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Estimating local interactions among many agents who observe their neighbors

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In various economic environments, people observe other people with whom they strategically interact. We can model such information-sharing relations as an information network, and the strategic interactions as a game on the network. When any two agents in the network are connected either directly or indirectly in a large network, empirical modeling using an equilibrium approach can be cumbersome, since the testable implications from an equilibrium generally involve all the players of the game, whereas a researcher's data set may contain only a fraction of these players in practice. This paper develops a tractable empirical model of linear interactions where each agent, after observing part of his neighbors' types, not knowing the full information network, uses best responses that are linear in his and other players' types that he observes, based on simple beliefs about the other players' strategies. We provide conditions on information networks and beliefs such that the best responses take an explicit form with multiple intuitive features. Furthermore, the best responses reveal how local payoff interdependence among agents is translated into local stochastic dependence of their actions, allowing the econometrician to perform asymptotic inference without having to observe all the players in the game or having to know the precise sampling process.

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1. Introduction

Interactions between agents—for example, through personal or business relations—generally lead to their actions being correlated. In fact, such correlated behaviors form the basis for identifying and estimating peer effects, neighborhood effects, or more generally, social interactions in the literature. (See Blume, Brock, Durlauf, and Ioannides (2010) and Durlauf and Ioannides (2010) for a review of this literature.)

Empirical modeling becomes nontrivial when one takes seriously the fact that people are often connected directly or indirectly on a large complex network, and observe some of their neighbors' types. Such strategic environments may be highly heterogeneous across agents, with each agent occupying a nearly "unique" position in the network. Information sharing potentially creates a complex form of cross-sectional dependence among the observed actions of agents, yet the econometrician typically observes only a fraction of the agents on the network, and rarely observes the entire network which governs the cross-sectional dependence structure.

The main contribution of this paper is to develop a tractable empirical model of linear interactions among agents with the following three major features. First, assuming a large game on a complex, exogenous network, our empirical model does not require the agents to observe the full network. Instead, we assume that each agent observes only a local network around herself and only part of the type information of those who are local to her.[1]

Second, our model explains strategic interdependence among agents through correlated observed behaviors. In this model, the cross-sectional local dependence structure among the observed actions reflects the network of strategic interdependence among the agents. Most importantly, unlike most incomplete information game models in the literature, our set-up allows for *information sharing on unobservables*, that is, each agent is allowed to observe his neighbors' payoff-relevant signals that are not observed by the econometrician.

Third, the econometrician does not need to observe the whole set of players in the game for inference. It suffices that he observe many (potentially) nonrandom samples of local interactions. The inference procedure that this paper proposes is asymptotically valid *independently* of the actual sampling process, as long as the sampling process satisfies certain weak conditions. Accommodating a wide range of sampling processes is useful because random sampling is rarely used for the collection of network data, and a precise formulation of the actual sampling process is often difficult in practice.

A standard approach for studying social interactions is to model them as a game, and use the game's equilibrium strategies to derive predictions and testable implications. Such an approach is cumbersome in our set-up. Since a particular realization of any agent's type affects all the other agents' equilibrium actions through a chain of information sharing, each agent needs to form a "correct" belief about the entire information graph. Apart from such an assumption being highly unrealistic, it also implies

[1] For example, a recent paper by Breza, Chandrasekhar, and Tahbaz-Salehi (2018) documents that people in a social network may lack substantial knowledge of the network and that such informational assumptions may have significant implications for the predictions of network models. Models assuming that agents possess only local knowledge have drawn interest in the literature on Bayesian learning on networks. For example, see a recent contribution by Li and Tan (2020) and references therein.

that predictions from an equilibrium that generate testable implications usually involve all the players in the game, when it is often the case that only a fraction of the players are observed in practice. Thus, an empirical analysis which regards the players in the researcher's sample as coincident with the actual set of players in the game may suffer from a lack of external validity when the target population is a large game involving many more players than those present in the actual sample.

Instead, this paper adopts an approach of behavioral modeling, where it is assumed that each agent, not knowing fully the information sharing relations, optimizes according to simple beliefs about the other players' strategies. The crucial part of our behavioral assumption is a primitive form of *belief projection* which says that each agent, not knowing the full set of information-sharing relations, projects his own beliefs about other players onto his payoff neighbors. More specifically, if agent i gives more weight to agent j than to agent k, agent i believes that each of his payoff neighbors s does the same in comparing agents j and k. Here, the "weights" represent the strategic importance of other players, and belief projection can be viewed as a rule-of-thumb for an agent who needs to form expectations of the actions of the players, not knowing who they observe. When the strategic importance of one player to another is based primarily on "vertical" characteristics such as skills or assets, the assumption of belief projection does not seem unrealistic.[2]

Our belief projection approach yields an explicit form of the best response which has intuitive features. For example, the best response is such that each agent i gives more weight to those agents with a higher *local centrality* to him, where the local centrality of agent j to agent i is said to be high if and only if a high fraction of agents whose actions affect agent i's payoff have their payoffs affected by agent j's action. Also, the best response is such that each agent responds to a change in his own type more sensitively when there are stronger strategic interactions, due to what we call the *reflection effect*. The reflection effect of player i captures the way that player i's type affects his own action through his payoff neighbors whose payoffs are affected by player i's types and actions.

The best responses reveal an explicit form of local dependence among the observed actions from which we can derive minimal conditions for feasible asymptotic inference. It turns out that the econometrician does not need to observe all the players in the game, nor does he need to know the precise sampling process. Furthermore, the best response from the belief assumption provides a testable implication for information sharing on unobservables in data. In fact, the cross-sectional correlation of residuals indicates information sharing on unobservables. (See the end of the replication file

[2]Belief projection in our paper can be viewed as connected, though loosely, to interpersonal projection studied in behavioral economics. A related behavioral concept is projection bias of Loewenstein, O'Donohue, and Rabin (2003) which refers to the tendency of a person projecting his own current taste to his future taste. See also Van Boven, Loewenstein, and Dunning (2003) who reported experimental results on the interpersonal projection of tastes onto other agents. Since an agent's belief formation is often tied to their information, belief projection is closely related to information projection in Madarász (2012), who focuses on the tendency of a person to project his information to other agents'. The main difference here is that our focus is to formulate the assumption in a way that is useful for inference using observational data on the actions of agents who interact on a network.

(Canen, Schwartz, and Song (2020)) for details on the testing procedure based on the cross-sectional correlation of residuals.)

It is instructive to compare the predictions from our behavioral model to those from an equilibrium model. When the payoff graph is comprised of multiple disjoint subgraphs that are complete, the behavioral strategies and equilibrium strategies coincide. Moreover, for a game on a general payoff graph, we show that as the rationality of agents deepens and their information expands, the behavioral strategies converge to the equilibrium strategies of an incomplete information game where each agent observes all the *sharable* types of every other agent.

We provide conditions under which the parameters are locally identified, but propose asymptotic inference in a general setting that does not require such conditions. We also investigate the finite sample properties of our asymptotic inference through Monte Carlo simulations using various payoff graphs. The results show reasonable performance of the inference procedures. In particular, the size and the power of the test for the strategic interaction parameter are good in finite samples. We apply our methods to an empirical application which studies the decision of state presence by municipalities, revisiting Acemoglu, García-Jimeno, and Robinson (2015). We consider an incomplete information game model which permits information sharing on unobservables. The fact that our best responses explicitly reveal the local dependence structure means that it is unnecessary to separately correct for spatial correlation following, for example, the procedure of Conley (1999).

The literature on social interactions often looks for evidence of interactions through correlated behaviors. For example, linear interactions models investigate correlation between the outcome of an agent i and the average outcome over agent i's neighbors. See, for example, Manski (1993), De Giorgi, Pellizzari, and Redaelli (2010), Bramoullé, Djebbari, and Fortin (2009), and Blume, Brock, Durlauf, and Ioannides (2015) for identification analysis in linear interactions models, and see Calvó-Armengol, Pattacchini, and Zenou (2009) for an application to the study of peer effects. Goldsmith-Pinkham and Imbens (2013) considered nonlinear interactions on a social network and discusses endogenous network formation. Such models often assume that the researcher observes many independent samples of such interactions, where each independent sample constitutes a game containing the entire set of the players in the game.

In the context of a complete information game, a linear interaction model on a large social network can generally be estimated without assuming independent samples. The outcome equations in such a setting frequently take the form of spatial autoregressive models, which have been actively studied in the spatial econometrics literature (Anselin (1988)). A recent study by Johnsson and Moon (2016) considers a model of linear interactions on a large social network which allows for endogenous network formation. Developing inference on a large game model with nonlinear interactions is more challenging. See Menzel (2016), Xu (2015), Song (2014), Xu and Lee (2015), and Yang and Lee (2016) for a large game model of nonlinear interactions. This large game approach is suitable when the data set does not have many independent samples of interactions. One of the

major issues in the large game approach is that the econometrician often observes only a subset of the agents from the original game of interest.[3]

Our empirical approach is based on a large game model which is close to models of linear interactions in the sense that it attempts to explain strategic interactions through the correlated behavior of neighbors. In our set-up, the cross-sectional dependence of the observed actions is not merely a nuisance that complicates asymptotic inference; it provides the very information that reveals the nature of strategic interdependence among agents. Such correlated behavior also arises in equilibrium in models of complete information games or games with types that are either privately or commonly observable. (See Bramoullé, Djebbari, and Fortin (2009) and Blume et al. (2015).) However, as emphasized before, such an approach can be cumbersome in our context of a large game primarily because the testable implications from the model typically involve the entire set of players, when in many applications the econometrician observes only a small subset of the game's players. After finishing the first draft of paper, we learned of a recent paper by Eraslan and Tang (2017) who model the interactions as a Bayesian game on a large network with private link information. They do not require the agents to observe the full network, and show identification of the model primitives adopting a Bayesian–Nash equilibrium as a solution concept. One of the major differences of our paper from theirs is that our paper permits information sharing on unobservables, so that the actions of neighboring agents are potentially correlated even after controlling for observables.

A departure from the equilibrium approach in econometrics is not new in the literature. Aradillas-Lopez and Tamer (2008) studied implications of various rationality assumptions for identification of the parameters in a game. Unlike their approach, our focus is on a large game where many agents interact with each other on a single complex network, and, instead of considering all the beliefs which rationalize observed choices, we consider a particular set of beliefs that satisfy a simple rule and yield an explicit form of best responses. (See also Goldfarb and Xiao (2011) and Hwang (2017) for empirical research adopting behavioral modeling for interacting agents.)

This paper is organized as follows. In Section 2, we introduce an incomplete information game of interactions with information sharing. This section derives the crucial result of best responses under simple belief rules. We also show the convergence of behavioral strategies to equilibrium strategies as the rationality of agents becomes higher and their information sets expand. Section 3 focuses on econometric inference. It explains the data set-up and a method for constructing confidence intervals. Section 4 investigates the finite sample properties of our inference procedure through a Monte Carlo study. Section 5 presents an empirical application on state capacity among municipalities. Section 6 concludes. Due to the space constraints, the technical proofs of the results are found in the Online Supplemental Material (Canen, Schwartz, and Song (2020)). Further materials including extensions to a model of information sharing among many

[3]Song (2014), Xu (2015), Johnsson and Moon (2016), Xu and Lee (2015), and Yang and Lee (2016) assumed that all the players in the large game are observed by the researcher. In contrast, Menzel (2016) allowed for observing i.i.d. samples from the many players, but assumes that each agent's payoff involves all the other agents' actions exchangeably.

agents over time, testing for information sharing on unobservables, and a model selection procedure for choosing among different behavioral models, are found in the replication file (Canen, Schwartz, and Song (2020)).

2. Strategic interactions with information sharing

2.1 *A model of interactions with information sharing*

Strategic interactions among a large number of information-sharing agents can be modeled as an incomplete information game. Let N be the set of a finite yet large number of players. Each player $i \in N$ is endowed with his type vector (τ_i, η_i), where η_i is a private type and τ_i a sharable type.[4] As we will elaborate later, information η_i is kept private to player i whereas τ_i is observed by his neighbors in a network which we define below.

To capture strategic interactions among players, let us introduce an *undirected* graph $G_P = (N, E_P)$, where E_P denotes the set of edges ij, $i, j \in N$ with $i \neq j$, and each edge $ij \in E_P$ represents that the action of player i affects player j's payoff.[5] We denote $N_P(j)$ to be the G_P-neighborhood of player j, that is, the collection of players whose actions affect the payoff of player j:

$$N_P(j) = \{i \in N : ij \in E_P\},$$

and let $n_P(j) = |N_P(j)|$. We define $\overline{N}_P(j) = N_P(j) \cup \{j\}$ and let $\overline{n}_P(j) = |\overline{N}_P(j)|$.

Player i choosing action $y_i \in \mathcal{Y}$ with the other players choosing $y_{-i} = (y_j)_{j \neq i}$ obtains payoff:

$$u_i(y_i, y_{-i}, \tau, \eta_i) = y_i(\tau_i + \beta_0 \overline{y}_i + \eta_i) - \frac{1}{2} y_i^2, \tag{2.1}$$

where $\tau = (\tau_i)_{i \in N}$, and

$$\overline{y}_i = \frac{1}{n_P(i)} \sum_{k \in N_P(i)} y_k,$$

if $N_P(i) \neq \emptyset$, and $\overline{y}_i = 0$ otherwise. Thus the payoff depends on other players' actions and types only through those of his G_P-neighbors. We call G_P the *payoff graph*.

The parameter β_0 measures the payoff externality among agents. As for β_0, we make the following assumption.

Assumption 2.1. $-1 < \beta_0 < 1$.

This assumption is commonly used to characterize a pure strategy equilibrium in the social interactions literature. (See, e.g., Bramoullé, Djebbari, and Fortin (2009) and Blume et al. (2015) for examples of its use.) When $\beta_0 > 0$, the game is called a game of strategic complements and, when $\beta_0 < 0$, a game of strategic substitutes.

[4] Later in a section devoted to econometric inference, we specify the sharable type τ_i to be a linear index of $(X_i', \varepsilon_i)'$, where X_i is a covariate vector observed by the econometrician and ε_i (together with the private type η_i) is not observed. Thus our framework permits *information sharing on unobservables* in the sense that "neighbors" of an agent i observe ε_i.

[5] A graph $G = (N, E)$ is undirected if $ij \in E$ whenever $ji \in E$ for all $i, j \in N$.

Let us introduce information sharing relations in the form of a *directed* graph (or a network) $G_I = (N, E_I)$ on N so that each ij in E_I represents the edge *from* player i *to* player j, where the presence of edge ij joining players i and j indicates that τ_i is observed by player j. Hence the presence of an edge ij between agents i and j represents information flow from i to j. This paper calls graph G_I the *information graph*. For each $j \in N$, define

$$N_I(j) = \{i \in N : ij \in E_I\},$$

that is, the set of G_I-neighbors observed by player j.[6] Also let $\overline{N}_I(i) = N_I(i) \cup \{i\}$, and $\overline{n}_I(i) = |\overline{N}_I(i)|$.

In this paper, we do not assume that each agent knows the whole information graph G_I and the payoff graph G_P. To be precise about each agent's information set, let us introduce some notation. For each $i \in N$, we set $\overline{N}_{P,1}(i) = \overline{N}_P(i)$ and $\overline{N}_{I,1}(i) = \overline{N}_I(i)$, and for $m \geq 2$, define recursively

$$\overline{N}_{P,m}(i) = \bigcup_{j \in \overline{N}_P(i)} \overline{N}_{P,m-1}(j), \quad \text{and} \quad \overline{N}_{I,m}(i) = \bigcup_{j \in \overline{N}_I(i)} \overline{N}_{I,m-1}(j).$$

Thus $\overline{N}_{P,m}(i)$ denotes the set of players which consist of player i and those players who are connected to player i through at most m edges in G_P, and similarly with $\overline{N}_{I,m}(i)$. Also, define $N_{P,m}(i) = \overline{N}_{P,m}(i) \setminus \{i\}$ and $N_{I,m}(i) = \overline{N}_{I,m}(i) \setminus \{i\}$.

For each player $i \in N$, let us introduce a local payoff graph $G_{P,m}(i) = (\overline{N}_{P,m}(i), E_{P,m}(i))$, where for $k_1, k_2 \in \overline{N}_{P,m}(i)$, $k_1k_2 \in E_{P,m}(i)$ if and only if $k_1k_2 \in E_P$. Define for $m \geq 1$,[7]

$$\mathcal{I}_{i,m-1} = \big(G_{P,m+1}(i), \overline{N}_{I,m}(i), \tau_{\overline{N}_{I,m}(i)}, \eta_i\big), \tag{2.2}$$

where $\tau_{\overline{N}_{I,m}(i)} = (\tau_j)_{j \in \overline{N}_{I,m}(i)}$. We use $\mathcal{I}_{i,m}$ to represent the information set of agent i. For example, when agent i has $\mathcal{I}_{i,0}$ as his information set, it means that agent i knows the payoff subgraph $G_{P,2}$ among the agents $\overline{N}_{P,2}(i)$, the set of agents whose types he observes (i.e., $N_I(i)$), and his own private signal η_i. As for the payoff graph and information graph, we make the following assumption.

ASSUMPTION 2.2. *For each $i \in N$ and $m \geq 1$,*

$$\overline{N}_{P,m+1}(i) \subset \overline{N}_{I,m}(i).$$

This assumption requires for example that an agent with information $\mathcal{I}_{i,0}$ observes their G_P neighbors and their payoff relevant neighbors. The assumption on G_I only requires what each set $\overline{N}_{I,m}(i)$ should at least include but not what it should exclude. Hence all the results of this paper carry through even if we have $\overline{N}_{I,m}(i) = N$ for all $i \in N$, as in a complete information game. In other words, the incomplete information feature of our game is *permitted but not required* for our framework.

[6] More precisely, the neighbors in $N_I(j)$ are called *in-neighbors* and $n_I(j) = |N_I(j)|$ *in-degree*. Throughout this paper, we simply use the term neighbors and degrees, unless specified otherwise.

[7] The graph $G_{P,m}(i)$ is an *induced subgraph* of G_P induced by the vertex set $\overline{N}_{P,m}(i)$. Note also that while $\overline{N}_{P,1}(i) \subset \overline{N}_{P,2}(i)$, this does not imply that a player who knows the set $\overline{N}_{P,2}(i)$ knows what the set $\overline{N}_{P,1}(i)$ is. Our information set assumption requires them to know the local graph $G_{P,m}(i)$ rather than just $\overline{N}_{P,m}(i)$.

2.2 *Predictions from rationality*

Each player chooses a strategy that maximizes his expected payoff according to his beliefs. Given player i's strategy, information set $\mathcal{I}_i$, and his beliefs on the strategy of other players $s^i_{-i} = (s^i_k)_{k\in N\setminus\{i\}}$, the (interim) expected payoff of player i is defined as

$$U_i\big(s_i, s^i_{-i}; \mathcal{I}_i\big) = \mathbf{E}\big[u_i\big(s_i(\mathcal{I}_i), s^i_{-i}(\mathcal{I}_{-i}), \tau, \eta_i\big)|\mathcal{I}_i\big],$$

where $s^i_{-i}(\mathcal{I}_{-i}) = (s^i_k(\mathcal{I}_k))_{k\in N\setminus\{i\}}$, $\mathcal{I}_{-i} = (\mathcal{I}_k)_{k\neq i}$ and $\tau = (\tau_i)_{i\in N}$. A *best response* s_i^{BR} of player i corresponding to the strategies s^i_{-i} of the other players as expected by player i is such that for any strategy s_i,

$$U_i\big(s_i^{\mathsf{BR}}, s^i_{-i}; \mathcal{I}_i\big) \geq U_i\big(s_i, s^i_{-i}; \mathcal{I}_i\big), \quad \text{a.e.}$$

The quadratic payoff function and the information structure of the game implies that if player i has information set $\mathcal{I}_i$ and believes that each of her G_P-neighbors, say, k, plays a strategy $s^i_k(\mathcal{I}_k)$, her best response is given by

$$s_i^{\mathsf{BR}}(\mathcal{I}_i) = \tau_i + \frac{\beta_0}{n_P(i)} \sum_{k\in N_P(i)} \mathbf{E}\big[s^i_k(\mathcal{I}_k)|\mathcal{I}_i\big] + \eta_i. \tag{2.3}$$

This implies that the best responses will be linear in types τ_j as long as the conditional expectation is.

In order to generate predictions, one needs to deal with the beliefs (i.e., $s^i_k(\mathcal{I}_k)$) in the conditional expectation. There are three approaches. The first approach is an equilibrium approach, where we take the predicted strategies as a set of best response strategies s_i^{BNE} such that for any strategy s_i,

$$U_i\big(s_i^{\mathsf{BNE}}, s^{\mathsf{BNE}}_{-i}; \mathcal{I}_i\big) \geq U_i\big(s_i, s^{\mathsf{BNE}}_{-i}; \mathcal{I}_i\big), \quad \text{a.e.} \tag{2.4}$$

Hence in equilibrium strategies, each player believes that the other players' strategies coincide with the best response strategies by the agents in equilibrium. The second approach, rationalizability, considers all strategies that are rationalizable given some belief. The third approach is a behavioral approach where one considers a set of simple behavioral assumptions on the beliefs and focuses on the best responses to these beliefs.

There are pros and cons with each of the three approaches. The equilibrium approach requires that the beliefs of all the players be "correct" in equilibrium. However, since each player i generally does not know who each of his G_P-neighbors observes, a Bayesian player in an incomplete information game with rational expectations would need to know the distribution of the entire information graph G_I (or at least have a common prior on the information graph commonly agreed upon by all the players) to form a "correct" belief given his information. Given that the players are only partially observed and G_I is rarely observed with precision, producing a testable implication from such an equilibrium model appears far from a trivial task.

The rationalizability approach can be used to relax this rational expectations assumption by eliminating the requirement that the beliefs be correct. Such an approach

considers all the predictions that are rationalizable given *some* beliefs. However, the set of predictions from rationalizability can potentially be large and may fail to produce sharp predictions useful in practice.

This paper takes the approach of behavioral modeling. We adopt a set of simple behavioral assumptions on players' beliefs which can be incorrect from the viewpoint of a person with full knowledge of the distribution of the information graph, yet useful as a rule-of-thumb for an agent in a complex decision-making environment such as the one in our model. As we shall see later, this approach can give a sharp prediction that is intuitive and analytically tractable. Furthermore, the best responses from this approach coincide with equilibrium strategies for a special class of payoff graphs, and converge to equilibrium strategies for any payoff graph as the rationality of agents becomes deeper and their information set expands.

2.3 *Belief projection and best responses*

2.3.1 *A game with complete payoff subgraphs* Let us consider first a special case where the payoff graph $G_P = (N, E_P)$ is such that N is partitioned into subsets $N_1, \ldots, N_Q$, and for all $i, j \in N$ with $i \neq j$, $ij \in E_P$ if and only if $i, j \in N_q$ for some $q = 1, \ldots, Q$. In this game, there are multiple strategically disjoint subgames and agents in each subgame observe all other players' sharable types τ_j's in the subgame, but do not observe their private types η_j.[8] In this case, there exists a unique Bayesian–Nash equilibrium where the equilibrium strategies take the explicit form of a linear strategy: for each player i,

$$s_i(\mathcal{I}_{i,0}) = \sum_{j \in \overline{N}_P(i)} \tau_j w_{ij} + \eta_i,$$

and w_{ij} are weights such that for all i, j, k, ℓ in the same cluster,

$$w_{kk} = w_{ii}, \quad \text{and} \quad w_{k\ell} = w_{ij}, \quad \text{whenever } i \neq j \text{ and } k \neq \ell. \tag{2.5}$$

Thus the equilibrium is within-cluster symmetric in the sense that the equilibrium strategy is the same across all the agents in the same cluster. However, this symmetric equilibrium does not extend to a general payoff graph G_P.

2.3.2 *Belief projection* Our approach uses a weaker version of the symmetry restrictions (2.5) to specify initial beliefs, so that a best response function exists uniquely for any payoff graph configuration.[9] More specifically, we introduce the following symmetry restrictions on the belief formation.

Definition 2.1. We say that a player i with information set $\mathcal{I}_{i,0}$ *does BP* (*Belief projection*), if she believes that each of her G_P-neighbors, say, k, plays a linear strategy as

$$s_k^i(\mathcal{I}_{k,0}) = \sum_{j \in \overline{N}_P(k)} \tau_j w_{kj}^i + \eta_k, \tag{2.6}$$

[8] Each subgame is a special case of the Bayesian game in Blume et al. (2015).

[9] A best response function satisfying the symmetry restrictions in (2.5) may not exist for a general payoff graph G_P.

for some nonnegative weights w^i_{kj} in (2.6) that satisfy the following conditions:

(BP-a) $w^i_{kk} = w_{ii}$ and $w^i_{kj} = w_{ij}$ for $j \neq i, k$;

(BP-b) $w^i_{ki} = w_{ik}$.

Condition (2.6) assumes that player i believes that player k responds only to the types of those players $j \in N_P(k)$. This is a rule-of-thumb for player i to form expectations about player k while not observing $N_I(k)$.

In forming beliefs about other players' strategies, not knowing who they observe, BP assumes that each player *projects his own beliefs about himself and other players onto* his G_P neighbors, as epitomized by Conditions (BP-a)–(BP-b). The ranking of weights w_{ij} over j represents the relative strategic importance of player j to player i.

More specifically, Condition (BP-a) says that each player i believes that the weight his G_P-neighbor k attaches to himself or player $j \in N_P(k)$ is the same as the weight player i attaches to himself or the same player j. Thus player i's belief on his G_P neighbor k's weight to player j is formed in reference to his own weight to player j. In other words, without any information on how his G_P-neighbors rank other players, each player simply takes himself as a benchmark to form beliefs about his G_P neighbors' ranking of other players. This assumption does not seem unreasonable if the weights w_{ij} are based on the vertical characteristics such as skills or assets of agent j. Condition (BP-b) imposes a symmetry restriction that player i believes that player k gives the same weight to player i as the weight player i gives to player k.[10]

2.3.3 *Belief projection as a prior specification for a Bayesian decision maker* The best response of a player with quadratic utility and beliefs as in Definition 2.1 can be viewed as arising from a decision maker with Bayesian rationality with a prior that satisfies certain symmetry restrictions. More specifically, let $W = (W_{k,j})_{k,j \in N}$ be a weight matrix such that $W_{k,j}$ is the weight player k attaches to player j. Each player i believes that other players k play a linear strategy, say, $s_k(\mathcal{I}_{k,0}; w_k)$, $w_k = (w_{kj})_{j \in N}$, of the form in (2.6) of Definition 2.1. (We are simply making explicit the dependence of the strategy on w_k in our notation.) Suppose that each agent i is given information set $\mathcal{I}_{i,0}$ and a prior Q_i over $(\mathcal{I}_{j,0})_{j \in N}$ and W in the game. We assume that W and $(\mathcal{I}_{j,0})_{j \in N}$ are independent under Q_i.[11] Then the decision maker with Bayesian rationality proceeds as follows.

(Step 1) A mediator (or Nature) suggests a weight vector w_i to each player i.

(Step 2) Player i forms a posterior mean of his payoff given w_i and $\mathcal{I}_{i,0}$.

(Step 3) Unless there is an action that gives a strictly better posterior mean of his payoff than the action $s_i(\mathcal{I}_{i,0}; w_i)$, the player accepts w_i and chooses the latter action.

[10]It is important to note that we do not impose BP directly on the strategies of the players as predicted outcomes of the game. Instead, BP is used as an initial input for each player to form best response strategies with limited information on networks. It is these best response strategies that constitute the predicted outcomes from the game.

[11]This independence simply reflects that the strategies as a function of signals are distinct from the signals themselves.

Thus, the best response of this Bayesian decision maker is given by $s_i(\mathcal{I}_{i,0}; w_i)$ such that

$$\begin{aligned}&\mathbf{E}_{Q_i}\big[u_i\big(y_i, s_{-i}(\mathcal{I}_{-i,0}; W_{-i}); \mathcal{I}_{i,0}\big)|W_i = w_i, \mathcal{I}_{i,0}\big]\\ &\quad\le \mathbf{E}_{Q_i}\big[u_i\big(s_i(\mathcal{I}_{i,0}; W_i), s_{-i}(\mathcal{I}_{-i,0}; W_{-i}); \mathcal{I}_{i,0}\big)|W_i = w_i, \mathcal{I}_{i,0}\big], \quad \text{for all } y_i \in \mathcal{Y},\end{aligned}$$

where $\mathbf{E}_{Q_i}$ denotes the conditional expectation under Q_i.

Now, the belief projection of a player corresponds to a set of restrictions on the prior Q_i, where we specify Q_i as a Gaussian prior (centered at zero) over the weight matrix W such that the following restrictions hold for all $i = 1, \dots, n$ and $k \in N_P(i)$:

(a) $\mathbf{E}_{Q_i}[W_{kk}|W_i] = W_{ii}$ and $\mathbf{E}_{Q_i}[W_{kj}|W_i] = W_{ij}$ for all $j \in N_P(k)$.

(b) $\mathbf{E}_{Q_i}[W_{ki}|W_i] = W_{ik}$.

(c) $\mathbf{E}_{Q_i}[W_{kj}|W_i] = 0$ for all $j \notin \overline{N}_P(k)$.

Then it is not hard to see that the best response of this Bayesian player is the same as that of a simple type agent who behaves according to the belief projection assumption. (See the Online Supplemental Material for details.) The restriction (a) says that player i *projects* his weight W_{ii} and W_{ij} to W_{kk} and W_{kj}. To see the meaning of this restriction, suppose that

$$\mathrm{Var}_{Q_i}(W_{kk}) = \mathrm{Var}_{Q_i}(W_{ii}),$$

and that $\mathbf{E}_{Q_i}[W_{kk}|W_i] = \mathbf{E}_{Q_i}[W_{kk}|W_{ii}]$. Then the first statement of (a) requires that W_{kk} and W_{ii} are perfectly positively correlated under Q_i. The perfect correlation seems the only reasonable specification in this setting because if W_{kk} and W_{ii} are not perfectly correlated, it yields the odd implication that player i believes W_{kk} to be less than W_{ii} for all $k \in N_P(i)$ in absolute value. A similar observation can be made for the second statement of (a). The restriction (b) imposes that the weight that agent k is expected to attach to i is the same as the weight that agent i gives to the agent k. The restriction (c) means that player i does not consider the weight of player k attaches to player j when j is outside of $\overline{N}_P(k)$, because player i may not even know who j is, and from player i's perspective, player j can be any player in the large population outside of $\overline{N}_P(k)$.

2.3.4 *Best responses* Let us first define the type of players and games for which we derive best responses.

Definition 2.2. (i) Player $i \in N$ is said to be *of simple type* if she has information set $\mathcal{I}_{i,0}$, and does BP. Let Γ_0 denote the game populated by n players who are of simple type and have payoff functions in (2.1) and payoff graph G_P.

(ii) For each $m \ge 1$, player $i \in N$ is said to be *of the mth order sophisticated type*, if she has information set $\mathcal{I}_{i,m}$ and believes that the other players play the best response strategies from Γ_{m-1}. Let Γ_m denote the game populated by n players who are of the mth order sophisticated type and have payoff functions in (2.1) and payoff graph G_P.

The higher-order sophisticated type agents are analogous to agents in level-k models in behavioral economics. (See Chapter 5 of Camerer (2003) for a review.[12]) In experiments, a level-0 player chooses an action without considering strategic interactions, making them much simpler than our simple types. Our simple-type player already considers strategic interdependence and forms a best response. On the other hand, the level-k models allows the agents to be of different orders of rationality within the same game. In our set-up centered on observational data, identification of the unknown proportion of each rationality type appears far from trivial. Hence in this paper, we consider a game where all the agents have the same order of sophistication.[13]

Unlike in level-k models, the difference between the simple type and the first-order sophisticated type lies not only on their degree of rationality but also on their information set. In particular, the information requirements are stronger for more sophisticated agents. For example, the first-order sophisticated type knows who belongs to $\overline{N}_{P,3}(i)$, whereas a simple-type does not need to.

Below we give a unique explicit form of best responses from game Γ_0. First, define for all $i, j \in N$ with $i \neq j$,

$$\lambda_{ij} \equiv \frac{1}{1-\beta_0 c_{ij}}, \quad \text{and} \quad \overline{\lambda}_i \equiv \frac{1}{n_P(i)} \sum_{j \in N_P(i)} \lambda_{ij},$$

where

$$c_{ij} \equiv \frac{\left|N_P(i) \cap N_P(j)\right|}{n_P(i)}. \tag{2.7}$$

The theorem below gives the explicit form of the best response in this game.

THEOREM 2.1. *Suppose that Assumptions 2.1–2.2 hold and for each $i \in N$, and any $k \neq i$, $\mathbf{E}[\eta_k|\mathcal{I}_{i,0}] = 0$. Then each player i's best response s_i^{BR} from game Γ_0 takes the form $s_i^{\mathsf{BR}} = s_i^{[0]}$ with*

$$s_i^{[0]}(\mathcal{I}_{i,0}) = w_{ii}^{[0]}\tau_i + \sum_{j \in N_P(i)} w_{ij}^{[0]}\tau_j + \eta_i,$$

where, if $n_P(i) \geq 1$,

$$w_{ii}^{[0]} \equiv 1 + \frac{\beta_0^2 \overline{\lambda}_i}{n_P(i) - \beta_0^2 \overline{\lambda}_i}, \quad \textit{and}$$

$$w_{ij}^{[0]} \equiv \frac{\beta_0 \lambda_{ij}}{n_P(i)} w_{ii}^{[0]}, \quad \textit{for } i \neq j,$$

and if $n_P(i) = 0$, $w_{ii}^{[0]} = 1$ and $w_{ij}^{[0]} = 0$.

[12]Note that level-k models in behavioral economics are different from k-rationalizability models of Bernheim (1984) and Pearce (1984) which are studied by Aradillas-Lopez and Tamer (2008) in the context of econometrics.

[13]See Gillen (2010) and An (2017) for an application of level-k models to observational data from first-price auctions. One of the major distinctions between our model and their level-k models is that we focus on a set-up of a single large game populated by many players occupying strategically heterogenous positions, whereas their research centers on a set-up where the econometrician observes the same game played by a fixed number of agents many times.

It is worth noting that the unique best response in Theorem 2.1 always exists regardless of the configurations of the payoff graph G_P. (Indeed, we always have $n_P(i) - \beta_0^2\overline{\lambda}_i > 0$ as long as $\beta_0 \in (-1, 1)$. See Lemma A.1 in the Online Supplemental Material.) Furthermore, the best responses have several intuitive features. First, note that the behavioral strategies $s_i^{[0]}(\mathcal{I}_{i,0})$ maintain strategic interactions to be local around each player's G_P-neighbors, regardless of the magnitude of β_0, so that a player can have a strong interaction with his G_P-neighbors without being influenced by a change in the type of a far-away player.[14]

Second, the best response captures the network externality in an intuitive way. The quantity c_{ij} measures the proportion of player i's G_P-neighbors whose payoffs are influenced by the type and action of player j. Hence if $c_{ij} < c_{ik}$, player k is "strategically more important" to player i than player j. Note that λ_{ij} is an increasing function of c_{ij} with its slope increasing in β_0. We take λ_{ij} to represent the strategic *local centrality* of player j to player i. Then we have

$$\begin{aligned} \frac{\partial s_i^{[0]}(\mathcal{I}_{i,0})}{\partial \tau_i} &= 1 + \frac{\beta_0^2\overline{\lambda}_i}{n_P(i) - \beta_0^2\overline{\lambda}_i} \quad \text{and} \\ \frac{\partial s_i^{[0]}(\mathcal{I}_{i,0})}{\partial \tau_j} &= \frac{\beta_0\lambda_{ij}}{n_P(i)}\left(1 + \frac{\beta_0^2\overline{\lambda}_i}{n_P(i) - \beta_0^2\overline{\lambda}_i}\right), \quad \text{for } j \in N_P(i), \end{aligned} \tag{2.8}$$

both of which measure the response of actions of agent i to a change in the observed type change of his own and his G_P-neighbors. The second quantity captures the *network externality* in the strategic interactions.

The network externality for agent i from a particular agent j decreases in $n_P(i)$ and increases in β_0. More importantly, the network externality from one player to another is heterogeneous, depending on each player's "importance" to others in the payoff graph. This is seen from the network externality (2.8) being an increasing function of agent j's local centrality to agent i, that is, λ_{ij}, when the game is that of strategic complements (i.e., $\beta_0 > 0$). In other words, the larger the fraction of agent i's G_P-neighbors whose payoff is affected by agent j's action, the higher the network externality of agent i from agent j's type change becomes.

It is interesting to note that the network externality for agent i with respect to his own type τ_i is greater than 1. We call the additive term in (2.8),

$$\frac{\beta_0^2\overline{\lambda}_i}{n_P(i) - \beta_0^2\lambda_i},$$

the *reflection effect* which captures the way player i's type affects his own action through his G_P neighbors whose payoffs are affected by player i's types and actions. The reflection effect arises because each agent, in decision making, considers the fact that his type affects other G_P-neighbors' decision making. When there is no payoff externality (i.e.,

[14]This prediction is in contrast with that from the equilibrium strategies of a complete information version of the game. According to the equilibrium strategies, the influence of one player can reach a far-away player when β_0 is high; see Section 2.4.

$\beta_0 = 0$), the reflection effect is zero. However, when there is a strong strategic interactions or when a majority of player i's G_P-neighbors have a small G_P-neighborhood (i.e., for a majority of $j \in N_P(i)$, $N_P(j)$'s in the definition of c_{ij} in (2.7) have few elements), the reflection effect is large.

When G_P consists of disconnected complete subgraphs as in (2.3.1), the best responses $s_i^{[0]}$ coincide with Bayesian–Nash equilibrium strategies. More specifically, let $s_i^{\mathsf{BNE}}(\mathcal{I}_{i,0})$ be the Bayesian–Nash equilibrium strategies which satisfy (2.4). Then we can show that

$$s_i^{\mathsf{BNE}}(\mathcal{I}_{i,m}) = w_{ii}^{\mathsf{BNE}}\tau_i + \frac{\beta_0}{n_P(i)} \sum_{j \in N_P(i)} w_{ij}^{\mathsf{BNE}}\tau_j + \eta_i, \tag{2.9}$$

where

$$w_{ii}^{\mathsf{BNE}} \equiv 1 + \frac{\beta_0^2}{\big(n_P(i) + \beta_0\big)(1 - \beta_0)}, \quad \text{and}$$

$$w_{ij}^{\mathsf{BNE}} \equiv \frac{\beta_0}{\big(n_P(i) + \beta_0\big)(1 - \beta_0)}, \quad \text{for } i \neq j.$$

In this case with disconnected complete subgraphs, $n_P(i)$'s are all equal for i's in the same cluster, and $c_{ij} = (n_P(i) - 1)/n_P(i)$, yielding

$$\lambda_{ij} = \frac{n_P(i)}{n_P(i) - \beta_0\big(n_P(i) - 1\big)}, \quad \text{for all } j \in N_P(i).$$

Using this, it is not hard to check that

$$w_{ii}^{[0]} = w_{ii}^{\mathsf{BNE}}, \quad \text{and} \quad w_{ij}^{[0]} = w_{ij}^{\mathsf{BNE}}.$$

(See the Online Supplemental Material for the derivations.)

The following theorem shows that each Γ_m yields a unique, explicit form of best responses.

THEOREM 2.2. *Suppose that Assumptions 2.1–2.2 hold and for each $i \in N$ and $k \neq i$, $\mathbf{E}[\eta_k|\mathcal{I}_{i,m}] = 0$ for $m \geq 1$. Then each player i's best response s_i^{BR} from game Γ_m takes the form $s_i^{\mathsf{BR}} = s_i^{[m]}$ with*

$$s_i^{[m]}(\mathcal{I}_{i,m}) = w_{ii}^{[m]}\tau_i + \sum_{j \in N_{P,m+1}(i)} w_{ij}^{[m]}\tau_j + \eta_i,$$

where

$$w_{ii}^{[m]} = 1 + \frac{\beta_0}{n_P(i)} \sum_{k \in N_P(i)} w_{ki}^{[m-1]} \quad \textit{and}$$

$$w_{ij}^{[m]} = \frac{\beta_0}{n_P(i)} \sum_{k \in N_P(i)} w_{kj}^{[m-1]} 1\big\{j \in \overline{N}_{P,m}(k)\big\}.$$

As compared to game Γ_0, game Γ_1 predicts outcomes with broader network externality. Indeed, when $m = 1$,

$$s_i^{[1]}(\mathcal{I}_{i,1}) = \Bigg(1 + \frac{\beta_0}{n_P(i)} \sum_{k \in N_P(i)} w_{ki}^{[0]}\Bigg)\tau_i + \sum_{j \in N_{P,2}(i)} \Bigg(\frac{\beta_0}{n_P(i)} \sum_{k \in N_P(i)} w_{kj}^{[0]} 1\big\{j \in \overline{N}_P(k)\big\}\Bigg)\tau_j + \eta_i. \tag{2.10}$$

For example, the types of neighbors whose actions do not affect player i's payoff can affect his best response. More specifically, note that for $j \in N_{P,2}(i) \setminus N_P(i)$,

$$\frac{\partial s_i^{[1]}(\mathcal{I}_{i,1})}{\partial \tau_j} = \frac{\beta_0}{n_P(i)} \sum_{k \in N_P(i)} 1\big\{j \in \overline{N}_P(k)\big\} w_{kj}^{[0]}.$$

The externality from player j to player i is strong when player j has a high local centrality λ_{kj} to a large fraction of player i's G_P-neighbors k.

2.4 *Comparing equilibrium strategies and behavioral strategies*

2.4.1 *Convergence of behavioral strategies to equilibrium strategies* We show that as the information set expands and the order of sophistication becomes higher, the behavioral strategies converge to the equilibrium strategies from a game where all players observe all other players' sharable types. Let Γ_∞ be the game where players have the same payoff function and the same payoff graph as in Γ_0 except that the information set for each player i is given by $\mathcal{I}_{i,\infty} = (G_P, \tau, \eta_i)$. Thus each player i knows the whole payoff graph G_P, all sharable types, $\tau = (\tau_i)_{i \in N}$, and private information η_i. (This information structure is similar to Blume et al. (2015).) Let $s^{\mathsf{BNE}} = (s_i^{\mathsf{BNE}})_{i \in N}$ be the Bayesian–Nash equilibrium strategy profile from the game Γ_∞.

Below, we give a theorem which shows that the sequence of behavioral strategies $s_i^{[m]}$ converges to the equilibrium strategies s_i^{BNE} as $m \to \infty$.

THEOREM 2.3. *Suppose that the conditions of Theorem 2.1 hold and that*

$$\max_{i \in N} \mathbf{E}\big[\|\tau_i\|^2\big] < \infty. \tag{2.11}$$

Then, as $m \to \infty$,

$$\mathbf{E}\Big[\max_{i \in N}\big(s_i^{[m]}(\mathcal{I}_{i,m}) - s_i^{\mathsf{BNE}}(\mathcal{I}_{i,\infty})\big)^2\Big] \to 0.$$

Theorem 2.3 shows that as the order of sophistication deepens, the best response strategies from the behavioral model become closer to the equilibrium strategies. It is not hard to check that $s_i^{[m]}(\mathcal{I}_{i,m}) = s_i^{[m]}(\mathcal{I}_{i,\infty})$, that is, the best response remains the same if we expand the information set $\mathcal{I}_{i,m}$ to $\mathcal{I}_{i,\infty}$. Therefore, the convergence in Theorem 2.3 can be viewed as the convergence of the best responses $s_i^{[m]}(\mathcal{I}_{i,\infty})$ to equilibrium strategies $s_i^{\mathsf{BNE}}(\mathcal{I}_{i,\infty})$ while the information set is fixed to be $\mathcal{I}_{i,\infty}$.

2.4.2 *Comparison in terms of network externality* We compare the behavioral strategies and equilibrium strategies in terms of network externality which measures how sensitively an agent's action responds to a change in her neighbor's types. We also compare how this network externality changes as the network grows. Let Y_i be the observed outcome of player i as predicted from either of the two game models. For simplicity, we remove η_i's from the models so that the game Γ_∞ now becomes a complete information game.

The complete information game gives the following prediction for action Y_i of agent i:

$$Y_i = \frac{\beta_0}{n_P(i)} \sum_{j \in N_P(i)} Y_j + \tau_i,$$

where Y_i denotes the action of player i in equilibrium. Then the reduced form for Y_i's can be written as

$$y = (I - \beta_0 A)^{-1} \tau, \tag{2.12}$$

where $y = (Y_1, \ldots, Y_n)'$, $\tau = (\tau_1, \ldots, \tau_n)'$, and A is a row-normalized adjacency matrix of the payoff graph G_P, that is, the (i, j)-th entry of A is $1/n_P(i)$ if $j \in N_P(i)$ and zero otherwise. Thus when β_0 is close to one (i.e., the local interaction becomes strong), the equilibrium outcome can exhibit extensive cross-sectional dependence.

On the other hand, our behavioral model predicts the following:

$$Y_i = \left(1 + \frac{\beta_0^2 \overline{\lambda}_i}{n_P(i) - \beta_0^2 \overline{\lambda}_i}\right)\left(\tau_i + \sum_{j \in N_P(i)} \frac{\beta_0 \lambda_{ij}}{n_P(i)} \tau_j\right),$$

which comes from Theorem 2.1 without η_i's. When we compare this with (2.12), it is clear that the cross-sectional dependence structure of our behavioral model is different from that from the complete information equilibrium model. In the case of the complete information equilibrium model, it is possible that two actions Y_i and Y_j between two agents i and j can be correlated even if i and j are very far from each other in graph G_P. However, the cross-sectional dependence structure of the actions from the behavioral model closely follows the graph G_P: Y_i and Y_j can be correlated only if their G_P neighbors overlap.

For comparison purposes, for a given strategy $s_i(\mathcal{I}_i)$ for an agent i with information set $\mathcal{I}_i$, we introduce the average network externality (ANE):

$$\frac{1}{n} \sum_{j \in N} \sum_{i \in N : i \neq j} \frac{\partial s_i(\mathcal{I}_i)}{\partial \tau_j}. \tag{2.13}$$

The ANE measures the average impact of a change in the neighbors' type on the actions of the player. The ANE from equilibrium strategies of the complete information game is $\frac{1}{n} \sum_{j \in N} \sum_{i \in N : i \neq j} [(I - \beta_0 A)^{-1}]_{ij}$, where $[(I - \beta_0 A)^{-1}]_{ij}$ denotes the (i, j)-th entry of the matrix $(I - \beta_0 A)^{-1}$.

We consider the average of the ANE's over simulated payoff graphs. For the payoff graph G_P, we considered two different models for random graph generation. The first

TABLE 1. The characteristics of the payoff graphs.

	Erdős–Rényi			Barabási–Albert		
	Network A	Network B	Network C	Network A	Network B	Network C
n	164.9	783.4	3116.8	236.1	1521.0	4773.8
d_{mx}	11.14	12.74	14.12	70.00	124.4	135.4
d_{av}	2.046	2.307	3.198	1.563	2.057	2.568

Note: This table gives average characteristics of the payoff graphs, G_P, used in the simulation study, where the average was over 50 simulations. d_{av} and d_{mx} denote the average and maximum degrees of the payoff graphs.

kind of random graphs are Erdős–Rényi (ER) random graph with the probability equal to $5/n$ and the second kind of random graphs are Barabási–Albert (BA) random graph such that beginning with an Erdős–Rényi random graph of size 20 with each link forming with equal probability $1/19$ and grows by including each new node with two links formed with the existing nodes with probability proportional to the degree of the nodes.

For each random graph, we first generate a random graph of size 10,000, and then construct three subgraphs A, B, C such that network A is a subgraph of network B and the network B is a subgraph of network C. We generate these subgraphs as follows. First, we take a subgraph A to be one that consists of agents within distance k from agent $i = 1$. Then network B is constructed to be one that consists of the neighbors of the agents in network A and network C is constructed to be one that consists of the neighbors of the agents in network B. For an ER random graph, we took $k = 3$ and for a BA random graph, we took $k = 2$. We repeated the process 50 times to construct an average behavior of network externality as we increase the network. Table 1 shows the average network sizes and degree characteristics as we move from Networks A, B to C.

The ANEs from the equilibrium strategies from game Γ_∞, and the behavioral strategies from games Γ_m as m becomes higher are shown in Figures 1 and 2.[15] First, as m becomes larger, the ANEs from Γ_∞ and those from Γ_m get closer, as predicted by Theorem 2.3. Furthermore, the ANEs from the behavioral model are similar to that from the equilibrium model especially when β_0 is between -0.5 and 0.5. Finally, the network externalities from the games with simple type players are somewhat sensitive to the size of the networks when β_0 is very high or very low. This sensitivity is reduced substantially when we consider the game with first-order sophisticated agents. Finally, network externality tends to be much higher for equilibrium models than the behavioral models when β_0 is high. Hence using our behavioral approach as a proxy for an equilibrium approach makes sense only when strategic interdependence is not too high.

3. ECONOMETRIC INFERENCE

3.1 *General overview*

3.1.1 *Partial observation of interactions* A large network data set is often obtained through a nonrandom sampling process (see, e.g., Kolaczyk (2009)). The actual sampling

[15] We provide conditions for the local identification of β in Section 3.1.3 below.

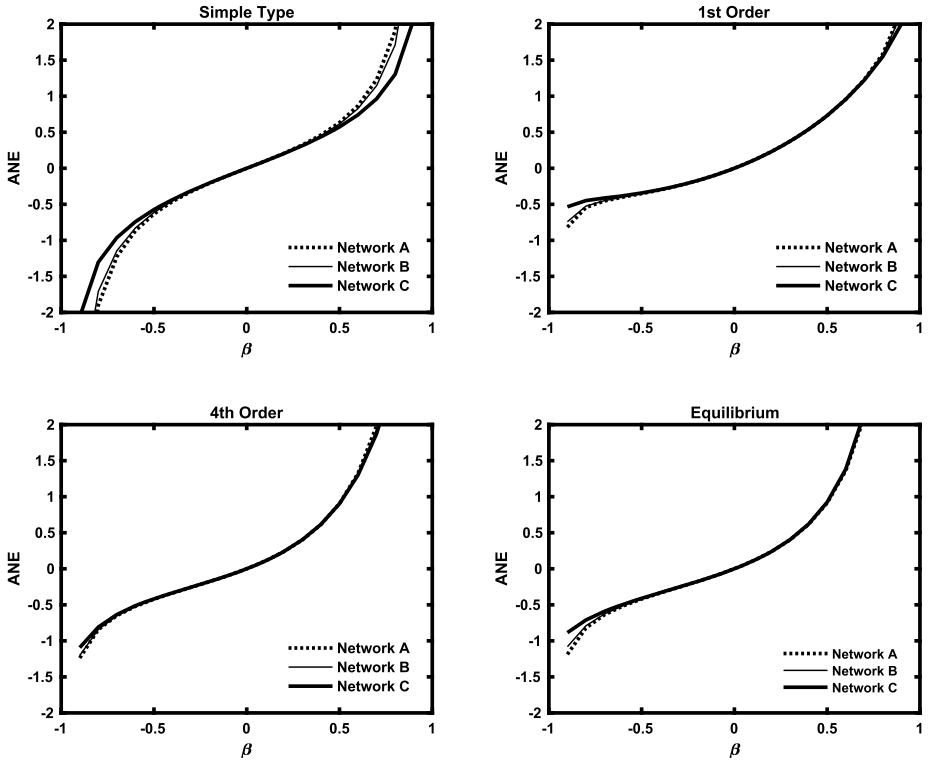

FIGURE 1. The average network externality comparison between equilibrium and behavioral models: Erdős–Rényi graphs.

process of network data is often unknown to the researcher. Our approach of empirical modeling can be useful in a situation where only a fraction of the players are observed through a certain nonrandom sampling scheme that is not precisely known to the researcher. In this section, we make explicit the data requirements for the econometrician and propose inference procedures. We mainly focus on the game where all the players in the game are of simple type. The inference for games with agents of first-order sophisticated type is found in the replication file of Canen, Schwartz, and Song (2020).

Suppose that the original game of interactions consists of a large number of agents whose set we denote by N. Let the set of players be on a payoff graph G_P and an information graph G_I, facing the strategic environment as described in the preceding section. Denote the best response as an observed dependent variable Y_i: for $i \in N$,

$$Y_i = s_i^{[0]}(\mathcal{I}_{i,0}),$$

where the sharable type τ_i is specified as

$$\tau_i = X_i'\rho_0 + \varepsilon_i, \tag{3.1}$$

and X_i is a d-dimensional vector of covariates pertaining to agent i observed by the econometrician, $\rho_0 \in \mathbf{R}^d$ is a coefficient vector, and ε_i is unobserved heterogeneity.

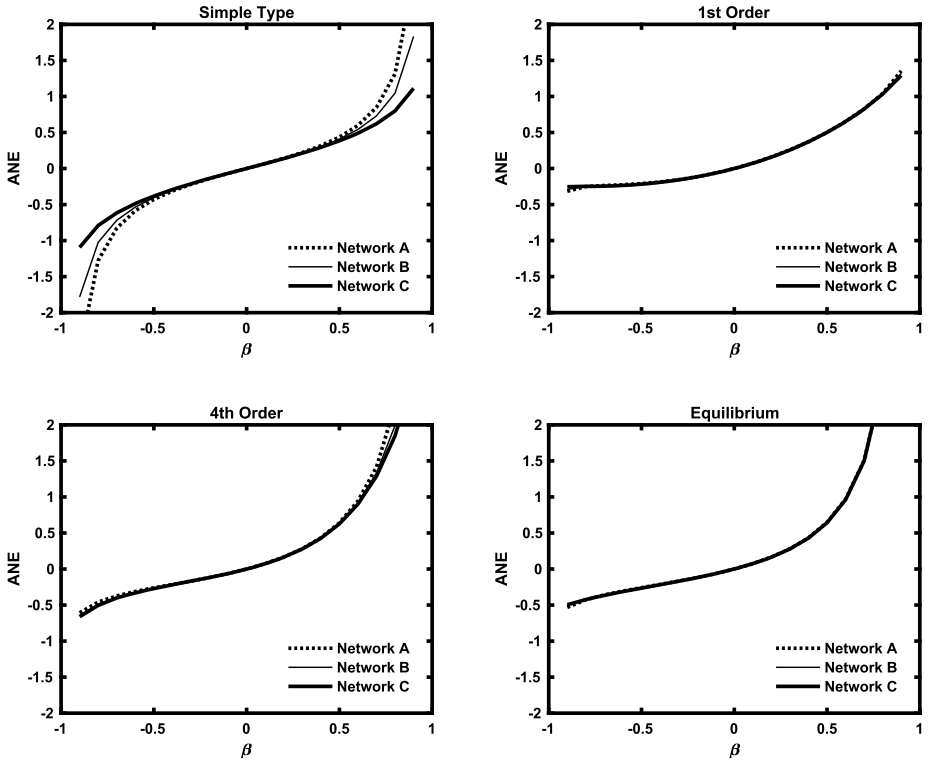

FIGURE 2. The average network externality comparison between equilibrium and behavioral models: Barabási–Albert graphs.

The covariate X_i can contain G_P-neighborhood averages of individual covariates. Let us make the following additional assumption on this original large game. Let us first define

$$\mathcal{F} = \sigma(X, G_P, G_I) \vee \mathcal{C},$$

that is, the σ-field generated by $X = (X_i)_{i\in N}$, G_P, G_I and $\mathcal{C}$ is a given common shock which is to be explained below.

ASSUMPTION 3.1. (i) *ε_i's and η_i's are conditionally i.i.d. across i's given $\mathcal{F}$.*

(ii) *$\{\varepsilon_i\}_{i=1}^n$ and $\{\eta_i\}_{i=1}^n$ are conditionally independent given $\mathcal{F}$.*

(iii) *For each $i \in N$, $\mathbf{E}[\varepsilon_i|\mathcal{F}] = 0$ and $\mathbf{E}[\eta_i|\mathcal{F}] = 0$.*

Condition (i) excludes preexisting cross-sectional dependence of unobserved heterogeneity in the payoffs once conditioned in $\mathcal{F}$. This condition implies that conditional on $\mathcal{F}$, the cross-sectional dependence of observed actions is due solely to the information sharing among the agents. Condition (ii) requires that conditional on $\mathcal{F}$, the unobserved payoff heterogeneities observed by other players and those that are private are independent. Condition (iii) excludes endogenous formation of G_P or G_I, because

the condition requires that the unobserved type components ε_i and η_i be conditionally mean independent of these graphs, given $X = (X_i)_{i\in N}$ and $\mathcal{C}$. However, the condition does not exclude the possibility that G_P and G_I are exogenously formed based on $(X, \mathcal{C})$. For example, suppose that $ij \in E_P$ if and only if

$$f_{ij}(X_i, X_j, a_i, a_j, u_{ij}) \geq 0,$$

where a_i represents degree heterogeneity, u_{ij}'s errors, and f_{ij} a given nonstochastic function. In this set-up, the econometrician does not observe a_i's or u_{ij}'s. This nests the dyadic regression model of Graham (2017) as a special case. Condition (iii) accommodates such a set-up, as long as $\{a_i\}_{i=1}^n$ and $\{u_{ij}\}_{i,j=1}^n$ are conditionally independent of $\{\varepsilon_i\}_{i=1}^n$ and $\{\eta_i\}_{i=1}^n$ given X. One simply has to take $\mathcal{C}$ to contain a_i's and u_{ij}'s.

The econometrician observes only a subset $N^* \subset N$ of agents and part of G_P through a potentially stochastic sampling process of unknown form. We assume for simplicity that $n^* \equiv |N^*|$ is nonstochastic. This assumption is satisfied, for example, if one collects the data for agents with predetermined sample size n^*. We assume that though being a small fraction of N, the set N^* is still a large set justifying our asymptotic framework that sends n^* to infinity. Most importantly, constituting only a small fraction of N, the observed sample N^* of agents induces a payoff subgraph which one has no reason to view as "approximating" or "similar to" the original payoff graph G_P. Let us make precise the data requirements.

Condition A. The stochastic elements of the sampling process are conditionally independent of $\{(\tau_i, \eta_i)'\}_{i\in N}$ given $\mathcal{F}$.

Condition B. For each $i \in N^*$, the econometrician observes $N_P(i)$ and (Y_i, X_i), and for each $j \in N_P(i)$, the econometrician observes $|N_P(i) \cap N_P(j)|$, $n_P(j)$ and X_j.

Condition C. Either of the following two conditions is satisfied:

(a) For $i, j \in N^*$ such that $i \neq j$, $N_P(i) \cap N_P(j) = \emptyset$.

(b) For each agent $i \in N^*$, and for any agent $j \in N^*$ such that $N_P(i) \cap N_P(j) \neq \emptyset$, the econometrician observes Y_j, $|N_P(j) \cap N_P(k)|$, $n_P(k)$ and X_k for all $k \in N_P(j)$.

Before we discuss the conditions, it is worth noting that these conditions are trivially satisfied when we observe the full payoff graph G_P and $N^* = N$. Condition A is satisfied, for example, if the sampling process is based on observed characteristics X and some characteristics of the strategic environment that is commonly observed by all the players. This condition is violated if the sampling is based on the outcomes Y_i's or unobserved payoff-relevant signals such as ε_i or η_i. Condition B essentially requires that in the data set, we observe (Y_i, X_i) of many agents i, and for each G_P-neighbor j of agent i, observe the number of the agents who are common G_P-neighbors of i and j and the size of G_P-neighborhood of j along with the observed characteristics X_j.[16] As for a G_P-

[16]Note that this condition is violated when the neighborhoods are top-coded in practice. For example, the maximum number of friends in the survey for a peer effects study can be set to be lower than the actual number of friends for many students. The impact of this top-coding upon the inference procedure is an interesting question on its own which deserves exploration in a separate paper.

neighbor j of agent $i \in N^*$, this condition does not require that the agent j's action Y_j or the full set of his G_P-neighbors are observed. Condition C(a) is typically satisfied when an initial sample of agents is randomly selected from a much larger set of agents so that no two agents have overlapping G_P-neighbors in the sample, and then their G_P neighbors are selected for each agent in the sample to constitute N^*.[17] In practice for use in inference, one can take the set N^* to include only those agents that satisfy Conditions A–C as long as N^* thereof is still large and the selection is based only on (X, G_P). One can simply use only those agents whose G_P-neighborhoods are not overlapping, as long as there are many such agents in the data.

3.1.2 *Moment conditions* In order to introduce inference procedures for β_0 and other payoff parameters, let us define for $i \in N$,

$$Z_i = \left(1 + \frac{\beta_0^2 \overline{\lambda}_i}{n_P(i) - \beta_0^2 \overline{\lambda}_i}\right)\left(X_i + \frac{\beta_0}{n_P(i)} \sum_{j \in N_P(i)} \lambda_{ij} X_j\right). \tag{3.2}$$

(Note that Z_i relies on β_0 although it is suppressed from notation for simplicity as we do frequently below for other quantities.) By Theorem 2.1 and (3.1), we can write

$$Y_i = Z_i' \rho_0 + v_i, \tag{3.3}$$

where

$$v_i = \left(1 + \frac{\beta_0^2 \overline{\lambda}_i}{n_P(i) - \beta_0^2 \overline{\lambda}_i}\right)\left(\varepsilon_i + \frac{\beta_0}{n_P(i)} \sum_{j \in N_P(i)} \lambda_{ij} \varepsilon_j\right) + \eta_i.$$

Note that the observed actions Y_i are cross-sectionally dependent (conditional on $\mathcal{F}$) due to information sharing on unobservables ε_i.

Suppose that φ_i is $M \times 1$ vector of instrumental variables (which potentially depend on β_0) with $M > d$ such that for all $i \in N$,

$$\mathbf{E}[v_i \varphi_i] = 0. \tag{3.4}$$

Note that the orthogonality condition above holds for any φ_i as long as for each $i \in N$, φ_i is $\mathcal{F}$-measurable, that is, once $\mathcal{F}$ is realized, there is no extra randomness in φ_i. This is the case, for example, when φ_i is a function of $X = (X_i)_{i \in N}$ and G_P.

While the asymptotic validity of our inference procedure admits a wide range of choices for φ_i's, one needs to choose them with care to obtain sharp inference on the payoff parameters. Especially, it is important to consider instrumental variables which involve the characteristics of G_P-neighbors to obtain sharp inference on payoff externality parameter β_0. This is because the cross-sectional dependence of observations carries substantial information for strategic interdependence among agents.

[17] This random selection does not need to be a random sampling from the population of agents. Note that the random sampling is extremely hard to implement in practice in this situation, because one needs to use the equal probability for selecting each agent into the collection N^*, but this equal probability will be feasible only when one has at least the catalog of the entire population N.

3.1.3 *Local identification* It is not hard to see that under regularity conditions (such as those preventing multicollinearity in Z_i), ρ_0 is identified up to β_0.[18] However, the moment function in (3.4) is nonlinear in β_0, and hence even local identification of β_0 is not guaranteed unless we impose further assumptions. Here we provide conditions for local identification, but for inference we propose later, we pursue asymptotically valid inference allowing the parameters to be only partially identified.

Let $\theta \equiv [\beta, \rho']'$ and write $v_i(\theta) = Y_i - Z_i'(\beta)\rho$, where $Z_i(\beta)$ is the same as Z_i except that β_0 is replaced by β. Let Θ be the parameter space for θ_0.

ASSUMPTION 3.2. (i) *For all $i = 1, \ldots, n$, φ_i does not depend on $\theta \in \Theta$, and the parameter space Θ is compact, and $\beta \in [-1+\nu, 1-\nu]$ for all β such that $[\beta, \rho']' \in \Theta$, for some small $\nu > 0$.*

(ii) *There exists $C > 0$ such that for all $n \geq 1$,*

$$\frac{1}{n}\sum_{i=1}^{n} \mathbf{E}\left[\|X_{i,n}\|^4 + \frac{1}{n_P(i)} \sum_{j \in N_P(i)} \|X_{j,1}\|^4 \Big| G_P \right] + \frac{1}{n}\sum_{i=1}^{n} \mathbf{E}\big[\|\varphi_i\|^4 | G_P\big] < C.$$

(iii) *There exists $c > 0$ such that the minimum eigenvalue of the matrix*

$$\sum_{m=1}^{M} \left(\frac{1}{n}\sum_{i=1}^{n} H_{i,m}(\theta_0) \right) \left(\frac{1}{n}\sum_{i=1}^{n} H_{i,m}(\theta_0) \right)'$$

is bounded from below by c for all $n \geq 2$, where

$$H_{i,m}(\theta) \equiv \mathbf{E}\left[\left[\begin{pmatrix} \partial v_i(\theta)/\partial \beta \\ -Z_i(\beta) \end{pmatrix} \varphi_{i,m} \right] \Big| G_P \right],$$

and $\varphi_{i,m}$ is the mth entry of φ_i.

(iv) $\max_{i \in N} n_P^2(i)/\sqrt{n} \to 0$, *as* $n \to \infty$.

Assumption 3.2(i) in regards to φ_i simplifies the identification arguments and is satisfied when the "instruments" φ_i consist only of observed variables. Assumption 3.2(ii) is a moment condition for the covariates. Assumption 3.2(iii) is a nontrivial condition and is violated if the parameter space for ρ_0 includes zero, because we have $\partial v_i(\beta, 0)/\partial \beta = 0$. Thus this assumption requires the researcher to know that the true parameter ρ_0 is away from zero. Assumption 3.2(iv) is a mild condition that requires that the payoff graph G_P is not overly dense.

Under this assumption, in combination with the conditions of Theorem 2.1, we can show that θ_0 is locally identified (i.e., consistently estimable over a neighborhood of θ_0.)

[18] A standard identification analysis centers on a "representative probability" from which we observe i.i.d. draws. A parameter is identified if it is uniquely determined under each representative probability. However, in our set-up, there is no such probability, as all observations exhibit heterogeneity and local dependence along a large, complex network. Here, "identification" simply means "consistent estimability" and "local identification" means "consistent estimability around a neighborhood of the true parameter."

THEOREM 3.1. *Suppose that Assumption 3.2 and the conditions of Theorem 2.1 hold. Then there exists $\varepsilon > 0$ such that if $\Theta = \overline{B}(\theta_0; \varepsilon)$, θ_0 is consistently estimable, where $B(\theta_0; \varepsilon)$ is the ε-neighborhood of θ_0 and $\overline{B}(\theta_0; \varepsilon)$ is its closure.*

Since consistent estimability of θ_0 requires that ρ_0 be away from zero, it is expected that as ρ_0 gets close to zero, β_0 is only "weakly (locally) identified." As a researcher is rarely *a priori* certain that ρ_0 is away from zero, we pursue inference that does not require this.

3.1.4 *Estimation and inference* We first estimate ρ_0 assuming knowledge of β_0. Define

$$S_{\varphi\varphi} = \varphi'\varphi/n^*, \quad \text{and} \quad \tilde{\varphi} = \varphi S_{\varphi\varphi}^{-1/2},$$

where φ is an $n^* \times M$ matrix whose i-th row is given by φ_i', $i \in N^*$. Define

$$\Lambda = \frac{1}{n^*} \sum_{i \in N^*} \sum_{j \in N^*} \mathbf{E}[v_i v_j | \mathcal{F}] \tilde{\varphi}_i \tilde{\varphi}_j', \tag{3.5}$$

where $\tilde{\varphi}_i$ represents the transpose of the ith row of $\tilde{\varphi}$, and let $\hat{\Lambda}$ be a consistent estimator of Λ. (We will explain how we construct this estimator in Section 3.1.7 below.) Define

$$S_{Z\tilde{\varphi}} = Z'\tilde{\varphi}/n^*, \quad \text{and} \quad S_{\tilde{\varphi}y} = \tilde{\varphi}'y/n^*,$$

where Z is an $n^* \times d$ matrix whose ith row is given by Z_i' and y is an $n^* \times 1$ vector whose ith entry is given by Y_i, $i \in N^*$. Then we estimate

$$\hat{\rho} = \left[S_{Z\tilde{\varphi}} \hat{\Lambda}^{-1} S_{Z\tilde{\varphi}}'\right]^{-1} S_{Z\tilde{\varphi}} \hat{\Lambda}^{-1} S_{\tilde{\varphi}y}. \tag{3.6}$$

Using this estimator, we construct a vector of residuals $\hat{v} = [\hat{v}_i]_{i \in N^*}$, where

$$\hat{v}_i = Y_i - Z_i'\hat{\rho}. \tag{3.7}$$

Finally, we form a profiled test statistic as follows:

$$T(\beta_0) = \frac{\hat{v}'\tilde{\varphi}\hat{\Lambda}^{-1}\tilde{\varphi}'\hat{v}}{n^*}, \tag{3.8}$$

making it explicit that the test statistic depends on β_0. Later we show that

$$T(\beta_0) \to_d \chi^2_{M-d}, \quad \text{as } n^* \to \infty,$$

where χ^2_{M-d} denotes the χ^2 distribution with degree of freedom $M - d$. Let $C^{\beta}_{1-\alpha}$ be the $(1-\alpha)100\%$ confidence set for β_0 defined as

$$C^{\beta}_{1-\alpha} \equiv \left\{\beta \in (-1, 1) : T(\beta) \leq c_{1-\alpha}\right\},$$

where $T(\beta)$ is computed as $T(\beta_0)$ with β_0 replaced by β and the critical value $c_{1-\alpha}$ is the $(1-\alpha)$-quantile of χ^2_{M-d}.

Let us now construct a confidence set for ρ_0. First, we establish that under regularity conditions

$$\sqrt{n^*}\hat{V}^{-1/2}(\hat{\rho}-\rho_0)\to_d N(0,I_d),$$

as $n^*\to\infty$, where

$$\hat{V}=\left[S_{Z\tilde{\varphi}}\hat{\Lambda}^{-1}S'_{Z\tilde{\varphi}}\right]^{-1}.$$

(See Section 3.2 below for conditions and formal results.) Using this estimator $\hat{\rho}$, we can construct a $(1-\alpha)100\%$ confidence interval for $a'\rho_0$ for any nonzero vector a. For this, define

$$\hat{\sigma}^2(a)=a'\hat{V}a.$$

Let $z_{1-(\alpha/4)}$ be the $(1-(\alpha/4))$-percentile of $N(0,1)$. Define for a vector a with the same dimension as ρ,

$$C^{\rho}_{1-(\alpha/2)}(\beta_0,a)=\left[a'\hat{\rho}-\frac{z_{1-(\alpha/4)}\hat{\sigma}(a)}{\sqrt{n}},a'\hat{\rho}+\frac{z_{1-(\alpha/4)}\hat{\sigma}(a)}{\sqrt{n}}\right].$$

Then the confidence set for $a'\rho$ is given by[19]

$$C^{\rho}_{1-\alpha}(a)=\bigcup_{\beta\in C^{\beta}_{1-(\alpha/2)}}C^{\rho}_{1-(\alpha/2)}(\beta,a).$$

Notice that since β runs in $(-1,1)$ and the estimator $\hat{\rho}$ has an explicit form, the confidence interval is not computationally costly to construct in general.

Often the eventual parameter of interest is one that captures how strongly the agents's decisions are interdependent through the network. For this, we can use the average network externality (ANE) introduced in (2.13). Let $s_i^{[0]}(\mathcal{I}_{i,0})$ be the best response of agent i having information set $\mathcal{I}_i$. Then the ANE with respect to $X_{i,r}$ (where $X_{i,r}$ represents the rth entry of X_i) is given by $\theta_1(\beta_0,\rho_{0,r})$, where

$$\begin{aligned}\theta_1(\beta_0,\rho_{0,r})&=\frac{1}{n^*}\sum_{i\in N^*}\sum_{j\in N_P(i)}\frac{\partial s_i^{[0]}(\mathcal{I}_{i,0})}{\partial x_{j,r}}\\&=\frac{1}{n^*}\sum_{i\in N^*}\sum_{j\in N_P(i)}\frac{\beta_0\lambda_{ij}}{n_P(i)}\left(1+\frac{\beta_0^2\overline{\lambda}_i}{n_P(i)-\beta_0^2\overline{\lambda}_i}\right)\rho_{0,r},\end{aligned}$$

and $\rho_{0,r}$ denotes the rth entry of ρ_0. See (2.8). Thus the confidence interval for $\theta_1(\beta_0,\rho_{0,r})$ can be constructed from the confidence interval for β_0 and ρ_0 as follows:

$$C^{\theta_1}_{1-\alpha}=\left\{\theta_1(\beta,\rho_r):\beta\in C^{\beta}_{1-\alpha/2},\text{ and }\rho_r\in C^{\rho_r}_{1-\alpha/2}\right\},\tag{3.9}$$

[19]Instead of the Bonferroni approach here, one could consider a profiling approach where one uses $T(\rho)=\sup_\beta T(\beta,\rho)$ as the test statistic, where $T(\beta,\rho)$ is the test statistic constructed using ρ in place of $\hat{\rho}$. The profiling approach is cumbersome to use here because one needs to simulate the limiting distribution of $T(\rho)$ for each ρ, which can be computationally complex when the dimension of ρ is large. Instead, this paper's Bonferroni approach is simple to use because β_0 takes values from $(-1,1)$.

where $C^{\beta}_{1-\alpha/2}$ and $C^{\rho_r}_{1-\alpha/2}$ denote the $(1-\alpha/2)100\%$ confidence intervals for β_0 and $\rho_{0,r}$, respectively.

3.1.5 *Downweighting players with high degree centrality* When there are players who are linked to many other players in G_P, the graph G_P tends to be denser, and it becomes difficult to obtain good variance estimators that perform stably in finite samples. (In particular, obtaining an estimator of Λ in (3.5) which performs well in finite samples can be difficult.) To remedy this situation, this paper proposes a downweighting of those players with high degree centrality in G_P. More specifically, in choosing an instrument vector φ_i, we may consider the following:

$$\varphi_i(X) = \frac{1}{\sqrt{\overline{n}_P(i)}} g_i(X), \tag{3.10}$$

where $g_i(X)$ is a function of X. This choice of φ_i downweights players i who have a large G_P-neighborhood. Thus we rely less on the variations of the characteristics of those players who have many neighbors in G_P.

Downweighting agents too heavily may hurt the power of inference because the actions of agents with high centrality contain information about the parameter of interest through the moment restrictions. On the other hand, downweighting them too lightly may hurt the finite sample stability of inference due to strong cross-sectional dependence they cause to the observations. Since a model with agents of higher-order sophisticated type results in observations with more extensive cross-sectional dependence, the role of downweighting can be important for finite sample stability of inference in such a model.

3.1.6 *Comparison with linear-in-means models* Let us compare our model with a linear-in-means model used in the literature, which is specified as follows:

$$Y_i = X_{i,1}'\gamma_0 + \overline{X}_{i,2}'\delta_0 + \beta_0 \mu_i^e(\overline{Y}_i) + v_i, \tag{3.11}$$

where $\mu_i^e(\overline{Y}_i)$ denotes the player i's expectation of $\overline{Y}_i$, and

$$\overline{Y}_i = \frac{1}{n_P(i)} \sum_{j \in N_P(i)} Y_j \quad \text{and} \quad \overline{X}_{i,2} = \frac{1}{n_P(i)} \sum_{j \in N_P(i)} X_{j,2}.$$

The literature assumes rational expectations by equating $\mu_i^e(\overline{Y}_i)$ to $\mathbf{E}[\overline{Y}_i|\mathcal{I}_i]$, and then proceeds to identification analysis of parameters γ, δ_0 and β_0. For actual inference, one needs to use an estimated version of $\mathbf{E}[\overline{Y}_i|\mathcal{I}_i]$. One standard way in the literature is to replace it by $\overline{Y}_i$ so that we have

$$Y_i = X_{i,1}'\gamma_0 + \overline{X}_{i,2}'\delta_0 + \beta_0 \overline{Y}_i + \tilde{v}_i,$$

where $\tilde{v}_i$ is an error term defined as $\tilde{v}_i = \beta_0(\mathbf{E}[\overline{Y}_i|\mathcal{I}_i] - \overline{Y}_i) + v_i$. The complexity arises

due to the presence of $\overline{Y}_i$ which is an endogenous variable that is involved in the error term $\tilde{v}_i$.[20]

As for dealing with endogeneity, there are two kinds of instrumental variables proposed in the literature. The first kind is a peers-of-peers type instrumental variable which is based on the observed characteristics of the neighbors of the neighbors. This strategy was proposed by Kelejian and Robinson (1993), Bramoullé, Djebbari, and Fortin (2009) and De Giorgi, Pellizzari, and Redaelli (2010). The second kind of an instrumental variable is based on observed characteristics excluded from the group characteristics as instrumental variables. (See Brock and Durlauf (2001) and Durlauf and Tanaka (2008).) However, finding such an instrumental variable in practice is not always a straightforward task in empirical research.

Our approach of empirical modeling is different in several aspects. Our modeling uses behavioral assumptions instead of rational expectations, and produces a reduced form for observed actions Y_i from using best responses. This reduced form gives a rich set of testable implications and makes explicit the source of cross-sectional dependence in relation to the payoff graph. Our inference approach permits any nontrivial functions of $\mathcal{F}$ to serve as instrumental variables. Furthermore, one does not need to observe many independent interactions for inference.

3.1.7 *Estimation of asymptotic covariance matrix* One needs to find estimators $\hat{\Lambda}$ and $\hat{V}$ to perform inference. First, let us find an expression for their population versions. After some algebra, it is not hard to see that the population version (conditional on $\mathcal{F}$) of $\hat{V}$ is given by

$$V = \left[S_{Z\tilde{\varphi}} \Lambda^{-1} S'_{Z\tilde{\varphi}}\right]^{-1}. \tag{3.12}$$

For estimation, it suffices to estimate Λ defined in (3.5). For this, we need to incorporate the cross-sectional dependence of the residuals v_i properly. From the definition of v_i, it turns out that v_i and v_j can be correlated if i and j are connected indirectly through two edges in G_P. One may construct an estimator of Λ that is similar to the HAC (Heteroskedasticity and Autocorrelation Consistent) estimator, simply by imposing the dependence structure and replacing v_i by $\hat{v}_i$. However, this standard method can lead to conservative inference with unstable finite sample properties, especially when each player has many players connected through two edges. Instead, this paper proposes an alternative estimator of Λ as follows. (See the Online Supplemental Material for more explanations for this estimator.)

Fixing a value for β_0, we first obtain a first-step estimator of ρ as follows:

$$\tilde{\rho} = \left[S_{Z\tilde{\varphi}} S'_{Z\tilde{\varphi}}\right]^{-1} S_{Z\tilde{\varphi}} S_{\tilde{\varphi}y}. \tag{3.13}$$

(Compare this with (3.6).) Using this estimator, we construct a vector of residuals $\tilde{v} = [\tilde{v}_i]_{i\in N^*}$, where

$$\tilde{v}_i = Y_i - Z'_i\tilde{\rho}. \tag{3.14}$$

[20]A similar observation applies in the case of a complete information version of the model, where one directly uses $\overline{Y}_i$ in place of $\mu_i^e(\overline{Y}_i)$ in (3.11). Still due to simultaneity of the equations, $\overline{Y}_i$ necessarily involve error terms v_i not only of agent i's own but other agents' as well.

Then we define

$$\hat{\Lambda}_1 = \frac{1}{n^*}\sum_{i\in N^*} \tilde{v}_i^2 \tilde{\varphi}_i \tilde{\varphi}_i', \quad \text{and}$$

$$\hat{\Lambda}_2 = \frac{\hat{s}_\varepsilon}{n^*}\sum_{i\in N^*}\sum_{j\in N^*_{-i}: N_P(i)\cap N_P(j)\neq\emptyset} q_{\varepsilon,ij}\tilde{\varphi}_i\tilde{\varphi}_j',$$

where

$$\hat{s}_\varepsilon = \frac{\sum_{i\in N^*}\sum_{j\in N_P(i)\cap N^*}\tilde{v}_i\tilde{v}_j}{\sum_{i\in N^*}\sum_{j\in N_P(i)\cap N^*} q_{\varepsilon,ij}}, \tag{3.15}$$

and

$$q_{\varepsilon,ij} = w_{ii}^{[0]}w_{jj}^{[0]}\left(\frac{\lambda_{ji}1\{i\in N_P(j)\}}{n_P(j)} + \frac{\lambda_{ij}1\{j\in N_P(i)\}}{n_P(i)} + \frac{\beta_0}{n_P(i)n_P(j)}\sum_{k\in N_P(i)\cap N_P(j)}\lambda_{ik}\lambda_{jk}\right).$$

(Note that the quantity $q_{\varepsilon,ij}$ can be evaluated once β_0 is fixed.) We construct an estimator of Λ as follows:[21]

$$\hat{\Lambda} = \hat{\Lambda}_1 + \hat{\Lambda}_2.$$

Using $\hat{\Lambda}$, we take the estimator for the covariance matrix V to be[22]

$$\hat{V} = \left[S_{Z\tilde{\varphi}}\hat{\Lambda}^{-1}S_{Z\tilde{\varphi}}'\right]^{-1}. \tag{3.16}$$

3.2 *Asymptotic theory*

In this section, we present the assumptions and formal results of asymptotic inference. We introduce some technical conditions.

ASSUMPTION 3.3. *There exists $c > 0$ such that for all $n^* \geq 1$, $\lambda_{\min}(S_{\varphi\varphi}) \geq c$, $\lambda_{\min}(S_{Z\tilde{\varphi}}S_{Z\tilde{\varphi}}') \geq c$, $\lambda_{\min}(S_{Z\tilde{\varphi}}\Lambda^{-1}S_{Z\tilde{\varphi}}') \geq c$, $\lambda_{\min}(\Lambda) \geq c$, $\sigma_\eta^2 > 0$, and*

$$\frac{1}{n^*}\sum_{i\in N^*}\frac{1}{n_P(i)}\sum_{j\in N_P(i)\cap N^*}\lambda_{ij} > c,$$

where $\lambda_{\min}(A)$ for a symmetric matrix A denotes the minimum eigenvalue of A.

[21] Under Condition C(a) for sample N^*, we have $\Lambda_2 = 0$ because the second sum in the expression for Λ_2 is empty. Hence in this case, we can simply set $\hat{\Lambda}_2 = 0$.

[22] In finite samples, $\hat{V}$ is not guaranteed to be positive definite. We can modify the estimator by using spectral decomposition similarly as in Cameron, Gelbach, and Miller (2011). More specifically, we first take a spectral decomposition $\hat{V} = \hat{B}\hat{A}\hat{B}'$, where $\hat{A}$ is a diagonal matrix of eigenvalues $\hat{a}_j$ of $\hat{V}$. We replace each $\hat{a}_j$ by the maximum between $\hat{a}_j$ and some small number $c > 0$ in $\hat{A}$ to construct $\hat{A}_*$. Then the modified version $\tilde{V} \equiv \hat{B}\hat{A}_*\hat{B}'$ is positive definite. For $c > 0$, one may take $c = 0.005$. In our simulation studies, this modification does not make much difference after all.

ASSUMPTION 3.4. *There exists a constant $C > 0$ such that for all $n^* \geq 1$,*

$$\max_{i \in N^\circ} \|X_i\| + \max_{i \in N^\circ} \|\tilde{\varphi}_i\| \leq C$$

and $\mathbf{E}[\varepsilon_i^4|\mathcal{F}] + \mathbf{E}[\eta_i^4|\mathcal{F}] < C$, where $n^\circ = |N^\circ|$ and

$$N^\circ = \bigcup_{i \in N^*} \overline{N}_P(i).$$

Assumption 3.3 is used to ensure that the asymptotic distribution is nondegenerate. This regularity condition is reasonable, because an asymptotic scheme that gives a degenerate distribution would not be adequate for approximating a finite sample, nondegenerate distribution of an estimator. Assumption 3.4 can be weakened at the expense of added complexity in the conditions and the proofs.

We introduce an assumption which requires the payoff graph to have a bounded degree over i in the observed sample N^*.

ASSUMPTION 3.5. *There exists $C > 0$ such that for all $n^* \geq 1$,*

$$\max_{i \in N^*} \left|N_P(i)\right| \leq C.$$

We may relax the assumption to a weaker, yet more complex condition at the expense of longer proofs, but in our view, this relaxation does not give additional insights. When N^* is large, one can remove very high-degree nodes to obtain stable inference. As such removal is solely based on the payoff graph G_P, the removal does not lead to any violation of the conditions in the paper.

The following theorem establishes the asymptotic validity of inference based on the best responses in Theorem 2.1, without using Assumption 3.2, that is, without requiring local identification of θ_0.

THEOREM 3.2. *Suppose that the conditions of Theorem 2.1 and Assumptions 3.1–3.5 hold. Then*

$$T(\beta_0) \to_d \chi^2_{M-d}, \quad \text{and} \quad \hat{V}^{-1/2}\sqrt{n^*}(\hat{\rho} - \rho_0) \to_d N(0, I_d),$$

as $n^* \to \infty$.

The theorem yields that the confidence sets for β_0 and ρ_0 that we proposed earlier are asymptotically valid. The proof is found in the Online Supplemental Material. At the center of the asymptotic derivation is noting first that v_i's have a conditional depenency graph in the sense that two sets $(v_j)_{j \in A}$ and $(v_j)_{j \in B}$ with G_P neighborhoods of A and B nonoverlapping are conditionally independent given $\mathcal{F}$) and then applying the central limit theorem for a sum of random variables that has a sparse conditional dependency graph. For the proof, we use a version of such a central limit theorem in Penrose (2003). The sparsity of such a graph is ensured by the bounded degree assumption (3.5). (Note that we can relax this assumption by letting the maximum degree increase slowly with

n, but as mentioned before, this relaxation does not add additional insights only lengthening the mathematical proofs.) The local dependence structure coming from the conditional dependency graph affects our inference through the estimated variance via the way $\hat{\Lambda}$ is constructed. If the payoff graph G_P is not sparse enough, the asymptotic approximation in Theorem 3.2 may perform poorly in finite samples.

4. A Monte Carlo simulation study

4.1 *Simulation design*

In this section, we investigate the finite sample properties of the asymptotic inference across various configurations of the payoff graph, G_P. (We present Monte Carlo simulation results for the game with the first-order sophisticated types in the replication file (Canen, Schwartz, and Song (2020).) The payoff graphs are generated according to two models of random graph formation, which we call Specifications 1 and 2. Specification 1 uses the Barabási–Albert model of preferential attachment, with m representing the number of edges each new node forms with existing nodes. The number m is chosen from $\{1, 2, 3\}$. Specification 2 is the Erdős–Rényi random graph with probability $p = \lambda/n$, where λ is also chosen from $\{1, 2, 3\}$.[23] In Table 2, we report degree characteristics of the payoff graphs used in the simulation study.

For the simulations, we set the following:

$$\tau_i = X_i'\rho_0 + \varepsilon_i,$$

where $\rho_0 = (2, 4, 1, 3, 4)'$ and $X_i = (X_{i,1}, \overline{X}_{i,2})'$, and

$$\overline{X}_{i,2} = \frac{1}{n_P(i)} \sum_{j \in N_P(i)} X_{j,2}.$$

We generate Y_i from the best response function in Theorem 2.1 (or as in (3.3)). We set and a to be a column of ones so that $a'\rho_0 = 14$. The variables ε and η are drawn i.i.d. from $N(0, 1)$. The first column of $X_{i,1}$ is a column of ones, while remaining columns of $X_{i,1}$ are drawn independently from $N(1, 1)$. The columns of $X_{i,2}$ are drawn independently from $N(3, 1)$.

For instruments, we use downweighting (3.10) as follows:

$$\varphi_i(X) = \frac{1}{\sqrt{\overline{n}_P(i)}} g_i(X),$$

where

$$g_i(X) = \left[\tilde{Z}_{i,1}, X_{i,1}^2, \overline{X}_{i,2}^2, \overline{X}_{i,2}^3\right]',$$

[23] Note that in Specification 1, the Barabási–Albert graph is generated with an Erdős–Rényi seed graph, where the number of nodes in the seed is set to equal the smallest integer above $5\sqrt{n}$. All graphs in the simulation study are undirected.

TABLE 2. The average and maximum degrees of graphs in the simulations.

		Specification 1			Specification 2		
n		$m=1$	$m=2$	$m=3$	$\lambda=1$	$\lambda=2$	$\lambda=3$
500	d_{mx}	17	21	30	5	8	11
	d_{av}	1.7600	3.2980	4.8340	0.9520	1.9360	2.9600
1000	d_{mx}	18	29	34	6	7	9
	d_{av}	1.8460	3.5240	5.2050	0.9960	1.9620	3.0020
5000	d_{mx}	32	78	70	7	10	11
	d_{av}	1.9308	3.7884	5.6466	0.9904	2.0032	3.0228

Note: This table gives characteristics of the payoff graphs, G_P, used in the simulation study. d_{av} and d_{mx} represent the average and maximum degrees of the networks respectively; that is, $d_{\text{av}} \equiv \frac{1}{n}\sum_{i\in N} n_P(i)$ and $d_{\text{mx}} \equiv \max_{i\in N} n_P(i)$.

where we define

$$\tilde{Z}_{i,1} \equiv \frac{1}{n_P(i)} \sum_{j\in N_P(i)} \lambda_{ij} X_{j,1}.$$

While the instruments $X_{i,1}^2$, $\overline{X}_{i,2}^2$, $\overline{X}_{i,2}^3$ capture the nonlinear impact of X_i's, the instrument $\tilde{Z}_{i,1}$ captures the cross-sectional dependence along the payoff graph. The use of this instrumental variable is crucial in obtaining a sharp inference for β_0. Note that since we have already concentrated out ρ_0 in forming the moment conditions, we cannot use linear combinations of $X_{i,1}$ and $\overline{X}_{i,2}$ as our instrumental variables. The nominal size in all the experiments is set at $\alpha = 0.05$. The Monte Carlo simulation number is set to 5000.

4.2 *Results*

The finite sample performance of the asymptotic inference is shown in Tables 3 and 4. Overall, the simulation results illustrate good power and size properties for the asymptotic inference on β_0 and $a'\rho_0$.

As for the size properties, the coverage probabilities of confidence intervals for β_0 are close to 95% nominal level, as shown in Table 3. The size properties are already good with $n = 500$, and thus, show little improvement as n increase to 5000. The coverage probabilities of confidence intervals for $a'\rho_0$ (as shown in Table 4) are a little conservative. This conservativeness is expected, given the fact that the interval is constructed using a Bonferroni approach. The conservativeness is alleviated as we increase the sample size.

As for the power properties, we consider the average length of the confidence intervals. The confidence intervals for β_0 are very short, with average length around 0.1–0.2 when $n = 500$ and 0–0.03 when $n = 5000$. As for $a'\rho_0$, the average length of the confidence intervals is around 2–17 when $n = 500$ and 0.4–2.5 when $n = 5000$. Since $a'\rho_0 = 14$, the average length shows good power properties of the inference.

TABLE 3. The empirical coverage probability and average length of confidence intervals for β_0 at 95% nominal level.

		Specification 1			Specification 2		
β_0		$m=1$	$m=2$	$m=3$	$\lambda=1$	$\lambda=2$	$\lambda=3$
				Coverage Probability			
	$n=500$	0.9642	0.9580	0.9648	0.9686	0.9638	0.9622
−0.5	$n=1000$	0.9638	0.9634	0.9574	0.9650	0.9644	0.9604
	$n=5000$	0.9596	0.9560	0.9530	0.9704	0.9608	0.9596
	$n=500$	0.9540	0.9536	0.9612	0.9608	0.9546	0.9568
−0.3	$n=1000$	0.9566	0.9568	0.9566	0.9564	0.9578	0.9548
	$n=5000$	0.9534	0.9548	0.9542	0.9636	0.9568	0.9546
	$n=500$	0.9504	0.9464	0.9554	0.9474	0.9478	0.9490
0	$n=1000$	0.9486	0.9508	0.9514	0.9498	0.9510	0.9526
	$n=5000$	0.9440	0.9490	0.9546	0.9516	0.9482	0.9478
	$n=500$	0.9548	0.9512	0.9584	0.9562	0.9552	0.9556
0.3	$n=1000$	0.9600	0.9558	0.9524	0.9598	0.9592	0.9590
	$n=5000$	0.9524	0.9536	0.9574	0.9604	0.9544	0.9522
	$n=500$	0.9648	0.9574	0.9618	0.9640	0.9610	0.9620
0.5	$n=1000$	0.9630	0.9604	0.9534	0.9710	0.9648	0.9634
	$n=5000$	0.9564	0.9598	0.9612	0.9700	0.9632	0.9584
				Average Length of CI			
	$n=500$	0.0834	0.1307	0.1947	0.1089	0.0751	0.0750
−0.5	$n=1000$	0.0490	0.0794	0.1038	0.0630	0.0438	0.0463
	$n=5000$	0.0053	0.0203	0.0303	0.0108	0.0026	0.0024
	$n=500$	0.0799	0.1216	0.1639	0.1083	0.0865	0.0910
−0.3	$n=1000$	0.0464	0.0758	0.0990	0.0639	0.0519	0.0577
	$n=5000$	0.0034	0.0187	0.0296	0.0116	0.0060	0.0075
	$n=500$	0.0785	0.1212	0.1572	0.1070	0.0970	0.1087
0	$n=1000$	0.0452	0.0753	0.0996	0.0638	0.0597	0.0700
	$n=5000$	0.0024	0.0182	0.0298	0.0113	0.0106	0.0155
	$n=500$	0.0713	0.1062	0.1384	0.0983	0.0685	0.0676
0.3	$n=1000$	0.0404	0.0640	0.0872	0.0562	0.0389	0.0412
	$n=5000$	0.0017	0.0155	0.0262	0.0076	0.0015	0.0013
	$n=500$	0.0495	0.0738	0.1085	0.0666	0.0289	0.0240
0.5	$n=1000$	0.0252	0.0337	0.0657	0.0328	0.0089	0.0079
	$n=5000$	0.0001	0.0055	0.0147	0.0004	0.0000	0.0000

Note: The first-half of the table reports the empirical coverage probability of the asymptotic confidence interval for β_0 and the second-half reports its average length. The simulated rejection probability at the true parameter is close to the nominal size of $\alpha = 0.05$ and the average lengths decrease with n. The simulation number is $R = 5000$.

TABLE 4. The empirical coverage probability and average length of confidence intervals for $a'\rho_0$ at 95% nominal level.

		Specification 1			Specification 2		
β_0		$m=1$	$m=2$	$m=3$	$\lambda=1$	$\lambda=2$	$\lambda=3$
				Coverage Probability			
	$n=500$	0.9848	0.9802	0.9860	0.9862	0.9834	0.9740
−0.5	$n=1000$	0.9616	0.9610	0.9680	0.9682	0.9670	0.9596
	$n=5000$	0.9596	0.9548	0.9606	0.9706	0.9668	0.9614
	$n=500$	0.9802	0.9772	0.9858	0.9832	0.9826	0.9794
−0.3	$n=1000$	0.9620	0.9756	0.9796	0.9772	0.9682	0.9692
	$n=5000$	0.9544	0.9510	0.9562	0.9588	0.9568	0.9556
	$n=500$	0.9740	0.9754	0.9868	0.9828	0.9792	0.9786
0	$n=1000$	0.9668	0.9770	0.9798	0.9746	0.9738	0.9756
	$n=5000$	0.9430	0.9500	0.9524	0.9546	0.9496	0.9494
	$n=500$	0.9804	0.9804	0.9866	0.9810	0.9778	0.9788
0.3	$n=1000$	0.9698	0.9794	0.9812	0.9816	0.9718	0.9758
	$n=5000$	0.9476	0.9524	0.9546	0.9572	0.9536	0.9498
	$n=500$	0.9824	0.9828	0.9858	0.9810	0.9722	0.9720
0.5	$n=1000$	0.9724	0.9786	0.9826	0.9810	0.9596	0.9594
	$n=5000$	0.9552	0.9536	0.9596	0.9626	0.9596	0.9542
				Average Length of CI			
	$n=500$	5.4643	10.2562	17.4488	7.0549	5.6285	6.0639
−0.5	$n=1000$	3.3501	5.9311	8.1165	4.2270	3.4243	3.8800
	$n=5000$	0.7489	1.8254	2.4970	1.1165	0.6337	0.6588
	$n=500$	4.2511	6.8326	9.5995	5.6856	4.9154	5.4091
−0.3	$n=1000$	2.5915	4.3335	5.6775	3.4711	3.0685	3.5331
	$n=5000$	0.5297	1.3514	1.9123	0.9505	0.7346	0.8560
	$n=500$	3.5812	5.3587	6.8508	4.7774	4.3760	4.8399
0	$n=1000$	2.1797	3.4445	4.4160	2.9651	2.7944	3.2136
	$n=5000$	0.4238	1.0993	1.5392	0.8365	0.7964	0.9882
	$n=500$	3.3664	4.5527	5.7822	4.4438	3.2458	3.1962
0.3	$n=1000$	2.0559	2.8904	3.6607	2.6872	2.0100	2.1074
	$n=5000$	0.4072	0.9592	1.2989	0.6961	0.4399	0.4713
	$n=500$	3.0230	4.1201	5.8780	3.7624	2.1350	1.9826
0.5	$n=1000$	1.7684	2.1613	3.4957	2.1027	1.1718	1.2075
	$n=5000$	0.3576	0.6764	1.0002	0.3995	0.3723	0.4081

Note: The true $a'\rho_0$ is equal to 14. The first-half of the table reports the empirical coverage probability of the asymptotic confidence interval and the second-half its average length for $a'\rho_0$. The empirical coverage probability of the confidence interval for $a'\rho_0$ is generally conservative which is expected from the use of the Bonferroni approach. Nevertheless, the length of the confidence interval is reasonably small. The simulation number, R, is 5000.

5. Empirical application: State presence across municipalities

5.1 *Motivation and background*

State capacity (i.e., the capacity of a country to provide public goods, basic services, and the rule of law) can be limited for various reasons. (See, e.g., Besley and Persson (2009) and Gennaioli and Voth (2015)).[24] A "weak state" may arise due to political corruption and clientelism, and result in spending inadequately on public goods (Acemoglu (2005)), accommodating armed opponents of the government (Powell (2013)), and war (McBride, Milante, and Skaperdas (2011)). Empirical evidence has shown how these weak states can persist from precolonial times, with higher state capacities apparently related to current level prosperity at the ethnic and national levels (Gennaioli and Rainer (2007) and Michalopoulos and Papaioannou (2013)).

Our empirical application is based on a recent study by Acemoglu, García-Jimeno, and Robinson (2015) who investigate the local choices of state capacity in Colombia, using a model of a complete information game on an exogenously formed network. In their set-up, municipalities choose a level of spending on public goods and state presence (as measured by either the number of state employees or state agencies). Network externalities in a municipality's choice exist because municipalities that are adjacent to one another can benefit from their neighbors' choices of public goods provisions, such as increased security, infrastructure, and bureaucratic connections. Thus, a municipality's choice of state capacity can be thought of as a strategic decision on a geographic network.

It is not obvious that public good provision in one municipality leads to higher spending on public goods in neighboring municipalities. Some neighbors may free-ride and underinvest in state presence if they anticipate others will invest highly. Rent-seeking by municipal politicians would also limit the provision of public goods. On the other hand, economies of scale could lend to complementarities in state presence across neighboring municipalities.

In our study, we extend the model in Acemoglu, García-Jimeno, and Robinson (2015) to an incomplete information game where information may be shared across municipalities. In particular, we do not assume that all municipalities know and observe all characteristics and decisions of the others. It seems reasonable that the decisions made across the country may not be observed or well known by those municipalities that are geographically remote.

5.2 *Empirical set-up*

Let y_i denote the state capacity in municipality i (as measured by the log number of public employees in municipality i) and G_P denote the geographic network, where an edge is defined on two municipalities that are geographically adjacent.[25] We assume that G_P is exogenously formed. The degree distribution of G_P is shown in Figure 3. We study the

[24] See also an early work by Brett and Pinkse (2000) for an empirical study on the spatial effects on municipal governments' decisions on business property tax rates.

[25] This corresponds to the case in of $\delta_1 = \delta_2 = 0$ in Acemoglu, García-Jimeno, and Robinson (2015).

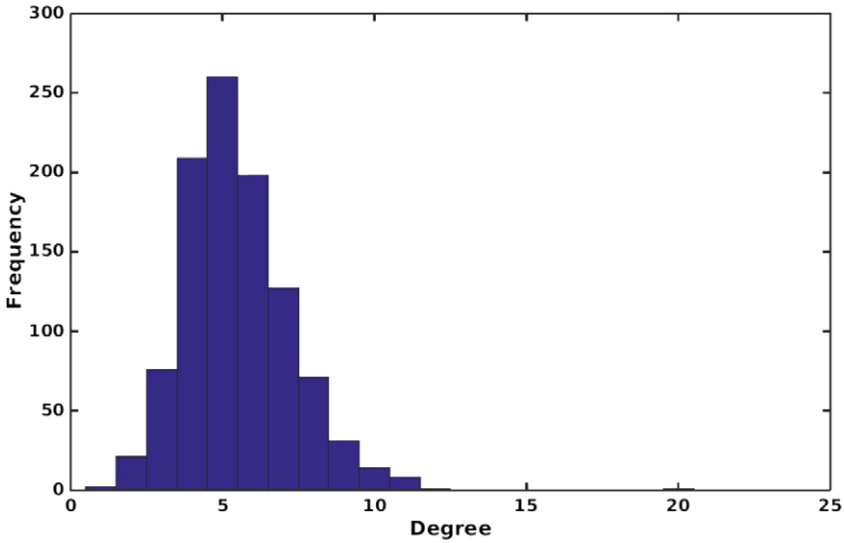

FIGURE 3. Degree distribution of G_P.

optimal choice of y_i, where y_i leads to a larger prosperity p_i. Prosperity in municipality i is modeled as

$$p_i = \left(\beta \bar{y}_i + x'_{1,i}\gamma + \eta_i + \varepsilon_i + \varsigma_i^D\right) y_i, \tag{5.1}$$

where ς_i^D is a district specific dummy variable, ε_i and η_i are our sharable and non-sharable private information, and $\overline{y}_i = \frac{1}{n_P(i)} \sum_{j \in N_P(i)} y_j$. The term $x_{1,i}$ represents municipality characteristics. These include geographic characteristics, such as land quality, altitude, latitude, rainfall; and municipal characteristics, such as distance to highways, distance to royal roads and Colonial State Presence.[26]

The welfare of a municipality is given by

$$u_i(y_i, y_{-i}, \tau, \eta_i) = p_i(y_i, \bar{y}_i, \tau, \eta_i) - \frac{1}{2} y_i^2, \tag{5.2}$$

where the second term refers to the cost of higher state presence, and the first term is the prosperity p_i.

We can rewrite the welfare of the municipality by substituting (5.1) into (5.2):

$$u_i(y_i, y_{-i}, \tau, \eta_i) = \left(\beta \bar{y}_i + x'_{1,i}\gamma + \eta_i + \varepsilon_i + \varsigma_i^D\right) y_i - \frac{1}{2} y_i^2. \tag{5.3}$$

We assume that municipalities (or the mayor in charge), wishes to maximize welfare by choosing state presence, given their beliefs about the types of the other municipalities.

[26]Note that p_i is only a function of terms that are multiplied by y_i. This is a simplification from their specification. We do so because we will focus on the best response equation. The best response equation, derived from the first-order condition to this problem, would not include any term that is not a function of y_i itself.

In our specification, we allow for incomplete information. This is reflected in the terms ε_i, η_i, which will be present in the best response function. The municipality, when choosing state presence y_i, will be able to observe ε_i of its neighbors and will use its beliefs over the types of the others to generate its best response. The best response will follow the results from Theorem 2.1.

5.3 *Model specification*

We closely follow Table 3 in Acemoglu, García-Jimeno, and Robinson (2015) for the choice of specifications and variables. First, we will consider the model with simple types.[27]

Throughout the specifications, we include longitude, latitude, surface area, elevation, annual rainfall, department fixed effects and a department capital dummy (all in X_1). We further consider the effect of variables distance to current highways, land quality, and presence of rivers in the municipality.

For the choice of instruments, we consider two separate types of instruments. The first is the sum of neighbor values (across G_P) of the historical variables (denoted as C_i).[28] The historical variables used are Total Crown Employees (also called Colonial State Officials), Distance to Royal Roads, Colonial State Agencies, and Historical Population, as well as Colonial State Presence Index squared and Distance to Royal Roads squared. Using the latter two additionally sharpens inference. We also use the variable $\tilde{Z}_i = n_P(i)^{-1} \sum_{j \in N_P(i)} \lambda_{ij} X_{j,1}$ as part of the instrumental variables, which was shown to perform well in the Monte Carlo Simulations in Section 4. This variable captures cross sectional dependence as a crucial source of variation for inference on the strategic interactions. We use downweighting of our instruments as explained in a preceding section and rescale instruments by multiplying them by $S_{\varphi\varphi}^{-1/2}$.

5.4 *Results*

The results across a range of specifications are presented in Table 5. In these results, we see that the effect is statistically different than 0 and stable across specifications. It indicates that there is complementarity in the provision of public goods and state presence ($\beta > 0$).

[27] In the replication file of this paper, we consider the empirical application with first-order sophisticated types (game Γ_1), as well as the model selection test between simple types and first-order sophisticated types. Since the simple type model is not rejected in the data and it is more parsimonious, we present it in the main text. The results for the first-order sophisticated case are more or less similar except that the confidence intervals of β_0 are wider. At 5%, the model selection procedure did not reject either of the sets of the moment conditions from the simple type and the first-order sophisticated players.

[28] For this, we assume the exclusion restriction in Acemoglu, García-Jimeno, and Robinson (2015), namely that historical variables only affect prosperity in the same municipality. This means that although one's historical variables (Total Crown Employees, Distance to Royal Roads, Colonial State Agencies and Historical Population, as well as functions thereof) can affect the same municipality's prosperity, it can only affect those of the neighbors by impacting the choice of state capacity in the first, which then impacts the choice of the state capacity in the neighbors.

TABLE 5. State presence and networks effects across Colombian municipalities.

	Outcome: The Number of State Employees			
	Baseline (1)	Distance to Highway (2)	Land Quality (3)	Rivers (4)
β_0	[0.16, 0.31]	[0.16, 0.32]	[0.17, 0.39]	[0.07, 0.38]
dy_i/d(Colonial State Officials)	[−0.060, 0.003]	[−0.048, −0.001]	[−0.051, 0.003]	[−0.034, 0.009]
Average dy_i/d(Colonial State Agencies)	[−1.323, 4.051]	[−1.249, 2.793]	[−0.972, 3.545]	[−4.186, 2.719]
Average dy_i/d(distance to Royal Roads)	[−0.010, 0.011]	[−0.009, 0.011]	[−0.008, 0.018]	[−0.010, 0.013]
n	1018	1018	1003	1003

Note: Confidence sets for β are presented in the table, obtained from inverting the test statistic $T(\beta)$ from Section 3 for first-order sophisticated types, with confidence level of 95%. The critical values in the first row come from the asymptotic statistic. Downweighting is used. The average marginal effects for historical variables upon state capacity are also shown. The marginal effect of Colonial State Officials is equal to its γ coefficient. The marginal effect for Distance to Royal Roads for municipality i equals $\gamma_{\text{Royal Roads}} + 2 * \gamma_{\text{Royal Roads}^2}(\text{Royal Roads})_i$, where $\gamma_{\text{Royal Roads}}$ is the γ coefficient of its linear term, and $\gamma_{\text{Royal Roads}^2}$ is the coefficient of its quadratic term, as this variable enters X_1 as a quadratic form. The analogous expression holds for the variable Colonial State Agencies. We show the average marginal effect for these two variables. We then present the confidence set for these marginal effects, computed by the inference procedure on $a'\gamma$ developed in Section 3. All specifications include controls of latitude, longitude, surface area, elevation, rainfall, as well as Department and Department capital dummies. Instruments are constructed from payoff neighbors' sum of the G_P neighbors values of the historical variables Total Crown Employees, Colonial State Agencies, Colonial State Agencies squared, population in 1843, distance to Royal Roads, distance to Royal Roads squared, together with the non-linear function $\tilde{Z}_i = n_P(i)^{-1} \sum_{j \in N_P(i)} \lambda_{ij} X_{j,1}$. Column (2) includes distance to current highway in X_1, Column (3) expands the specification of Column (2) by also including controls for land quality (share in each quality level). Column (4) controls for rivers in the municipality and land quality, in addition to those controls from Column (1). One can see that the results are very stable across specifications.

Let us compare our results to those in Acemoglu, García-Jimeno, and Robinson (2015). There, the authors report the average marginal effects over their weighted graph. The (weighted) average degree is 0.0329, so our results can be compared in an approximation, by considering $0.0329\,\hat{\beta}$.

In general, our estimates have the same sign and significance as those of Acemoglu, García-Jimeno, and Robinson (2015). Our estimates are in the range of [0.002, 0.013], after reweighting as mentioned before, somewhat comparable to theirs of [0.016, 0.022] (in the case of the outcome of the number of public employees, in Table 3 in their paper). Hence, we find similar qualitative effects, although a smaller magnitude. Recall that our confidence set is built without assuming that β_0 is consistently estimable.

In Figure 4, we show the results of our estimated network externalities for the estimates from Table 5, for the importance of being a department capital. The average network externality (ANE) is computed as

$$\frac{1}{N} \sum_{i \in N} \sum_{j \in N_P(i)} \frac{\beta_0 \hat{\gamma}_{dc}}{n_P(i)(1 - \beta_0 c_{ij})} \left(1 + \frac{\beta_0^2 \overline{\lambda}_i}{n_P(i) - \beta_0^2 \overline{\lambda}_i}\right),$$

where $\hat{\gamma}_{dc}$ is the estimated parameter of the X_1 variable department capital. The parameter is defined in Section 3.1.4, and captures the average effect of a neighbor being a department capital. We construct a confidence interval as in (3.9).

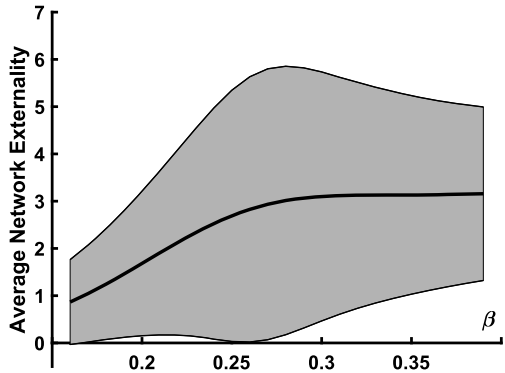

FIGURE 4. Average network externality from being a department capital.

The figure shows that there is a strong and increasing network externality from being a department capital over the range of the confidence set of β. This indicates that the effect of being a capital has spillovers on other municipalities: since $\beta > 0$, and one expects that department capitals have more state presence and resources, being a department capital yields increasing returns the stronger the complementarity.

6. CONCLUSION

This paper proposes a new approach of empirical modeling for interactions among many agents when the agents observe the types of their neighbors in a single large network. The main challenge arises from the fact that the information sharing relations are typically connected among a large number of players whereas the econometrician observes only a fraction of those agents. Using a behavioral model of belief formation, this paper produces an explicit form of best responses from which an asymptotic inference procedure for the payoff parameters is developed. As we showed in our paper, this explicit form gives a reduced form for the observed actions, and exhibits various intuitive features. For example, the best responses show that network externality is heterogeneous across agents depending on the relations of their payoff neighbors.

The main advantage of our paper's approach is two-fold. First, the empirical modeling according to our approach accommodates a wide range of sampling processes. Such a feature is crucial because the econometrician rarely has precise knowledge about the actual sampling process through which data are generated. Second, the model can be used when only a fraction of the players are observed from a large connected network of agents. This can be quite useful as the econometrician typically does not observe the entire set of agents who interact with each other.

An interesting extension from this research is to consider the situation where the network formation is endogenous, that is, the randomness of G_P is correlated with the unobserved payoff heterogeneities of agents even after conditioning on the covariates. It is not trivial to extend our framework to this situation, because we need to explicitly consider the randomness of G_P in relation to other variables instead of conditioning our inference on G_P as we do in this paper. Such consideration should be properly made in

the construction of moment conditions and the proposal of appropriate instrumental variables. We relegate this extension to future research.

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