Technical Appendix

In this appendix we extensively use a special case of the Fortunin, Kasteleyn and Ginibre (1971) inequality [1] (henceforth FKG) in Harris (1960). This inequality (adapted to our setting) states that for any probability measure on R, and increasing functions $f(x)$ and $g(x)$, $\int_R f(x)g(x)\,d\mu(x) \geq \int_R f(x)\,d\mu(x) \int_R g(x)\,d\mu(x)$.

Proof of Proposition 2.1

a. Property II $\Rightarrow$ Property I

$$\operatorname{var}\left(\tilde{x}_t + \tilde{x}_{t+1}\right) = \operatorname{var}\left(\tilde{x}_t\right) + \operatorname{var}\left(\tilde{x}_{t+1}\right) + 2\operatorname{cov}\left(\tilde{x}_t, \tilde{x}_{t+1}\right) < 2\operatorname{var}\left(\tilde{x}_t\right) \text{ (by Property II)}$$

$$\Rightarrow \operatorname{var}\left(\tilde{x}_{t+1}\right) + 2\operatorname{cov}\left(\tilde{x}_t, \tilde{x}_{t+1}\right) < \operatorname{var}\left(\tilde{x}_t\right).$$

Since $\operatorname{var}\left(\tilde{x}_{t+1}\right) = \operatorname{var}\left(\tilde{x}_t\right)$, this implies that $\operatorname{cov}\left(\tilde{x}_t, \tilde{x}_{t+1}\right) < 0$,

Hence Property I holds.

b. Property I together with the covariance condition (3) implies Property II.

The proof is by induction.

Let $j = 1$, $\operatorname{var}\left(\tilde{x}_t + \tilde{x}_{t+1}\right) = \operatorname{var}\left(\tilde{x}_t\right) + \operatorname{var}\left(\tilde{x}_{t+1}\right) + 2\operatorname{cov}\left(\tilde{x}_t, \tilde{x}_{t+1}\right) < 2\operatorname{var}\left(\tilde{x}_t\right)$

(this follows by Property I and the fact that $\operatorname{var}\left(\tilde{x}_t\right) = \operatorname{var}\left(\tilde{x}_{t+1}\right)$)

Let $j = 2$, $\operatorname{var}\left(\tilde{x}_t + \tilde{x}_{t+1} + \tilde{x}_{t+2}\right)$

$$= \sum_{j=0}^{2} \operatorname{var}\left(\tilde{x}_{t+j}\right) + 2\operatorname{cov}\left(\tilde{x}_t, \tilde{x}_{t+1}\right) + 2\operatorname{cov}\left(\tilde{x}_t, \tilde{x}_{t+2}\right) + 2\operatorname{cov}\left(\tilde{x}_{t+1}, \tilde{x}_{t+2}\right)$$

[1] We thank Awi Federgruen for bringing the FKG inequality to our attention.

By Property I, $2\operatorname{cov}\left(\tilde{x}_t, \tilde{x}_{t+1}\right) < 0$

and by condition (3), $2\operatorname{cov}\left(\tilde{x}_t, \tilde{x}_{t+1}\right) + 2\operatorname{cov}\left(\tilde{x}_t, \tilde{x}_{t+2}\right) < 0$

Again by Property I, $2\operatorname{cov}\left(\tilde{x}_{t+1}, \tilde{x}_{t+2}\right) < 0$. Therefore,

$$\operatorname{var}\left(\tilde{x}_t + \tilde{x}_{t+1} + \tilde{x}_{t+2}\right) < \sum_{j=0}^{2} \operatorname{var}\left(\tilde{x}_{t+j}\right) = 3\operatorname{var}\left(\tilde{x}_t\right).$$

Suppose, by Property I and condition (3), $\operatorname{var}\left(\tilde{x}_t + \tilde{x}_{t+1} + ... + \tilde{x}_{t+j-1}\right) < j\operatorname{var}\left(\tilde{x}_t\right)$ holds. We show by induction that this implies

$$\operatorname{var}\left(\tilde{x}_t + \tilde{x}_{t+1} + ... + \tilde{x}_{t+j}\right) < \left(j+1\right)\operatorname{var}\left(\tilde{x}_t\right).$$

$$\operatorname{var}\left(\tilde{x}_t + \tilde{x}_{t+1} + \tilde{x}_{t+2} + ... + \tilde{x}_{t+j}\right)$$

$$= \operatorname{var}\left(\tilde{x}_t + \tilde{x}_{t+1} + ... + \tilde{x}_{t+j-1}\right) + \operatorname{var}\left(\tilde{x}_{t+j}\right) + 2\sum_{s=0}^{j-2}\operatorname{cov}\left(\tilde{x}_{t+s}, \tilde{x}_{t+j}\right) + 2\operatorname{cov}\left(\tilde{x}_{t+j-1,}, \tilde{x}_{t+j}\right)$$

$$< \left(j\right)\operatorname{var}\left(\tilde{x}_t\right) + \operatorname{var}\left(\tilde{x}_{t+j}\right) + 2\sum_{s=0}^{j-2}\operatorname{cov}\left(\tilde{x}_{t+s}, \tilde{x}_{t+j}\right) + 2\operatorname{cov}\left(\tilde{x}_{t+j-1,}, \tilde{x}_{t+j}\right)$$

(by induction)

By Property I, $\operatorname{cov}\left(\tilde{x}_{t+j-1,}, \tilde{x}_{t+j}\right) < 0$. Thus by condition (3),

$$+2\sum_{s=0}^{j-2}\operatorname{cov}\left(\tilde{x}_{t+s}, \tilde{x}_{t+j}\right) + 2\operatorname{cov}\left(\tilde{x}_{t+j-1,}, \tilde{x}_{t+j}\right) < 0.$$

Therefore, since $\operatorname{var}\left(\tilde{x}_{t+j}\right) = \operatorname{var}\left(\tilde{x}_t\right)$

$$\operatorname{var}\left(\tilde{x}_t + \tilde{x}_{t+1} + ... + \tilde{x}_{t+j}\right) < \left(j+1\right)\operatorname{var}\left(\tilde{x}_t\right).$$

For I $\Rightarrow$ II, the persistence of the series must rapidly decline.

Proof of Proposition 3.1

a. The equity price relationship follows from an application of Jensen's inequality:

$$\mathrm{cov}\left(\tilde{p}^e_{t-1},\, \tilde{p}^e_t\right) = \mathrm{cov}\left(\tilde{k}_t, \tilde{k}_{t+1}\right) = \mathrm{cov}\left(\tilde{k}_t,\, \alpha\beta\tilde{k}^\alpha_t\tilde{\lambda}_t\right)$$

$$= \alpha\beta E\left(\tilde{\lambda}_t\right)\left\{E\left(\tilde{k}^{1+\alpha}_t\right) - E\left(\tilde{k}_t\right)E\left(\tilde{k}^\alpha_t\right)\right\}$$

$$> \alpha\beta E\left(\tilde{\lambda}_t\right)\left\{E\left(\tilde{k}^{1+\alpha}_t\right) - E\left(\tilde{k}_t\right)\left(E\left(\tilde{k}_t\right)\right)^\alpha\right\}$$, since k^α_t is concave and $E\left(\tilde{k}^\alpha_t\right) < \left(E\left(\tilde{k}_t\right)\right)^\alpha$ by Jensen's inequality.

$$= \alpha\beta E\left(\tilde{\lambda}_t\right)\left\{E\left(\tilde{k}^{1+\alpha}_t\right) - \left(E\left(\tilde{k}_t\right)\right)^{1+\alpha}\right\} > 0\,.$$

b. Mean aversion in dividends

By (17), $\mathrm{cov}\left(\tilde{d}_t,\, \tilde{d}_{t+1}\right) = \mathrm{cov}\left(\alpha\left(1-\beta\right)\tilde{k}^\alpha_t\tilde{\lambda}_t, \alpha\left(1-\beta\right)\tilde{k}^\alpha_{t+1}\tilde{\lambda}_{t+1}\right)$

$$= \left(\alpha\left(1-\beta\right)\right)^2 \mathrm{cov}\left(\tilde{k}^\alpha_t\tilde{\lambda}_t, \left(\alpha\beta\tilde{k}^\alpha_t\tilde{\lambda}_t\right)^\alpha \tilde{\lambda}_{t+1}\right)$$

$$= \left(\alpha\left(1-\beta\right)\right)^2 \left(\alpha\beta\right)^\alpha \mathrm{cov}\left(\tilde{k}^\alpha_t\, \tilde{\lambda}_t, \tilde{k}^{\alpha^2}_t\tilde{\lambda}^\alpha_t\tilde{\lambda}_{t+1}\right)$$

$$= \left(\alpha\left(1-\beta\right)\right)^2 \left(\alpha\beta\right)^\alpha \left\{E\left(\tilde{k}^{\alpha+\alpha^2}_t\tilde{\lambda}^{1+\alpha}_t\tilde{\lambda}_{t+1}\right)\right.$$

$$\left. -E\left(\tilde{k}^\alpha_t\tilde{\lambda}_t\right)E\left(\tilde{k}^{\alpha^2}_t\tilde{\lambda}^\alpha_t\tilde{\lambda}_{t+1}\right)\right\}$$

$$= \left(\alpha\left(1-\beta\right)\right)^2 \left(\alpha\beta\right)^\alpha E\left(\tilde{\lambda}_{t+1}\right)\left\{E\left(\tilde{k}^{\alpha+\alpha^2}_t\right)E\left(\tilde{\lambda}^{1+\alpha}_t\right) - E\left(\tilde{k}^\alpha_t\right)E\left(\tilde{\lambda}_t\right)E\left(\tilde{k}^{\alpha^2}_t\right)E\left(\tilde{\lambda}^\alpha_t\right)\right\}.$$

By FKG or the Harris inequality

$E\left(\tilde{k}^{\alpha+\alpha^2}_t\right) \geq E\left(\tilde{k}^\alpha_t\right)E\left(\tilde{k}^{\alpha^2}_t\right)$, and

$E\left(\tilde{\lambda}^{1+\alpha}_t\right) \geq E\left(\tilde{\lambda}_t\right)E\left(\tilde{\lambda}^\alpha_t\right)$. Thus,

$$\left(\alpha\left(1-\beta\right)\right)^{2}\left(\alpha\beta\right)^{\alpha} E\left(\tilde{\lambda}_{t+1}\right)\left\{E\left(\tilde{k}_{t}^{\alpha+\alpha^{2}}\right)E\left(\tilde{\lambda}_{t}^{1+\alpha}\right)-E\left(\tilde{k}_{t}^{\alpha}\right)E\left(\tilde{\lambda}_{t}\right)E\left(\tilde{k}_{t}^{\alpha^{2}}\right)E\left(\tilde{\lambda}_{t}^{\alpha}\right)\right\}\geq 0.$$

c. Derivation of risk free bond price.

$$p_{t}^{b}=\beta\int\frac{u_{1}\left(\tilde{c}_{t+1}\right)}{u_{1}\left(c_{t}\right)}dF\left(\tilde{c}_{t+1},c_{t}\right)$$

$$=\beta\int\frac{\left(1-\alpha\beta\right)k_{t}^{\alpha}\lambda_{t}}{\left(1-\alpha\beta\right)\left[\alpha\beta k_{t}^{\alpha}\lambda_{t}\right]^{\alpha}\tilde{\lambda}_{t+1}}\,dF\left(\tilde{\lambda}_{t+1}\right)$$

$$=\beta\int\frac{1}{\left(\alpha\beta\right)^{\alpha}\left[k_{t}^{\alpha}\lambda_{t}\right]^{\alpha-1}\tilde{\lambda}_{t+1}}\,dF\left(\tilde{\lambda}_{t+1}\right)$$

$$=\frac{\beta}{\left(\alpha\beta\right)^{\alpha}}k_{t}^{\alpha(1-\alpha)}\lambda_{t}^{1-\alpha}E\left(\lambda_{t+1}^{-1}\right),\text{ since }\left\{\tilde{\lambda}_{t}\right\}\text{ is i.i.d.}$$

Thus, $p_{t}^{b}=\frac{\beta E\left(\tilde{\lambda}_{t+1}^{-1}\right)}{\left(\alpha\beta\right)^{\alpha}}k_{t}^{\alpha\left(1-\alpha\right)}\lambda_{t}^{1-\alpha}$, where $E\left(\tilde{\lambda}_{t+1}^{-1}\right)$ is constant for all t. As a result, we henceforth omit the time subscript from this term.

$$\operatorname{cov}\left(\tilde{p}_{t}^{b},\tilde{p}_{t+1}^{b}\right)=\operatorname{cov}\left(\left(\frac{\beta E\left(\tilde{\lambda}^{-1}\right)}{\left(\alpha\beta\right)^{\alpha}}\right)\tilde{k}_{t}^{\alpha\left(1-\alpha\right)}\tilde{\lambda}_{t}^{1-\alpha},\ \frac{\beta E\left(\tilde{\lambda}^{-1}\right)}{\left(\alpha\beta\right)^{\alpha}}\tilde{k}_{t+1}^{\alpha\left(1-\alpha\right)}\tilde{\lambda}_{t+1}^{1-\alpha}\right)$$

$$=\left[\frac{\beta E\left(\tilde{\lambda}^{-1}\right)}{\left(\alpha\beta\right)^{\alpha}}\right]^{2}\operatorname{cov}\left(\tilde{k}_{t}^{\alpha-\alpha^{2}}\tilde{\lambda}_{t}^{1-\alpha},\ \tilde{k}_{t+1}^{\alpha\left(1-\alpha\right)}\tilde{\lambda}_{t+1}^{1-\alpha}\right)$$

$$=\left[\frac{\beta E\left(\tilde{\lambda}^{-1}\right)}{\left(\alpha\beta\right)^{\alpha}}\right]^{2}\operatorname{cov}\left(\tilde{k}_{t}^{\alpha-\alpha^{2}}\tilde{\lambda}_{t}^{1-\alpha},\left(\alpha\beta\right)^{\alpha-\sigma^{2}}\tilde{k}_{t}^{\alpha^{2}-\alpha^{3}}\tilde{\lambda}_{t}^{\alpha-\alpha^{2}}\tilde{\lambda}_{t+1}^{1-\alpha}\right),\text{ since }k_{t+1}=\alpha\beta k_{t}^{\alpha}\lambda_{t}.$$

$$= \left[\frac{\beta E\left(\tilde{\lambda}^{-1}\right)}{\left(\alpha\beta\right)^{\alpha}} \right]^2 \left(\alpha\beta\right)^{\alpha-\sigma^2} \left\{ E\left(\tilde{k}_t^{\alpha-\alpha^3} \tilde{\lambda}_t^{1-\alpha^2} \tilde{\lambda}_{t+1}^{1-\alpha}\right) - E\left(\tilde{k}_t^{\alpha-\alpha^2} \tilde{\lambda}_t^{1-\alpha}\right) E\left(\tilde{k}_t^{\alpha^2-\alpha^3} \tilde{\lambda}_t^{\alpha-\alpha^2} \tilde{\lambda}_{t+1}^{1-\alpha}\right) \right\}.$$

Since $\{\lambda_t\}$ is i.i.d. and the fact that k_t is determined in period $t-1$ independent of λ_t or λ_{t+1}, we may equivalently write

$$\operatorname{cov}\left(\tilde{p}_t^b, \tilde{p}_{t+1}^b\right) = \left[\frac{\beta E\left(\tilde{\lambda}^{-1}\right)}{\left(\alpha\beta\right)^{\alpha}} \right]^2 \left(\alpha\beta\right)^{\alpha-\alpha^2} \left\{ E\left(\tilde{k}_t^{\alpha-\alpha^3}\right) E\left(\tilde{\lambda}_t^{1-\alpha^2}\right) E\left(\tilde{\lambda}_{t+1}^{1-\alpha}\right) \right.$$

$$\left. - E\left(\tilde{k}_t^{\alpha-\alpha^2}\right) E\left(\tilde{\lambda}_t^{1-\alpha}\right) E\left(\tilde{k}_t^{\alpha^2-\alpha^3}\right) E\left(\tilde{\lambda}_t^{\alpha-\alpha^3}\right) E\left(\tilde{\lambda}_{t+1}^{1-\alpha}\right) \right\}.$$

Let $f_1\left(k\right) = k^{\alpha-\alpha^2}$ and $f_2\left(k\right) = k^{\alpha^2-\alpha^3}$.

Since both $f_1\left(k\right)$ and $f_2\left(k\right)$ are increasing functions of k, and $f_1\left(k\right) f_2\left(k\right) = k^{\alpha-\alpha^3}$,

by FKG or the Harris inequality, $E\left(\tilde{k}_t^{\alpha-\alpha^3}\right) \geq E\left(\tilde{k}_t^{\alpha-\alpha^2}\right) E\left(\tilde{k}_t^{\alpha^2-\alpha^3}\right)$.

Similarly,

$$E\left(\tilde{\lambda}_t^{1-\alpha^2}\right) \geq E\left(\tilde{\lambda}_t^{1-\alpha}\right) E\left(\tilde{\lambda}_t^{\alpha-\alpha^2}\right).$$

Thus, $\operatorname{cov}\left(\tilde{p}_t^b, \tilde{p}_{t+1}^b\right) = \left[\frac{\beta E\left(\tilde{\lambda}^{-1}\right)}{\left(\alpha\beta\right)^{\alpha}} \right]^2 \left(\alpha\beta\right)^{\alpha-\alpha^2} E\left(\tilde{\lambda}_{t+1}^{1-\alpha}\right) \geq 0$

The inequality is strict if $\left\{\tilde{\lambda}_t\right\}$ is log-normally distributed. The proof in this case follows identically.

Proof of Proposition 3.2

a) $\operatorname{cov}\left(\tilde{k}_t,\, \tilde{k}_{t+1}\right)$

$= \operatorname{cov}\left(\tilde{k}_t,\, i\left(\tilde{k}_t,\, \tilde{\lambda}_t\right) + \left(1-\Omega\right)\tilde{k}_t\right)$

$= \int\int\left(\tilde{k}_t, -\overline{k}\right)\left(i\left(\tilde{k}_t,\, \tilde{\lambda}_t\right) + \left(1-\Omega\right)\tilde{k}_t - \overline{k}\right)dF\left(\tilde{k}_t\right)dG\left(\tilde{\lambda}_t\right)$

Let $f_1\left(\tilde{k}_t, \tilde{\lambda}_t\right) = \tilde{k}_t - \overline{k}$, and $f_2\left(\tilde{k}_t, \tilde{\lambda}_t\right) = i\left(\tilde{k}_t, \tilde{\lambda}_t\right) + \left(1-\Omega\right)\tilde{k}_t - \overline{k}$.

Both $f_1(\ )$ and $f_2(\ )$ are increasing functions of their arguments by assumption. Hence by FKG or Harris inequality,

$\int\int\left(\tilde{k}_t, -\overline{k}\right)\left(i\left(\tilde{k}_t,\, \tilde{\lambda}_t\right) + \left(1-\Omega\right)\tilde{k}_t - \overline{k}\right)dF\left(\tilde{k}_t\right)dG\left(\tilde{\lambda}_t\right)$

$\geq \int\int\left(\tilde{k}_t, -\overline{k}\right)dF\left(\tilde{k}_t\right)dG\left(\tilde{\lambda}_t\right) \times \int\int\left(i\left(\tilde{k}_t,\, \tilde{\lambda}_t\right) + \left(1-\Omega\right)\tilde{k}_t - \overline{k}\right)dF\left(\tilde{k}_t\right)dG\left(\tilde{\lambda}_t\right) = 0$

Thus, $\operatorname{cov}\left(p^e_{t-1}, p^e_t\right) = \operatorname{cov}\left(\tilde{k}_t, \tilde{k}_{t+1}\right) \geq 0$.

In the case of p^b_t ,

$\operatorname{cov}\left(p^b_t, p^b_{t+1}\right) = \operatorname{cov}\left(h\left(\tilde{k}_t, \tilde{\lambda}_t\right), h\left(\tilde{k}_{t+1}, \tilde{\lambda}_{t+1}\right)\right)$ where $h\left(\tilde{k}_t,\, \lambda_t\right)$ is given by (10).

$= \operatorname{cov}\left(-h\left(\tilde{k}_t, \tilde{\lambda}_t\right),\, -h\left(i\left(\tilde{k}_t, \tilde{\lambda}_t\right) + \left(1-\Omega\right)\tilde{k}_t, \tilde{\lambda}_{t+1}\right)\right)$.

In general $h\left(\tilde{k}_t,\, \lambda_t\right)$ will be a decreasing function of each of its arguments, as it is for specification (15). However, if $h\left(\tilde{k}_t,\, \tilde{\lambda}_t\right)$ is a decreasing function $-h\left(\tilde{k}_t,\, \tilde{\lambda}_t\right)$ is increasing. Furthermore,

$$\operatorname{cov}\left(h\left(\tilde{k}_t,\, \tilde{\lambda}_t\right),\, h\left(\tilde{k}_{t+1},\, \tilde{\lambda}_{t+1}\right)\right) = \operatorname{cov}\left(-h\left(\tilde{k}_t,\, \tilde{\lambda}_t\right),\, -h\left(\tilde{k}_{t+1},\, \tilde{\lambda}_{t+1}\right)\right),$$

and the argument above may be applied.

Proof of Proposition 3.3

a) $$\operatorname{cov}\left(\tilde{r}_t^e,\, \tilde{r}_{t+1}^e,\, \right) = \operatorname{cov}\left(\alpha \tilde{k}_t^{a-1}\tilde{\lambda}_t,\, \alpha\left[\alpha\beta\tilde{k}_t^{\alpha}\tilde{\lambda}_t\right]^{\alpha-1}\tilde{\lambda}_{t+1}\right)$$

$$= \alpha^2\left(\alpha\beta\right)^{\alpha-1} E\left(\tilde{\lambda}_{t+1}\right)\left\{E\left(\tilde{k}_t^{a^2-1}\tilde{\lambda}_t^{\alpha}\right) - E\left(\tilde{k}_t^{\alpha-1}\tilde{\lambda}_t\right)E\left(\tilde{k}_t^{a^2-\alpha}\tilde{\lambda}_t^{\alpha-1}\right)\right\}$$

(by independence of $\left\{\tilde{\lambda}_t\right\}$).

$$= \alpha^2\left(\alpha\beta\right)^{\alpha-1} E\left(\tilde{\lambda}_{t+1}\right)\left\{E\left(\tilde{k}_t^{a^2-1}\right)E\left(\tilde{\lambda}_t^{\alpha}\right) - E\left(\tilde{k}_t^{\alpha-1}\right)E\left(\tilde{k}_t^{a^2-\alpha}\right)E\left(\tilde{\lambda}_t\right)E\left(\tilde{\lambda}_t^{\alpha-1}\right)\right\}. \text{ (TA 1)}$$

We wish first to explore constituents of the preceding expression:

$E\left(\tilde{k}_t^{a^2-1}\right)$ vs. $E\left(\tilde{k}_t^{a-1}\right)E\left(\tilde{k}_t^{a^2-\alpha}\right)$.

These expressions are of the general form

$E\left(\tilde{k}_t^{\gamma_0+\gamma_1}\right)$ and $E\left(\tilde{k}_t^{\gamma_0}\right)E\left(\tilde{k}_t^{\gamma_1}\right)$ where $\gamma_0 < 0,\, \gamma_1 < 0$.

Define $\tilde{x}_t = \tilde{k}_t^{\gamma_0}$, and $g\left(\tilde{x}_t\right) = \tilde{x}_t^{\left(\gamma_1/\gamma_0\right)}$.

Since $\left(\gamma_1 / \gamma_0\right) > 0,\, g\left(\tilde{x}_t\right)$ is an increasing function of $\tilde{x}_t$, and $g\left(\tilde{x}_t\right) = \tilde{k}_t^{\gamma_1}$.

Thus, $E\left(\tilde{k}_t^{\gamma_0+\gamma_1}\right) = E\left(\tilde{x}_t\, g\left(\tilde{x}_t\right)\right) > E\left(\tilde{x}_t\right)E\left(g\left(\tilde{x}_t\right)\right) = E\left(\tilde{k}_t^{\gamma_0}\right)E\left(\tilde{k}_t^{\gamma_1}\right)$

by the FKG or Harris inequality.

Accordingly, $E\left(\tilde{k}_t^{\alpha^2-1}\right) \geq E\left(\tilde{k}_t^{\alpha-1}\right)E\left(\tilde{k}_t^{\alpha^2-\alpha}\right)$.

We may thus conclude that expression (TA 1) above is

$$\operatorname{cov}\left(\tilde{r}_t^e,\, \tilde{r}_{t+1}^e,\, \right) \geq \alpha^2\left(\alpha\beta\right)^{\alpha-1} E\left(\tilde{\lambda}_{t+1}\right)\left\{E\left(\tilde{k}_t^{a^2-1}\right)E\left(\tilde{\lambda}_t^{\alpha}\right) - E\left(\tilde{k}_t^{a^2-1}\right)E\left(\tilde{\lambda}_t\right)E\left(\tilde{\lambda}_t^{\alpha-1}\right)\right\}$$

$$\geq \alpha^2\left(\alpha\beta\right)^{\alpha-1} E\left(\tilde{\lambda}_{t+1}\right)E\left(\tilde{k}_t^{a^2-1}\right)\left\{E\left(\tilde{\lambda}_t^{\alpha}\right) - E\left(\tilde{\lambda}_t\right)E\left(\tilde{\lambda}_t^{\alpha-1}\right)\right\}$$

$$\geq \alpha^2\left(\alpha\beta\right)^{\alpha-1} E\left(\tilde{\lambda}_{t+1}\right)E\left(\tilde{k}_t^{a^2-1}\right)\left\{E\left(\tilde{\lambda}_t^{\alpha}\right) - E\left(\tilde{\lambda}_t\right)\left(E\left(\tilde{\lambda}_t\right)\right)^{\alpha-1}\right\}$$

(by Jensen's inequality, since $\tilde{\lambda}^{\alpha-1}$, $0 < \alpha < 1$ is a convex function of $\tilde{\lambda}$)

$$\operatorname{cov}\left(\tilde{r}_t^e,\, \tilde{r}_{t+1}^e,\, \right) \geq \alpha^2\left(\alpha\beta\right)^{\alpha-1} E\left(\tilde{\lambda}_{t+1}\right)E\left(\tilde{k}_t^{a^2-1}\right)\left\{E\left(\tilde{\lambda}_t^{\alpha}\right) - \left(E\left(\tilde{\lambda}_t\right)\right)^{\alpha}\right\}$$

Again by Jensen's inequality, since $\tilde{\lambda}^{\alpha}$, $0<\alpha<1$ is a concave function of $\tilde{\lambda}$

$$E\left(\tilde{\lambda}_t^{\alpha}\right)-\left(E\left(\tilde{\lambda}_t\right)\right)^{\alpha}<0$$

Hence, $\operatorname{cov}\left(\tilde{r}_t^e,\ \tilde{r}_{t+1}^e,\ \right)\geq -M,\ M>0$. $\operatorname{cov}\left(\tilde{r}_t^e,\ \tilde{r}_{t+1}^e,\ \right)$ can be negative.

b) $\operatorname{cov}\left(\tilde{r}_t^b,\tilde{r}_{t+1}^b\right)\geq 0$.

$$\operatorname{cov}\left(\tilde{r}_t^b,\tilde{r}_{t+1}^b\right)=\operatorname{cov}\left(\frac{(\alpha\beta)^2}{\beta E\left(\tilde{\lambda}^{-1}\right)}\tilde{k}_t^{\alpha(\alpha-1)}\tilde{\lambda}_t^{\alpha-1},\ \frac{(\alpha\beta)^2}{\beta E\left(\tilde{\lambda}^{-1}\right)}\tilde{k}_{t+1}^{\alpha(\alpha-1)}\tilde{\lambda}_{t+1}^{\alpha-1},\right)$$

$$=\left(\frac{(\alpha\beta)^2}{\beta E\left(\tilde{\lambda}^{-1}\right)}\right)^2\operatorname{cov}\left(\tilde{k}_t^{\alpha(\alpha-1)}\tilde{\lambda}_t^{\alpha-1},\ \left(\alpha\beta\tilde{k}_t^{\alpha}\tilde{\lambda}_t\right)^{\alpha(\alpha-1)}\tilde{\lambda}_{t+1}^{\alpha-1}\right)$$

$$=(\alpha\beta)^{\alpha(\alpha-1)}\left(\frac{(\alpha\beta)^2}{\beta E\left(\tilde{\lambda}^{-1}\right)}\right)^2\operatorname{cov}\left(\tilde{k}_t^{\alpha^2-\alpha}\tilde{\lambda}_t^{\alpha-1},\ \tilde{k}_t^{\alpha^3-\alpha^2}\tilde{\lambda}_t^{\alpha^2-\alpha}\tilde{\lambda}_{t+1}^{\alpha-1}\right)$$

By the properties of the covariance function and that $\tilde{k}_t$, $\tilde{\lambda}_t$, and $\tilde{\lambda}_{t+1}$ are all independent of one another, the RHS expression becomes

$$=M\ E\left(\tilde{\lambda}_{t+1}^{\alpha-1}\right)\left\{E\left(\tilde{k}_t^{\alpha^3-\alpha}\right)E\left(\tilde{\lambda}_t^{\alpha^2-1}\right)\right\}-E\left(\tilde{k}_t^{\alpha^2-\alpha}\right)E\left(\tilde{\lambda}_t^{\alpha-1}\right)E\left(\tilde{k}_t^{\alpha^3-\alpha^2}\right)E\left(\tilde{\lambda}_t^{\alpha^2-\alpha}\right),$$

for some positive constant M. Let $f_1\left(\tilde{k}\right)=\tilde{k}^{\alpha^2-\alpha}$ and $f_2\left(\tilde{k}\right)=\tilde{k}^{\alpha^3-\alpha^2}$; each is a decreasing function of k, furthermore, $f_1\left(\tilde{k}\right)f_2\left(\tilde{k}\right)=\tilde{k}^{\alpha^3-\alpha}$ which is also decreasing in k. By the FKG or Harris inequality

$$E\left(\tilde{k}_t^{\alpha^3-\alpha}\right)\geq E\left(\tilde{k}_t^{\alpha^2-\alpha}\right)E\left(\tilde{k}_t^{\alpha^3-\alpha^2}\right) \text{ and}$$

$$E\left(\tilde{\lambda}_t^{\alpha^2-1}\right)\geq E\left(\tilde{\lambda}_t^{\alpha-1}\right)E\left(\tilde{\lambda}_t^{\alpha^2-\alpha}\right).$$

Thus $\operatorname{cov}\left(\tilde{r}_t^b,\ \tilde{r}_{t+1}^b\right)\geq 0$.

We have been unable to derive any definitive result for the premium.

Proof of Proposition 4.1

This follows simply from the construction of the derived process $\left\{\tilde{\gamma}_t^{AB}\right\}$.

Every period the process $\left\{\tilde{\gamma}_t\right\}$ assumes a value above its mean, it is assuming a value in set γ^A. Furthermore, if the process $\left\{\tilde{\gamma}_t^{AB}\right\}$ is in set γ^A, then it is assuming a value above its mean. Thus the average number of periods the process $\left\{\tilde{\gamma}_t\right\}$ is above its mean $\left(\tilde{\gamma}_t \in \gamma^A\right)$must coincide with the average number of periods it is in set γ^A. Thus $ACT^A_{\{\tilde{\gamma}_t\}} = ACT^A_{\{\tilde{\gamma}_t^{AB}\}}$. A similar identification establishes that $ACT^B_{\{\tilde{\gamma}_t\}} = ACT^B_{\{\tilde{\gamma}_t^{AB}\}}$.

Properties of $\left\{\tilde{\gamma}^{AB}\right\}$

We first restrict attention to a consideration of arbitrary two state Markov chains. We adopt the convention that any measurement of the 'time to crossing' includes the crossing period itself.

A. Consider an arbitrary two state Markov chain with states γ_1, γ_2 and transition matrix:

$$\begin{array}{cc} & \begin{array}{cc} \gamma_1 & \gamma_2 \end{array} \\ \begin{array}{c} \gamma_1 \\ \gamma_2 \end{array} & \begin{bmatrix} \varphi_1 & 1-\varphi_1 \\ 1-\varphi_2 & \varphi_2 \end{bmatrix} \end{array}$$

where $0 < \varphi_1 < 1$ and $0 < \varphi_2 < 1$. The associated ergodic probability distribution (π_1, π_2) satisfies

$$\left[\pi_1, \pi_2\right] = \left[\pi_1, \pi_2\right] \begin{bmatrix} \varphi_1 & 1-\varphi_1 \\ 1-\varphi_2 & \varphi_2 \end{bmatrix},$$

with solution $\pi_1 = \dfrac{1-\phi_2}{2-\left(\phi_1+\phi_2\right)}$ and $\pi_2 = \dfrac{1-\phi_1}{2-\left(\phi_1+\phi_2\right)}$

B. Suppose the process is in state $\tilde{\gamma} = \gamma_1$. The average time to crossing to state 2, ACT_1, is given by:

$$ACT_1 = \sum_{n=1}^{\infty} nProb\left(\begin{matrix} \tilde{\gamma} = \gamma_2 \\ in\ step\ n \end{matrix} \;\middle|\; \begin{matrix} \tilde{\gamma} = \gamma_1,\ for \\ the\ n-1\ prior\ steps \end{matrix} \right)$$

$$= \sum_{n=1}^{\infty} n\varphi_1^{n-1}\left(1-\varphi_1\right) = \left(1-\varphi_1\right)\sum_{n=1}^{\infty} n\varphi_1^{n} = \left(1-\varphi_1\right)\left(\frac{1}{\left(1-\varphi_1\right)^2}\right) = \frac{1}{1-\varphi_1}$$

Similarly, $ACT_2 = \dfrac{1}{1-\varphi_2}$

.

Accordingly, the average crossing time, ACT, satisfies

$$ACT = \pi_1 ACT_1 + \pi_2 ACT_2 = \pi_1\left(\frac{1}{1-\varphi_1}\right) + \pi_2\left(\frac{1}{1-\varphi_2}\right)$$

$$= \frac{1}{2-\left(\phi_1+\phi_2\right)}\left[\frac{1-\phi_2}{1-\phi_1} + \frac{1-\phi_1}{1-\phi_2}\right]$$

C. We compute $corr\left(\tilde{\gamma}_t, \tilde{\gamma}_{t+1}\right) = \dfrac{cov(\tilde{\gamma}_t, \tilde{\gamma}_{t+1})}{\sigma_{\tilde{\gamma}_t}\sigma_{\tilde{\gamma}_{t+1}}}$

$$= \frac{E(\tilde{\gamma}_t\tilde{\gamma}_{t+1}) - E(\tilde{\gamma}_t)E(\tilde{\gamma}_{t+1})}{\sigma_{\tilde{\gamma}_t}\sigma_{\tilde{\gamma}_{t+1}}}$$

Without loss of generality, we assume $\gamma_1 = 1$ and $\gamma_2 = -1$, since the $corr\left(\tilde{\gamma}_t, \tilde{\gamma}_{t+1}\right)$ is determined by the structure of the transition matrix and not the specific values assumed by $\tilde{\gamma}_t$.[2]

[2] Consider a stochastic process $\tilde{\gamma}_t$; then for any $a > 0$ and any $b \in \mathbb{R}$, $corr\left(\tilde{\gamma}_t, \tilde{\gamma}_{t+1}\right) = corr\left(a\tilde{\gamma}_t + b,\ a\tilde{\gamma}_{t+1} + b\right)$.

The correlation computation requires the following constituents:

$$\text{(i)}\quad E(\tilde{\gamma}_t) = \gamma_1\left(\frac{1-\phi_2}{2-\left(\phi_1+\phi_2\right)}\right) + \gamma_2\left(\frac{1-\phi_1}{2-\left(\phi_1+\phi_2\right)}\right)$$

$$= \frac{\phi_1-\phi_2}{2-\left(\phi_1+\phi_2\right)}$$

$$\text{(ii)}\quad E(\tilde{\gamma}_{t+1}) = \left(\frac{\phi_1-\phi_2}{2-\left(\phi_1+\phi_2\right)}\right)$$

$$\text{(iii)}\quad \operatorname{var}(\tilde{\gamma}_t) = E\left(\tilde{\gamma}_t - E(\tilde{\gamma}_t)\right)^2$$

$$= \left(\frac{1-\phi_2}{2-\left(\phi_1+\phi_2\right)}\right)\left(1-\frac{\phi_1-\phi_2}{2-\left(\phi_1+\phi_2\right)}\right)^2$$

$$+\left(\frac{1-\phi_1}{2-\left(\phi_1+\phi_2\right)}\right)\left(-1-\frac{\phi_1-\phi_2}{2-\left(\phi_1+\phi_2\right)}\right)^2$$

$$= \left(\frac{1}{2-\left(\phi_1+\phi_2\right)}\right)^3\left[\left(1-\phi_2\right)\left(2-2\phi_1\right)^2 + \left(1-\phi_1\right)\left(-2+2\phi_2\right)^2\right]$$

$$= \frac{4\left(1-\phi_1\right)\left(1-\phi_2\right)}{\left[2-\left(\phi_1+\phi_2\right)\right]^3}\left\{2-\left(\phi_1+\phi_2\right)\right\}$$

$$= \frac{4\left(1-\phi_1\right)\left(1-\phi_2\right)}{\left[2-\left(\phi_1+\phi_2\right)\right]^2} = \operatorname{var}(\tilde{\gamma}_{t+1})$$

We can thus assume $\gamma_1 = 1$ and $\gamma_2 = -1$, if constants a, b as per the above correlation equality satisfy $a\gamma_1 + b = 1$ and $a\gamma_2 + b = -1$. The solution to this simple system of equations is $a = \dfrac{2}{\gamma_1 - \gamma_1}$ and $b = \left(-1 + \gamma_2 / \gamma_1\right)/\left(1 - \gamma_2 / \gamma_1\right)$.

(iv) $$E(\tilde{\gamma}_t\,\tilde{\gamma}_{t+1}) = \sum_{\gamma_t} \text{Prob}\left(\gamma_t\right)\left[\sum_{\gamma_{t+1}} \gamma_{t+1}\,\text{Prob}\left(\gamma_{t+1}\middle|\,\gamma_t\right)\right]\left(\gamma_t\right)$$

$$= 1\left[\frac{1-\phi_2}{2-\left(\phi_1+\phi_2\right)}\right]\left[1\phi_1 + \left(1-\phi_1\right)\left(-1\right)\right]$$

$$+\left(-1\right)\left[\frac{1-\phi_1}{2-\left(\phi_1+\phi_2\right)}\right]\left[1\left(1-\phi_2\right)\left(-1\phi_2\right)\right]$$

$$= \frac{1}{2-\left(\phi_1+\phi_2\right)}\left\{\left(1-\phi_2\right)\left[-1+2\phi_1\right]-\left(1-\phi_1\right)\left[1-2\phi_2\right]\right\}$$

$$= \frac{1}{2-\left(\phi_1+\phi_2\right)}\left\{-2+3\phi_1+3\phi_2-4\phi_1\phi_2\right\}$$

(v) $$corr\left(\tilde{\gamma}_t,\,\tilde{\gamma}_{t+1}\right) = \frac{E\left(\tilde{\gamma}_t\tilde{\gamma}_{t+1}\right)-E\left(\tilde{\gamma}_1\right)E\left(\tilde{\gamma}_{t+1}\right)}{\sigma_{\tilde{\gamma}_t}\sigma_{\tilde{\gamma}_{t+1}}}$$

$$= \frac{\dfrac{1}{2-\left(\phi_1+\phi_2\right)}\left[-2+3\left(\phi_1+\phi_2\right)-4\phi_1\phi_2\right]-\left[\dfrac{\phi_1-\phi_2}{2-\left(\phi_1+\phi_2\right)}\right]^2}{\dfrac{4\left(1-\phi_1\right)\left(1-\phi_2\right)}{\left(2-\left(\phi_1+\phi_2\right)\right)^2}}$$

$$= \frac{\left[2-\left(\phi_1+\phi_2\right)\right]\left[-2+3\left(\phi_1+\phi_2\right)-4\phi_1\phi_2\right]-\left[\phi_1-\phi_2\right]^2}{4\left(1-\phi_1\right)\left(1-\phi_2\right)}$$

(after considerable simplification)

$$= \frac{-4+\left(8+4\phi_1\phi_2\right)\left[\phi_1+\phi_2\right]-12\phi_1\phi_2-4\phi_1^2-4\phi_2^2}{4\left(1-\phi_1\right)\left(1-\phi_2\right)}$$

$$= \frac{-\left(1+\phi_1\phi_2\right)+\left(\phi_1+\phi_2\right)\left[1+\left(1-\phi_1\right)\left(1-\phi_2\right)\right]}{\left(1-\phi_1\right)\left(1-\phi_2\right)}$$

$$= \left(\phi_1+\phi_2\right)+\left[\frac{-\left(1-\phi_1\right)\left(1-\phi_2\right)}{\left(1-\phi_1\right)\left(1-\phi_2\right)}\right] = \left(\phi_1+\phi_2\right)-1 \qquad \text{(A)}$$

D. Proposition 4.2. To show $ACT_{\{\tilde{\gamma}_t\}} \leq 2$ implies $corr\left(\tilde{\gamma}_t,\, \tilde{\gamma}_{t+1}\right) \leq 0$.

We show $ACT \leq 2$ implies $\left(\phi_1 + \phi_2\right) \leq 1$.

$$ACT_{\{\tilde{\gamma}_t\}} \leq 2 \quad \Rightarrow$$

$$\frac{1}{2-\left(\phi_1+\phi_2\right)}\left[\frac{1-\phi_1}{1-\phi_2}+\frac{1-\phi_2}{1-\phi_1}\right] \leq 2$$

The second term in the ACT expression is of the form $x + \frac{1}{x}$, which assumes a minimum at $x = 1$. Therefore,

$$\frac{1}{2-\left(\phi_1+\phi_2\right)}\left[2\right] \leq ACT_{\{\tilde{\gamma}_t\}} \leq 2 \text{, or}$$

$$\frac{2}{2-\left(\phi_1+\phi_2\right)} \leq 2 \text{; equivalently}$$

$$1 \leq 2-\left(\phi_1+\phi_2\right) \text{ or}$$

$$\left(\phi_1+\phi_2\right) \leq 1.$$

Suppose $\phi_1 = \phi_2$ and $\left(\phi_1+\phi_2\right) \leq 1$.

Then $ACT_{\{\tilde{\gamma}_t\}} = \frac{1}{2-\left(\phi_1+\phi_2\right)}\left[\frac{1-\phi_1}{1-\phi_2}+\frac{1-\phi_2}{1-\phi_1}\right] \leq \left[\frac{1-\phi_1}{1-\phi_2}+\frac{1-\phi_2}{1-\phi_1}\right] = 2$

E. When $\phi_1 = \phi_2 = \phi$, the ACT reduces to

$$ACT = \frac{1}{2-2\phi}\left[1+1\right] = \frac{1}{1-\phi} \text{ , and}$$

$$corr\left(\tilde{\gamma}_t, \tilde{\gamma}_{t+1}\right) = 2\phi - 1 \text{ . Thus}$$

$corr\left(\gamma_t^x, \gamma_{t+1}^x\right) \geq corr\left(\gamma_t^y,\ \gamma_{t+1}^y\right)$, if and only if

$2\phi^x - 1 > 2\phi^y - 1$, if and only if

$\phi^x > \phi^y$, if and only if

$\dfrac{1}{1-\phi^x} > \dfrac{1}{1-\phi^y}$, if and only if

$ACT_{\left\{\gamma_t^x\right\}} > ACT_{\left\{\gamma_t^y\right\}}$.

F. The region A is computed by searching for the pairs $\left(0,\ 0\right) < \left(\phi_1,\ \phi_2\right) < \left(1,\ 1\right)$ such that

$$\partial ACT / \partial\phi_1 > 0 \text{ and } \partial ACT / \partial\phi_2 > 0.$$

The indicated region (Figure 2) was constructed numerically.

Proof of Proposition 5.1

(a) We first offer the proof for $\left\{\tilde{p}_t^e\right\}$; $\left\{\tilde{d}_t\right\}$ is analyzed similarly, since

$d_t = \left(\left(1-\beta\right)/\beta\right)p_t^e$

$$\begin{aligned}
\text{cov}\left(\tilde{p}_t^e,\ \tilde{p}_{t+1}^e\right) &= \text{cov}\left(\tilde{k}_{t+1},\ \tilde{k}_{t+2}\right) \\
&= \text{cov}\left(\alpha\beta\tilde{k}_t^{\alpha}e^{\tilde{\lambda}_t},\ \alpha\beta\left[\alpha\beta\tilde{k}_t^{\alpha}e^{\tilde{\lambda}_t}\right]^{\alpha} e^{\tilde{\lambda}_{t+1}}\right) \\
&= \text{cov}\left(\alpha\beta\tilde{k}_t^{\alpha}e^{\tilde{\lambda}_t},\ \left(\alpha\beta\right)^{1+\alpha}\tilde{k}_t^{\alpha^2}e^{\alpha\tilde{\lambda}_t}e^{\left(\rho\tilde{\lambda}_t+\tilde{\varepsilon}_{t+1}\right)}\right) \\
&= \left(\alpha\beta\right)^{2+\alpha}\text{cov}\left(\tilde{k}_t^{\alpha}e^{\tilde{\lambda}_t},\ \tilde{k}_t^{\alpha^2}e^{\left(\alpha+\rho\right)\tilde{\lambda}_t}\cdot e^{\tilde{\varepsilon}_{t+1}}\right) \\
&= \left(\alpha\beta\right)^{2+\alpha}E\left(e^{\tilde{\varepsilon}_{t+1}}\right)\left\{E\left(\tilde{k}_t^{\alpha+\alpha^2}e^{\left(\alpha+\rho+1\right)\tilde{\lambda}_t}\right) - E\left(\tilde{k}_t^{\alpha}e^{\tilde{\lambda}_t}\right)E\left(\tilde{k}_t^{\alpha^2}e^{\left(\alpha+\rho\right)\tilde{\lambda}_t}\right)\right\}.
\end{aligned}$$

Let $g^1\left(k_t,\lambda_t\right) = \tilde{k}_t^{\alpha}e^{\tilde{\lambda}_t}$, $g_1^1 > 0$; $g_2^1 > 0$

$$g^2\left(k_t,\lambda_t\right)=\tilde{k}_t^{\alpha^2}e^{(\alpha+\rho)\tilde{\lambda}_t}\ \ ,\ g_1^2>0;\ g_2^2>0$$

By FKG or the Harris inequality

$$E\left(g^1\left(\tilde{k}_t,\,\tilde{\lambda}_t\right)g^2\left(\tilde{k}_t,\,\tilde{\lambda}_t\right)\right)$$

$$\geq E\left(g^1\left(\tilde{k}_t,\,\tilde{\lambda}_t\right)\right)E\left(g^2\left(\tilde{k}_t,\,\tilde{\lambda}_t\right)\right)$$

Thus, $\operatorname{cov}\left(\tilde{p}_t^e,\,\tilde{p}_{t+1}^e\right)=\operatorname{cov}\left(\tilde{k}_{t+1}^e,\,\tilde{k}_{t+2}^e\right)\geq 0$.

(b) $\operatorname{cov}\left(\tilde{p}_t^b,\,\tilde{p}_{t+1}^b\right)$

where $p_t^b=\beta e^{\sigma_\varepsilon^2/2}\left(\alpha\beta\right)^{-2}k_t^{\alpha-\alpha^2}e^{(1-\alpha-\rho)\tilde{\lambda}_t}$

$$\operatorname{cov}\left(\tilde{p}_t^b,\,\tilde{p}_{t+1}^b\right)=\beta^2e^{\sigma_\varepsilon^2}\left(\alpha\beta\right)^{-2\alpha}\operatorname{cov}\left(\tilde{k}_t^{\alpha-\alpha^2}e^{(1-\alpha-\rho)\tilde{\lambda}_t},\,\left(\alpha\beta\tilde{k}_t^{\alpha}e^{\tilde{\lambda}_t}\right)^{\alpha-\alpha^2}e^{(1-\alpha-\rho)\left(\rho\tilde{\lambda}_t+\tilde{\varepsilon}_{t+1}\right)}\right)$$

$$=\underbrace{\beta^2e^{\sigma_\varepsilon^2}\left(\alpha\beta\right)^{-2\alpha}\left(\alpha\beta\right)^{\alpha-\alpha^2}}_{\mathrm{M}}\operatorname{cov}\left(\tilde{k}_t^{\alpha-\alpha^2}e^{(1-\alpha-\rho)\tilde{\lambda}_t},\,\tilde{k}_t^{\alpha^2-\alpha^3}e^{\left(\alpha-\alpha^2\right)\tilde{\lambda}_t}\,e^{(1-\alpha-\rho)\left(\rho\tilde{\lambda}_t+\tilde{\varepsilon}_{t+1}\right)}\right)$$

$$=\mathrm{M}\left(\operatorname{cov}\tilde{k}_t^{\alpha-\alpha^2}e^{(1-\alpha-\rho)\tilde{\lambda}_t},\,\tilde{k}_t^{\alpha^2-\alpha^3}e^{\left(\alpha-\alpha^2+\rho(1-\alpha-\rho)\right)\tilde{\lambda}_t}e^{(1-\alpha-\rho)\tilde{\varepsilon}_{t+1}}\right)$$

$$=Me^{(1-\alpha-\rho)^2\sigma_\varepsilon^2/2}\left\{E\left(\tilde{k}_t^{\alpha-\alpha^3}e^{\left(\alpha-\alpha^2+(1+\rho)(1-\alpha-\rho)\tilde{\lambda}_t\right)}\right)-E\left(\tilde{k}_t^{\alpha-\alpha^2}e^{(1-\alpha-\rho)\tilde{\lambda}_t}\right)E\left(\tilde{k}_t^{\alpha^2-\alpha^3}e^{\left(\alpha-\alpha^2+\rho(1-\alpha-\rho)\right)\tilde{\lambda}_t}\right)\right\}$$

If we let $g\left(\tilde{k}_t,\tilde{\lambda}_t\right)=\tilde{k}_t^{\alpha-\alpha^2}e^{(1-\alpha-\rho)\tilde{\lambda}_t}$

and

$$f\left(\tilde{k}_t,\,\tilde{\lambda}_t\right)=\tilde{k}_t^{\alpha^2-\alpha^3}e^{\left(\alpha-\alpha^2+\rho(1-\alpha-\rho)\right)\tilde{\lambda}_t},$$

then, if $\alpha+\rho<1$,

$$f_1\left(\quad\right)>0\quad f_2\left(\quad\right)>0,\ g_1\left(\quad\right)>0\quad g_2\left(\quad\right)>0.$$

By the FKG or the Harris inequality we conclude immediately that

$\operatorname{cov}\left(\tilde{p}_t^b,\ \tilde{p}_{t+1}^b\right) \geq 0$, provided $\alpha + \rho < 1$.

Proof of Proposition 5.2

(a) $\operatorname{cov}\left(\tilde{r}_t^b, \tilde{r}_{t+1}^b\right) = \operatorname{cov}\left(\tilde{r}_{t+1}^b, \tilde{r}_{t+2}^b\right)$

$$= \underbrace{\left[\frac{\left(\alpha\beta\right)^{\alpha}}{\beta} e^{-\sigma_\varepsilon^2/2}\right]^2}_{L} \operatorname{cov}\left(\tilde{k}_t^{\alpha^2-\alpha}\, e^{(\alpha+\rho-1)\tilde{\lambda}_t}, \tilde{k}_{t+1}^{\alpha^2-\alpha} e^{(\alpha+\rho-1)\tilde{\lambda}_{t+1}}\right)$$

But ` $k_{t+1} = \alpha\beta k_t^{\alpha}\, e^{\tilde{\lambda}_t}$ and $\tilde{\lambda}_{t+1} = \rho\tilde{\lambda}_t + \tilde{\varepsilon}_{t+1}$; thus:

$$= L\left(\alpha\beta\right)^{\alpha^2-\alpha} \operatorname{cov}\left(\tilde{k}_t^{\alpha^2-\alpha}\, e^{(\alpha+\rho-1)\tilde{\lambda}_t}, \tilde{k}_t^{\alpha^3-\alpha^2}\, e^{\tilde{\lambda}_t\left(\alpha^2-\alpha\right)}\, e^{(\alpha+\rho-1)\rho\tilde{\lambda}_t\, +\, (\alpha+\rho-1)\tilde{\varepsilon}_{t+1}}\right)$$

$$= L\left(\alpha\beta\right)^{\alpha^2-\alpha} \operatorname{cov}\left(\tilde{k}_t^{\alpha^2-\alpha}\, e^{(\alpha+\rho-1)\tilde{\lambda}_t}, \tilde{k}_t^{\alpha^3-\alpha^2}\, e^{\left[\left(\alpha^2-\alpha\right)+(\alpha+\rho-1)\rho\right]\tilde{\lambda}_t} \cdot e^{(\alpha+\rho-1)\tilde{\varepsilon}_{t+1}}\right)$$

$$= L\left(\alpha\beta\right)^{\alpha^2-\alpha} e^{(\alpha+\rho-1)^2\sigma_\varepsilon^2/2} \left\{E\left(\tilde{k}_t^{\alpha^3-\alpha} e^{\left[\left(\alpha^2-\alpha\right)+2(\alpha+\rho-1)\right]\tilde{\lambda}_t}\right) - E\left(\tilde{k}_t^{\alpha^2-\alpha}\, e^{(\alpha+\rho-1)\tilde{\lambda}_t}\right) E\left(\tilde{k}_t^{\alpha^3-\alpha^2}\, e^{\left[\left(\alpha^2-\alpha\right)+(\alpha+\rho-1)\rho\right]\tilde{\lambda}_t}\right)\right\}$$

Let $g^1\left(\tilde{k}, \tilde{\lambda}\right) = \left(\tilde{k}_t^{\alpha^2-\alpha}\right) e^{(\alpha+\rho-1)\tilde{\lambda}_t}$

$g^2\left(\tilde{k}, \tilde{\lambda}\right) = \tilde{k}_t^{\alpha^3-\alpha^2}\, e^{\left[\left(\alpha^2-\alpha\right)+(\alpha+\rho-1)\rho\right]\tilde{\lambda}_t}$

$g_1^1\left(\tilde{k}, \tilde{\lambda}\right) < 0 \qquad g_2^1\left(\tilde{k}, \tilde{\lambda}\right) < 0$ if $\alpha + \rho < 1$

$g_1^2\left(\tilde{k}, \tilde{\lambda}\right) < 0 \qquad g_2^2\left(\tilde{k}, \tilde{\lambda}\right) < 0$ if $\alpha + \rho < 1$.

Hence, $-g^1\left(\tilde{k}_t, \tilde{\lambda}_t\right)$ is increasing

and $-g^2\left(\tilde{k}_t, \tilde{\lambda}_t\right)$ is increasing. Thus,

$$\int -g^1\left(\tilde{k}_t, \tilde{\lambda}_t\right)\left(-g^2\left(\tilde{k}_t, \tilde{\lambda}_t\right)\right) dF\left(\tilde{k}_t, \tilde{\lambda}_t\right) \geq \int -g^1\left(\tilde{k}_t, \tilde{\lambda}_t\right) dF\left(\tilde{k}_t, \tilde{\lambda}_t\right) \int -g^2\left(\tilde{k}_t, \tilde{\lambda}_t\right) dF\left(\tilde{k}_t, \tilde{\lambda}_t\right)$$

$$\int g^1\left(\tilde{k}_t,\tilde{\lambda}_t\right) g^2\left(\tilde{k}_t,\tilde{\lambda}_t\right) dF\left(\tilde{k}_t,\tilde{\lambda}_t\right) \geq \int g^1\left(\tilde{k}_t,\tilde{\lambda}_t\right) dF\left(\tilde{k}_t,\tilde{\lambda}_t\right) \int g^2\left(\tilde{k}_t,\tilde{\lambda}_t\right) dF\left(\tilde{k}_t,\tilde{\lambda}_t\right),$$

(by the FKG or the Harris inequality).

Thus, $\operatorname{cov}\left(\tilde{r}_t^b,\, \tilde{r}_{t+1}^b\right) \geq 0$.

(b) $\operatorname{cov}\left(\tilde{r}_t^e,\, \tilde{r}_{t+1}^e\right)$

$$= \operatorname{cov}\left(\alpha \tilde{k}_t^{\alpha-1} e^{\tilde{\lambda}_t},\, \alpha \tilde{k}_{t+1}^{\alpha-1} e^{\tilde{\lambda}_{t+1}}\right)$$

$$= \operatorname{cov}\left(\alpha \tilde{k}_t^{\alpha-1} e^{\tilde{\lambda}_t},\, \alpha\left(\alpha\beta\, \tilde{k}_t^{\alpha} e^{\tilde{\lambda}_t}\right)^{\alpha-1} e^{\rho\tilde{\lambda}_t+\tilde{\varepsilon}_{t+1}}\right)$$

$$= \operatorname{cov}\left(\alpha \tilde{k}_t^{\alpha-1} e^{\tilde{\lambda}_t},\, \alpha^{\alpha}\beta^{\alpha-1}\, \tilde{k}_t^{\alpha^2-\alpha}\, e^{(\alpha-1)\tilde{\lambda}_t}\, e^{\rho\tilde{\lambda}_t+\tilde{\varepsilon}_{t+1}}\right)$$

$$= \underbrace{\alpha^{1+\alpha}\beta^{\alpha-1}}_{\mathrm{M}\,>\,0} \operatorname{cov}\left(\tilde{k}_t^{\alpha-1} e^{\tilde{\lambda}_t},\, \tilde{k}_t^{\alpha^2-\alpha}\, e^{(\alpha+\rho-1)\tilde{\lambda}_t}\, e^{\tilde{\varepsilon}_{t+1}}\right)$$

$$= M\left\{E\left(\tilde{k}_t^{\alpha^2-1} e^{(\alpha+\rho)\tilde{\lambda}_t} e^{\tilde{\varepsilon}_{t+1}}\right) - E\left(\tilde{k}_t^{\alpha-1} e^{\tilde{\lambda}_t}\right) E\left(\tilde{k}_t^{\alpha^2-\alpha} e^{(\alpha+\rho-1)\tilde{\lambda}_t} e^{\tilde{\varepsilon}_{t+1}}\right)\right\}$$

$$= M\, e^{\sigma_\varepsilon^2/2}\left\{E\left(\tilde{k}_t^{\alpha^2-1} e^{(\alpha+\rho)\tilde{\lambda}_t}\right) - E\left(\tilde{k}_t^{\alpha-1} e^{\tilde{\lambda}_t}\right) E\left(\tilde{k}_t^{\alpha^2-\alpha} e^{(\alpha+\rho-1)\tilde{\lambda}_t}\right)\right\} \qquad \text{(TA 2)}$$

(i) From the proof of Proposition 3.3,

$$E\left(\tilde{k}_t^{\alpha^2-1}\right) > E\left(\tilde{k}_t^{\alpha-1}\right) E\left(\tilde{k}_t^{\alpha^2-\alpha}\right).$$

(ii) If $(\alpha+\rho) > 1$, then $e^{(\alpha+\rho)\tilde{\lambda}_t}$, $e^{\tilde{\lambda}_t}$ and $e^{(\alpha+\rho-1)\tilde{\lambda}_t}$ are all increasing functions of $\tilde{\lambda}_t$. Thus, by the FKG or the Harris inequality,

$$E\left(e^{(\alpha+\rho)\tilde{\lambda}_t}\right) \geq E\left(e^{\tilde{\lambda}_t}\right) E\left(e^{(\alpha+\rho-1)\tilde{\lambda}_t}\right).$$

Since (TA 2) is equivalent to (TA 3)

$$= M e^{\sigma_\varepsilon^2/2}\left\{E\left(\tilde{k}_t^{\alpha^2-1}\right) E\left(e^{(\alpha+\rho)\tilde{\lambda}_t}\right) - E\left(\tilde{k}_t^{\alpha-1}\right) E\left(e^{\tilde{\lambda}_t}\right) E\left(\tilde{k}_t^{\alpha^2-\alpha}\right) E\left(e^{(\alpha+\rho-1)\tilde{\lambda}_t}\right)\right\} \qquad \text{(TA 3)}$$

then by relationship (i), (ii) noted previously, we have

$\operatorname{cov}\left(\tilde{r}_t^e,\ \tilde{r}_{t+1}^e\right) \geq 0$, provided $\left(\alpha+\rho\right) > 1$.

Proof of Proposition 5.3 (This result is due to Sergio Villar)

This proof uses the fact that if $\tilde{y} \sim N\left(0, v\right)$, then $E\left[\exp(\tilde{y})\right] = \frac{v}{2}$

For the $AR(1)$ process, $\operatorname{Var}\left(\tilde{x}_t\right) = \sigma^2\left(\frac{1-\rho^{2t}}{1-\rho^2}\right)$:

$$\operatorname{Cov}\left(\tilde{\lambda}_t, \tilde{\lambda}_{t+1}\right) = \operatorname{Cov}\left(\exp\left(\tilde{x}_t\right), \exp\left(\rho\tilde{x}_t + \tilde{\varepsilon}_{t+1}\right)\right)$$

$$= E\left[\exp\left(\tilde{x}_t\right)\exp\left(\rho\tilde{x}_t + \tilde{\varepsilon}_{t+1}\right)\right] - E\left[\exp\left(\tilde{x}_t\right)\right]E\left[\exp\left(\rho\tilde{x}_t + \tilde{\varepsilon}_{t+1}\right)\right]$$

$$= E\left[\exp\left(\left(\rho+1\right)\tilde{x}_t + \tilde{\varepsilon}_{t+1}\right)\right] - E\left[\exp\left(\tilde{x}_t\right)\right]E\left[\exp\left(\rho\tilde{x}_t + \tilde{\varepsilon}_{t+1}\right)\right]$$

$$= E\left[\exp\left(\left(\rho+1\right)\tilde{x}_t\right)\right]E\left[\exp\left(\tilde{\varepsilon}_{t+1}\right)\right] - E\left[\exp\left(\tilde{x}_t\right)\right]E\left[\exp\left(\rho\tilde{x}_t\right)\right]E\left[\exp\left(\tilde{\varepsilon}_{t+1}\right)\right]$$

$$= E\left[\exp\left(\tilde{\varepsilon}_{t+1}\right)\right]\left(E\left[\exp\left(\left(\rho+1\right)\tilde{x}_t\right)\right]\right) - E\left[\exp\left(\tilde{x}_t\right)\right]E\left[\exp\left(\rho\tilde{x}_t\right)\right]$$

$$= \exp\left(\frac{\sigma^2}{2}\right)\left(\exp\left(\frac{1}{2}\left(\rho+1\right)^2\sigma^2\left(\frac{1-\rho^{2t}}{1-\rho^2}\right)\right) - \exp\left(\frac{1}{2}\sigma^2\left(\frac{1-\rho^{2t}}{1-\rho^2}\right)\right)\exp\left(\frac{1}{2}\rho^2\sigma^2\left(\frac{1-\rho^{2t}}{1-\rho^2}\right)\right)\right)$$

$$= \exp\left(\frac{\sigma^2}{2}\right)\left(\exp\left(\frac{1}{2}\left(\rho^2+2\rho+1\right)\sigma^2\left(\frac{1-\rho^{2t}}{1-\rho^2}\right)\right) - \exp\left(\frac{1}{2}\left(\rho^2+1\right)\sigma^2\left(\frac{1-\rho^{2t}}{1-\rho^2}\right)\right)\right)$$

$$= \exp\left(\frac{\sigma^2}{2}\right)\exp\left(\frac{1}{2}\left(\rho^2+1\right)\sigma^2\left(\frac{1-\rho^{2t}}{1-\rho^2}\right)\right)\left(\exp\left(\rho\sigma^2\left(\frac{1-\rho^{2t}}{1-\rho^2}\right)\right)-1\right)$$

$$= \exp\left(\frac{\sigma^2}{2} + \frac{1}{2}\left(\rho^2+1\right)\sigma^2\left(\frac{1-\rho^{2t}}{1-\rho^2}\right)\right)\left(\exp\left(\rho\sigma^2\left(\frac{1-\rho^{2t}}{1-\rho^2}\right)\right)-1\right)$$

$$= \exp\left(\frac{\sigma^2}{2}\left(1+\left(\rho^2+1\right)\left(\frac{1-\rho^{2t}}{1-\rho^2}\right)\right)\right)\left(\exp\left(\rho\sigma^2\left(\frac{1-\rho^{2t}}{1-\rho^2}\right)\right)-1\right)$$

Thus $\operatorname{Cov}\left(\tilde{\lambda}_t, \tilde{\lambda}_{t+1}\right) = \exp\left(\frac{\sigma^2}{2}\left(1+\left(\rho^2+1\right)\left(\frac{1-\rho^{2t}}{1-\rho^2}\right)\right)\right)\left(\exp\left(\rho\sigma^2\left(\frac{1-\rho^{2t}}{1-\rho^2}\right)\right)-1\right)$.

Now, clearly, the first element is positive since it is an exponent. Further, since

$$\sigma^2\left(\frac{1-\rho^{2t}}{1-\rho^2}\right) > 0,$$

we know that the second element is a strictly increasing function of ρ, reaching a value of zero at $\rho = 0$. Therefore, for $\rho < 0$, the expression is negative, while for $\rho > 0$ the expression is positive.

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