equilibria is defined as the complement of the cases of indetermiancy given in Definition 3 below.

4 Generic Indeterminacy of the Walras-von Neumann Steady-State Equilibrium

In this section, we will show that each **W-N** $\boldsymbol{SE}$ is indeterminate whenever it is regular. Moreover, we will also show that for each economy and for almost all positive interest rates, the associated **W-N** $\boldsymbol{SE}$s are *regular*. In summary, we will show that for each economy, any non-regular equilibrium is *non-generic*, in that the set of positive interest rates whose associated **W-N** $\boldsymbol{SE}$s are non-regular is of measure zero.

An interesting feature of **W-N** $\boldsymbol{SE}$s is that the regularity of such equilibria not only warrants the indeterminacy of those equilibria but also eliminates the possibility of the local uniqueness of such equilibria. This feature is a sharp contrast with the case of the standard Walrasian competitive equilibria in static production economies, where the regularity warrants the determinacy and the local uniqueness of such equilibria.

First let us introduce the definition of indeterminacy, due to Mandler (1999).

Definition 3: For an economy $E = \langle (A, B, L)\,; \omega_l; z\rangle$, a Walras-von Neumann steady-state equilibrium $((p, w, r)\,; y)$ is *indeterminate* if for any $\varepsilon > 0$, there is another Walras-Neumann steady-state equilibrium $((p', w', r')\,; y')$ under this economy such that $(p', w', r') \neq (p, w, r)$ and $\|(p', w', r') - (p, w, r)\| < \varepsilon$.

Let us focus on a **W-N** $\boldsymbol{SE}$ $(p(r), w(r), r; y(r))$ with $(p(r), w(r), r) > \mathbf{0}$. This restriction is reasonable, given our assumption that the aggregate demand function z satsifies **A4**. Then, for this equilibrium, we should have $By(r) > \mathbf{0}$, as $z(p(r), w(r), r) > \mathbf{0}$. In this case, the equilibrium system of inequalities is given by:

$$\begin{aligned} z(p(r), w(r), r) - [B - A]\, y(r) &= \mathbf{0};\ (1) \\ p(r)\,[B - (1 + r) A] - w(r)\, L &\leqq \mathbf{0},\ (2) \end{aligned}$$

where there are $n + m$ inequalities while there are $n + 1 + m$ unknown variables, assuming that one commodity is used as numeraire.

To develop the analysis for generic indeterminacy, we may need to reduce the inequalities (2) to a system of equations. Let $(A(p, w, r), B(p, w, r), L(p, w, r))$ be the subsystem of (A, B, L) in that $(A(p, w, r), B(p, w, r), L(p, w, r))$ collects the actually operated processes at prices (p, w, r), each of which achieves the profit maximization at prices (p, w, r). Then, by definition, (2) can be reduced to:

$$p\,[B(p, w, r) - (1 + r) A(p, w, r)] - wL(p, w, r) = \mathbf{0}.\ (2)^*$$

Now, we may get a mapping F as follows:

$$F(p, w, r, y) \equiv \begin{cases} z(p, w, r) - [B - A]\, y \\ p\,[B(p, w, r) - (1 + r) A(p, w, r)] - wL(p, w, r) \end{cases}.$$

The map F has a regular value if $F(p, w, r, y) = \mathbf{0}$ holds for some $(p, w, r; y) \in \mathbb{R}^n_+ \times \mathbb{R}_+ \times (0, R) \times \mathbb{R}^m_+$.

Let $B(p, w, r)$ and $A(p, w, r)$ be respectively $n \times k$ matrices, and $L(p, w, r)$ be a $1 \times k$ row vector, where k $(\leqq m)$ is the number of processes which are actually operated at prices (p, w, r). Without loss of generality, assume in the following that, whenever $F(p, w, r, y) = \mathbf{0}$ holds and k is the number of actually operated processes in $y \geq \mathbf{0}$ at prices (p, w, r), then for any process $j = 1, \ldots, m$, $y_j = 0$ if and only if $[p(B - (1+r)A) - wL]\,\boldsymbol{e}_j < 0$, and the number k is *minimal*, in that no other vector $y' \geq \mathbf{0}$ with a less than k number of positive components can be an equilibrium activity vector at prices (p, w, r).

Note that, the system of equations $F(p, w, r, y) = \mathbf{0}$ is reduced to $F(\bar{p}, w, r, \overline{y}) = \mathbf{0}$, where $\bar{p} \equiv (\frac{p_1}{p_n}, ..., \frac{p_{n-1}}{p_n}, 1)$ is obtained from p by choosing commodity n as the numeraire, while $\overline{y} \in \mathbb{R}^k_{++}$ is obtained by deducting the $m-k$ zero components of the $m \times 1$ column vector $y \in \mathbb{R}^m_+$. In this reduced system of equations, there are $k+n$ equations, while there are $k+n+1$ number of unknowns.

Given a **W-N** $\boldsymbol{SE}$ $((p, w, r), y)$, define $\bar{p} \equiv (\frac{p_1}{p_n}, ..., \frac{p_{n-1}}{p_n}, 1)$ and the associated continuously differentiable mapping as follows[12]:

$$F(\bar{p}, w, r, \overline{y}) \equiv \begin{bmatrix} z(\bar{p}, w, r) - [B(\bar{p}, w, r) - A(\bar{p}, w, r)]\,\overline{y} \\ (\bar{p}B(\bar{p}, w, r) - (1+r)\bar{p}A(\bar{p}, w, r) - wL(\bar{p}, w, r))^T \end{bmatrix},$$

where the superscript T means *transpose*. By the definition of **W-N** $\boldsymbol{SE}$, $F(\bar{p}, w, r, \overline{y}) = \mathbf{0}$ holds. Then, let us introduce the notion of regular equilibria.

Definition 4: Let $E = \langle (A, B, L); \omega_l; z \rangle$ be an economy as specified above. Then, a Walras-von Neumann steady-state equilibrium $((p, w, r), y)$ under this economy is *regular* if the Jacobian of $F(\bar{p}, w, r, \overline{y}) = \mathbf{0}$ has full row rank.

Now, we are ready to argue the indeterminacy of **W-N** $\boldsymbol{SE}$s, which is summarized as follows:

Theorem 2: Let $E = \langle (A, B, L); \omega_l; z \rangle$ be an overlapping generation economy as specified above. Then, for almost all $r \in (0, R)$, its associated Walras-von Neumann steady-state equilibrium $((p, w, r), y)$ under this economy is *regular* and *indeterminate*.

To illustrate this theorem, a simple numerical example of an economy is given as follows.

Example 1 *Assume that $n = 2$, $m = 3$, $\omega_l = 1$, and the aggregate Marshallian demand function is derived from the following form of utility function: for any $(z_b, z_a) \in \mathbb{R}^2_+ \times \mathbb{R}^2_+$,*

$$u(z_b, z_a) \equiv \left[(z_{b1})^{\alpha} \cdot (z_{b2})^{1-\alpha}\right] \cdot \left[(z_{a1})^{\alpha} \cdot (z_{a2})^{1-\alpha}\right]$$

where $\alpha \in (0, 1)$. Let a von Neumann production technology be given by:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 3 & 5 & 0 \\ 3 & 0 & 5 \end{bmatrix}, \; \textit{and } L = (1, 1, 1).$$

[12]The continuous differentiability of the excess demand functions

$$z(\overline{p}, w, r) - [B(\overline{p}, w, r) - A(\overline{p}, w, r)]\,\overline{y}$$

is warranted at least in a small neighborhood of each equilibrium $(\overline{p}, w, r, \overline{y})$, as argued in Kehoe (1980, 1982). Indeed, the matrix $[B(\overline{p}, w, r) - A(\overline{p}, w, r)]$ is invariant with respect to a small change of $(\overline{p}, w, r)$ at each equilibrium, since a sufficiently small change of each of $\overline{p}, w, r$ induces a sufficiently small change of $\overline{y}$ such that the all of the k positive components in $\overline{y}$ are still positive, while the remaining $n-k$ processes are still inactive as any process that did not achieve the zero profit condition still will not after such a small change of prices. Therefore, the actually operated k processes remain constant for such a small change of prices, which implies that $[B(\overline{p}, w, r) - A(\overline{p}, w, r)]$ is invariant with respect to a small change of prices. Then, the derivatives of the functions $z(\overline{p}, w, r) - [B(\overline{p}, w, r) - A(\overline{p}, w, r)]\,\overline{y}$ with respect to prices are identical to those of the demand functions $z(\overline{p}, w, r)$ with respect to each of $\overline{p}, w, r$.

Thus, an economy is specified by $E = \langle (A, B, L); 1; \alpha \rangle$*. Then, the set of* **W-N** ***SE****s can be specified as follows:*

(1) for any economy E *with* $\alpha \in \left(0, \frac{2}{5}\right)$,

$$\left\{ ((p, w, r), y) \in \triangle \times [0, 2) \times \mathbb{R}^3_+ \mid (p, w) = \left(\left(\frac{0.4}{3-r}, \frac{0.6}{3-r} \right), \frac{2-r}{3-r} \right), y = \left(\frac{1+5\alpha}{3}, 0, \frac{2-5\alpha}{3} \right)^T \right\};$$

(2) for any economy E *with* $\alpha \in \left[\frac{2}{5}, \frac{3}{5}\right]$,

$$\left\{ ((p, w, r), y) \in \triangle \times [0, 2) \times \mathbb{R}^3_+ \mid (p, w) = \left(\left(\frac{\alpha}{3-r}, \frac{1-\alpha}{3-r} \right), \frac{2-r}{3-r} \right), y = (1, 0, 0)^T \right\}; \text{ and}$$

(3) for any economy E *with* $\alpha \in \left(\frac{3}{5}, 1\right)$,

$$\left\{ ((p, w, r), y) \in \triangle \times [0, 2) \times \mathbb{R}^3_+ \mid (p, w) = \left(\left(\frac{0.6}{3-r}, \frac{0.4}{3-r} \right), \frac{2-r}{3-r} \right), y = \left(\frac{6-5\alpha}{3}, \frac{5\alpha-3}{3}, 0 \right)^T \right\}.$$

■

In this example, every **W-N** ***SE*** is a continuous function of interest rates within $[0, 2)$, and thus the set of **W-N** ***SE***s constitutes an one-dimensional continuum at every economy.

The proof of Theorem 2 is relegated to Appendix A, where we will show that, for any given economy E, and *for almost all* $r \in (0, R)$, the Jacobian of F at the equilibrium associated with this r has *full row rank.* Once this claim is shown, then by applying the implicit function theorem, it can be shown that for any given economy E, and *for almost all* $r \in (0, R)$, every **W-N** ***SE*** associated with this r is indeterminate, due to Definition 3.

Theorem 2 is quite appealing as it claims that for each and every economy, almost all **W-N** ***SE***s are indeterminate and each of them can be represented as the image of a single-valued continuous mapping of interest rates. Such a unique feature of indeterminacy is generically observed without imposing any stringent assumption on the class of economies.

Combining with Proposition 1, Theorem 2 implies that for each economy E, the associated equilibrium graph correspondence Ψ_E has the following properties. First, for almost all $r \in (0, R)$, the corresponding image of Ψ_E, $\Psi_E(r)$, consists of a finite number of associated **W-N** ***SE***s, each of which is locally isolated within $\Psi_E(r)$. Therefore, Ψ_E essentially consists of a finite number of distinct single-valued continuous mappings. Second, the whole set of **W-N** ***SE***s in each economy E constitutes a closed graph $\Psi_E((0, R)) = \{(p(r), w(r), r, y(r)) \in \Psi_E(r) \mid r \in (0, R)\}$, in which at most finite number of continuous curves appears to represent each and every equilibrium $(p(r), w(r), r, y(r))$ at almost all r in $(0, R)$.

However, all of these features are of the image of the equilibrium set for a *given* economy. To argue this point in more detail, denote the set of economies satisfying the assumptions in this paper by $\mathcal{E}$. Then, define a correspondence $\Psi^{WN} : \mathcal{E} \twoheadrightarrow \mathbb{R}^n_{++} \times \mathbb{R}_{++} \times \mathbb{R}_+ \times \mathbb{R}^m_{++}$ such that for each $E \in \mathcal{E}$, $\Psi^{WN}(E) = \Psi_E((0, R_E))$, where R_E comes from the maximal Frobenius eigenvalue $\frac{1}{1+R_E}$ specified for the economy E. Call this mapping the *Walras-von Neumann correspondence* (W-N correspondence).

With a suitable topology on $\mathcal{E}$, it can be shown that W-N correspondence is upper hemi-continuous, like the case of the *Walrasian correspondence*[13] in static economies. However, it is well-known that the image of the Walrasian correspondence at each given (regular) economy consists of a finite number of distinct and locally unique Walrasian equilibria. In contrast, the image $\Psi^{WN}(E)$ of W-M correspondence at each given economy E *generically* constitutes a finite number of distinct *curves* (one-dimensional continuums) of the **W-N** $\boldsymbol{SE}$s in that economy. Here, we say "generically" because there may be at most finite number of interest rates, whose corresponding images of Ψ_E contain *non-regular* **W-N** $\boldsymbol{SE}$s.

Insert Figures 1a and 1b aroud here.

As in the standard literature of regular economies, let us say that an economy $E \in \mathcal{E}$ is *regular* if and only if its corresponding image $\Psi^{WN}(E)$ contains only regular **W-N** $\boldsymbol{SE}$s. Then, as discussed in section 4.2 below, we can see that the set of regular economies is of full measure, so almost all economies are regular. In addition, Theorem 2 suggests that *even within a non-regular economy*, the *set of non-regular* **W-N** $\boldsymbol{SE}$s is *of Lebesgue measure zero*.

A short remark may be given to the issue of the index theorem in this context. Because of the generic indeterminacy result here, it is meaningless to count the number of all **W-N** $\boldsymbol{SE}$s even in a regular economy. However, it may be relevant to count the finite number of continuous curves of those equilibria in the regular economy. For this issue, given a regular economy E, for each interest rate $r \in (0, R_E)$, we may apply an 'index theorem' similar to Proposition 6.4.1 in Mas-Colell (1985; p.250) into $\Psi_E(r)$, in order to verify that the number of equilibria within $\Psi_E(r)$ is odd.[14] This may lead us to conclude that the number of continuous curves of **W-N** $\boldsymbol{SE}$s in a regular economy is odd.

4.1 Ψ^{WN} in the No Joint Production Case

The structure of $\Psi^{WN}(E)$ examined in the previous section reveals even a sharper feature when the production technology (A, B, L) in E is reduced to that with *no joint production*. In this subsection, let us examine the set of **W-N** $\boldsymbol{SE}$s in the specific case of economies with no possibility of joint production.

As a preliminary step, let $\mathbf{e}_i$ be a $n \times 1$ i-th unit vector consisting of 1 at its i-th component and 0 at any other components, where $i = 1, \ldots, n$. Given a von Neumann production technology (A, B, L) specified in section 2, each production process in (A, B, L) can produce *only one type of commodity* as gross outputs if and only if for each production process $j = 1, \ldots, m$, there exists a commodity $i = 1, \ldots, n$ such that $B_j = \mathbf{e}_i$ holds. In this case, let us introduce n production sectors, where sector $i = 1, \ldots, n$ is the collection of processes which produce commodity i alone. Then, each process j can be classified as a *Leontief process of sector* i if and only if $B_j = \mathbf{e}_i$.

[13] The Walrasian correspondence is a mapping that associates to each economy the set of competitive equilibria.

[14] Perhaps, Bidard and Erreygers (1998) may be regarded to be related to this issue, as it considers a restricted version of the von Neumann production model, in that it has a given composition of final demands as well as a given uniform rate of profits, and then shows that the total number of balanced growth equilibria associated with the given profit rate is odd. However, as their analysis does not address the effect of consumers' behavior on the determination of market equilibria, we cannot apply the main theorem of Bidard and Erreygers (1998) to the indexation problem discussed here.

After such classification, let us pick up one process from each sector $i = 1, \ldots, n$, then we can constitute one *Leontief production technique* $(A^\sigma, I^\sigma, L^\sigma)$, where A^σ is a $n \times n$ non-negative square matrix of material input coefficients; I^σ is a $n \times n$ identity matrix; and L^σ is a $1 \times n$ positive vector of labor input coefficients. Denote the number of all such Leontief production techniques derived from (A, B, L) by Υ.

Assume that every such Leontief production technique $(A^\sigma, I^\sigma, L^\sigma)$, $\sigma = 1, \ldots, \Upsilon$, derived from (A, B, L) is productive and indecomposable. Then, each available Leontief production technique $\sigma \in \{1, \ldots, \Upsilon\}$ has its associated maximal eigenvalue $\frac{1}{1+R^\sigma} < 1$. Among them, let $\sigma^* \in \{1, \ldots, \Upsilon\}$ be the technique whose associated maximal eigenvalue is minimal. Then, let $R_E \equiv R^{\sigma^*}$.

In this way, a profile of multiple Leontief production techniques with no joint production $\{(A^\sigma, I^\sigma, L^\sigma)\}_{\sigma=1,\ldots,\Upsilon}$ can be derived as a special case of von Neumann production technology (A, B, L). Denote an economy with such a special case of production technology with no joint production by $E = \left\langle \{(A^\sigma, I^\sigma, L^\sigma)\}_{\sigma=1,\ldots,\Upsilon}; \omega_l; z \right\rangle$.

Obviously, for such an economy $E = \left\langle \{(A^\sigma, I^\sigma, L^\sigma)\}_{\sigma=1,\ldots,\Upsilon}; \omega_l; z \right\rangle$, Theorems 1 and 2 and Proposition 1 can apply. In addition, a further sharper characterization about the set of **W-N *SE***s can be observed:

Corollary 1: Let $E = \left\langle \{(A^\sigma, I^\sigma, L^\sigma)\}_{\sigma=1,\ldots,\Upsilon}; \omega_l; z \right\rangle$ be an overlapping generation economy as specified above. Then, $\{(p(r), w(r), r) \mid (p(r), w(r), r, y(r)) \in \Psi_E(r)\}$ is singleton for each $r \in [0, R_E)$.

Proof. It is well-known that for each $r \in [0, R_E)$, there is a unique price vector $(p(r), w(r), r)$ associated with the cost minimizing Leontief production technique (A^r, I^r, L^r) at that prices (see Kurz and Salvadori (1995, p. 131, Theorem 5.1)): that is,

$$
\begin{aligned}
& (p(r), w(r), r) \\
\equiv & \left(\frac{L^r [I^r - (1+r) A^r]^{-1}}{\sum_{i=1}^n L^r \left([I^r - (1+r) A^r]^{-1} \right)_i + 1}, \frac{1}{\sum_{i=1}^n L^r \left([I^r - (1+r) A^r]^{-1} \right)_i + 1}, r \right)
\end{aligned}
$$

where $\left([I^r - (1+r) A^r]^{-1} \right)_i$ is the i-th column vector of the matrix $[I^r - (1+r) A^r]^{-1}$.

By Theorem 1, there should exist a **W-N *SE*** associated with this r. In such an equilibrium, its equilibrium price vector must be the above defined unique price vector $(p(r), w(r), r)$.

Summarizing the above arguments, it follows that

$$\{(p(r), w(r), r) \mid (p(r), w(r), r, y(r)) \in \Psi_E(r)\}$$

is singleton for each $r \in [0, R_E)$. ■

That is, the set of steady-state equilibrium prices is represented by a *single continuous curve*.

Note that for each $r \in [0, R_E)$, $(p(r), w(r), r, y(r)) \in \Psi_E(r)$ is shown to be regular, as in Yoshihara and Kwak (2024).

4.2 On Genericity of Regular Economies

In this section, we will show that regular economies are generic even though the solution concept is of **W-N** $\boldsymbol{SE}$. Precisely speaking, we will define a parameter set of economies and then define *regular economies* within such a parameter set. More precisely speaking, we will examine the openness and genericity of parameter set of economies in which every **W-N** $\boldsymbol{SE}$ is regular. The openness and genericity is related to the stability and coverage of indeterminacy in the perturbation of parameters characterizing the set of economies.

For the demand function of two generations z^a, z^b, labor endowment ω_ℓ and for $h = (h_1, h_2, ..., h_n, h^o) \in \mathbb{R}^{n+1}$, define a perturbed demand function with similar form in Mandler (1999a) as

$$z_i(h) \equiv z_i^b(h) + z_i^a(h)$$

where

$$z_i^b(h) \equiv z_i^b(p, w, r) + \frac{w}{p_i} h_i, \;\; z_i^a(h) \equiv z_i^a(p, w, r) + \frac{w}{p_i} h^o$$

for each $i = 1, 2, ..., n$. In order to preserve Walras' law and homogeneity, the perturbation of labor endowment is given as $\omega_l(h) \equiv \omega_l + \sum_{i=1}^n h_i + \frac{nh^o}{1+r}$.

Now define a function F on the space of $n+1$ price variables $(\bar{p}, w, r)$ where $\bar{p} \equiv (p_1, ...p_{n-1}, 1)$, n quantity variables $(y_1, y_2, ...y_n)$, and adding the parameter set (A, B, L, h) to $\mathbb{R}^{2n}$, *i.e.*

$$F : \mathbb{R}_{++}^{n-1} \times \mathbb{R}_{++} \times \mathbb{R}_{+} \times \mathbb{R}_{++}^{k} \times \mathbb{R}_{+}^{nm} \times \mathbb{R}_{+}^{nm} \times \mathbb{R}_{++}^{m} \times \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+k}$$

such that

$$F(\bar{p}, w, r, \overline{y}, A, B, L, h) = \begin{bmatrix} z(h) - \left[B\left(\overline{p}, w, r\right) - A\left(\overline{p}, w, r\right)\right]\overline{y} \\ \left(\bar{p}B\left(\overline{p}, w, r\right) - (1+r)\bar{p}A\left(\overline{p}, w, r\right) - wL\left(\overline{p}, w, r\right)\right)^T \end{bmatrix}.$$

Definition 5: An *economy* is a profile of (A, B, L, h) where (A, B, L) is a von Neumann production technique, in which A is $n \times m$ non-negative matrix of material input coefficients, B is $n \times m$ non-negative matrix of gross output coefficients, L is $1 \times m$ positive row vector of direct labor coefficients, and $h = (h_1, h_2, ..., h_n, h^o) \in \mathbb{R}^{n+1}$ for perturbation.

An economy (A, B, L, h) is *regular* if every **W-N** $\boldsymbol{SE}$ $((p, w, r), y)$ is regular, that is, the Jacobian $\boldsymbol{D}F$ has full-rank at $(\bar{p}, w, r, \overline{y})$. Denote the set of economies as P and the set of regular economies as P_R. Then, the following theorem is obtained via some routine works.

Theorem 3: P_R is open and has full measure in P.

The proof of Theorem 3 can be obtained in a similar way to the proof of Theorem 2 in Yoshihara and Kwak (2024).

5 Further Discussions

5.1 Distinctive features of the Walras-von Neumann equilibrium manifold

The main results developed in sections 3 and 4 shed light on distinctive features of the **W**-**N** $\boldsymbol{SE}$s in comparison to the standard Walrasian equilibria. Remember that the finite and discrete features of the Walrasian competitive equilibria in static economies and of the perfectly farsighted equilibria in intertemporal economies are no longer warranted whenever these underlying economies are not regular. Hence, verification of full measure of the regularity in those economies are crucial to establish the generic determinacy of those equilibria, as argued by Debreu (1970), Mas-Colell (1975), Kehoe (1980, 1982), and Kehoe and Levine (1985). In contrast, the generic indeterminacy of **W**-**N** $\boldsymbol{SE}$s is established essentially without reference to the regularity of economies. Therefore, the investigation of regular economies is less significant in the case of **W**-**N** $\boldsymbol{SE}$s.

The generic indeterminacy of **W**-**N** $\boldsymbol{SE}$s is also a sharp contrast with the finite and discrete features of steady states in the standard literature of OLG economies, such as Kehoe and Levine (1984) and Calvo (1978). Such finiteness and discreteness in the standard literature would come from some specific features of their OLG economic models. For instance, the steady states in OLG pure exchange economies have a quite similar structure to Walrasian equilibria in static pure exchange economies, as argued in Kehoe and Levine (1984). In the case of OLG production economies, Calvo's (1978) neoclassical two-sector model is so specific that the stationary level of capital stock and the corresponding stationary production activities can be solved entirely independent of the price system, that is the source of the finiteness of steady states in his model.[15]

Kehoe (1985; section VI; Theorem 7) considered essentially the same class of production technologies as that in section 4.1, and then showed that each of such economies is regular and has a *unique* (steady-state) equilibrium, by applying the *nonsubstitution theorem*.[16] This result may look incompatible with Corollary 1 of this paper, but it is not. This is because what Kehoe (1985; section VI) focused to examine is the case of steady-state equilibrium associated with $r = 0$ alone, while we also consider the cases of steady-state equilibria associated with $r > 0$. Indeed, our Corollary 1 is reduced to Kehoe (1985; section VI; Theorem 7) if the admissible domain of interested rates is restricted to $\{0\}$.

5.2 Further Implications of Main Theorems

What kinds of lessons can we obtain from the generic indeterminacy of Walras-Neumann equilibria? Remember that Debreu (1970, 1976) considered that equilibrium indeterminacy is a non-desirable feature for warranting the explanatory power of the theory. However, such an interpretation may be inappropriate.

Rather, it may indicate that the classical and Marxian views that the functional income distribution is determined at least partly by some historical, institutional, and sociopo-

[15] See Yoshihara and Kwak (2024) for more detailed comments about the source of Calvo's (1978) results about the steady states.

[16] Note that the class of economies in Kehoe (1985; section VI) does not need to have the OLG structure, but this point is not essential in the following argument. Indeed, though we assume the OLG structure in this paper, our Corollary 1 does not depend on this structure.

litical schemes are indeed compatible with the standard general equilibrium reasoning. In other words, it may suggest, in determining the long-period equilibrium position, the necessity of an *equilibrium selection mechanism* that is *non-market competitive* in nature and is applied before implementing the competitive market mechanism. That is, a *two-stage comprehensive resource allocation mechanism* should be constituted, in which the first stage consists of a *non-market scheme* to determine a functional income distribution, that is, to select either an interest rate or a wage rate. Then the second stage is the competitive market mechanism which determines a **W-N** $\boldsymbol{SE}$, associated with the selected interest rate (or the wage rate).

Though the issue of what kind of non-market scheme would be relevant is beyond the scope of this paper, it might involve even *democratic decision-making* about an *appropriate social welfare function*, that can specify an optimal equilibrium selection among infinitely many **W-N** $\boldsymbol{SE}$s. Such an equilibrium selection may be relevant to the selection of monetary policy by the central bank to influence the long-term interest rate. Or, given that we can choose wage rates, rather than interest rates, as the parameter of market competition in Theorems 1 and 2, the centralized collective bargaining system, like those in Nordic countries, could be regarded as such an equilibrium selection mechanism. Recently, Oxfam (2024) suggests, as prescriptions for fighting inequality, "ensure no share dividend payments before living wages" as well as "limit top pay to no more than 20 times that of the average (median) worker", which could also serve as examples of the first-stage equilibrium selection mechanism.

6 Concluding Remarks

In the above sections, we have considered a class of von Neumann production economies with a simple OLG structure to investigate Walras-von Neumann steady-state equilibria. Then, we have established the following distinctive features of the set of Walras-von Neumann steady-state equilibria. That is, a non-empty set of such equilibria exists for every production economy E, which can be described as a closed graph of the equilibrium-graph correspondence Ψ_E defined over the admissible domain, $[0, R_E)$, of non-negative interest rates. Moreover, for almost every interest rate in the domain $[0, R_E)$, the corresponding Walras-von Neumann steady-state equilibrium is regular and indeterminate. Therefore, except for a negligible subset of the domain $[0, R_E)$, the closed graph of the correspondence Ψ_E consists of a finite number of one-dimensional continuous curves of regular Walras-von Neumann steady-state equilibria. This feature is observed regardless of whether economy E is regular or not, though full measure of the regularity of economies is also warranted. As mentioned in section 1 and developed in Appendix B, these main results are preserved even if the economy exhibits population growth and/or it has no OLG structure.

Owing to the main results in this paper, new windows would open for some new research agenda. For instance, as suggested in section 5.2, the design problem of an optimal income (re)distribution policy may be formulated as that of the first-stage scheme within an appropriate comprehensive two-stage resource allocation mechanism. Perhaps, the design problem of such a non-market scheme would require a proper view about which of infinitely many steady-states equilibria, the intertemporal equilibrium paths starting from the first-stage selection would approach. Hence, it would be interesting to examine how the so-called turnpike theorem might be robust or required to be revised, given the existence of a finite number of one-dimensional continuums of Walras-von Neumann steady-state equilibria. We leave them for future research.

7 References

Balasko, Y. (2009) *The Equilibrium Manifold: Postmodern Developments in the Theory of General Economic Equilibrium*, The MIT Press, Cambridge, Massachusetts, London, England.

Bidard, C. (2004) *Prices, Reproduction, Scarcity*, Cambridge Univ. Press, Cambridge, UK.

Bidard, C. and Erreygers, G. (1998) The Number and Type of Long-term Equilibria, *Journal of Economics*, 67, 181-205.

Bidard, C. and Hosoda, E. (1987) On Consumption Baskets in a Generalized von Neumann Model, *International Economic Review*, 28, 509-519.

Border, K. (1985) *Fixed point theorems with applications to economics and gema theory*, Cambridge Univ. Press, New York.

Benhabib, J. and Farmer, R. (1994) Indeterminacy and Increasing Returns, *Journal of Economic Theory*, 63, 19-41.

Benhabib, J. and Nishimura, K. (1998) Indeterminacy and Sunspots with Constant Returns, *Journal of Economic Theory*, 81, 58-96.

Calvo, G. A. (1978) On the Indeterminacy of Interest Rates and Wages with Perfect Foresight, *Journal of Economic Theory*, 19, 321-337.

Debreu, G. (1970) Economies with finite set of equilibria, *Econometrica*, 38, 387-393.

Debreu, G. (1976) Regular Differentiable Economies, *American Economic Review*, 66, 280-287.

Fujimoto, T. and Krause, U. (1988) More theorems on joint production, *Journal of Economics*, 48, 189-196.

Gale, D. (1960) *The Theory of Linear Economic Models*, The University of Chicago Press, Chicago and London.

Hahn, F. (1982) The Neo-Ricardians, *Cambridge Journal of Economics*, 6, 353-374.

Kehoe, T. (1980) An index theorem for general equilibrium models with production, *Econometrica*, 48, 1211-1232.

Kehoe, T. (1982) Regular production economies, *Journal of Mathematical Economics*, 10, 147-176.

Kehoe, T. (1985) Multiplicity of Equilibria and Comparative Statics, *Quarterly Journal of Economics*, 100, 119-147.

Kehoe, T. and Levine, D. K. (1984) Regularity in overlapping generations exchange economies, *Journal of Mathematical Economics*, 13, 69-93.

Kehoe, T. and Levine, D. K. (1985) Comparative statics and perfect foresight in infinite horizon economies, *Econometrica*, 53, 433-454.

Kurz, H. D. and Salvadori, N. (1995) *Theory of Production: A Long-Period Analysis*, Cambridge Univ. Press, Cambridge, UK.

Malinvaud, E. (1972) *Lectures on Microeconomic Theory*, North-Holland, Amsterdam.

Mandler, M. (1995) Sequential indeterminacy in production economies, *Journal of Economic Theory*, 66, 406-436.

Mandler, M. (1997) Sequential regularity in smooth production economies, *Journal of Mathematical Economics*, 27, 487-504.

Mandler, M. (1999a) Sraffian indeterminacy in general equilibrium, *Review of Economic Studies*, 66, 693-711.

Mandler, M. (1999b) *Dilemmas in Economic Theory*, Oxford University Press, New York.

Mandler, M. (2002) Classical and neoclassical indeterminacy in one-shot versus ongoing equilibria, *Metroeconomica*, 53, 203-222.

Mangasarian, O. L. (1971) Perron-Frobenius properties of $Ax = \lambda Bx$, *Journal of Mathematical Analysis and Applications*, 36, 86-102.

Mas-Colell, A. (1975) On the continuity of equilibrium prices in constant return production economies, *Journal of Mathematical Economics*, 2, 21-33.

Mas-Colell, A. (1983) *The Theory of General Economic Equilibrium: A Differentiable Approach*, Cambridge Univ. Press, Cambridge, UK.

Morishima, M. (1960) Economic Expansion and the Interest Rate in Generalized von Neumann Models, *Econometrica*, 28, 352-363.

Morishima, M. (1969) *Theory of Economic Growth*, Clarendon Press, Oxford.

Neumann, J. von (1945) A Model of General Economic Equilibrium, *Review of Economic Studies*, 13, 1-9.

Nishimura, K and Venditti, A. (2006) Indeterminacy in Discrete-Time Infinite-Horizon Models, in Dana, R-A., Le Van, C., Mitra, T. and Nishimura, K. (eds), *Handbook on Optimal Growth 1 Discrete Time*, 273-296, Springer Berlin, Heidelberg.

Oxfam (2024) *Inequality Inc. How corporative power divides our world and the need for a new era of public action*, Oxfam GB, Oxfam House, John Smith Drive, Cowley, Oxford, OX4 2JY, UK.

Radner, R. (1972) Existence of equilibrium of plans, prices, and price expecta- tions in a sequence of markets, *Econometrica*, 40, 289-303.

Salvadori, N. (1980) On a Generalized von Neumann Model, *Metroeconomica*, 32, 51-62.

Sraffa, P. (1960) *Production of Commodities by Means of Commodities*: *Prelude to a Critique of Economic Theory*, Cambridge University Press, Cambridge.

Yoshihara, N. and Kwak, S. H. (2023) A Simple Example of Sraffian Indeter-minacy in Overlapping Generation Production Economies, *Communications in Economics and Mathematical Sciences* 2, 44-57.

Yoshihara, N. and Kwak, S. H. (2024) Sraffian indeterminacy of steady-state equilibria in the Walrasian general equilibrium framework, *Metroeconomica*, 75, 377-395.

8 Appendix

8.1 Appendix A: Proofs of Main Theorems

8.1.1 Proof of Theorem 1

Let us fix wage rates as *unity*: $w = 1$. Then, by condition (a) of Definition 1, any equilibrium commodity price vector associated with an interest rate $r \in [0, R)$ must belong to the following set:

$$\triangle^r \equiv \left\{ p \in \mathbb{R}^n_+ \mid p \left[B - (1+r) A \right] \leqq L \right\}.$$

As $L > \mathbf{0}$, $\triangle^r$ is non-empty, compact, and convex. Then for each $p \in \triangle^r$, specify

$$\triangle^{(p,r)} \equiv \left\{ p' \in \mathbb{R}^n_+ \mid p' \left[B - A \right] \leqq rpA + L \right\}.$$

As $p \in \triangle^{(p,r)}$ follows from $p \in \triangle^r$, $\triangle^{(p,r)}$ is non-empty. It is also compact and convex. Define $\triangle^{(\triangle^r,r)} \equiv \cup_{p \in \triangle^r} \triangle^{(p,r)}$.

Define the domain of commodity price vectors by the following set:

$$\triangle^K \equiv \left\{ p \in \mathbb{R}^n_+ \mid \sum_{i=1}^n p_i \leqq K \right\},$$

where $K > 0$ is sufficiently large. Let $\partial \triangle^K$ be the upper boundary of $\triangle^K$. Then, as K is sufficiently large, for each $p \in \triangle^{(\triangle^r,r)}$, there is a $t \geqq 1$ such that $tp \in \partial \triangle^K$. This implies that $\triangle^K \supset \triangle^{(\triangle^r,r)}$. It also follows that for any $p \in \partial \triangle^K$, there exists $\lambda \in (0,1)$ such that $\lambda p \in \triangle^{(\triangle^r,r)}$.

Given $r \in [0, R)$ and for each $p \in \triangle^K$, the Marshallian demand vectors $z_b(p,1,r)$ and $z_a(p,1,r)$ are uniquely identified at prices $(p,1,r)$. Then, given such fixed $z_b(p,1,r)$ and $z_a(p,1,r)$, define the following problem:

$$(MP_1(p)) \quad \max_{p' \in \triangle^K} p' \cdot \left[z_b(p,1,r) + z_a(p,1,r) \right] \text{ subject to } p' \left[B - A \right] \leqq rpA + L.$$

Let us denote the set of optimal solutions to $(MP_1(p))$ by $\psi^r(p)$. Then:

Lemma 1: The correspondence $\psi^r : \triangle^K \twoheadrightarrow \triangle^K$ has a fixed point.

Proof. It is easy to see that $\psi^r(p)$ is non-empty, compact, and convex for each $p \in \triangle^K$. Moreover, by Berge's maximum theorem, ψ^r is upper hemi-continuous. Therefore, by the Kakutani fixed point theorem, there exists $p^* \in \triangle^K$ such that $p^* \in \psi^r(p^*)$. ■

Lemma 1 implies that there exists $p^* \in \triangle^K$ such that

$$p^* \in \arg \max_{p' \in \triangle^K;\ p'[B-A] \leqq rp^*A+L} p' \cdot z(p^*,1,r).$$

Note that $z_b(p^*,1,r) + z_a(p^*,1,r) \geq \mathbf{0}$ follows from $w = 1$. Moreover, by A4, $p^* > \mathbf{0}$ holds.

Given this datum $p^* \in \triangle^K$, let us define the following linear programing problems:

$$(MP_1^*) \quad \max_{p' \in \mathbb{R}^n_+} p' \cdot z(p^*,1,r) \text{ subject to } p' \left[B - A \right] \leqq rp^*A + L.$$

and

$$(MP_2^*)\ \min_{y\in\mathbb{R}_+^m\backslash\{\mathbf{0}\}} (rp^*A+L)\,y \text{ subject to } [B-A]\,y \geqq z\,(p^*,1,r)\,.$$

Then:

Lemma 2: $p^* \in \triangle^K$ is an optimal solution to (MP_1^*).

Proof. To show the claim, it is sufficient to verify that for any $p' \in \mathbb{R}_+^n$, if $p'\,[B-A] \leqq rp^*A+L$, then $p' \in \triangle^K$. Let

$$\triangle^{(p^*,r)} \equiv \left\{p \in \mathbb{R}_+^n \mid p\,[B-A] \leqq rp^*A+L\right\}.$$

Then, by definition, $\triangle^{(p^*,r)} \subseteq \triangle^{(\triangle^r,r)} \subset \triangle^K$ holds. Therefore, for any $p \in \mathbb{R}_+^n\backslash\triangle^K$, $p \notin \triangle^{(p^*,r)}$ holds, thus $p\,[B-A] \nleqq rp^*A+L$. Thus, for $p \in \mathbb{R}_+^n\backslash\triangle^K$, p cannot be a solution to (MP_1^*). As $p^* \in \triangle^K$ is a solution to $(MP_1\,(p^*))$, according to Lemma 1, it is also a solution to (MP_1^*). ■

Next, consider (MP_2^*). By A1, there is a feasible solution $y \in \mathbb{R}_+^m\backslash\{\mathbf{0}\}$ such that $[B-A]\,y \geqq z\,(p^*,1,r) \geq \mathbf{0}$ holds. Therefore, there exists a solution $y^* \in \mathbb{R}_+^m\backslash\{\mathbf{0}\}$ to (MP_2^*).

Thus, we can have

$$p^*\cdot z\,(p^*,1,r) \leqq p^*\,[B-A]\,y^* \leqq (rp^*A+L)\,y^*.$$

Then:

Lemma 3: $p^*\cdot z\,(p^*,1,r) = p^*\,[B-A]\,y^* = (rp^*A+L)\,y^*$.

Proof. The claim follows from the duality theorem. ■

Proof of Theorem 1: We can see that $((p^*,1,r)\,;y^*)$ satisfies conditions (a) and (b) of Definition 1. Then, it can be shown by means of Walras' law $(*)$ and Lemma 3 that $((p^*,1,r)\,;y^*)$ also satisfies condition (c) of Definition 1 with equality. Thus, $((p^*,1,r)\,;y^*)$ is a **W-N *SE*** associated with the positive interest rate $r>0$ in the economy $E = \langle (A,B,L)\,;\omega_l;u\rangle$. ■

8.1.2 Proof of Theorem 2

Given an economy E, for each $r \in (0,R)$, let $((p,w,r)\,,y) \in \mathbb{R}_{++}^n\times\mathbb{R}_{++}\times(0,R)\times\mathbb{R}_+^m$ be an associated **W-N *SE*** in E, whose existence is ensured by Theorem 1. Then, $F\,(\bar{p},w,r,\overline{y}) = \mathbf{0}$ holds, where $\bar{p} \equiv (\frac{p_1}{p_n},...,\frac{p_{n-1}}{p_n},1)$ is obtained from p by choosing commodity n as the numeraire, while $\overline{y} \in \mathbb{R}_{++}^k$ is obtained by deducting the $m-k$ zero components of the $m\times 1$ column vector $y \in \mathbb{R}_+^m$. In the following argument, we will show that, for any given economy E and *for almost all* $r \in (0,R)$, the Jacobian of F at the equilibrium associated with this r has full row rank.

The Jacobian of F at the equilibrium $((p,w,r),y)$ is given by:

$$\begin{aligned}&\boldsymbol{D}_{(y,\overline{p},w,r)}(F(\overline{p},w,r,\overline{y}))\\ =&\left[\begin{array}{cccc} [A-B](\overline{p},w,r) & \boldsymbol{D}_{\overline{p}}z(\overline{p},w,r) & \boldsymbol{D}_{w}z(\overline{p},w,r) & \boldsymbol{D}_{r}z(\overline{p},w,r) \\ \mathbf{0} & [\boldsymbol{b}_{-n}-(1+r)\boldsymbol{a}_{-n}]^{T}(\overline{p},w,r) & -L(\overline{p},w,r)^{T} & -(\overline{p}A(\overline{p},w,r))^{T}\end{array}\right],\end{aligned}$$

where

$$\begin{aligned}[A-B](\overline{p},w,r) &\equiv [A(\overline{p},w,r)-B(\overline{p},w,r)],\\ [\boldsymbol{b}_{-n}-(1+r)\boldsymbol{a}_{-n}]^{T}(\overline{p},w,r) &\equiv [\boldsymbol{b}_{-n}(\overline{p},w,r)-(1+r)\boldsymbol{a}_{-n}(\overline{p},w,r)]^{T},\end{aligned}$$

and $[\boldsymbol{b}_{-n}(\overline{p},w,r)-(1+r)\boldsymbol{a}_{-n}(\overline{p},w,r)]^{T}$ is the $k\times(n-1)$ matrix obtained by deleting the n-th column of the matrix $[B(\overline{p},w,r)-(1+r)A(\overline{p},w,r)]^{T}$. We will verify that $rank\left[\boldsymbol{D}_{(y,\overline{p},w,r)}(F(\overline{p},w,r,\overline{y}))\right]=n+k$ holds for almost all r within $(0,R)$.

First, we will verify the following claim:

Lemma 4: $rank\left[\begin{array}{cccc}[A-B](\overline{p},w,r) & \boldsymbol{D}_{\overline{p}}z(\overline{p},w,r) & \boldsymbol{D}_{w}z(\overline{p},w,r) & \boldsymbol{D}_{r}z(\overline{p},w,r)\end{array}\right]=n.$

Proof. The claim follows from

$$rank\left[\begin{array}{cc}\boldsymbol{D}_{\overline{p}}z(\overline{p},w,r) & \boldsymbol{D}_{w}z(\overline{p},w,r)\end{array}\right]=n,$$

since the $n\times n$ matrix $\left[\begin{array}{cc}\boldsymbol{D}_{\overline{p}}z(\overline{p},w,r) & \boldsymbol{D}_{w}z(\overline{p},w,r)\end{array}\right]$ is invertible (see 1.11 on page 6 in Balasko (2009)). ■

Second, we need to verify that

$$rank\left[\begin{array}{cccc}\mathbf{0} & [\boldsymbol{b}_{-n}-(1+r)\boldsymbol{a}_{-n}]^{T}(\overline{p},w,r) & -L(\overline{p},w,r)^{T} & -(\overline{p}A(\overline{p},w,r))^{T}\end{array}\right]=k.$$

Note that this claim holds whenever the row vectors in $[B-(1+r)A]^{T}(\overline{p},w,r)$ are linearly independent. To verify it, we need to do some preliminary analysis:

Lemma 5: The columns of $[B(\overline{p},w,r)-A(\overline{p},w,r)]$ are linear independent and $k\leqq n$.

Proof. Suppose not. Then, there is $\overline{x}\in\mathbb{R}^{k}\backslash\{\mathbf{0}\}$ such that $[B(\overline{p},w,r)-A(\overline{p},w,r)]\overline{x}=\mathbf{0}$. Therefore, there is another equilibrium activity vector $\overline{y}+\lambda\overline{x}\in\mathbb{R}_{+}^{k}$ for some $\lambda\in\mathbb{R}$ such that $[B(\overline{p},w,r)-A(\overline{p},w,r)](\overline{y}+\lambda\overline{x})=z(\bar{p},w,r)$ holds. Then, by properly choosing λ, the number of positive components of $(\overline{y}+\lambda\overline{x})$ can be less than k. However, this contradicts that k is the minimal number of actually operated processes at $(\overline{p},w,r)$. Therefore, the columns of $[B(\overline{p},w,r)-A(\overline{p},w,r)]$ are linearly independent and $k\leqq n$ must hold. ■

By Lemma 5, there is a $k\times k$ submatrix of $[B(\overline{p},w,r)-A(\overline{p},w,r)]$, which is invertible. Denote this submatrix by

$$(B(\overline{p},w,r)-A(\overline{p},w,r))^{(k)}$$

and its determinant by

$$\det(B(\overline{p},w,r)-A(\overline{p},w,r))^{(k)}.$$

Likewise, define a $k \times k$ submatrix of $[B(\overline{p}, w, r) - (1+r)A(\overline{p}, w, r)]$ by:

$$(B(\overline{p}, w, r) - (1+r)A(\overline{p}, w, r))^{(k)} \equiv (B(\overline{p}, w, r) - A(\overline{p}, w, r))^{(k)} - rA(\overline{p}, w, r)^{(k)},$$

where $A(\overline{p}, w, r)^{(k)}$ is the $k \times k$ submatrix of $A(\overline{p}, w, r)$, which appears in $(B(\overline{p}, w, r) - A(\overline{p}, w, r))^{(k)}$. Then:

Lemma 6: $\det(B(\overline{p}, w, r) - (1+r)A(\overline{p}, w, r))^{(k)} \neq 0$ if and only if

$$rank\left[\begin{array}{cccc} \mathbf{0} & [\boldsymbol{b}_{-n} - (1+r)\boldsymbol{a}_{-n}]^T(\overline{p}, w, r) & -L(\overline{p}, w, r)^T & -(\overline{p}A(\overline{p}, w, r))^T \end{array}\right] = k.$$

Proof. There are two possibilities: (i) $k < n$; and (ii) $k = n$.

Consider case (i): $k < n$. Then, $\left([B(\overline{p}, w, r) - (1+r)A(\overline{p}, w, r)]^T\right)^{(k)}$ is a submatrix of $[\boldsymbol{b}_{-n} - (1+r)\boldsymbol{a}_{-n}]^T(\overline{p}, w, r)$, so the claimed equivalence relation holds.

Consider case (ii): $k = n$. First, the equilibrium condition (2)* in section 4 can be rewritten as:

$$\sum_{i=1}^{n-1} \frac{\overline{p}_i}{w}(\boldsymbol{b}_i(\overline{p}, w, r) - (1+r)\boldsymbol{a}_i(\overline{p}, w, r))^T + \frac{1}{w}(\boldsymbol{b}_n(\overline{p}, w, r) - (1+r)\boldsymbol{a}_n(\overline{p}, w, r))^T = L(\overline{p}, w, r)^T,$$

where $(\boldsymbol{b}_i(\overline{p}, w, r) - (1+r)\boldsymbol{a}_i(\overline{p}, w, r))^T$ (resp. $(\boldsymbol{b}_n(\overline{p}, w, r) - (1+r)\boldsymbol{a}_n(\overline{p}, w, r))^T$) is the i-the column vector (resp. the n-th column vector) of $\left([B(\overline{p}, w, r) - (1+r)A(\overline{p}, w, r)]^T\right)^{(k)}$. Therefore, we have the following property:

$$\begin{aligned} & \det\left[\begin{array}{cc} \left([\boldsymbol{b}_{-n}(\overline{p}, w, r) - (1+r)\boldsymbol{a}_{-n}(\overline{p}, w, r)]^T\right)^{(k)} & -L(\overline{p}, w, r)^T \end{array}\right] \\ = & -\frac{1}{w}\det\left([B(\overline{p}, w, r) - (1+r)A(\overline{p}, w, r)]^T\right)^{(k)} \\ & + \sum_{i=1}^{n-1}\det\left[\left([\boldsymbol{b}_{-n}(\overline{p}, w, r) - (1+r)\boldsymbol{a}_{-n}(\overline{p}, w, r)]^T\right)^{(k)}, -\frac{\overline{p}_i}{w}(\boldsymbol{b}_i(\overline{p}, w, r) - (1+r)\boldsymbol{a}_i(\overline{p}, w, r))^T\right] \\ = & -\frac{1}{w}\det\left([B(\overline{p}, w, r) - (1+r)A(\overline{p}, w, r)]^T\right)^{(k)}. \end{aligned}$$

Hence, we have:

$$\det\left[\begin{array}{cc} \left([\boldsymbol{b}_{-n}(\overline{p}, w, r) - (1+r)\boldsymbol{a}_{-n}(\overline{p}, w, r)]^T\right)^{(k)} & -L(\overline{p}, w, r)^T \end{array}\right] \neq 0$$

if and only if

$$\det(B(\overline{p}, w, r) - (1+r)A(\overline{p}, w, r))^{(k)} \neq 0.$$

Thus, the claimed equivalence relation holds. ■

By Lemma 6, it is sufficient to show $\det(B(\overline{p}, w, r) - (1+r)A(\overline{p}, w, r))^{(k)} \neq 0$ (equivalently, $rank[B(\overline{p}, w, r) - (1+r)A(\overline{p}, w, r)]^T = k$) for almost all r within $(0, R)$. To show this claim, let us define:

$$\mathcal{R} \equiv \left\{r \in [0, R) \mid rank[B(\overline{p}, w, r) - (1+r)A(\overline{p}, w, r)]^T \text{ is not full row rank}\right\}.$$

We will verify that the set $\mathcal{R}$ is of Lebesgue measure zero in $[0, R)$. To show this claim, assume $r \in \mathcal{R}$, and then show that for a small open neighborhood $\mathcal{N}(r) \subset [0, R)$ of r, $\mathcal{N}(r) \cap \mathcal{R} = \{r\}$.

As a preliminary step, let $\mathcal{N}^{(\overline{p}, w, r, \overline{y})} \subset \mathbb{R}^n_{++} \times \mathbb{R}_{++} \times (0, R) \times \mathbb{R}^k_{++}$ be a sufficiently small open neighborhood of $(\bar{p}, w, r, \overline{y})$ such that for any $(\overline{p}', w', r', \overline{y}') \in \mathcal{N}^{(\overline{p}, w, r, \overline{y})}$ being a **W-N *SE*** in E, $B(\overline{p}', w', r') = B(\overline{p}, w, r)$, $A(\overline{p}', w', r') = A(\overline{p}, w, r)$, and $L(\overline{p}', w', r') = L(\overline{p}, w, r)$ hold.[17] Moreover, let $\mathcal{N}^{(\overline{p}, w, r, \overline{y})}(r) \subset (0, R)$ be a sufficiently small open neighborhood of r such that $\mathcal{N}^{(\overline{p}, w, r, \overline{y})}(r)$ is the projection of $\mathcal{N}^{(\overline{p}, w, r, \overline{y})}$ on $(0, R)$. Then, for any $r' \in \mathcal{N}^{(\overline{p}, w, r, \overline{y})}(r)$ and any **W-N *SE*** $(\overline{p}(r'), w(r'), r', \overline{y}(r')) \in \mathcal{N}^{(\overline{p}, w, r, \overline{y})}$, it follows that $B(\overline{p}(r'), w(r'), r') = B(\overline{p}, w, r)$, $A(\overline{p}(r'), w(r'), r') = A(\overline{p}, w, r)$, and $L(\overline{p}(r'), w(r'), r') = L(\overline{p}, w, r)$.

Define a real-valued function $\Xi^{(\overline{p}, w, r)} : \mathcal{N}^{(\overline{p}, w, r, \overline{y})}(r) \to \mathbb{R}$ by: for each $r' \in \mathcal{N}^{(\overline{p}, w, r, \overline{y})}(r)$,

$$\Xi^{(\overline{p}, w, r)}(r') \equiv \det\left(B(\overline{p}, w, r) - (1 + r') A(\overline{p}, w, r)\right)^{(k)}.$$

Then, we have:

Lemma 7: Let $r \in \mathcal{R}$, and $(\bar{p}, w, r, \overline{y})$ be a **W-N *SE*** associated with this r in E. Then, $\Xi^{(\overline{p}, w, r)}(r) = 0$ and there is a small open neighborhood $\mathcal{N}(r) \subseteq \mathcal{N}^{(\overline{p}, w, r, \overline{y})}(r)$ of r such that for any $r' \in \mathcal{N}(r) \setminus \{r\}$, $\Xi^{(\overline{p}, w, r)}(r') \neq 0$.

Proof. As $r \in \mathcal{R}$, $rank[B(\overline{p}, w, r) - (1 + r)A(\overline{p}, w, r)]^T < k$ holds, so there is a $k \times k$ submatrix $(B(\overline{p}, w, r) - (1 + r)A(\overline{p}, w, r))^{(k)}$ of $B(\overline{p}, w, r) - (1 + r)A(\overline{p}, w, r)$ such that $\Xi^{(\overline{p}, w, r)}(r) = \det(B(\overline{p}, w, r) - (1 + r)A(\overline{p}, w, r))^{(k)} = 0$ holds. Let $\left(\Xi^{(\overline{p}, w, r)}\right)^{-1} : \mathbb{R} \twoheadrightarrow \mathcal{N}^{(\overline{p}, w, r, \overline{y})}(r)$ be the inverse mapping of $\Xi^{(\overline{p}, w, r)}$. Then, as $\Xi^{(\overline{p}, w, r)}$ is a polynomial function defined over $\mathcal{N}^{(\overline{p}, w, r, \overline{y})}(r)$ with at most k degree, the set $\left(\Xi^{(\overline{p}, w, r)}\right)^{-1}(0)$ is at most finite. Therefore, by choosing a sufficiently small open neighborhood $\mathcal{N}(r) \subseteq \mathcal{N}^{(\overline{p}, w, r, \overline{y})}(r)$ of r, it follows that for any $r' \in \mathcal{N}(r) \setminus \{r\}$, $\Xi^{(\overline{p}, w, r)}(r') \neq 0$. ■

Then:

Lemma 8: Let $r \in \mathcal{R}$, and $(\bar{p}, w, r, \overline{y})$ be a **W-N *SE*** associated with this r in E. Then, there is a small open neighborhood $\mathcal{N}(r) \subseteq \mathcal{N}^{(\overline{p}, w, r, \overline{y})}(r)$ of r such that for any $r' \in \mathcal{N}(r) \setminus \{r\}$ and any **W-N *SE*** $(\overline{p}(r'), w(r'), r', \overline{y}(r')) \in \mathcal{N}^{(\overline{p}, w, r, \overline{y})}$, $[B(\overline{p}(r'), w(r'), r') - (1 + r') A(\overline{p}(r'), w(r'), r')]^T$ has the full row rank.

Proof. By construction of $\mathcal{N}^{(\overline{p}, w, r, \overline{y})}$, for any $r' \in \mathcal{N}^{(\overline{p}, w, r, \overline{y})}(r)$ and any **W-N *SE*** $(\overline{p}(r'), w(r'), r', \overline{y}(r')) \in \mathcal{N}^{(\overline{p}, w, r, \overline{y})}$, we have $B(\overline{p}(r'), w(r'), r') = B(\overline{p}, w, r)$, $A(\overline{p}(r'), w(r'), r') = A(\overline{p}, w, r)$, and $L(\overline{p}(r'), w(r'), r') = L(\overline{p}, w, r)$. By Lemma 7, we have a small open neighborhood $\mathcal{N}(r) \subseteq \mathcal{N}^{(\overline{p}, w, r, \overline{y})}(r)$ of r. Moreover, for any $r' \in \mathcal{N}(r) \setminus \{r\}$, $\Xi^{(\overline{p}, w, r)}(r') \neq 0$. Then, by construction of $\mathcal{N}^{(\overline{p}, w, r, \overline{y})}$, for any $r' \in \mathcal{N}(r) \setminus \{r\}$ and any **W-N *SE*** $(\overline{p}(r'), w(r'), r', \overline{y}(r')) \in \mathcal{N}^{(\overline{p}, w, r, \overline{y})}$,

$$\det\left(B(\overline{p}(r'), w(r'), r') - (1 + r') A(\overline{p}(r'), w(r'), r')\right)^{(k)} = \Xi^{(\overline{p}, w, r)}(r') \neq 0.$$

Thus, $rank[B(\overline{p}(r'), w(r'), r') - (1 + r') A(\overline{p}(r'), w(r'), r')]^T = k$ holds. ■

[17] The existence of such a small neighborhood is warranted, since $[B(\overline{p}, w, r) - (1 + r) A(\overline{p}, w, r)]$ is invariant with respect to a small change of prices, as argued in footnote 12.

With these preliminary analyses, we can conclude that:

Lemma 9: $\mathcal{R}$ is of Lebesgue measure zero in $[0, R)$.

Proof. By Lemma 8, for any $r \in \mathcal{R}$, there is a sufficiently small open neighborhood $\mathcal{N}(r) \subset (0, R)$ of r such that $\mathcal{N}(r) \cap \mathcal{R} = \{r\}$ holds. Hence, the set $\mathcal{R}$ is discrete. As $\mathcal{R}$ is a subset of a compact set $[0, R]$, it is at most finite. Thus, the set $\mathcal{R}$ is of Lebesgue measure zero within $[0, R)$. ■

Now, we can complete the proof of Theorem 2.

Proof of Theorem 2: By Lemma 9, $rank[B(\overline{p}, w, r) - (1+r)A(\overline{p}, w, r)]^T = k$ holds for almost all $r \in [0, R)$. Thus, for almost all $r \in (0, R)$, the corresponding **W-N** $\boldsymbol{SE}$ $(\bar{p}, w, r, \overline{y})$ under this economy is *regular*, according to Definition 4.

Then, by the implicit function theorem, for almost all $r \in (0, R)$, there exist an open neighborhood $\mathcal{O}(r) \subset (0, R)$ of r and also an open neighborhood $\mathcal{O}(\overline{p}, w, \overline{y}) \subset \mathbb{R}^n_{++} \times \mathbb{R}_{++} \times \mathbb{R}^k_{++}$ of $(\overline{p}, w, \overline{y}) \in \mathbb{R}^n_{++} \times \mathbb{R}_{++} \times \mathbb{R}^k_{++}$ such that there exists a continuous single-valued mapping $\eta : \mathcal{O}(r) \to \mathcal{O}(\overline{p}, w, \overline{y})$ such that for any $r' \in \mathcal{O}(r)$, there exists $(\overline{p}', w', \overline{y}') = \eta(r')$ with $F(\overline{p}', w', r', \overline{y}') = \mathbf{0}$. Note that as $(\overline{p}, w, \overline{y}) \in \mathbb{R}^n_{++} \times \mathbb{R}_{++} \times \mathbb{R}^k_{++}$, we can properly choose an open neighborhood of $(\overline{p}, w, \overline{y})$ as a subset of $\mathbb{R}^n_{++} \times \mathbb{R}_{++} \times \mathbb{R}^k_{++}$.

By the definition of the mapping F, $F(\overline{p}', w', r', \overline{y}') = \mathbf{0}$ implies that $z(\overline{p}', w', r') = [B(\overline{p}', w', r') - A(\overline{p}', w', r')]\overline{y}'$ and $\overline{p}'B(\overline{p}', w', r') = (1+r')\overline{p}'A(\overline{p}', w', r') + w'L(\overline{p}', w', r')$. Then, by means of Walras' law, it also follows that $L(\overline{p}', w', r')\overline{y}' = \omega_l$. Thus, $(\overline{p}', w', r', \overline{y}')$ is a **W-N** $\boldsymbol{SE}$ associated with $r' \in \mathcal{O}(r)$. ■

8.2 Appendix B: An Extension: Generic Indeterminacy of Balanced Growth Equilibria

In this section, consider a general von Neumann economic model with an exogenous population growth rate. Because of the growing population, a von Neumann equilibrium must be defined as a balanced growth equilibrium associated with the equilibrium rate of capital accumulation being equal to the population growth rate. Here, an economic environment is specified by a list $E = \langle (A, B, L); \omega_l, g; z \rangle$, where $g \geqq 0$ represents a rate of population growth, ω_l the present population size, and z the aggregate Marshallian demand function. As ω_l is the population size of the present period, it becomes $(1+g)\omega_l$ in the next period of production.

Unlike the model in section 2, we will leave the underlying microeconomic structure of consumers' behavior unspecified, except that z must meet the *aggregate budget constraint.* However, as discussed below, the last point invites more complexity in the analysis of von Neumann equilibria under such a general economic model than under the model with an OLG structure.

8.2.1 Walras-von Neumann Balanced Growth Equilibria with a General Aggregate Demand Functions

Let the Marshallian demand function $z(p, w, r, I)$ represent the aggregate demands of the whole population, where I represents the aggregate income distributed to the whole population after when firms deduct the investment funds for capital accumulation, so that the *aggregate budget constraint* $p \cdot z(p, w, r, I) = I$ must hold.[18] Note that the aggregate income I is a continuous function of the price system (p, w, r), the aggregate production plan $y \in \mathbb{R}_+^m$, and the aggregate labor endowment ω_l : $I = I(p, w, r, y, \omega_l)$. Therefore, the Marshallian demand function is reduced to:

$$\begin{aligned} z(p,w,r,I) &= z(p,w,r,I(p,w,r,y,\omega_l)) \\ &= z(p,w,r,I(p,w,r,y)) \text{ as } \omega_l \text{ is fixed throughout the analysis,} \\ &= z(p,w,r,y). \end{aligned}$$

In the following, we will denote the Marshallian demand function by $z(p, w, r, y)$. Note that $p \cdot z(p, w, r, y) = (r - g) pAy + \omega_l$ holds by the definition of this function.

This definition differs from that of the Marshallian demand function in section 2, as z in section 2 is a function of only prices. Remember that the underlying budget constraint in section 2 was given by the wage revenue $w\omega_l$, other than the price information (p, w, r). Therefore, as ω_l is fixed throughout the whole periods, the Marshallian aggregate demands in section 2 can be defined as a function of *only* price information. Such treatment is no longer possible in this section, so the Marshallian aggregate demands should also depend on the information of production plan y.

In this case, for each $r \in (0, R)$, a *Walras-von Neumann balanced growth equilibrium* (in short, **W-N *BGE***) is a profile $((p, w, r); y)$ which is a solution to the following system of inequalities:

$$\begin{aligned} pB &\leqq (1+r)pA + wL; \text{ (a)} \\ By &\geqq (1+g)Ay + z(p,w,r,y), \text{ (b)} \\ Ly &\leqq \omega_l. \text{ (c)} \end{aligned}$$

Here, Walras' law is correspondingly represented by:

$$p \cdot [By - (1+g)Ay - z(p,w,r,y)] + w(\omega_l - Ly) = 0.$$

By multiplying y to (a) from the right, we have:

$$pBy = (1+r)pAy + wLy. \text{ (a*)}$$

Then, Walras' law together with (a*) can be reduced to:

$$pz(p,w,r,y) - (r-g)pAy - w\omega_l = 0.$$

Next, by multiplying p to (b) from the left, we have:

$$pBy = (1+g)pAy + pz(p,w,r,y). \text{ (b*)}$$

By (a*) and (b*), we have $pz(p, w, r, y) = wLy + (r - g) pAy$. Then, together with the reduced form of Walras' law, we can obtain

$$Ly = \omega_l$$

Thus, we can get rid of condition (c).

[18] A typical example of the aggregate income function is $I(p, w, r, y) = (r - g) pAy + w\omega_l$. Here, $(r - g) pAy$ represents the aggregate net profits income distributed to the whole households which is equal to the residual of the aggregate profit revenue $rpAy$ after firms deducting the investment funds for capital accumulation $gpAx$, given that a production plan $y \in \mathbb{R}_+^m$ is activated at a present period.

8.2.2 Existence of a Closed Graph Relationship between Balanced Growth Equilibria and Interest Rates

As discussed above, an aggregate Marshallian demand vector cannot be fixed by the price information alone but also by the information of (ex-ante) production plan. However, as observed below, an equilibrium production plan will be determined corresponding to the given aggregate demand vector. This equilibrium production plan should also stipulate a balanced growth equilibrium path of capital accumulation. Therefore, under a balanced growth equilibrium, the aggregate Marshallian demand vector must be fixed by the information of the balanced growth equilibrium production plan, but the latter is a variable determined corresponding to the former. Thus, the existence problem of a balanced growth equilibrium in this section should involve one more complication, in that the 'ex-ante' production plan must coincide with the 'ex-post' equilibrium production plan.

Keeping the last point in mind, consider the existence of a balanced growth equilibrium for each $r \in (0, R)$. Assume again that $w = 1$. Define $X \equiv \left\{x \in \mathbb{R}_+^m \mid Lx = \omega_l\right\}$. Let

$$p^m A x^m \equiv \arg \max_{p^m \in \triangle^K;\ x \in X} pAx,$$

and then let

$$I^m \equiv r p^m A x^m + \omega_l.$$

Finally, define

$$X^m \equiv \left\{x \in \mathbb{R}_+^m \mid \exists p \in \triangle^K :\ rpAx + Lx \leqq I^m\right\}.$$

Note that X^m is compact and convex. The latter property holds, since $\triangle^K$ is convex.

Pick any $x \in X^m$ to represent an ax ante plan for capital accumulation. In other words, gAx represents an ax ante demands of capital goods for new investment.

For each $r \in [0, R)$ and each $p \in \triangle^K$, the aggregate demand vector is determined by $z(p, 1, r, x)$. Then, for such fixed $z(p, 1, r, x)$ and $x \in X^m$, define the following program:

$$(MP_1) \quad \max_{p' \in \triangle^K} p' \cdot [z(p, 1, r, x) + gAx] \ \text{ subject to } p'[B - A] \leqq rpA + L.$$

As already shown in the proof of Theorem 1, there exists $p^*(x) \in \triangle^K$ such that

$$p^*(x) \in \arg \max_{p' \in \triangle^K;\ p'[B-A] \leqq rp^*(x)A + L} p' \cdot [z(p^*(x), 1, r, x) + gAx].$$

Then, given this datum $p^*(x) > \mathbf{0}$, define the following problems:

$$(MP_1^*) \quad \max_{p' \in \mathbb{R}_+^n} p' \cdot [z(p^*(x), 1, r, x) + gAx] \ \text{ subject to } p'[B - A] \leqq rp^*(x)A + L,$$

and

$$(MP_2) \min_{y \in \mathbb{R}_+^m \setminus \{\mathbf{0}\}} [rp^*(x)A + L]\, y \text{ subject to } [B - A]\, y \geqq z(p^*(x), 1, r, x) + gAx.$$

As we already observed above, $p^*(x) > \mathbf{0}$ is a solution to (MP_1^*). Let $y^*(x) \in \mathbb{R}_+^m \setminus \{\mathbf{0}\}$ be a solution to this (MP_2). Note that the existence of such a solution is warranted by **A1**.

In this case, we can have

$$p^*(x) \cdot [z(p^*(x), 1, r, x) + gAx] \leqq p^*(x)[B - A]\, y^*(x) \leqq [rp^*(x)A + L]\, y^*(x).$$

By the duality theorem, it follows that

$$p^*(x) \cdot [z(p^*(x), 1, r, x) + gAx] = p^*(x)[B - A]y^*(x) = [rp^*(x)A + L]y^*(x). \quad (**)$$

By the way, the definition of $z(p^*(x), 1, r, x)$ implies that

$$\begin{aligned} p^*(x)[z(p^*(x), 1, r, x) + gAx] &= gp^*(x)Ax + I(p^*(x), 1, r, x) \\ &= rp^*(x)Ax + \omega_l \leqq \max_{p" \in \triangle^K} rp"Ax + \omega_l \leqq I^m. \end{aligned}$$

Thus, as $p^*(x) \cdot [z(p^*(x), 1, r, x) + gAx] = [rp^*(x)A + L]y^*(x)$, we have

$$[rp^*(x)A + L]y^*(x) \leqq I^m,$$

which implies $y^*(x) \in X^m$.

Define a correspondence $\phi : X^m \twoheadrightarrow X^m$ such that

$$\phi(x) \equiv \left\{ y^*(x) \in X^m \mid y^*(x) \in \arg \min_{[B-A]y \geqq z(p^*(x),1,r,x)+gAx} [rp^*(x)A + L]y \right\}.$$

Note that this correspondence is upper hemi-continuous and convex-valued. Thus, as X^m is compact and convex, by the Kakutani fixed point theorem, there exists a fixed point $y^* \in X^m$ such that $y^* \in \phi(y^*)$. Then, let $p^* \equiv p^*(y^*)$. By definition,

$$\begin{aligned} p^* &\in \arg \max_{p' \in \mathbb{R}^n_+;\ p'[B-A] \leqq rp^*A+L} p' \cdot [z(p^*, 1, r, y^*) + gAy^*] ; \\ y^* &\in \arg \min_{y \in \mathbb{R}^m_+ \setminus \{\mathbf{0}\};\ [B-A]y \geqq z(p^*,1,r,y^*)+gAy^*} [rp^*A + L]y \end{aligned}$$

As argued above, it follows that

$$p^* \cdot [z(p^*, 1, r, y^*) + gAy^*] = p^*[B - A]y^* = [rp^*A + L]y^* \quad (***)$$

by the duality theorem. Finally, as $p^* \cdot [z(p^*, 1, r, y^*) + gAy^*] = rp^*Ay^* + \omega_l$ holds by definition of the demand function, we have $Ly^* = \omega_l$ from (***). Likewise, by Walras' law, it follows that

$$(p^* \cdot [z(p^*, 1, r, y^*) + gAy^*] - p^*[B - A]y^*) + (Ly^* - \omega_l) = 0.$$

Hence, again by (***), we have $Ly^* = \omega_l$.

Therefore, we have a **W-N *BGE*** $((p^*, 1, r), y^*)$ associated with $r \in [0, R)$ for the economy $\langle (A, B, L); \omega_l; g; z \rangle$.

Theorem 4: Let $E = \langle (A, B, L); \omega_l; g; z \rangle$ be an economy as specified above. Then, for *each interest rate* $r \in [0, R)$, there exists a *Walras-von Neumann balanced growth equilibrium* $((p^*, 1, r), y^*)$ under this economy.

8.2.3 Generic Indeterminacy of Balanced Growth Equilibria

Generic indeterminacy of balanced growth equilibria can be shown as argued in section 4. First, a continuously differentiable mapping F is given in this section, as follows:[19]

$$F(\overline{p}, w, r, \overline{y}) \equiv \left\{ \begin{array}{l} [B(\overline{p}, w, r) - (1+g) A(\overline{p}, w, r)] \overline{y} - z(\overline{p}, w, r, \overline{y}) \\ p[B(\overline{p}, w, r) - (1+r) A(\overline{p}, w, r)] - wL(\overline{p}, w, r) \end{array} \right. .$$

Here, the definitions of $\overline{p}$ and $\overline{y}$ are the same as defined in section 4. So, $\overline{y} \in \mathbb{R}^k_{++}$ is obtained by deducting the $m-k$ zero components of the $m \times 1$ column vector $y \in \mathbb{R}^m_+$, where k is the minimal number of processes which can meet the equilibrium condition (b) in equality.

Then, the corresponding Jacobian of $F(\overline{p}, w, r, \overline{y}) = 0$ at a **W-N** $\boldsymbol{BGE}$ $(\overline{p}, w, r, \overline{y})$ is given as follows:

$$\begin{array}{l} \boldsymbol{D}_{(y,\overline{p},w,r)}(F(\overline{p}, w, r, \overline{y})) \\ = \left[\begin{array}{cccc} JF_{n\times k} & \boldsymbol{D}_{\overline{p}} z(\overline{p}, w, r, \overline{y}) & \boldsymbol{D}_w z(\overline{p}, w, r, \overline{y}) & \boldsymbol{D}_r z(\overline{p}, w, r, \overline{y}) \\ \mathbf{0} & [\boldsymbol{b}_{-n} - (1+r)\boldsymbol{a}_{-n}]^T (\overline{p}, w, r) & -L(\overline{p}, w, r)^T & -(\overline{p}A(\overline{p}, w, r))^T \end{array} \right], \end{array}$$

where

$$JF_{n\times k} \equiv [(1+g) A - B](\overline{p}, w, r) + \boldsymbol{D}_{\overline{y}} z(\overline{p}, w, r, \overline{y}).$$

As shown in the proof of Theorem 2, the first n row vectors

$$\left[\begin{array}{cccc} [(1+g) A - B](\overline{p}, w, r) + \boldsymbol{D}_{\overline{y}} z(\overline{p}, w, r, \overline{y}) & \boldsymbol{D}_{\overline{p}} z(\overline{p}, w, r, \overline{y}) & \boldsymbol{D}_w z(\overline{p}, w, r, \overline{y}) & \boldsymbol{D}_r z(\overline{p}, w, r, \overline{y}) \end{array} \right]$$

are linearly independent as $\left[\begin{array}{cc} \boldsymbol{D}_{\overline{p}} z(\overline{p}, w, r, \overline{y}) & \boldsymbol{D}_w z(\overline{p}, w, r, \overline{y}) \end{array} \right]$ is invertible. Moreover, the next k row vectors

$$\left[\begin{array}{cccc} \mathbf{0} & [\boldsymbol{b}_{-n} - (1+r)\boldsymbol{a}_{-n}]^T (\overline{p}, w, r) & -L(\overline{p}, w, r)^T & -(\overline{p}A(\overline{p}, w, r))^T \end{array} \right]$$

are shown to be linearly independent for almost all $r \in (0, R)$. First, as shown in the proof of Theorem 2, we can see that

$$rank[B - (1+g) A]^T (\overline{p}, w, r) = k.$$

Then, as shown in the proof of Theorem 2, we can see: for almost all $r \in (0, R)$,

$$rank \left[\begin{array}{cc} [\boldsymbol{b}_{-n} - (1+r)\boldsymbol{a}_{-n}]^T (\overline{p}, w, r) & -L(\overline{p}, w, r)^T \end{array} \right] = k.$$

In summary, we can obtain the following result:

Theorem 5: Let $E = \langle (A, B, L); \omega_l; g; z \rangle$ be a general von Neumann production economy specified as above. Then, for almost all $r \in (0, R)$, its associated Walras-von Neumann balanced growth equilibrium $((p, w, r), y)$ under this economy is *indeterminate*.

[19] As in the case of steady-state equilibria discussed in section 4, the continuous differentiability of the excess demand functions

$$[B(\overline{p}, w, r) - (1+g) A(\overline{p}, w, r)] \overline{y} - z(\overline{p}, w, r, \overline{y})$$

is warranted at least in a small neighborhood of each equilibrium $(\overline{p}, w, r, \overline{y})$.

8.2.4 Remark on the existence theorems of the balanced growth equilibrium in the present literature

Morishima (1960, 1969) also considered von Neumann production economies with Mashallian demand functions, and then Salvadori (1980) and Bidard and Hosoda (1988) examined the existence of a balanced growth equilibrium in that model. While assuming the wage payment in advance of production, Morishima (1960) introduced the exogenous saving rate of the capitalist class as s_c with $0 < s_c < 1$. Then, a balanced growth equilibrium is defined by the following system of equilibrium inequalities:

$$\begin{aligned} pB &\leqq (1+r)[pA + wL]; \text{ (a)} \\ By &\geqq (1+g)[Ay + wLy \cdot d^w(p,w,r)] + r[pAy + wLy] \cdot d^c(p,w,r), \text{ (b)} \\ g &= rs_c, \text{ (c)} \end{aligned}$$

where $d^c(p,w,r)$ represents the Capitalist class's Marshallian consumption demands per $1 - s_c$ units of expenditure: for any $p \in \mathbb{R}^n_+$, $p \cdot d^c(p,w,r) = 1 - s_c$ holds. Likewise, $d^w(p,w,r)$ represents the Working class's Marshallian consumption demands per unit of expenditure: for any $p \in \mathbb{R}^n_+$, $p \cdot d^w(p,w,r) = 1$ holds.[20]

Here, condition (c) represents the Cambridge equation, where Morishima (1960) assumed that population growth $g > 0$ is exogenously given. Then, the system of inequalities consists of $n + k + 1 + 1$ equations (n equations of excess demand conditions for commodity markets (b); k equations of zero profit conditions for the operated processes (a); the Cambridge equation (c); and the commodity price normalization equation, $\sum_{i=1}^n p_i = 1$. In contrast, the unknowns are p, y, w, and r. Here, Walras' law is represented by:

$$\begin{aligned} pBy &= (1+g)p[Ay + wLy \cdot d^w(p,w,r)] + r[pAy + wLy] \cdot pd^c(p,w,r) \\ &= (1+g)[pAy + wLy] + r[pAy + wLy](1 - s_c). \end{aligned}$$

Then, by multiplying y in (a) from the right, we have:

$$pBy = (1+r)[pA + wL]y. \text{ (a*)}$$

Then, by means of Walras' law and (a*), the Cambridge equation (c) is derived. Thus, we can get rid of (c). However, r can also be removed from the list of the unknowns due to the derived Cambridge equation, given that the population growth rate g is given. Therefore, there are $n + k$ independent equations while there are $n + k$ unknowns. So, we cannot observe the freedom of degree one feature in Morishima's (1960) system of equilibrium inequalities. Thus, a balanced growth equilibrium in the Morishima (1960) model is determinate. Indeed, while r can be determined by the derived Cambridge equation, the equilibrium wage rate w is determined by the intersection of the growth rate curves of labor demand and of labor supply, where the former is derived from the wage-interest-rates frontier of this economy and is downward sloping with respect to wage rates and the latter is a flat curve drawn at the point g.

[20] Morishima (1969) also provided a more generalized von Neumann model than Morishima (1960) by introducing an exogenous rate of saving for the working class. Since then, Salvadori (1980) and Bidard and Hosoda (1988) studied the existence of a balanced growth equilibrium in the Morishima (1969) model.

However, to highlight the deterministic feature of the equilibrium in the Morishima type of von Neumann models, examining the original Morishima (1960) model is sufficient.

In contrast, the model discussed in Appendix B of this paper also introduces an exogenous rate of population growth but does not have a fixed saving rate as a parameter. However, the model in Appendix B introduces the labor market equilibrium condition that does not appear in Morishima's (1960) system of equilibrium inequalities. As we observed, the labor equilibrium condition is also removed by the application of Walras' law, so that there are also $n+k$ independent equations in Appendix B, whereas the unknown variables cannot be further reduced. As a consequence, given the population growth rate, the system of equilibrium inequalities in Appendix B cannot determine the equilibrium interest rate, and thus the continuum of the Neumann equilibria can be observed for the exogenously given population growth rate.

8.3 Appendix C: A simple microeconomic model of OLG economies

In this appendix, a simple microeconomic model of OLG economies is presented, where an individual optimization problem for her economic decision-making is specified in order to provide an underlying structure of economic data E.

Given the basic information about the OLG structure of each generation with the labor endowment ω_l and a profile of von Neumann production technology (A, B, L) specified in section 2, let $u : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathbb{R}$ be a welfare function of lifetime consumption activities, which is common to all generations. As usual, u is assumed to be continuous and strongly monotonic. Thus, an *overlapping generation economy* is given by a profile $E = \langle (A, B, L)\,; \omega_l; u \rangle$.

Assume also, for each generation t, $l^t \in \mathbb{R}_+$ represents t's labor supplied at the beginning of the young age; $\omega^{t+1} \in \mathbb{R}^n_+$ represents a commodity bundle for the purpose of saving monetary value $p_t\omega^{t+1}$, which will be chosen by generation t at the end of the young age and will be used in the old age; $\delta^{t+1} \in \mathbb{R}^n_+$ represents a commodity bundle purchased for the purpose of speculative activities by generation t at the end of the old age; $y^{t+1} \in \mathbb{R}^m_+$ represents a production activity vector decided by generation t at the beginning of the old age; z^t_b is the consumption bundle which will be consumed by the generation t at the young age; and z^t_a is the consumption bundle which will be consumed by generation t at the old age.

Each generation t at the young age is faced with the following optimization program MP^t: for a given sequence of price vectors $\{(p_t, w_t, r_t), (p_{t+1}, w_{t+1}, r_{t+1})\}$,

$$\max_{l^t, \omega^{t+1}, \delta^{t+1}, y^{t+1}, z^t_b, z^t_a} u\left(z^t_b, z^t_a\right)$$

subject to

$$\begin{aligned} p_t z^t_b + p_t\omega^{t+1} &\leqq w_t l^t, \\ l^t &\leqq \omega^t_l, \\ p_t\delta^{t+1} + p_t A y^{t+1} &= p_t\omega^{t+1}, \text{ and} \\ p_{t+1} z^t_a &\leqq p_{t+1}\delta^{t+1} + p_{t+1} B y^{t+1} - w_{t+1} L y^{t+1}. \end{aligned}$$

That is, each generation t can supply l^t amount of labor at the young age as a worker employed by generation $t-1$. From the wage income $w_t l^t$ earned at the end of the young age, she can save $p_t\omega^{t+1}$ amount of money and can purchase a consumption bundle z^t_b. By using the saved money $p_t\omega^{t+1}$, the generation t at the beginning of the old age can purchase δ^{t+1} for the speculative purpose and can purchase a vector of capital goods Ay^{t+1} for the productive investment. As an industrial capitalist, she can employ Ly^{t+1} amount of generation $t+1$'s labor. Then, at the end of the old age, she can earn $p_{t+1}\delta^{t+1}$ as the revenue of the speculative investment and can earn $p_{t+1}By^{t+1} - w_{t+1}Ly^{t+1}$ as the return of the productive investment. From these revenues, she can purchase a consumption bundle z_a^t.

Let $\left(l^t, \omega^{t+1}, \delta^{t+1}, y^{t+1}, z_b^t, z_a^t\right)$ be a solution to the optimization program MP^t for each generation t. In optimum, all of the weak inequalities in the above constraints should hold in equality, given the assumption of u. That is,

$$\begin{aligned} p_t z_b^t + p_t \omega^{t+1} &= w_t l^t, \\ l^t &= \omega_l^t, \text{ and} \\ p_{t+1} z_a^t &= p_{t+1}\delta^{t+1} + p_{t+1}By^{t+1} - w_{t+1}Ly^{t+1}. \end{aligned}$$

Note that the production activity vector y^t planned by generation $t-1$ at the beginning of the old age should satisfy profit maximization condition. As market prices should satisfy the zero-profit condition in equilibrium, the following condition:

$$p_t B \leqq (1 + r_t)\, p_{t-1} A + w_t L$$

holds for every period t. Therefore, the profit maximization condition in equilibrium for every period t is represented by:

$$p_t B y^t = (1 + r_t)\, p_{t-1} A y^t + w_t L y^t.$$

Thus, the revenue constraint $p_{t+1} z_a^t = p_{t+1}\delta^{t+1} + p_{t+1}By^{t+1} - w_{t+1}Ly^{t+1}$ of generation t at the end of the old age can be reduced to

$$p_{t+1} z_a^t = p_{t+1}\delta^{t+1} + (1 + r_{t+1})\, p_t A y^{t+1}.$$

Given a pair of sequence of price vectors $(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}) \equiv \{(p_t, w_t, r_t)\}_{t \geq 0}$, let $(z_b^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}), z_a^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}))$ be a solution of the generation $t = 1, 2, \ldots,$ to the above mentioned problem MP^t of the utility maximization under the budget constraint. Then, a competitive equilibrium can be formulated as follows.

Definition A1: A *competitive equilibrium* under the overlapping generation economy $E = \langle (A, B, L); \omega_l; u \rangle$ is a pair of sequence of price vectors $(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}) \equiv \{(p_t, w_t, r_t)\}_{t \geq 0}$ and sequence of each generation's optimal actions

$$\left\{ \left(\omega^{t+1}, y^{t+1}, \delta^{t+1}, z_b^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}), z_a^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}) \right) \right\}_{t \geq 0}$$

satisfying the following conditions:

$$\begin{aligned} p_t B &\leqq (1 + r_t)\, p_{t-1} A + w_t L \ \ (\forall t); \quad \text{(A1.1)} \\ \delta^t + By^t &\geqq z^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}) + \omega^{t+1} \ \ (\forall t); \quad \text{(A1.2)} \end{aligned}$$

where $z^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}) \equiv z_b^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}) + z_a^{t-1}(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r})$ is the aggregate consumption demands at each t;

$$\begin{aligned} \delta^t + Ay^t &\leqq \omega^t \ \ (\forall t); \quad \text{(A1.3)} \\ \text{and } Ly^t &\leqq \omega_l^t \ \ (\forall t). \quad \text{(A1.4)} \end{aligned}$$

In the above definition, the excess demand condition in commodity markets is given by (A1.2). In each period t, the aggregate consumption demand vector is given by $z^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}) = z_b^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}) + z_a^{t-1}(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r})$. It may contain some zero components. For such a commodity i as $z_i^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}) = 0$, it follows that in equilibrium, $\delta_i^t + y_i^t \geqq \omega_i^{t+1}$. In the above inequality of excess demand condition (A1.2), y^t is the gross output vector which is planned by generation $t-1$ at the beginning of period t and is harvested at the end of this period, while δ^t is the commodity bundle purchased by generation $t-1$ at the beginning of period t and is sold by generation $t-1$ at the end of period t.

In each period t, the capital market equilibrium condition is given by (A1.3) of Definition A1. Note that the choice between the speculative investment δ^t and the productive investment Ay^t is given by generation $t-1$ at the beginning of the old age. Moreover, the saving of commodity bundle ω^t is given by generation $t-1$ at the end of the young age.

In each period t, the labor market equilibrium condition is given by (A1.4) of Definition A1. Note that the aggregate labor demand Ly^t is chosen by generation $t-1$ at the old age, while the aggregate labor supply ω_l^t is given by generation t at the young age.

A specific long-period feature of competitive equilibrium is given as a steady-state equilibrium, where all of the investment activities are simply of the replacements. In such a case, given a pair of sequence of price vectors $(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}) \equiv \{(p_t, w_t, r_t)\}_{t\geq 0}$ such that $(p_t, w_t, r_t) = (p, w, r)$ for every period t, the solution $(z_b^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}), z_a^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}))$ of the generation $t = 1, 2, \ldots,$ to the optimization problem MP^t can be represented by $z_b^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}) = z_b(p, w, r)$ and $z_a^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}) = z_a(p, w, r)$ for every generation t. Correspondingly, the aggregate demand function $z^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}) = z_b^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}) + z_a^{t-1}(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r})$ for every period t can be represented by $z^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}) = z(p, w, r)$. Such an equilibrium is given as follows.

Definition A2: A *steady-state equilibrium* under the overlapping economy $E = \langle (A, B, L); \omega_l; u \rangle$ is a competitive equilibrium $(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r})$ associated with

$$\left\{\left(\omega^{t+1}, y^{t+1}, \delta^{t+1}, z_b^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}), z_a^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r})\right)\right\}_{t\geq 0},$$

such that there exists a profile of a stationary price vector (p, w, r), a gross output vector $y \geq \mathbf{0}$, and a speculative activity vector $\delta \geqq \mathbf{0}$, satisfying $(p_t, w_t, r_t) = (p, w, r)$, $y^{t+1} = y$, $\delta^{t+1} = \delta$, $\omega^{t+1} = Ay + \delta$, $z_b^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}) = z_b(p, w, r)$, and $z_a^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}) = z_a(p, w, r)$ for every t, and the following conditions hold:

$$\begin{aligned} pB &\leqq (1+r)\,pA + wL; \text{ (A.a)} \\ By + \delta &\geqq z(p, w, r) + \omega, \text{ (A.b)} \\ \text{where } z(p, w, r) &= z_b(p, w, r) + z_a(p, w, r); \text{ and} \\ Ly &\leqq \omega_l, \text{ (A.c)} \\ Ay + \delta &= \omega, \text{ (A.d)}. \end{aligned}$$

In the above definitions of two types of equilibria, the choice between speculative investment δ^{t+1} and productive investment Ay^{t+1} is the consequence of each generation's optimal action in the program MP^t. Therefore, $\delta = \mathbf{0}$ can be optimal under the steady-state equilibrium whenever the equilibrium interest rate r is non-negative.

To see the last point, let us consider under what conditions the market equilibrium holds with no speculative activity, $\delta^t = \mathbf{0}$ $(\forall t)$, in general. Note that if the whole monetary wealth $p_{t-1}\omega^t$ of generation $t-1$ is used for the productive investment, then she would earn $(1+r_t)\,p_{t-1}\omega^t$, while it is used for the speculative investment, then she would earn $p_t\omega^t$. Therefore, the productive investment of the whole monetary wealth is an optimal action for generation $t-1$ at the beginning of the old age if and only if $(1+r_t)\,p_{t-1}\omega^t \geqq p_t\omega^t$. In general, if

$$(1+r_t)\,p_{t-1} \geqq p_t$$

holds for every period $t = 1, \ldots$, then $\delta^t = \mathbf{0}$ is an optimal action for every generation $t-1$ at the beginning of the old age. Thus, under the steady-state equilibrium, this inequality condition holds automatically, as $(1+r)\,p \geqq p$ holds whenever $r \geqq 0$.

In contrast, under a stationary price system associated with $r < 0$, every generation would devote all of her wealth to speculative investments. Then, no production takes place in every period, and so no consumption good can be supplied in every period. Thus, if a steady-state equilibrium is associated with $r < 0$, it would be only a trivial one. As we are interested in the non-trivial case of equilibria, we focus on the case with $r \geqq 0$ as well as $\delta = \mathbf{0}$ throughout the whole analysis of this paper.

Thus, we can introduce a specific case of a steady-state equilibrium, which is defined as follows.

Definition A3: A *steady-state equilibrium* $((p, w, r), y, \omega)$ under the overlapping economy $E = \langle (A, B, L); \omega_l; u \rangle$ is called *Walras-von Neumann* (**W-N**) if and only if $(p, w, 1+r) > \mathbf{0}$ with $r \geqq 0$ and condition (A.c) in Definition A2 holds in equality.

Thus, given a steady-state price vector $(p, w, r) \in \mathbb{R}^n_+ \times \mathbb{R}_{++} \times \mathbb{R}_+$, each and every generation t at the young age is faced with the following optimization program MP:

$$\max_{\omega, z_b, z_a} u(z_b, z_a)$$

subject to

$$\begin{aligned} pz_b + p\omega &= w\omega_l, \text{ and} \\ pz_a &= (1+r)\,p\omega. \end{aligned}$$

Let $z_b(p, w, r)$ and $z_a(p, w, r)$ be the Marshallian demand functions which are defined as the optimal solutions to the above problem MP at the price system (p, w, r).

Finally, **Definition A3** can be reduced to **Definition 2** in section 2 by getting rid of ω from the list of optimal actions at the steady-state price vector, since the equilibrium choice of ω is automatically fixed whenever the equilibrium activity vector y is determined, due to condition (A.d) of **Definition A3**.

8.4 Appendix D: An alternative proof of Lemma 3

Lemma 3: $p^* \cdot z(p^*, 1, r) = p^*[B - A]\,y^* = (rp^*A + L)\,y^*$.

Proof. To show this claim, it is sufficient to show $p^* \cdot z(p^*, 1, r) \geqq (rp^*A + L)\,y^*$. That is,

$$\begin{bmatrix} \mathbf{0} & (B-A)^T \\ -(B-A) & \mathbf{0} \\ (rp^*A + L) & -z(p^*, 1, r)^T \end{bmatrix} \begin{pmatrix} y^* \\ p^{*T} \end{pmatrix} \leqq \begin{pmatrix} (rp^*A + L)^T \\ -z(p^*, 1, r) \\ 0 \end{pmatrix}$$

must hold, where the superscript T means *transpose*.

To show the last inequalities, let us take any $(\boldsymbol{x}, \boldsymbol{q}, \psi) \in \mathbb{R}_+^m \times \mathbb{R}_+^n \times \mathbb{R}_+$ satisfying

$$(\boldsymbol{x}, \boldsymbol{q}, \psi) \begin{bmatrix} \mathbf{0} & (B-A)^T \\ -(B-A) & \mathbf{0} \\ (rp^*A+L) & -z(p^*,1,r)^T \end{bmatrix} \geqq (\mathbf{0}, \mathbf{0}).$$

This implies

$$\begin{aligned} -\boldsymbol{q}(B-A) + \psi(rp^*A+L) &\geqq \mathbf{0}; \\ (B-A)\boldsymbol{x}^T - \psi z(p^*,1,r) &\geqq \mathbf{0}. \end{aligned}$$

Let $\psi > 0$. Then, $\psi(rp^*A+L)\boldsymbol{x}^T \geqq \boldsymbol{q}(B-A)\boldsymbol{x}^T \geqq \psi\boldsymbol{q}z(p^*,1,r)$. Thus, $(rp^*A+L)\boldsymbol{x}^T - \boldsymbol{q}z(p^*,1,r) \geqq 0$ holds.

Let $\psi = 0$. Then, $\boldsymbol{q}(B-A) \leqq \mathbf{0}$ and $(B-A)\boldsymbol{x}^T \geqq \mathbf{0}$ hold. By the way, $[B-A]y^* \geqq z(p^*,1,r)$ and $\boldsymbol{q}(B-A)\boldsymbol{x}^T \leqq \mathbf{0}$ for any $\boldsymbol{x} \in \mathbb{R}_+^m$ together imply that

$$\boldsymbol{q}z(p^*,1,r) \leqq \boldsymbol{q}(B-A)y^* \leqq 0.$$

Also, $p^*[B-A] \leqq rp^*A+L$ and $\boldsymbol{p}(B-A)\boldsymbol{x}^T \geqq 0$ for any $\boldsymbol{p} \in \mathbb{R}_+^n$ together imply that

$$0 \leqq p^*[B-A]\boldsymbol{x}^T \leqq (rp^*A+L)\boldsymbol{x}^T.$$

Thus, $(rp^*A+L)\boldsymbol{x}^T - \boldsymbol{q}z(p^*,1,r) \geqq 0$ holds.

In summary, $(rp^*A+L)\boldsymbol{x}^T - \boldsymbol{q}z(p^*,1,r) \geqq 0$ holds. Then, according to Gale (1960, p. 47; Theorem 2.8), there exists a non-negative solution $(\boldsymbol{y}, \boldsymbol{p}) \in \mathbb{R}_+^m \times \mathbb{R}_+^n$ to the following system of inequalities:

$$\begin{bmatrix} \mathbf{0} & (B-A)^T \\ -(B-A) & \mathbf{0} \\ (rp^*A+L) & -z(p^*,1,r)^T \end{bmatrix} \begin{pmatrix} y \\ p^T \end{pmatrix} \leqq \begin{pmatrix} (rp^*A+L)^T \\ -z(p^*,1,r) \\ 0 \end{pmatrix}.$$

Thus, we have $\boldsymbol{p} \cdot z(p^*,1,r) \geqq (rp^*A+L)\boldsymbol{y}$.

Note that by (MP_2), $(rp^*A+L)y^* \leqq (rp^*A+L)\boldsymbol{y}$ holds. Next, as p^* is a solution to (MP_1^*) and $\boldsymbol{p}(B-A) \leqq (rp^*A+L)$, it follows that $p^* \cdot z(p^*,1,r) \geqq \boldsymbol{p} \cdot z(p^*,1,r)$. Therefore, we can conclude $p^* \cdot z(p^*,1,r) \geqq (rp^*A+L)y^*$.

Hence, $((p^*,1,r); y^*)$ is a **W-N** $\boldsymbol{SE}$ associated with the positive interest rate $r > 0$ in the economy $E = \langle (A,B,L); \omega_l; u \rangle$. ■

<table>
<tr><th colspan="2">Economic Domains</th><th>Generic Determinacy</th><th>Generic Indeterminacy</th></tr>
<tr><td>Static</td><td>Pure Exchange: Debreu (1970)
Production with CRS Technologies:
Mas-Collel (1975), Kehoe (1980,1982).</td><td>regular economies: full measure.
regular equilibria: finite, locally unique Walrasian CEs.</td><td></td></tr>
<tr><td rowspan="6">Intertemporal</td><td>a finite number of infinitely lived agents:
Kehoe and Levine (1985).</td><td>regular economies: full measure.
regular equilibria: finite, locally unique perfectly farsighted equilibria.</td><td></td></tr>
<tr><td>OLG Pure Exchange: Kehoe and Levine (1985).</td><td rowspan="2">regular economies: full measure.
regular equilibria: finite, locally unique steady states.</td><td rowspan="3">Local Indeterminacy:
the existence of a continuum of nearby steady-state equilibrium paths converging to the same steady-state.</td></tr>
<tr><td>OLG two-sector neoclassical production:
Calvo (1978)</td></tr>
<tr><td>a finite number of infinitely lived agents with market imperfection:
Nishimura and Venditti (2016), etc.</td><td>finite, locally unique steady states.</td></tr>
<tr><td>a finite number of finitely lived agents:
Mandler (1995, 1999).</td><td></td><td>Generic Indeterminacy of Sequential Equilibria:
the second-period continuation equilibria are indeterminate for almost every induced second-period economy.</td></tr>
<tr><td>OLG with simple Leontief production:
Yoshihara and Kwak (2023, 2024).</td><td></td><td>Generic Indeterminacy of non-trivial steady state equilibria.</td></tr>
</table>

Table 1: Literature on Generic (In)determinacy

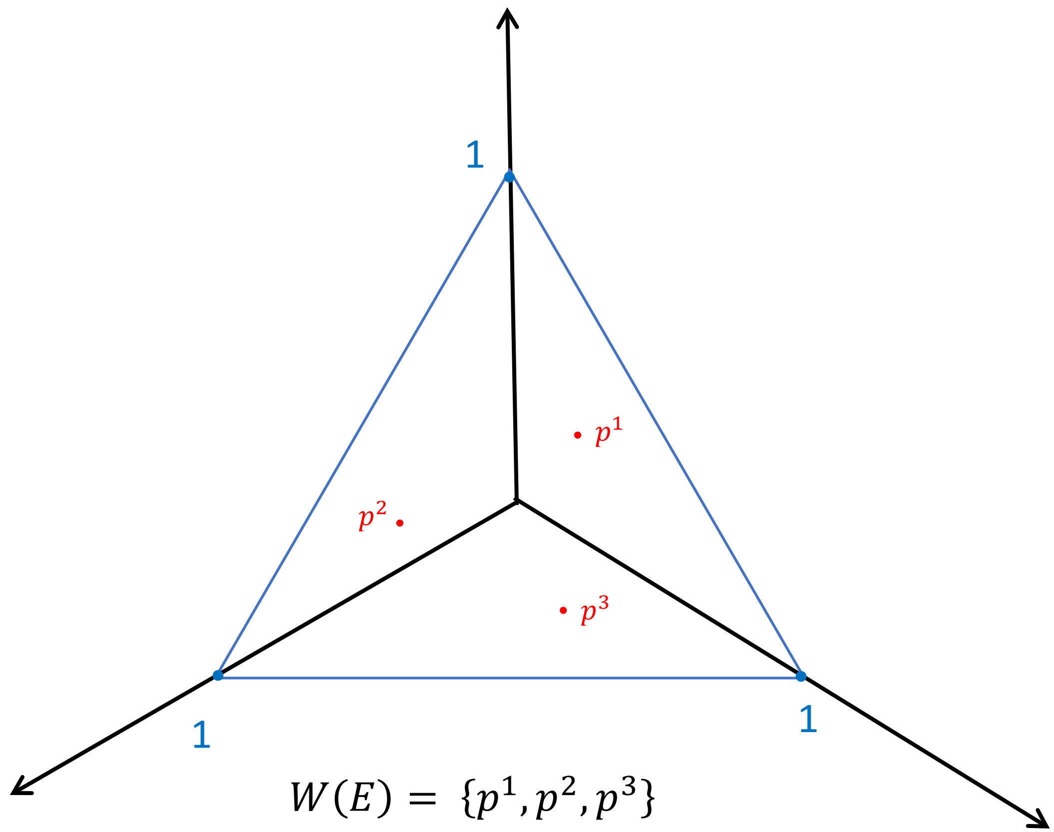

Figure 1a: the set of the standard Walrasian competitive equilibria in a regular economy *E*

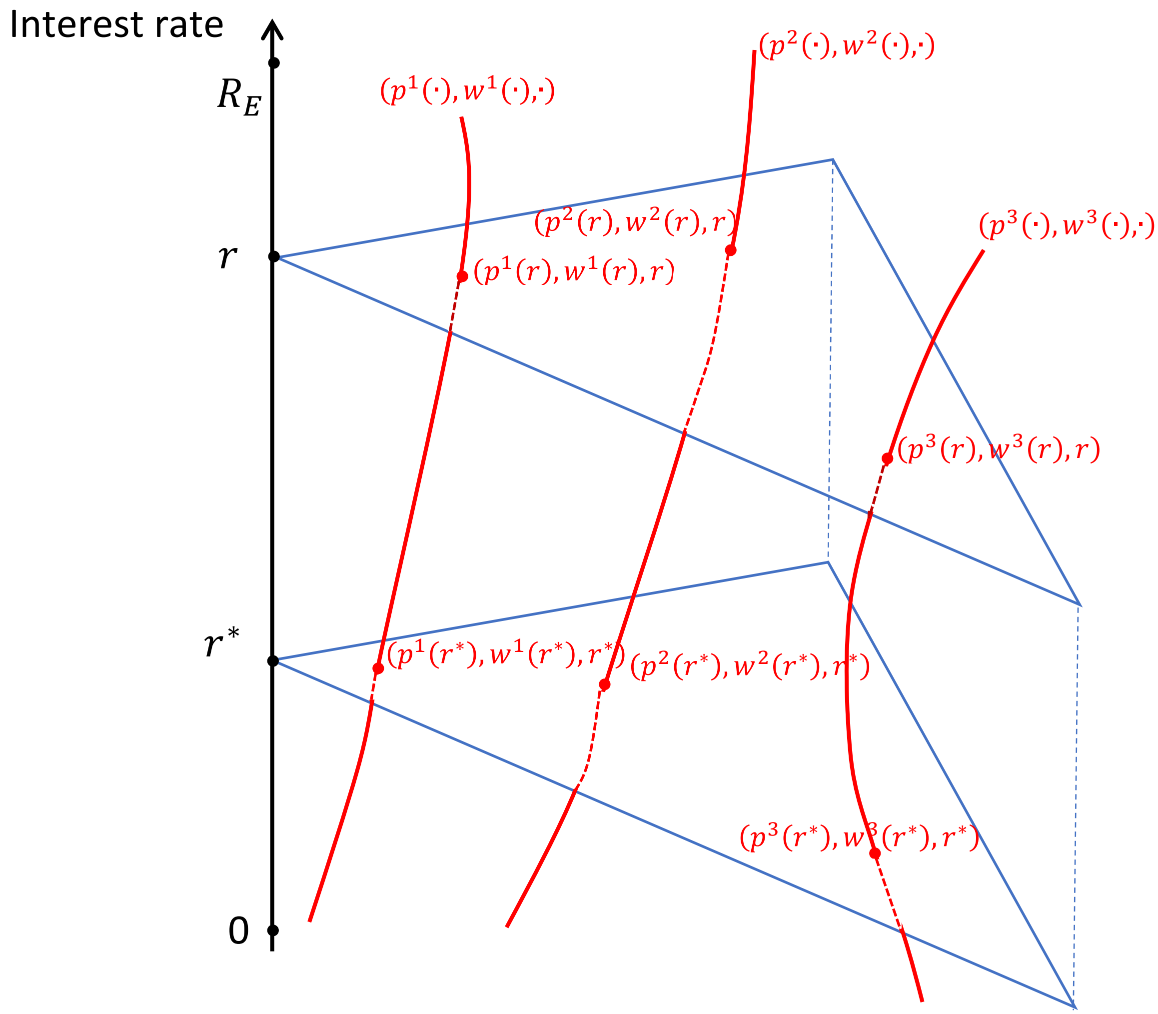

Figure 1b: the set of *W*-*N* equilibria in a regular economy *E*

$$\Psi^{WN}(E) = \{(p^1(r'), w^1(r'), r'), (p^2(r'), w^2(r'), r'), (p^3(r'), w^3(r'), r') | r' \in (0, R_E)\}$$