Supplement to "A more powerful subvector Anderson Rubin test in linear instrumental variables regression"

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This online supplement contains additional tables of critical values, computational details, and additional numerical results.

Appendix C: Additional tables of critical values

10%, 5%, and 1% conditional critical values $c_{1-\alpha}(\hat{\kappa}_1, k - m_W)$ were computed by numerically integrating the density (2.12) at different values of the conditioning variable $\hat{\kappa}_1$ for the cases $k - m_W = 6, \dots, 20$ (cases $k - m_W = 1, \dots, 5$ are reported in the main paper). The results are reported in Tables 8 to 22. The conditional quantiles are rounded upwards to one decimal place, and the initial value of $\hat{\kappa}_1$ in each table is the smallest $\hat{\kappa}_1$ for which the rounded quantile is less than $\hat{\kappa}_1$.

Appendix D: Computational details

D.1 *Computation of the hypergeometric function*

The function ${}_0F_1^{(2)}$ of two matrix arguments, which appears in the kernel of the density (A.16), involves an infinite series of Jack functions that converge very slowly and it is notoriously hard to compute accurately. We use the recently developed algorithm of Koev and Edelman (2006) which is efficient and fast. The algorithm approximates ${}_0F_1^{(2)}$ using a finite sum of terms M terms, so we need to choose M large enough for an accurate approximation. By extensive experimentation with different values of M up to 500, we found that $M = 200$ seems to be sufficiently large for all the cases we considered, because the results are unchanged when M is increased further. Hence, we used $M = 200$ in all calculations.

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TABLE 8. $1-\alpha$ quantile of the conditional distribution, with density given in (2.12), $\text{cv} = c_{1-\alpha}(\hat{\kappa}_1, k-m_W)$ at different values of the conditioning variable $\hat{\kappa}_1$. Computed by numerical integration.

$k-m_W=6$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
$\alpha=10\%$													
1.1	1.0	2.6	2.4	4.6	4.0	6.8	5.6	9.5	7.2	13.7	8.8	39.4	10.4
1.2	1.1	2.9	2.6	4.9	4.2	7.1	5.8	9.9	7.4	14.5	9.0	81.9	10.6
1.3	1.2	3.1	2.8	5.1	4.4	7.4	6.0	10.3	7.6	15.5	9.2	1000	10.634
1.5	1.4	3.4	3.0	5.4	4.6	7.7	6.2	10.8	7.8	16.6	9.4	∞	10.645
1.7	1.6	3.6	3.2	5.7	4.8	8.0	6.4	11.3	8.0	18.1	9.6		
2.0	1.8	3.8	3.4	5.9	5.0	8.4	6.6	11.8	8.2	20.1	9.8		
2.2	2.0	4.1	3.6	6.2	5.2	8.7	6.8	12.4	8.4	23.0	10.0		
2.4	2.2	4.3	3.8	6.5	5.4	9.1	7.0	13.0	8.6	28.2	10.2		
$\alpha=5\%$													
1.6	1.5	3.3	3.1	5.4	4.9	7.7	6.7	10.4	8.5	14.4	10.3	30.9	12.1
1.7	1.6	3.5	3.3	5.6	5.1	7.9	6.9	10.8	8.7	15.1	10.5	41.6	12.3
1.8	1.7	3.7	3.5	5.9	5.3	8.2	7.1	11.1	8.9	15.8	10.7	75.0	12.5
2.0	1.9	4.0	3.7	6.1	5.5	8.5	7.3	11.5	9.1	16.7	10.9	1000	12.579
2.2	2.1	4.2	3.9	6.4	5.7	8.8	7.5	11.9	9.3	17.7	11.1	∞	12.592
2.4	2.3	4.4	4.1	6.6	5.9	9.1	7.7	12.4	9.5	18.9	11.3		
2.6	2.5	4.7	4.3	6.9	6.1	9.4	7.9	12.8	9.7	20.4	11.5		
2.9	2.7	4.9	4.5	7.1	6.3	9.7	8.1	13.3	9.9	22.5	11.7		
3.1	2.9	5.1	4.7	7.4	6.5	10.1	8.3	13.9	10.1	25.7	11.9		
$\alpha=1\%$													
3.7	3.6	5.5	5.4	7.8	7.5	10.3	9.6	13.1	11.7	17.1	13.8	28.7	15.9
3.8	3.7	5.9	5.7	8.2	7.8	10.7	9.9	13.6	12.0	17.9	14.1	35.5	16.2
4.0	3.9	6.2	6.0	8.5	8.1	11.1	10.2	14.1	12.3	18.8	14.4	52.4	16.5
4.3	4.2	6.5	6.3	8.9	8.4	11.5	10.5	14.6	12.6	19.8	14.7	159.8	16.8
4.6	4.5	6.8	6.6	9.2	8.7	11.9	10.8	15.2	12.9	21.1	15.0	1000	16.796
4.9	4.8	7.2	6.9	9.6	9.0	12.3	11.1	15.7	13.2	22.8	15.3	∞	16.812
5.2	5.1	7.5	7.2	9.9	9.3	12.7	11.4	16.4	13.5	25.1	15.6		

D.2 *Size calculations*

The computation of the NRP in Section 2 was conducted using numerical integration of the exact density (A.17). Their accuracy depends in part on the accuracy of the computation of ${}_0F_1^{(2)}$. To assess that, we compare in Figure 7 the NRP computed using Monte Carlo integration with 1 million replications to the one reported in Figure 2. The results are essentially identical to 3 decimals.

Further results on the size of conditional subvector AR test are given in Section E.1.

D.3 *Power bounds*

In this section, we explain how we compute bounds to the power of the rank testing problem in Section 2.4 using the methods of Andrews, Moreira, and Stock (2008, Sec-

TABLE 9. $1-\alpha$ quantile of the conditional distribution, with density given in (2.12), cv $= c_{1-\alpha}(\hat{\kappa}_1, k-m_W)$ at different values of the conditioning variable $\hat{\kappa}_1$. Computed by numerical integration.

$k-m_W=7$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
$\alpha=10\%$													
1.2	1.1	2.9	2.7	5.1	4.5	7.5	6.3	10.3	8.1	14.7	9.9	38.5	11.7
1.3	1.2	3.2	2.9	5.3	4.7	7.7	6.5	10.7	8.3	15.5	10.1	65.0	11.9
1.4	1.3	3.4	3.1	5.6	4.9	8.0	6.7	11.1	8.5	16.3	10.3	112.1	12.0
1.6	1.5	3.6	3.3	5.8	5.1	8.3	6.9	11.5	8.7	17.3	10.5	1000	12.005
1.8	1.7	3.9	3.5	6.1	5.3	8.7	7.1	12.0	8.9	18.4	10.7	∞	12.017
2.0	1.9	4.1	3.7	6.4	5.5	9.0	7.3	12.4	9.1	19.9	10.9		
2.3	2.1	4.3	3.9	6.6	5.7	9.3	7.5	12.9	9.3	21.8	11.1		
2.5	2.3	4.6	4.1	6.9	5.9	9.6	7.7	13.5	9.5	24.7	11.3		
2.7	2.5	4.8	4.3	7.2	6.1	10.0	7.9	14.1	9.7	29.3	11.5		
$\alpha=5\%$													
1.9	1.8	3.8	3.6	6.1	5.6	8.6	7.6	11.5	9.6	15.9	11.6	35.6	13.6
2.0	1.9	4.0	3.8	6.3	5.8	8.9	7.8	11.9	9.8	16.5	11.8	49.0	13.8
2.1	2.0	4.3	4.0	6.6	6.0	9.1	8.0	12.2	10.0	17.2	12.0	94.6	14.0
2.3	2.2	4.5	4.2	6.8	6.2	9.4	8.2	12.6	10.2	18.0	12.2	1000	14.053
2.5	2.4	4.7	4.4	7.1	6.4	9.7	8.4	13.0	10.4	19.0	12.4	∞	14.067
2.7	2.6	4.9	4.6	7.3	6.6	10.0	8.6	13.4	10.6	20.1	12.6		
2.9	2.8	5.2	4.8	7.5	6.8	10.3	8.8	13.9	10.8	21.4	12.8		
3.1	3.0	5.4	5.0	7.8	7.0	10.6	9.0	14.3	11.0	23.2	13.0		
3.4	3.2	5.6	5.2	8.1	7.2	10.9	9.2	14.8	11.2	25.6	13.2		
3.6	3.4	5.9	5.4	8.3	7.4	11.2	9.4	15.3	11.4	29.3	13.4		
$\alpha=1\%$													
4.2	4.1	6.4	6.2	9.0	8.6	11.8	11.0	15.1	13.4	20.1	15.8	62.0	18.2
4.3	4.2	6.7	6.5	9.3	8.9	12.1	11.3	15.6	13.7	21.0	16.1	117.1	18.4
4.5	4.4	7.0	6.8	9.6	9.2	12.5	11.6	16.1	14.0	22.2	16.4	1000	18.459
4.8	4.7	7.3	7.1	10.0	9.5	12.9	11.9	16.6	14.3	23.6	16.7	∞	18.475
5.1	5.0	7.6	7.4	10.3	9.8	13.3	12.2	17.2	14.6	25.5	17.0		
5.4	5.3	8.0	7.7	10.7	10.1	13.7	12.5	17.8	14.9	28.1	17.3		
5.7	5.6	8.3	8.0	11.0	10.4	14.2	12.8	18.5	15.2	32.4	17.6		
6.0	5.9	8.6	8.3	11.4	10.7	14.6	13.1	19.2	15.5	40.5	17.9		

tion 4.2) and Elliott, Müller, and Watson (2015) (henceforth AMS and EMW respectively). The testing problem is

$$H_0 : \kappa_2 = 0, \qquad \kappa_1 \geq 0 \quad \text{versus} \quad H_1 : \kappa_2 > 0, \qquad \kappa_1 \geq \kappa_2,$$

where κ_1, κ_2 are the eigenvalues of the noncentrality parameter $\kappa_i(\mathcal{M}'\mathcal{M})$ of the 2×2 noncentral Wishart matrix $\Xi'\Xi \sim \mathcal{W}_2(k, I_2, \mathcal{M}'\mathcal{M})$, and $\hat{\kappa}_i = \kappa_i(\Xi'\Xi)$. The joint density of the eigenvalues $f_{\hat{\kappa}_1,\hat{\kappa}_2}(x_1, x_2; \kappa_1, \kappa_2)$ is given in (A.17).

All simulations in this section are performed using importance sampling. The parameter space for κ_1 under H_0 is discretized into $N_{\kappa_1} = 42$ points, in the same way as for the size calculations before, that is, $\kappa_1 \in \{\kappa_{1,1}, \ldots, \kappa_{1,N_{\kappa_1}}\}$, where $\kappa_{1,j}$ are equally

TABLE 10. $1-\alpha$ quantile of the conditional distribution, with density given in (2.12), cv $= c_{1-\alpha}(\hat{\kappa}_1, k-m_W)$ at different values of the conditioning variable $\hat{\kappa}_1$. Computed by numerical integration.

$k-m_W=8$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
						$\alpha=10\%$							
1.4	1.3	3.3	3.1	5.7	5.1	8.3	7.1	11.4	9.1	16.1	11.1	47.8	13.1
1.5	1.4	3.6	3.3	5.9	5.3	8.6	7.3	11.8	9.3	16.9	11.3	93.3	13.3
1.6	1.5	3.8	3.5	6.2	5.5	8.8	7.5	12.2	9.5	17.7	11.5	1000	13.348
1.8	1.7	4.0	3.7	6.4	5.7	9.1	7.7	12.6	9.7	18.6	11.7	∞	13.362
2.0	1.9	4.3	3.9	6.7	5.9	9.4	7.9	13.0	9.9	19.7	11.9		
2.2	2.1	4.5	4.1	6.9	6.1	9.7	8.1	13.4	10.1	21.0	12.1		
2.4	2.3	4.7	4.3	7.2	6.3	10.1	8.3	13.9	10.3	22.7	12.3		
2.7	2.5	5.0	4.5	7.5	6.5	10.4	8.5	14.4	10.5	25.1	12.5		
2.9	2.7	5.2	4.7	7.7	6.7	10.7	8.7	14.9	10.7	28.7	12.7		
3.1	2.9	5.4	4.9	8.0	6.9	11.1	8.9	15.5	10.9	34.8	12.9		
						$\alpha=5\%$							
2.1	2.0	4.2	4.0	6.7	6.2	9.4	8.4	12.5	10.6	17.1	12.8	37.6	15.0
2.2	2.1	4.4	4.2	6.9	6.4	9.6	8.6	12.9	10.8	17.7	13.0	50.0	15.2
2.3	2.2	4.6	4.4	7.1	6.6	9.9	8.8	13.2	11.0	18.4	13.2	86.5	15.4
2.5	2.4	4.9	4.6	7.4	6.8	10.2	9.0	13.6	11.2	19.1	13.4	155.6	15.5
2.7	2.6	5.1	4.8	7.6	7.0	10.4	9.2	13.9	11.4	19.9	13.6	1000	15.492
2.9	2.8	5.3	5.0	7.9	7.2	10.7	9.4	14.3	11.6	20.9	13.8	∞	15.507
3.1	3.0	5.5	5.2	8.1	7.4	11.0	9.6	14.7	11.8	22.0	14.0		
3.3	3.2	5.8	5.4	8.4	7.6	11.3	9.8	15.1	12.0	23.4	14.2		
3.5	3.4	6.0	5.6	8.6	7.8	11.6	10.0	15.6	12.2	25.2	14.4		
3.8	3.6	6.2	5.8	8.9	8.0	11.9	10.2	16.1	12.4	27.7	14.6		
4.0	3.8	6.4	6.0	9.1	8.2	12.2	10.4	16.6	12.6	31.4	14.8		
						$\alpha=1\%$							
4.7	4.6	6.8	6.7	9.4	9.1	12.2	11.5	15.2	13.9	19.3	16.3	28.5	18.7
4.8	4.7	7.2	7.0	9.7	9.4	12.5	11.8	15.7	14.2	19.9	16.6	31.8	19.0
5.0	4.9	7.5	7.3	10.1	9.7	12.9	12.1	16.1	14.5	20.6	16.9	37.2	19.3
5.3	5.2	7.8	7.6	10.4	10.0	13.2	12.4	16.6	14.8	21.4	17.2	48.4	19.6
5.6	5.5	8.1	7.9	10.8	10.3	13.6	12.7	17.0	15.1	22.4	17.5	82.9	19.9
5.9	5.8	8.4	8.2	11.1	10.6	14.0	13.0	17.6	15.4	23.4	17.8	118.4	20.0
6.2	6.1	8.8	8.5	11.4	10.9	14.4	13.3	18.1	15.7	24.7	18.1	1000	20.073
6.5	6.4	9.1	8.8	11.8	11.2	14.8	13.6	18.7	16.0	26.3	18.4	∞	20.090

spaced in log-scale between 0 and 100. We will compute point-optimal power bounds over a grid of point alternatives. Let F denote a distribution over H_1, so that $H_{1,F} \in H_1$ is a point alternative, and let g denote the density of the data under $H_{1,F}$. For the power envelope, we consider one-point distributions F, whose support varies over the range $\kappa_2 \in [0.1, \bar{\kappa}_2(k)]$, discretized into 30 equally spaced points, and $\kappa_1 - \kappa_2 \in \{0, 1, 2, 4, 8, 16, 32, 64\}$. We do not consider greater values of $\kappa_1 - \kappa_2$ because the power curves of κ_2 are already indistinguishable at $\kappa_1 - \kappa_2 = 64$. The upper bound of κ_2 under H_1, $\bar{\kappa}_2(k)$, is chosen to be about just high enough for the power of the conditional subvector AR test φ_c to be above 0.99, and it necessarily varies with k (larger values are

TABLE 11. $1-\alpha$ quantile of the conditional distribution, with density given in (2.12), cv $= c_{1-\alpha}(\hat{\kappa}_1, k-m_W)$ at different values of the conditioning variable $\hat{\kappa}_1$. Computed by numerical integration.

$k-m_W=9$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
$\alpha=10\%$													
1.5	1.4	3.4	3.2	5.7	5.2	8.2	7.2	11.0	9.2	14.6	11.2	21.6	13.2
1.6	1.5	3.6	3.4	5.9	5.4	8.4	7.4	11.3	9.4	15.1	11.4	23.0	13.4
1.7	1.6	3.9	3.6	6.2	5.6	8.7	7.6	11.6	9.6	15.6	11.6	24.8	13.6
1.9	1.8	4.1	3.8	6.4	5.8	9.0	7.8	11.9	9.8	16.1	11.8	27.3	13.8
2.1	2.0	4.3	4.0	6.7	6.0	9.2	8.0	12.3	10.0	16.6	12.0	31.0	14.0
2.3	2.2	4.5	4.2	6.9	6.2	9.5	8.2	12.6	10.2	17.3	12.2	37.3	14.2
2.5	2.4	4.8	4.4	7.2	6.4	9.8	8.4	13.0	10.4	17.9	12.4	50.3	14.4
2.8	2.6	5.0	4.6	7.4	6.6	10.1	8.6	13.4	10.6	18.7	12.6	91.8	14.6
3.0	2.8	5.2	4.8	7.7	6.8	10.4	8.8	13.8	10.8	19.5	12.8	1000	14.669
3.2	3.0	5.5	5.0	7.9	7.0	10.7	9.0	14.2	11.0	20.5	13.0	∞	14.684
$\alpha=5\%$													
2.3	2.2	4.5	4.3	7.2	6.7	10.0	9.1	13.4	11.5	18.1	13.9	36.9	16.3
2.4	2.3	4.8	4.6	7.5	7.0	10.4	9.4	13.8	11.8	18.9	14.2	53.5	16.6
2.6	2.5	5.1	4.9	7.8	7.3	10.8	9.7	14.3	12.1	19.9	14.5	154.4	16.9
2.9	2.8	5.5	5.2	8.2	7.6	11.2	10.0	14.9	12.4	21.0	14.8	1000	16.903
3.2	3.1	5.8	5.5	8.6	7.9	11.6	10.3	15.4	12.7	22.4	15.1	∞	16.919
3.5	3.4	6.1	5.8	8.9	8.2	12.0	10.6	16.0	13.0	24.1	15.4		
3.8	3.7	6.5	6.1	9.3	8.5	12.5	10.9	16.6	13.3	26.5	15.7		
4.2	4.0	6.8	6.4	9.7	8.8	12.9	11.2	17.3	13.6	30.2	16.0		
$\alpha=1\%$													
5.2	5.1	7.6	7.5	10.5	10.2	13.6	12.9	17.1	15.6	22.1	18.3	44.1	21.0
5.3	5.2	8.0	7.8	10.9	10.5	14.0	13.2	17.6	15.9	22.9	18.6	61.9	21.3
5.5	5.4	8.3	8.1	11.2	10.8	14.4	13.5	18.1	16.2	23.8	18.9	143.4	21.6
5.8	5.7	8.6	8.4	11.5	11.1	14.7	13.8	18.5	16.5	24.8	19.2	1000	21.647
6.1	6.0	8.9	8.7	11.9	11.4	15.1	14.1	19.0	16.8	26.0	19.5	∞	21.666
6.4	6.3	9.2	9.0	12.2	11.7	15.5	14.4	19.6	17.1	27.5	19.8		
6.7	6.6	9.6	9.3	12.6	12.0	15.9	14.7	20.1	17.4	29.4	20.1		
7.0	6.9	9.9	9.6	12.9	12.3	16.3	15.0	20.7	17.7	32.2	20.4		
7.3	7.2	10.2	9.9	13.3	12.6	16.7	15.3	21.4	18.0	36.4	20.7		

needed for higher k). With some experimentation, we picked $\bar{\kappa}_2(2)=25$, $\bar{\kappa}_2(5)=30$, $\bar{\kappa}_2(10)=38$, $\bar{\kappa}_2(20)=46$. We index the density of the data under the alternative by $r=1,\ldots,N_r=30\times 8$, so that $g_r(\cdot)=f_{\hat{\kappa}_1,\hat{\kappa}_2}(\cdot;\kappa_{1,r},\kappa_{2,r})$.

Let $\boldsymbol{x}_{i,j}\in\Re^2$ denote a draw from $\mathcal{W}_2(k,I_2,\mathrm{diag}(\kappa_{1,j},0))$. We draw N_0 simulations from each of N_{κ_1} data generating processes (DGPs). We abbreviate by $f_l(\boldsymbol{x}_{i,j})$ the joint density (A.17) at parameter l evaluated at the ith draw $x_{i,j}$ from DGP j, that is,

$$\begin{aligned} f_l(\boldsymbol{x}_{i,j}) &= f_{\hat{\kappa}_1,\hat{\kappa}_2}(x_{1,i,j},x_{2,i,j};\kappa_{1,l},0),\\ x_{s,i,j} &= \kappa_s(X_i),\quad s=1,2,\qquad X_i\sim\mathcal{W}_2\big(k,I_2,\mathrm{diag}(\kappa_{1,j},0)\big),\\ l,j &= 1,\ldots,N_{\kappa_1}, i=1,\ldots,N_0. \end{aligned}$$

TABLE 12. $1-\alpha$ quantile of the conditional distribution, with density given in (2.12), cv $= c_{1-\alpha}(\hat{\kappa}_1, k-m_W)$ at different values of the conditioning variable $\hat{\kappa}_1$. Computed by numerical integration.

$k-m_W=10$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
$\alpha=10\%$													
1.7	1.6	3.9	3.7	6.7	6.1	9.7	8.5	13.2	10.9	18.4	13.3	54.4	15.7
1.8	1.7	4.3	4.0	7.0	6.4	10.1	8.8	13.7	11.2	19.4	13.6	98.3	15.9
2.0	1.9	4.6	4.3	7.4	6.7	10.5	9.1	14.2	11.5	20.5	13.9	1000	15.971
2.3	2.2	4.9	4.6	7.8	7.0	10.9	9.4	14.8	11.8	21.9	14.2	∞	15.987
2.6	2.5	5.3	4.9	8.1	7.3	11.3	9.7	15.4	12.1	23.6	14.5		
2.9	2.8	5.6	5.2	8.5	7.6	11.8	10.0	16.1	12.4	26.0	14.8		
3.3	3.1	6.0	5.5	8.9	7.9	12.2	10.3	16.8	12.7	29.7	15.1		
3.6	3.4	6.3	5.8	9.3	8.2	12.7	10.6	17.5	13.0	36.6	15.4		
$\alpha=5\%$													
2.5	2.4	5.0	4.8	8.0	7.5	11.2	10.2	15.0	12.9	20.8	15.6	183.9	18.3
2.6	2.5	5.3	5.1	8.3	7.8	11.6	10.5	15.5	13.2	21.8	15.9	1000	18.289
2.8	2.7	5.6	5.4	8.7	8.1	12.0	10.8	16.0	13.5	23.0	16.2	∞	18.307
3.1	3.0	6.0	5.7	9.0	8.4	12.4	11.1	16.6	13.8	24.4	16.5		
3.4	3.3	6.3	6.0	9.4	8.7	12.8	11.4	17.1	14.1	26.3	16.8		
3.7	3.6	6.6	6.3	9.7	9.0	13.2	11.7	17.7	14.4	28.9	17.1		
4.0	3.9	7.0	6.6	10.1	9.3	13.6	12.0	18.4	14.7	32.9	17.4		
4.4	4.2	7.3	6.9	10.5	9.6	14.1	12.3	19.1	15.0	40.4	17.7		
4.7	4.5	7.6	7.2	10.8	9.9	14.5	12.6	19.9	15.3	59.3	18.0		
$\alpha=1\%$													
5.7	5.6	8.1	8.0	11.0	10.7	14.0	13.4	17.4	16.1	21.6	18.8	30.6	21.5
5.8	5.7	8.5	8.3	11.3	11.0	14.4	13.7	17.8	16.4	22.2	19.1	33.0	21.8
6.0	5.9	8.8	8.6	11.7	11.3	14.7	14.0	18.2	16.7	22.9	19.4	36.7	22.1
6.3	6.2	9.1	8.9	12.0	11.6	15.1	14.3	18.6	17.0	23.6	19.7	42.7	22.4
6.6	6.5	9.4	9.2	12.3	11.9	15.5	14.6	19.1	17.3	24.4	20.0	54.9	22.7
6.9	6.8	9.7	9.5	12.7	12.2	15.8	14.9	19.6	17.6	25.2	20.3	90.9	23.0
7.2	7.1	10.0	9.8	13.0	12.5	16.2	15.2	20.0	17.9	26.2	20.6	221.2	23.2
7.5	7.4	10.4	10.1	13.3	12.8	16.6	15.5	20.5	18.2	27.4	20.9	1000	23.190
7.8	7.7	10.7	10.4	13.7	13.1	17.0	15.8	21.1	18.5	28.8	21.2	∞	23.209

The rejection probability of any test $\varphi(\boldsymbol{x})$, under DGP j, $RP_j(\varphi)$, is computed by Monte Carlo integration with importance sampling using the formula

$$\widehat{RP}_j(\varphi) = \frac{1}{N_{\kappa_1} N_0} \sum_{l=1}^{N_{\kappa_1}} \sum_{i=1}^{N_0} \frac{f_j(\boldsymbol{x}_{i,l})}{\bar{f}(\boldsymbol{x}_{i,l})} \varphi(\boldsymbol{x}_{i,l}), \tag{D.1}$$

where $\bar{f}(\cdot) = N_{\kappa_1}^{-1} \sum_{j=1}^{N_{\kappa_1}} f_j(\cdot)$.

Let Λ denote a distribution over the space of the nuisance parameter κ_1, that is, a distribution over H_0. A point null hypothesis $H_{0,\Lambda} \in H_0$ is defined by the distribution $\int f_{\kappa_1} \, d\Lambda$, and is approximated here by $\sum_{l=1}^{N_{\kappa_1}} f_l(\cdot) w_{l,\Lambda}$, where $w_{l,\Lambda}$, $l = 1, \ldots, N_{\kappa_1}$ are the weights over the (discretized) support of Λ. A least favorable distribution Λ^{LF} for test-

TABLE 13. $1-\alpha$ quantile of the conditional distribution, with density given in (2.12), cv $= c_{1-\alpha}(\hat{\kappa}_1, k-m_W)$ at different values of the conditioning variable $\hat{\kappa}_1$. Computed by numerical integration.

$k-m_W=11$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
$\alpha=10\%$													
1.8	1.7	4.0	3.8	6.7	6.2	9.6	8.6	12.8	11.0	17.1	13.4	25.7	15.8
1.9	1.8	4.3	4.1	7.1	6.5	10.0	8.9	13.3	11.3	17.7	13.7	28.3	16.1
2.1	2.0	4.7	4.4	7.4	6.8	10.3	9.2	13.8	11.6	18.5	14.0	32.4	16.4
2.4	2.3	5.0	4.7	7.8	7.1	10.7	9.5	14.2	11.9	19.3	14.3	40.1	16.7
2.7	2.6	5.3	5.0	8.1	7.4	11.1	9.8	14.8	12.2	20.2	14.6	60.3	17.0
3.0	2.9	5.7	5.3	8.5	7.7	11.6	10.1	15.3	12.5	21.2	14.9	112.6	17.2
3.4	3.2	6.0	5.6	8.8	8.0	12.0	10.4	15.8	12.8	22.4	15.2	1000	17.258
3.7	3.5	6.4	5.9	9.2	8.3	12.4	10.7	16.4	13.1	23.9	15.5	∞	17.275
$\alpha=5\%$													
2.8	2.7	5.3	5.1	8.2	7.8	11.4	10.5	14.9	13.2	19.7	15.9	32.8	18.6
2.9	2.8	5.6	5.4	8.6	8.1	11.7	10.8	15.4	13.5	20.4	16.2	38.3	18.9
3.1	3.0	5.9	5.7	8.9	8.4	12.1	11.1	15.8	13.8	21.2	16.5	49.8	19.2
3.4	3.3	6.3	6.0	9.3	8.7	12.5	11.4	16.3	14.1	22.0	16.8	86.7	19.5
3.7	3.6	6.6	6.3	9.6	9.0	12.9	11.7	16.8	14.4	23.0	17.1	127.1	19.6
4.0	3.9	6.9	6.6	10.0	9.3	13.3	12.0	17.3	14.7	24.1	17.4	1000	19.656
4.3	4.2	7.2	6.9	10.3	9.6	13.7	12.3	17.9	15.0	25.5	17.7	∞	19.675
4.6	4.5	7.6	7.2	10.7	9.9	14.1	12.6	18.4	15.3	27.2	18.0		
5.0	4.8	7.9	7.5	11.0	10.2	14.5	12.9	19.1	15.6	29.4	18.3		
$\alpha=1\%$													
6.1	6.0	8.9	8.8	12.3	12.0	16.0	15.2	20.1	18.4	26.5	21.6	208.0	24.7
6.2	6.1	9.4	9.2	12.8	12.4	16.4	15.6	20.8	18.8	27.7	22.0	1000	24.705
6.5	6.4	9.8	9.6	13.2	12.8	16.9	16.0	21.4	19.2	29.3	22.4	∞	24.725
6.9	6.8	10.2	10.0	13.7	13.2	17.4	16.4	22.1	19.6	31.3	22.8		
7.3	7.2	10.6	10.4	14.1	13.6	17.9	16.8	22.8	20.0	34.2	23.2		
7.7	7.6	11.1	10.8	14.6	14.0	18.5	17.2	23.6	20.4	38.9	23.6		
8.1	8.0	11.5	11.2	15.0	14.4	19.0	17.6	24.4	20.8	48.3	24.0		
8.5	8.4	11.9	11.6	15.5	14.8	19.6	18.0	25.4	21.2	75.7	24.4		

ing H_0 against a particular point alternative $H_{1,F}$ (if it exists) is such that the α-level Neyman–Pearson test of $H_{0,\Lambda^{LF}}$ against $H_{1,F}$ has size α under H_0.

The least favorable distribution Λ^{LF} is not known in this application. As shown in Elliott, Müller, and Watson (2015, Lemma 1), any Neyman–Pearson test φ_Λ of size α under $H_{0,\Lambda}$ will provide an upper bound on the power of tests of H_0. But the power bound may be quite conservative in the sense that it could be far above the least upper bound. The procedures in AMS and EMW are designed to produce bounds that are close to the least upper bound obtained using Λ^{LF}. AMS consider one-point distributions Λ, and provide upper and lower bounds on the power envelope. The upper bound is obtained by looking for the (one-point) distribution Λ^* that gives the smallest size under H_0, that is, $\max_{\kappa_1} E_{\kappa_1}(\varphi_{\Lambda^*}) \leq \max_{\kappa_1} E_{\kappa_1}(\varphi_\Lambda)$ for all one-point distributions Λ, where $E_{\kappa_1}(\cdot)$ is expectation w.r.t. the Null distribution indexed by κ_1. When the size of φ_{Λ^*} exceeds α this

TABLE 14. $1-\alpha$ quantile of the conditional distribution, with density given in (2.12), cv $= c_{1-\alpha}(\hat{\kappa}_1, k-m_W)$ at different values of the conditioning variable $\hat{\kappa}_1$. Computed by numerical integration.

						$k-m_W=12$							
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
						$\alpha=10\%$							
2.0	1.9	4.5	4.3	7.5	7.0	10.8	9.7	14.5	12.4	19.7	15.1	37.6	17.8
2.1	2.0	4.9	4.6	7.9	7.3	11.2	10.0	15.0	12.7	20.4	15.4	49.3	18.1
2.3	2.2	5.2	4.9	8.2	7.6	11.6	10.3	15.5	13.0	21.3	15.7	89.6	18.4
2.6	2.5	5.5	5.2	8.6	7.9	12.0	10.6	16.0	13.3	22.3	16.0	139.0	18.5
2.9	2.8	5.9	5.5	9.0	8.2	12.4	10.9	16.5	13.6	23.4	16.3	1000	18.531
3.2	3.1	6.2	5.8	9.3	8.5	12.8	11.2	17.1	13.9	24.8	16.6	∞	18.549
3.6	3.4	6.5	6.1	9.7	8.8	13.2	11.5	17.7	14.2	26.5	16.9		
3.9	3.7	6.9	6.4	10.0	9.1	13.6	11.8	18.3	14.5	28.8	17.2		
4.2	4.0	7.2	6.7	10.4	9.4	14.1	12.1	18.9	14.8	32.0	17.5		
						$\alpha=5\%$							
3.0	2.9	5.8	5.6	9.1	8.6	12.5	11.6	16.6	14.6	22.2	17.6	56.7	20.6
3.1	3.0	6.1	5.9	9.4	8.9	12.9	11.9	17.0	14.9	23.0	17.9	109.1	20.9
3.3	3.2	6.4	6.2	9.7	9.2	13.3	12.2	17.5	15.2	23.9	18.2	181.6	21.0
3.6	3.5	6.8	6.5	10.1	9.5	13.7	12.5	18.0	15.5	25.0	18.5	1000	21.006
3.9	3.8	7.1	6.8	10.4	9.8	14.1	12.8	18.5	15.8	26.2	18.8	∞	21.026
4.2	4.1	7.4	7.1	10.8	10.1	14.5	13.1	19.0	16.1	27.6	19.1		
4.5	4.4	7.7	7.4	11.1	10.4	14.9	13.4	19.6	16.4	29.5	19.4		
4.8	4.7	8.1	7.7	11.5	10.7	15.3	13.7	20.2	16.7	32.0	19.7		
5.2	5.0	8.4	8.0	11.8	11.0	15.7	14.0	20.8	17.0	35.8	20.0		
5.5	5.3	8.7	8.3	12.2	11.3	16.1	14.3	21.5	17.3	42.4	20.3		
						$\alpha=1\%$							
6.6	6.5	9.4	9.3	12.8	12.5	16.4	15.7	20.4	18.9	25.7	22.1	45.5	25.3
6.7	6.6	9.9	9.7	13.2	12.9	16.8	16.1	20.9	19.3	26.7	22.5	61.7	25.7
7.0	6.9	10.3	10.1	13.7	13.3	17.3	16.5	21.5	19.7	27.7	22.9	136.8	26.1
7.4	7.3	10.7	10.5	14.1	13.7	17.8	16.9	22.1	20.1	28.9	23.3	232.0	26.2
7.8	7.7	11.1	10.9	14.6	14.1	18.3	17.3	22.7	20.5	30.4	23.7	1000	26.197
8.2	8.1	11.5	11.3	15.0	14.5	18.8	17.7	23.4	20.9	32.2	24.1	∞	26.217
8.6	8.5	12.0	11.7	15.5	14.9	19.3	18.1	24.1	21.3	34.8	24.5		
9.0	8.9	12.4	12.1	15.9	15.3	19.8	18.5	24.9	21.7	38.6	24.9		

bound may be too high. We will report here only the upper bound of AMS, because it is close to, and often indistinguishable from, the bound obtained by the ALFD method of EMW.

D.3.1 *AMS bound* The AMS algorithm for the upper bound on power, with a slight modification to do importance sampling, is as follows:

1. For each j, $j=1,\ldots,N_{\kappa_1}$, generate N_0 draws $\boldsymbol{x}_{i,j}$, $i=1,\ldots,N_0$ with density f_j. The draws must be independent across i and j.

2. Compute and store the importance sampling weights $\omega_{l,i,j}=f_l(\boldsymbol{x}_{i,j})/\bar{f}(\boldsymbol{x}_{i,j})$, $l,j=1,\ldots,N_{\kappa_1}$, $i=1,\ldots,N_0$.

TABLE 15. $1-\alpha$ quantile of the conditional distribution, with density given in (2.12), $\text{cv} = c_{1-\alpha}(\hat{\kappa}_1, k-m_W)$ at different values of the conditioning variable $\hat{\kappa}_1$. Computed by numerical integration.

						$k-m_W=13$							
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
						$\alpha=10\%$							
2.1	2.0	4.9	4.7	8.3	7.7	11.9	10.7	16.1	13.7	22.2	16.7	109.7	19.7
2.2	2.1	5.3	5.0	8.6	8.0	12.2	11.0	16.5	14.0	23.1	17.0	192.2	19.8
2.4	2.3	5.6	5.3	9.0	8.3	12.6	11.3	17.0	14.3	24.1	17.3	1000	19.793
2.7	2.6	5.9	5.6	9.3	8.6	13.0	11.6	17.6	14.6	25.4	17.6	∞	19.812
3.0	2.9	6.3	5.9	9.7	8.9	13.4	11.9	18.1	14.9	26.8	17.9		
3.3	3.2	6.6	6.2	10.0	9.2	13.8	12.2	18.7	15.2	28.7	18.2		
3.6	3.5	6.9	6.5	10.4	9.5	14.3	12.5	19.3	15.5	31.1	18.5		
4.0	3.8	7.3	6.8	10.7	9.8	14.7	12.8	19.9	15.8	34.8	18.8		
4.3	4.1	7.6	7.1	11.1	10.1	15.1	13.1	20.6	16.1	41.2	19.1		
4.6	4.4	7.9	7.4	11.5	10.4	15.6	13.4	21.4	16.4	55.3	19.4		
						$\alpha=5\%$							
3.2	3.1	6.0	5.8	9.2	8.8	12.6	11.8	16.4	14.8	21.3	17.8	32.0	20.8
3.3	3.2	6.3	6.1	9.5	9.1	13.0	12.1	16.8	15.1	21.9	18.1	34.9	21.1
3.5	3.4	6.6	6.4	9.9	9.4	13.3	12.4	17.3	15.4	22.6	18.4	39.3	21.4
3.8	3.7	6.9	6.7	10.2	9.7	13.7	12.7	17.7	15.7	23.3	18.7	47.4	21.7
4.1	4.0	7.3	7.0	10.6	10.0	14.1	13.0	18.2	16.0	24.0	19.0	66.2	22.0
4.4	4.3	7.6	7.3	10.9	10.3	14.5	13.3	18.6	16.3	24.9	19.3	154.5	22.3
4.7	4.6	7.9	7.6	11.2	10.6	14.8	13.6	19.1	16.6	25.9	19.6	1000	22.341
5.0	4.9	8.2	7.9	11.6	10.9	15.2	13.9	19.6	16.9	27.0	19.9	∞	22.362
5.3	5.2	8.6	8.2	11.9	11.2	15.6	14.2	20.2	17.2	28.3	20.2		
5.7	5.5	8.9	8.5	12.3	11.5	16.0	14.5	20.7	17.5	29.9	20.5		
						$\alpha=1\%$							
7.1	7.0	9.9	9.8	13.3	13.0	16.8	16.2	20.7	19.4	25.5	22.6	35.6	25.8
7.2	7.1	10.3	10.2	13.7	13.4	17.3	16.6	21.2	19.8	26.2	23.0	38.9	26.2
7.5	7.4	10.8	10.6	14.2	13.8	17.7	17.0	21.7	20.2	27.1	23.4	44.4	26.6
7.9	7.8	11.2	11.0	14.6	14.2	18.2	17.4	22.3	20.6	28.0	23.8	55.8	27.0
8.3	8.2	11.6	11.4	15.0	14.6	18.7	17.8	22.9	21.0	29.0	24.2	90.7	27.4
8.7	8.6	12.0	11.8	15.5	15.0	19.2	18.2	23.5	21.4	30.2	24.6	162.1	27.6
9.1	9.0	12.4	12.2	15.9	15.4	19.6	18.6	24.1	21.8	31.6	25.0	1000	27.668
9.5	9.4	12.9	12.6	16.4	15.8	20.1	19.0	24.8	22.2	33.3	25.4	∞	27.688

3. Set $r=1$.

4. Compute $LR_l(\boldsymbol{x}_{i,j}) = g_r(\boldsymbol{x}_{i,j})/f_l(\boldsymbol{x}_{i,j})$, $l, j = 1, \ldots, N_{\kappa_1}$, $i = 1, \ldots, N_0$.

5. Computation of cvs under H_0: For each $l = 1, \ldots, N_{\kappa_1}$, find $\varkappa_l$ by solving $\widehat{RP}_l(\varphi_l) = \alpha$, where $\varphi_l := \mathbf{1}[LR_l > \varkappa_l]$ is the LR test of f_l against g_r with critical value $\varkappa_l$, and $\widehat{RP}_l(\varphi_l)$ is the Monte Carlo estimate (D.1) with the weights $\omega_{l,i,j}$ computed in step 2.

6. Computation of size of each test: For each $l, j = 1, \ldots, N_{\kappa_1}$, compute $\widehat{RP}_j(\varphi_l)$, and obtain $D_l = \max_{j\in\{1,\ldots,N_{\kappa_1}\}}[\widehat{RP}_j(\varphi_l) - \alpha]$.

7. Find test with size closest to α: $l^* = \arg\min_{l\in\{1,\ldots,N_{\kappa_1}\}} D_l$.

TABLE 16. $1-\alpha$ quantile of the conditional distribution, with density given in (2.12), cv $= c_{1-\alpha}(\hat{\kappa}_1, k-m_W)$ at different values of the conditioning variable $\hat{\kappa}_1$. Computed by numerical integration.

$k-m_W=14$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
$\alpha=10\%$													
2.3	2.2	5.1	4.9	8.4	7.9	11.9	10.9	15.9	13.9	21.1	16.9	34.9	19.9
2.4	2.3	5.4	5.2	8.8	8.2	12.3	11.2	16.4	14.2	21.8	17.2	39.9	20.2
2.6	2.5	5.8	5.5	9.1	8.5	12.7	11.5	16.8	14.5	22.6	17.5	49.5	20.5
2.9	2.8	6.1	5.8	9.5	8.8	13.1	11.8	17.3	14.8	23.4	17.8	75.4	20.8
3.2	3.1	6.4	6.1	9.8	9.1	13.5	12.1	17.8	15.1	24.3	18.1	145.2	21.0
3.5	3.4	6.8	6.4	10.1	9.4	13.8	12.4	18.3	15.4	25.3	18.4	1000	21.044
3.8	3.7	7.1	6.7	10.5	9.7	14.2	12.7	18.8	15.7	26.4	18.7	∞	21.064
4.2	4.0	7.4	7.0	10.9	10.0	14.6	13.0	19.3	16.0	27.8	19.0		
4.5	4.3	7.8	7.3	11.2	10.3	15.1	13.3	19.9	16.3	29.5	19.3		
4.8	4.6	8.1	7.6	11.6	10.6	15.5	13.6	20.5	16.6	31.7	19.6		
$\alpha=5\%$													
3.4	3.3	6.5	6.3	10.0	9.6	13.8	12.9	18.0	16.2	23.7	19.5	43.4	22.8
3.5	3.4	6.8	6.6	10.4	9.9	14.1	13.2	18.4	16.5	24.3	19.8	53.7	23.1
3.7	3.6	7.1	6.9	10.7	10.2	14.5	13.5	18.9	16.8	25.1	20.1	80.3	23.4
4.0	3.9	7.4	7.2	11.0	10.5	14.9	13.8	19.3	17.1	25.9	20.4	145.9	23.6
4.3	4.2	7.8	7.5	11.4	10.8	15.2	14.1	19.8	17.4	26.8	20.7	1000	23.663
4.6	4.5	8.1	7.8	11.7	11.1	15.6	14.4	20.3	17.7	27.9	21.0	∞	23.685
4.9	4.8	8.4	8.1	12.0	11.4	16.0	14.7	20.8	18.0	29.1	21.3		
5.2	5.1	8.7	8.4	12.4	11.7	16.4	15.0	21.3	18.3	30.5	21.6		
5.5	5.4	9.1	8.7	12.7	12.0	16.8	15.3	21.8	18.6	32.3	21.9		
5.9	5.7	9.4	9.0	13.1	12.3	17.2	15.6	22.4	18.9	34.6	22.2		
6.2	6.0	9.7	9.3	13.4	12.6	17.6	15.9	23.0	19.2	38.0	22.5		
$\alpha=1\%$													
7.6	7.5	10.8	10.7	14.6	14.3	18.6	17.9	23.1	21.5	29.3	25.1	75.0	28.7
7.7	7.6	11.3	11.1	15.1	14.7	19.1	18.3	23.6	21.9	30.3	25.5	216.0	29.1
8.0	7.9	11.7	11.5	15.5	15.1	19.5	18.7	24.2	22.3	31.5	25.9	1000	29.120
8.4	8.3	12.1	11.9	15.9	15.5	20.0	19.1	24.8	22.7	32.8	26.3	∞	29.141
8.8	8.7	12.5	12.3	16.4	15.9	20.5	19.5	25.5	23.1	34.4	26.7		
9.2	9.1	12.9	12.7	16.8	16.3	21.0	19.9	26.1	23.5	36.6	27.1		
9.6	9.5	13.4	13.1	17.2	16.7	21.5	20.3	26.8	23.9	39.5	27.5		
10.0	9.9	13.8	13.5	17.7	17.1	22.0	20.7	27.6	24.3	44.2	27.9		
10.4	10.3	14.2	13.9	18.2	17.5	22.5	21.1	28.4	24.7	53.0	28.3		

8. Compute the AMS upper bound $\bar{\pi}_r = N_1^{-1}\sum_{i=1}^{N_1}\varphi_{l^*}(\boldsymbol{x}_i)$, where $\boldsymbol{x}_i$ are i.i.d. draws of $(\hat{\kappa}_1, \hat{\kappa}_2)$ with density g_r.

9. If $r < N_r$, set $r = r+1$ and go to step 4.

All the reported results are based on $N_0 = 10{,}000$ and $N_1 = 100{,}000$. (We can use a smaller number of simulations under H_0 for a similar level of precision due to importance sampling.)

TABLE 17. $1-\alpha$ quantile of the conditional distribution, with density given in (2.12), cv $= c_{1-\alpha}(\hat{\kappa}_1, k-m_W)$ at different values of the conditioning variable $\hat{\kappa}_1$. Computed by numerical integration.

$k-m_W=15$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
$\alpha = 10\%$													
2.4	2.3	5.5	5.3	9.2	8.6	13.0	11.9	17.4	15.2	23.5	18.5	55.6	21.8
2.5	2.4	5.9	5.6	9.5	8.9	13.4	12.2	17.9	15.5	24.3	18.8	91.2	22.1
2.7	2.6	6.2	5.9	9.8	9.2	13.8	12.5	18.3	15.8	25.1	19.1	224.8	22.3
3.0	2.9	6.5	6.2	10.2	9.5	14.1	12.8	18.8	16.1	26.1	19.4	1000	22.286
3.3	3.2	6.8	6.5	10.5	9.8	14.5	13.1	19.3	16.4	27.1	19.7	∞	22.307
3.6	3.5	7.2	6.8	10.9	10.1	14.9	13.4	19.8	16.7	28.4	20.0		
3.9	3.8	7.5	7.1	11.2	10.4	15.3	13.7	20.4	17.0	29.9	20.3		
4.2	4.1	7.8	7.4	11.6	10.7	15.7	14.0	20.9	17.3	31.7	20.6		
4.6	4.4	8.1	7.7	11.9	11.0	16.1	14.3	21.5	17.6	34.2	20.9		
4.9	4.7	8.5	8.0	12.3	11.3	16.6	14.6	22.1	17.9	37.8	21.2		
5.2	5.0	8.8	8.3	12.6	11.6	17.0	14.9	22.8	18.2	43.7	21.5		
$\alpha = 5\%$													
3.7	3.6	7.0	6.8	10.9	10.4	14.9	14.0	19.6	17.6	26.2	21.2	104.1	24.8
3.8	3.7	7.4	7.2	11.3	10.8	15.4	14.4	20.2	18.0	27.2	21.6	146.4	24.9
4.1	4.0	7.8	7.6	11.7	11.2	15.9	14.8	20.8	18.4	28.5	22.0	1000	24.973
4.5	4.4	8.3	8.0	12.2	11.6	16.4	15.2	21.4	18.8	29.9	22.4	∞	24.996
4.9	4.8	8.7	8.4	12.6	12.0	16.9	15.6	22.1	19.2	31.7	22.8		
5.3	5.2	9.1	8.8	13.1	12.4	17.4	16.0	22.8	19.6	34.1	23.2		
5.7	5.6	9.5	9.2	13.5	12.8	18.0	16.4	23.5	20.0	37.5	23.6		
6.1	6.0	10.0	9.6	14.0	13.2	18.5	16.8	24.3	20.4	43.4	24.0		
6.6	6.4	10.4	10.0	14.5	13.6	19.0	17.2	25.2	20.8	56.2	24.4		
$\alpha = 1\%$													
8.1	8.0	11.3	11.2	15.1	14.8	19.0	18.4	23.4	22.0	29.0	25.6	44.5	29.2
8.2	8.1	11.7	11.6	15.5	15.2	19.5	18.8	23.9	22.4	29.8	26.0	51.8	29.6
8.5	8.4	12.2	12.0	16.0	15.6	20.0	19.2	24.5	22.8	30.7	26.4	67.8	30.0
8.9	8.8	12.6	12.4	16.4	16.0	20.4	19.6	25.0	23.2	31.6	26.8	129.2	30.4
9.3	9.2	13.0	12.8	16.8	16.4	20.9	20.0	25.6	23.6	32.8	27.2	185.6	30.5
9.7	9.6	13.4	13.2	17.3	16.8	21.4	20.4	26.2	24.0	34.1	27.6	1000	30.557
10.1	10.0	13.8	13.6	17.7	17.2	21.9	20.8	26.8	24.4	35.7	28.0	∞	30.578
10.5	10.4	14.3	14.0	18.1	17.6	22.4	21.2	27.5	24.8	37.7	28.4		
10.9	10.8	14.7	14.4	18.6	18.0	22.9	21.6	28.2	25.2	40.4	28.8		

Because the size of the test φ_{l^*} can exceed α, the AMS upper bound $\bar{\pi}$ may be higher than the least upper bound. To gauge this, Figure 8 plots (Monte Carlo estimates of) the size of φ_{l^*} across the different alternatives r. Note that for most alternatives the size of the test φ_{l^*} is close to α, so $\bar{\pi}_r$ could be close to the least upper bound in those cases. However, for alternatives close to H_0 the size of φ_{l^*} deviates substantially from α, and the AMS upper bound $\bar{\pi}_r$ may be higher than the least upper bound. These are precisely the cases in which the conditional subvector AR test has the highest deviations from the power bound.

TABLE 18. $1-\alpha$ quantile of the conditional distribution, with density given in (2.12), cv $= c_{1-\alpha}(\hat{\kappa}_1, k-m_W)$ at different values of the conditioning variable $\hat{\kappa}_1$. Computed by numerical integration.

$k-m_W=16$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
$\alpha=10\%$													
2.6	2.5	5.7	5.5	9.3	8.8	13.1	12.1	17.3	15.4	22.7	18.7	34.9	22.0
2.7	2.6	6.0	5.8	9.6	9.1	13.5	12.4	17.8	15.7	23.4	19.0	38.0	22.3
2.9	2.8	6.4	6.1	10.0	9.4	13.8	12.7	18.2	16.0	24.0	19.3	42.9	22.6
3.2	3.1	6.7	6.4	10.3	9.7	14.2	13.0	18.6	16.3	24.7	19.6	51.8	22.9
3.5	3.4	7.0	6.7	10.7	10.0	14.6	13.3	19.1	16.6	25.5	19.9	73.1	23.2
3.8	3.7	7.3	7.0	11.0	10.3	15.0	13.6	19.6	16.9	26.3	20.2	184.3	23.5
4.1	4.0	7.7	7.3	11.4	10.6	15.3	13.9	20.0	17.2	27.2	20.5	1000	23.519
4.4	4.3	8.0	7.6	11.7	10.9	15.7	14.2	20.5	17.5	28.3	20.8	∞	23.542
4.8	4.6	8.3	7.9	12.0	11.2	16.1	14.5	21.0	17.8	29.5	21.1		
5.1	4.9	8.7	8.2	12.4	11.5	16.5	14.8	21.6	18.1	30.9	21.4		
5.4	5.2	9.0	8.5	12.8	11.8	16.9	15.1	22.1	18.4	32.6	21.7		
$\alpha=5\%$													
3.9	3.8	7.2	7.0	11.0	10.6	15.0	14.2	19.5	17.8	25.3	21.4	40.8	25.0
4.0	3.9	7.6	7.4	11.5	11.0	15.5	14.6	20.1	18.2	26.1	21.8	48.1	25.4
4.3	4.2	8.0	7.8	11.9	11.4	16.0	15.0	20.6	18.6	27.0	22.2	65.4	25.8
4.7	4.6	8.4	8.2	12.3	11.8	16.5	15.4	21.2	19.0	28.0	22.6	153.6	26.2
5.1	5.0	8.9	8.6	12.8	12.2	17.0	15.8	21.8	19.4	29.1	23.0	1000	26.272
5.5	5.4	9.3	9.0	13.2	12.6	17.4	16.2	22.4	19.8	30.5	23.4	∞	26.296
5.9	5.8	9.7	9.4	13.7	13.0	17.9	16.6	23.1	20.2	32.0	23.8		
6.3	6.2	10.2	9.8	14.1	13.4	18.5	17.0	23.8	20.6	34.0	24.2		
6.8	6.6	10.6	10.2	14.6	13.8	19.0	17.4	24.5	21.0	36.7	24.6		
$\alpha=1\%$													
8.5	8.4	12.0	11.9	16.2	15.9	20.6	19.9	25.5	23.9	32.5	27.9	175.4	31.9
8.6	8.5	12.6	12.4	16.8	16.4	21.2	20.4	26.2	24.4	33.8	28.4	1000	31.979
9.0	8.9	13.1	12.9	17.3	16.9	21.8	20.9	27.0	24.9	35.4	28.9	∞	32.000
9.5	9.4	13.6	13.4	17.8	17.4	22.4	21.4	27.7	25.4	37.4	29.4		
10.0	9.9	14.1	13.9	18.4	17.9	23.0	21.9	28.5	25.9	40.0	29.9		
10.5	10.4	14.6	14.4	18.9	18.4	23.6	22.4	29.4	26.4	44.0	30.4		
11.0	10.9	15.2	14.9	19.5	18.9	24.2	22.9	30.3	26.9	51.4	30.9		
11.5	11.4	15.7	15.4	20.0	19.4	24.9	23.4	31.4	27.4	69.6	31.4		

D.3.2 *EMW bound* The PO power bound reported in Figure 3 is based on the ALFD approach of EMW. The ALFD is designed to produce tests that are at most ε away from the true (unknown) power envelope. We apply the algorithm with a slight modification to allow ε to vary across alternatives—for some alternatives, we may be able to get closer to the least upper bound than for others.

We use the following modified version of the algorithm in Elliott, Müller, and Watson (2015, Appendix A.2) without switching, assuming the parameter space for the nuisance parameter is compact. The modification relates to steps 6 to 8 of the original algorithm and cannot underestimate the true power bound.

TABLE 19. $1-\alpha$ quantile of the conditional distribution, with density given in (2.12), cv $= c_{1-\alpha}(\hat{\kappa}_1, k-m_W)$ at different values of the conditioning variable $\hat{\kappa}_1$. Computed by numerical integration.

$k-m_W=17$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
$\alpha=10\%$													
2.7	2.6	6.0	5.8	9.9	9.4	14.1	13.0	18.7	16.6	24.8	20.2	44.6	23.8
2.8	2.7	6.4	6.2	10.4	9.8	14.5	13.4	19.3	17.0	25.7	20.6	58.2	24.2
3.1	3.0	6.9	6.6	10.8	10.2	15.0	13.8	19.8	17.4	26.7	21.0	112.6	24.6
3.5	3.4	7.3	7.0	11.3	10.6	15.5	14.2	20.5	17.8	27.8	21.4	166.2	24.7
3.9	3.8	7.7	7.4	11.7	11.0	16.0	14.6	21.1	18.2	29.1	21.8	1000	24.745
4.3	4.2	8.2	7.8	12.2	11.4	16.5	15.0	21.7	18.6	30.6	22.2	∞	24.769
4.7	4.6	8.6	8.2	12.6	11.8	17.1	15.4	22.4	19.0	32.5	22.6		
5.2	5.0	9.0	8.6	13.1	12.2	17.6	15.8	23.2	19.4	34.9	23.0		
5.6	5.4	9.5	9.0	13.6	12.6	18.1	16.2	24.0	19.8	38.5	23.4		
$\alpha=5\%$													
4.1	4.0	7.4	7.2	11.2	10.8	15.2	14.4	19.5	18.0	24.7	21.6	34.3	25.2
4.2	4.1	7.8	7.6	11.6	11.2	15.6	14.8	20.0	18.4	25.5	22.0	36.5	25.6
4.5	4.4	8.2	8.0	12.1	11.6	16.1	15.2	20.5	18.8	26.2	22.4	39.6	26.0
4.9	4.8	8.6	8.4	12.5	12.0	16.6	15.6	21.1	19.2	27.0	22.8	44.5	26.4
5.3	5.2	9.1	8.8	12.9	12.4	17.0	16.0	21.6	19.6	27.9	23.2	53.8	26.8
5.7	5.6	9.5	9.2	13.4	12.8	17.5	16.4	22.2	20.0	28.8	23.6	78.9	27.2
6.1	6.0	9.9	9.6	13.8	13.2	18.0	16.8	22.8	20.4	29.9	24.0	167.6	27.5
6.5	6.4	10.3	10.0	14.3	13.6	18.5	17.2	23.4	20.8	31.1	24.4	1000	27.562
7.0	6.8	10.8	10.4	14.7	14.0	19.0	17.6	24.1	21.2	32.5	24.8	∞	27.587
$\alpha=1\%$													
9.0	8.9	12.5	12.4	16.7	16.4	21.1	20.4	25.8	24.4	32.1	28.4	55.9	32.4
9.1	9.0	13.0	12.9	17.2	16.9	21.6	20.9	26.5	24.9	33.2	28.9	79.5	32.9
9.5	9.4	13.6	13.4	17.8	17.4	22.2	21.4	27.2	25.4	34.4	29.4	298.9	33.4
10.0	9.9	14.1	13.9	18.3	17.9	22.8	21.9	27.9	25.9	35.8	29.9	1000	33.388
10.5	10.4	14.6	14.4	18.8	18.4	23.4	22.4	28.6	26.4	37.4	30.4	∞	33.409
11.0	10.9	15.1	14.9	19.4	18.9	24.0	22.9	29.4	26.9	39.6	30.9		
11.5	11.4	15.7	15.4	19.9	19.4	24.6	23.4	30.3	27.4	42.5	31.4		
12.0	11.9	16.2	15.9	20.5	19.9	25.2	23.9	31.1	27.9	47.1	31.9		

1. For each j, $j=1,\dots,N_{\kappa_1}$, generate N_0 draws $\boldsymbol{x}_{i,j}$, $i=1,\dots,N_0$ with density f_j. The draws must be independent across i and j.

2. Compute and store $f_l(\boldsymbol{x}_{i,j})$ and $\bar{f}(\boldsymbol{x}_{i,j})$, $l,j=1,\dots,N_{\kappa_1}$, $i=1,\dots,N_0$.

3. Set $r=1$.

4. Set $\mu^{(0)}=(-2,\dots,-2)\in\Re^{N_{\kappa_1}}$.

5. Compute $\mu^{(s+1)}$ from $\mu^{(s)}$ via $\mu_j^{(s+1)}=\mu_j^{(s)}+\omega(\widehat{RP}_j(\varphi^{(s)})-\alpha)$ and $\omega=2$, where $\varphi^{(s)}:=\mathbf{1}[g_r>\sum_{i=1}^{N_{\kappa_1}}\exp(\mu_i^{(s)})f_i]$ and $\widehat{RP}_l(\varphi^{(s)})$ is the Monte Carlo estimate (D.1) with weights computed in step 2, and repeat this step $O=600$ times. Denote the resulting element in the simplex by $\hat{\Lambda}^*=(\hat{\lambda}_1,\dots,\hat{\lambda}_{N_{\kappa_1}})$, where $\hat{\lambda}_j=\exp(\mu_j^{(O)})/\sum_{i=1}^{N_{\kappa_1}}\exp(\mu_i^{(O)})$.

TABLE 20. $1-\alpha$ quantile of the conditional distribution, with density given in (2.12), cv $= c_{1-\alpha}(\hat{\kappa}_1, k-m_W)$ at different values of the conditioning variable $\hat{\kappa}_1$. Computed by numerical integration.

						$k-m_W=18$							
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
						$\alpha=10\%$							
2.9	2.8	6.2	6.0	10.1	9.6	14.2	13.2	18.6	16.8	24.2	20.4	35.2	24.0
3.0	2.9	6.6	6.4	10.5	10.0	14.6	13.6	19.2	17.2	25.0	20.8	38.2	24.4
3.3	3.2	7.1	6.8	11.0	10.4	15.1	14.0	19.7	17.6	25.8	21.2	42.8	24.8
3.7	3.6	7.5	7.2	11.4	10.8	15.6	14.4	20.3	18.0	26.7	21.6	51.6	25.2
4.1	4.0	7.9	7.6	11.9	11.2	16.1	14.8	20.9	18.4	27.6	22.0	75.2	25.6
4.5	4.4	8.4	8.0	12.3	11.6	16.6	15.2	21.5	18.8	28.7	22.4	158.1	25.9
4.9	4.8	8.8	8.4	12.8	12.0	17.1	15.6	22.1	19.2	29.9	22.8	1000	25.965
5.4	5.2	9.2	8.8	13.2	12.4	17.6	16.0	22.8	19.6	31.3	23.2	∞	25.989
5.8	5.6	9.7	9.2	13.7	12.8	18.1	16.4	23.5	20.0	33.0	23.6		
						$\alpha=5\%$							
4.3	4.2	8.0	7.8	12.2	11.8	16.7	15.8	21.6	19.8	27.9	23.8	48.8	27.8
4.4	4.3	8.4	8.2	12.7	12.2	17.1	16.2	22.1	20.2	28.8	24.2	61.3	28.2
4.7	4.6	8.8	8.6	13.1	12.6	17.6	16.6	22.7	20.6	29.7	24.6	101.2	28.6
5.1	5.0	9.2	9.0	13.5	13.0	18.1	17.0	23.2	21.0	30.7	25.0	190.8	28.8
5.5	5.4	9.7	9.4	14.0	13.4	18.5	17.4	23.8	21.4	31.8	25.4	1000	28.843
5.9	5.8	10.1	9.8	14.4	13.8	19.0	17.8	24.4	21.8	33.1	25.8	∞	28.869
6.3	6.2	10.5	10.2	14.8	14.2	19.5	18.2	25.1	22.2	34.7	26.2		
6.7	6.6	10.9	10.6	15.3	14.6	20.0	18.6	25.7	22.6	36.6	26.6		
7.1	7.0	11.4	11.0	15.7	15.0	20.5	19.0	26.4	23.0	39.1	27.0		
7.6	7.4	11.8	11.4	16.2	15.4	21.0	19.4	27.2	23.4	42.7	27.4		
						$\alpha=1\%$							
9.5	9.4	13.0	12.9	17.2	16.9	21.5	20.9	26.2	24.9	32.0	28.9	45.2	32.9
9.6	9.5	13.5	13.4	17.7	17.4	22.1	21.4	26.8	25.4	32.9	29.4	50.5	33.4
10.0	9.9	14.1	13.9	18.2	17.9	22.6	21.9	27.5	25.9	33.9	29.9	61.4	33.9
10.5	10.4	14.6	14.4	18.8	18.4	23.2	22.4	28.1	26.4	35.0	30.4	93.8	34.4
11.0	10.9	15.1	14.9	19.3	18.9	23.8	22.9	28.8	26.9	36.3	30.9	1000	34.785
11.5	11.4	15.6	15.4	19.9	19.4	24.4	23.4	29.6	27.4	37.7	31.4	∞	34.805
12.0	11.9	16.1	15.9	20.4	19.9	25.0	23.9	30.3	27.9	39.5	31.9		
12.5	12.4	16.7	16.4	20.9	20.4	25.6	24.4	31.1	28.4	41.9	32.4		

6. Compute the number $\varkappa^*$ such that the test $\varphi_{\hat{\Lambda}^*} := \mathbf{1}[g_r > \varkappa^* \sum_{i=1}^{N_{\kappa_1}} \hat{\lambda}_i^* f_i]$ is exactly of (Monte Carlo) level α when $\boldsymbol{x}$ is drawn with density $\sum_{i=1}^{N_{\kappa_1}} \hat{\lambda}_i^* f_i$, that is, solve $\sum_{j=1}^{N_{\kappa_1}} \hat{\lambda}_j^* (\widehat{RP}_j(\varphi_{\hat{\Lambda}^*}) - \alpha) = 0$.

7. Compute the estimate of the power bound $\bar{\pi}_r = N_1^{-1} \sum_{i=1}^{N_1} \varphi_{\hat{\Lambda}^*}(\boldsymbol{x}_i)$, where $\boldsymbol{x}_i$ are i.i.d. draws of $(\hat{\kappa}_1, \hat{\kappa}_2)$ with density g_r.

8. Compute the number $\varkappa$ such that the test $\tilde{\varphi}_{\hat{\Lambda}^*} = \mathbf{1}[g_r > \varkappa \sum_{i=1}^{N_{\kappa_1}} \hat{\lambda}_i^* f_i]$ is exactly of (Monte Carlo) level α when $\boldsymbol{x}$ is drawn with density f_i, $i = 1, \ldots, N_{\kappa_1}$, that is, solve $\max_{j \in \{1,\ldots,N_{\kappa_1}\}} (\widehat{RP}_j(\tilde{\varphi}_{\hat{\Lambda}^*}) - \alpha) = 0$.

TABLE 21. $1 - \alpha$ quantile of the conditional distribution, with density given in (2.12), cv = $c_{1-\alpha}(\hat{\kappa}_1, k - m_W)$ at different values of the conditioning variable $\hat{\kappa}_1$. Computed by numerical integration.

$k - m_W = 19$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
						$\alpha = 10\%$							
3.0	2.9	6.7	6.5	11.0	10.5	15.6	14.5	20.7	18.5	27.4	22.5	57.3	26.5
3.1	3.0	7.1	6.9	11.5	10.9	16.1	14.9	21.2	18.9	28.4	22.9	90.4	26.9
3.4	3.3	7.6	7.3	11.9	11.3	16.5	15.3	21.8	19.3	29.4	23.3	280.9	27.2
3.8	3.7	8.0	7.7	12.4	11.7	17.0	15.7	22.4	19.7	30.5	23.7	1000	27.178
4.2	4.1	8.4	8.1	12.8	12.1	17.5	16.1	23.0	20.1	31.8	24.1	∞	27.204
4.6	4.5	8.9	8.5	13.3	12.5	18.0	16.5	23.7	20.5	33.3	24.5		
5.0	4.9	9.3	8.9	13.7	12.9	18.5	16.9	24.3	20.9	35.1	24.9		
5.5	5.3	9.7	9.3	14.2	13.3	19.0	17.3	25.1	21.3	37.5	25.3		
5.9	5.7	10.2	9.7	14.6	13.7	19.6	17.7	25.8	21.7	40.9	25.7		
6.3	6.1	10.6	10.1	15.1	14.1	20.1	18.1	26.6	22.1	46.3	26.1		
						$\alpha = 5\%$							
4.5	4.4	8.2	8.0	12.4	12.0	16.8	16.0	21.6	20.0	27.4	24.0	39.0	28.0
4.6	4.5	8.6	8.4	12.8	12.4	17.2	16.4	22.1	20.4	28.1	24.4	41.9	28.4
4.9	4.8	9.0	8.8	13.3	12.8	17.7	16.8	22.6	20.8	28.9	24.8	46.3	28.8
5.3	5.2	9.4	9.2	13.7	13.2	18.2	17.2	23.1	21.2	29.7	25.2	54.0	29.2
5.7	5.6	9.8	9.6	14.1	13.6	18.6	17.6	23.7	21.6	30.6	25.6	71.5	29.6
6.1	6.0	10.3	10.0	14.6	14.0	19.1	18.0	24.3	22.0	31.5	26.0	146.7	30.0
6.5	6.4	10.7	10.4	15.0	14.4	19.6	18.4	24.8	22.4	32.6	26.4	229.3	30.1
6.9	6.8	11.1	10.8	15.4	14.8	20.1	18.8	25.4	22.8	33.8	26.8	1000	30.116
7.3	7.2	11.5	11.2	15.9	15.2	20.6	19.2	26.1	23.2	35.2	27.2	∞	30.144
7.8	7.6	12.0	11.6	16.3	15.6	21.0	19.6	26.7	23.6	36.9	27.6		
						$\alpha = 1\%$							
10.0	9.9	14.0	13.9	18.7	18.4	23.6	22.9	29.1	27.4	36.9	31.9	201.8	36.1
10.1	10.0	14.6	14.4	19.3	18.9	24.2	23.4	29.8	27.9	38.2	32.4	1000	36.171
10.5	10.4	15.1	14.9	19.8	19.4	24.8	23.9	30.5	28.4	39.8	32.9	∞	36.191
11.0	10.9	15.6	15.4	20.3	19.9	25.4	24.4	31.2	28.9	41.7	33.4		
11.5	11.4	16.1	15.9	20.9	20.4	25.9	24.9	32.0	29.4	44.3	33.9		
12.0	11.9	16.6	16.4	21.4	20.9	26.5	25.4	32.8	29.9	48.1	34.4		
12.5	12.4	17.1	16.9	22.0	21.4	27.2	25.9	33.7	30.4	54.5	34.9		
13.0	12.9	17.7	17.4	22.5	21.9	27.8	26.4	34.7	30.9	68.2	35.4		
13.5	13.4	18.2	17.9	23.1	22.4	28.4	26.9	35.7	31.4	116.7	35.9		

9. Compute another estimate of the power bound $\tilde{\pi}_r = N_1^{-1} \sum_{i=1}^{N_1} \tilde{\varphi}_{\hat{\Lambda}^*}(\boldsymbol{x}_i)$, where $\boldsymbol{x}_i$ are the i.i.d. draws in step 7, and $\varepsilon_r = \bar{\pi}_r - \tilde{\pi}_r$.

10. If $r < N_r$, set $r = r + 1$ and go to step 4.

All the reported results are based on $N_0 = 10{,}000$ and $N_1 = 100{,}000$. (We can use a smaller number of simulations under H_0 for a similar level of precision due to importance sampling.)

Up to step 7, the algorithm is identical to EMW (Appendix A.2.1). The difference is in step 7, which replaces steps 6 to 8 of the original algorithm. The number ε_r is an estimate

TABLE 22. $1-\alpha$ quantile of the conditional distribution, with density given in (2.12), cv $= c_{1-\alpha}(\hat{\kappa}_1, k-m_W)$ at different values of the conditioning variable $\hat{\kappa}_1$. Computed by numerical integration.

$k-m_W=20$													
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
$\alpha = 10\%$													
3.1	3.0	6.8	6.6	11.1	10.6	15.6	14.6	20.5	18.6	26.6	22.6	39.9	26.6
3.2	3.1	7.2	7.0	11.5	11.0	16.0	15.0	21.0	19.0	27.4	23.0	43.8	27.0
3.5	3.4	7.7	7.4	12.0	11.4	16.5	15.4	21.6	19.4	28.2	23.4	50.3	27.4
3.9	3.8	8.1	7.8	12.4	11.8	17.0	15.8	22.1	19.8	29.1	23.8	64.4	27.8
4.3	4.2	8.5	8.2	12.9	12.2	17.5	16.2	22.7	20.2	30.1	24.2	114.8	28.2
4.7	4.6	8.9	8.6	13.3	12.6	18.0	16.6	23.3	20.6	31.1	24.6	273.1	28.4
5.1	5.0	9.4	9.0	13.8	13.0	18.5	17.0	23.9	21.0	32.3	25.0	1000	28.385
5.5	5.4	9.8	9.4	14.2	13.4	19.0	17.4	24.6	21.4	33.7	25.4	∞	28.412
6.0	5.8	10.2	9.8	14.7	13.8	19.5	17.8	25.2	21.8	35.3	25.8		
6.4	6.2	10.7	10.2	15.1	14.2	20.0	18.2	25.9	22.2	37.3	26.2		
$\alpha = 5\%$													
4.8	4.7	8.9	8.7	13.5	13.1	18.4	17.5	23.8	21.9	30.8	26.3	64.7	30.7
4.9	4.8	9.3	9.1	14.0	13.5	18.8	17.9	24.3	22.3	31.7	26.7	101.7	31.1
5.2	5.1	9.7	9.5	14.4	13.9	19.3	18.3	24.8	22.7	32.6	27.1	299.7	31.4
5.6	5.5	10.1	9.9	14.8	14.3	19.8	18.7	25.4	23.1	33.7	27.5	1000	31.382
6.0	5.9	10.6	10.3	15.3	14.7	20.3	19.1	26.0	23.5	34.8	27.9	∞	31.410
6.4	6.3	11.0	10.7	15.7	15.1	20.7	19.5	26.6	23.9	36.2	28.3		
6.8	6.7	11.4	11.1	16.1	15.5	21.2	19.9	27.2	24.3	37.8	28.7		
7.2	7.1	11.8	11.5	16.6	15.9	21.7	20.3	27.9	24.7	39.8	29.1		
7.6	7.5	12.2	11.9	17.0	16.3	22.2	20.7	28.5	25.1	42.4	29.5		
8.1	7.9	12.7	12.3	17.5	16.7	22.7	21.1	29.3	25.5	46.1	29.9		
8.5	8.3	13.1	12.7	17.9	17.1	23.2	21.5	30.0	25.9	52.3	30.3		
$\alpha = 1\%$													
10.4	10.3	14.4	14.3	19.1	18.8	24.0	23.3	29.3	27.8	36.3	32.3	72.0	36.8
10.5	10.4	14.9	14.8	19.6	19.3	24.5	23.8	29.9	28.3	37.3	32.8	126.7	37.3
10.9	10.8	15.5	15.3	20.2	19.8	25.1	24.3	30.6	28.8	38.5	33.3	231.5	37.5
11.4	11.3	16.0	15.8	20.7	20.3	25.7	24.8	31.3	29.3	39.9	33.8	1000	37.547
11.9	11.8	16.5	16.3	21.2	20.8	26.2	25.3	32.0	29.8	41.5	34.3	∞	37.566
12.4	12.3	17.0	16.8	21.8	21.3	26.8	25.8	32.8	30.3	43.6	34.8		
12.9	12.8	17.5	17.3	22.3	21.8	27.4	26.3	33.6	30.8	46.3	35.3		
13.4	13.3	18.1	17.8	22.9	22.3	28.0	26.8	34.4	31.3	50.2	35.8		
13.9	13.8	18.6	18.3	23.4	22.8	28.6	27.3	35.3	31.8	57.1	36.3		

of the maximum distance of the power bound $\tilde{\pi}_r$ from the unknown least upper bound. $\tilde{\pi}_r$ is the PO power bound used in Figure 3. Figure 9 plots ε_r across all alternatives. In most cases ε_r is equal to zero to 3 decimals, indicating that the ALFD upper bound is essentially least favorable. The only exceptions are for a handful of alternatives very close to the null. Hence, the ALFD upper bound is arguably a good approximation of the PO power envelope.

The bound $\bar{\pi}_r$ (which is obtained from step 5 in the original EMW algorithm) can also serve as an upper bound on the power, similar to the AMS bound in the previous

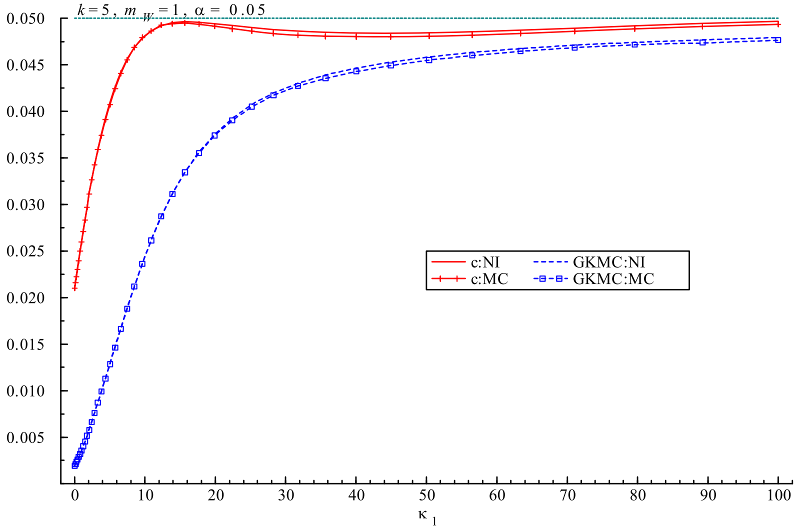

FIGURE 7. Comparison of estimates of NRP obtained by numerical integration (NI) and Monte Carlo simulation (MC) with 1 million draws. NRP of 5% level conditional (2.15) (red solid) and GKMC subvector AR (blue dotted) tests as a function of the nuisance parameter κ_{m_W}. The number of instruments is $k = 5$ and the number of nuisance parameters is $m_W = 1$.

section. The only difference is the use of a distribution Λ^* with full support on the discretized H_0, as opposed to a one-point candidate least favorable distribution in AMS. But as for AMS, $\bar{\pi}_r$ can be far from the least upper bound if $\varphi_{\hat{\Lambda}^*}$ is oversized under H_0. To gauge this, Figure 10 plots (Monte Carlo estimates of) the size of φ_{Λ^*} across the different alternatives r. The figure is directly comparable to Figure 8 for the AMS algorithm in the previous subsection. Compared to AMS, the EMW procedure has size much closer to α across most (but not all) alternatives.

Let $\bar{\pi}_r^{\text{AMS}}$ and $\bar{\pi}_r^{\text{EMW}}$ denote the power bounds obtained from the AMS and EMW algorithms, respectively. Since they are both upper bounds to the true PO power envelope, so is their minimum, $\bar{\pi}_r^{\min} = \min(\bar{\pi}_r^{\text{AMS}}, \bar{\pi}_r^{\text{EMW}})$. We can therefore use $\bar{\pi}_r^{\min}$ as a possibly tighter upper bound on the power envelope.

D.4 *The ACZ test reported in Section 4*

The test is constructed as follows:

1. Compute $\mathcal{C}_\gamma = \{\gamma : AR(\beta_0, \gamma) < \chi^2_{k,1-\alpha_1}\}$.
2. Reject H_0 if $\min_{\gamma \in \mathcal{C}_\gamma} \text{AR}_\beta(\beta_0; \gamma) > \chi^2_{1,1-\alpha_2}$.

The $\text{AR}_\beta(\beta_0; \gamma)$ statistic is a $C(\alpha)$ score test in the present case because the model is just-identified. It is defined as

$$\text{AR}(\theta) = n\tilde{g}(\theta)'\tilde{g}(\theta),$$

$$\text{AR}_\beta(\beta_0; \gamma) = n\tilde{g}(\beta_0, \gamma)' M_{\tilde{D}(\beta_0, \gamma)} \tilde{g}(\beta_0, \gamma),$$

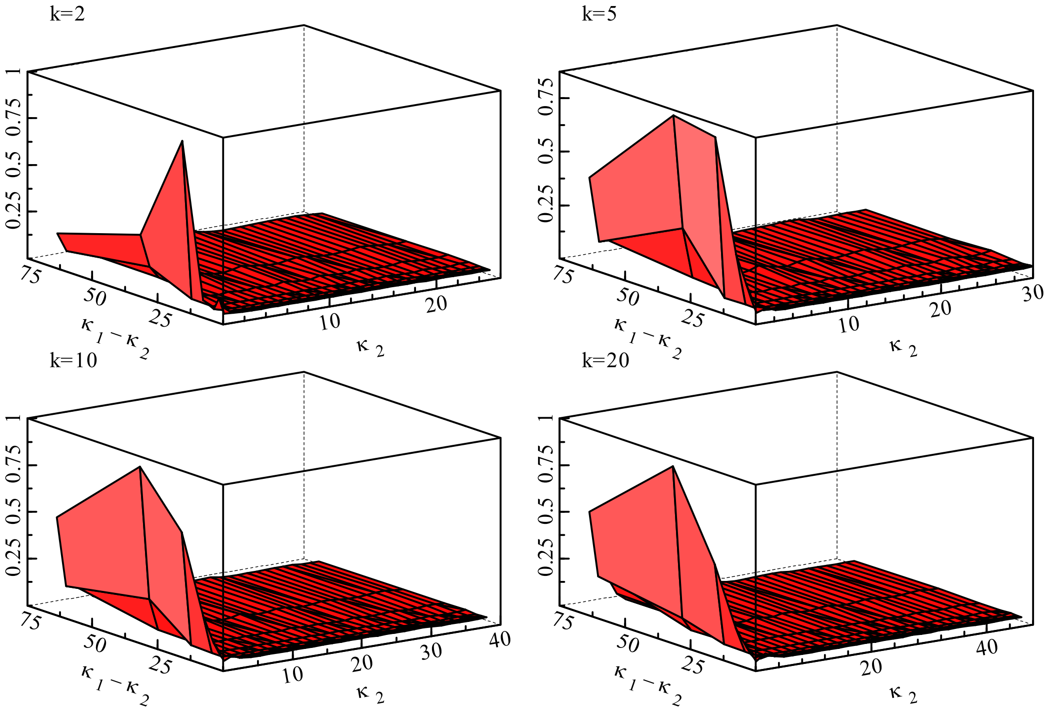

FIGURE 8. Size of the AMS test for each point under H_1, $\kappa_2 \in [0.1, \bar{\kappa}_2(k)]$, $\bar{\kappa}_2(2) = 25$, $\bar{\kappa}_2(5) = 30$, $\bar{\kappa}_2(10) = 38$, $\bar{\kappa}_2(20) = 46$, discretized into 30 equally spaced points, and $\kappa_1 - \kappa_2 \in \{0, 1, 2, 4, 8, 16, 32, 64\}$. Calculated over 42 points of the nuisance parameter under H_0, using 10,000 Monte Carlo replications with importance sampling.

$$\tilde{g}(\beta, \gamma) = \hat{\Sigma}(\theta)^{-1/2} \sum_{i=1}^{n} g_i(\theta)/n, \quad \theta = (\beta, \gamma)',$$

$$g_i(\theta) = Z_i(y_i - Y_i\beta - W_i\gamma),$$

$$\hat{g}_n(\theta) = n^{-1} \sum_{i=1}^{n} Z_i(y_i - Y_i\beta - W_i\gamma),$$

$$\hat{\Sigma}(\theta) = n^{-1} \sum_{i=1}^{n} \big(g_i(\theta) - \hat{g}_n(\theta)\big)\big(g_i(\theta) - \hat{g}_n(\theta)\big)',$$

$$\tilde{D}(\theta) = \hat{\Sigma}(\theta)^{-1/2} D(\theta),$$

$$D(\theta) = -n^{-1} Z'W - \hat{\Gamma}(\theta)\hat{\Sigma}(\theta)^{-1}\hat{g}_n(\theta),$$

$$\hat{\Gamma}(\theta) = -n^{-1} \sum_{i=1}^{n} \big(Z_i W_i - n^{-1} Z'W\big) g_i(\theta)'.$$

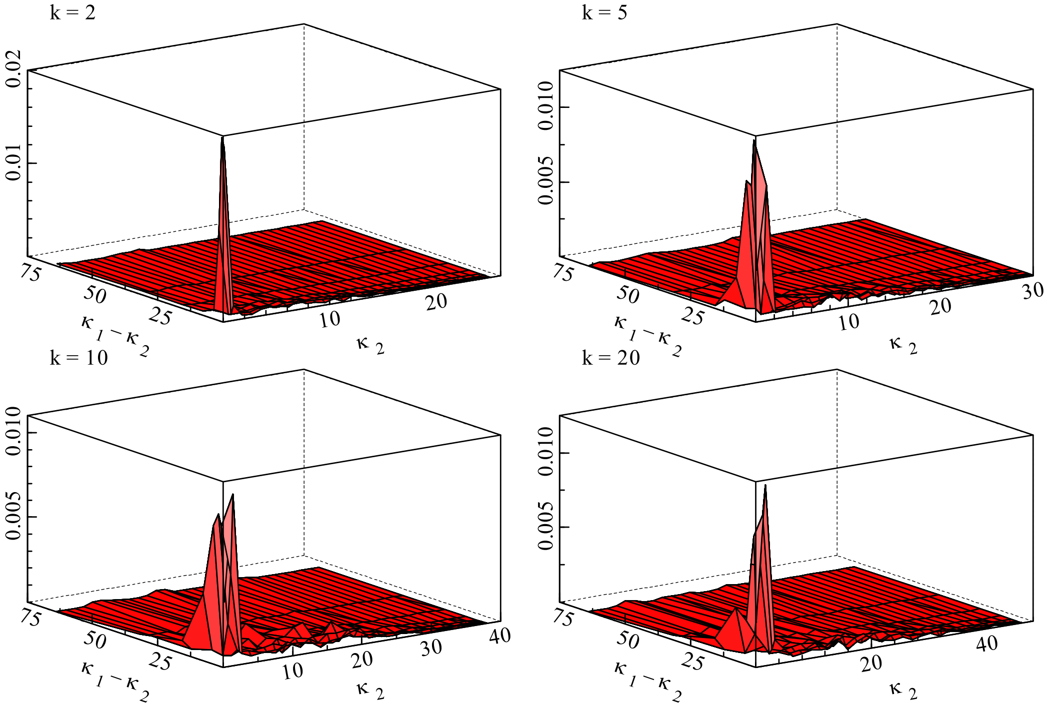

FIGURE 9. Estimates of the distance of the ALFD power bound from the least favorable bound, $\varepsilon = \bar{\pi} - \tilde{\pi}$, for each point under H_1, $\kappa_2 \in [0.1, \bar{\kappa}_2(k)]$, $\bar{\kappa}_2(2) = 25$, $\bar{\kappa}_2(5) = 30$, $\bar{\kappa}_2(10) = 38$, $\bar{\kappa}_2(20) = 46$, discretized into 30 equally spaced points, and $\kappa_1 - \kappa_2 \in \{0, 1, 2, 4, 8, 16, 32, 64\}$. Calculated over 42 points of the nuisance parameter under H_0, using 10,000 Monte Carlo replications with importance sampling.

The second step size α_2 is chosen as

$$\alpha_2 = \begin{cases} \alpha - \alpha_1, & \text{if ICS} \leq K_L \\ \alpha, & \text{if ICS} > K_L, \end{cases}$$

where

$$\text{ICS} = \frac{\left(W' Z \hat{\Sigma}(\beta_0, \gamma)^{-1} Z' W\right)^{1/2}}{n \hat{\sigma}_\gamma},$$

$$\hat{\sigma}_\gamma^2 = n^{-1} \sum_{i=1}^{n} \left(\| Z_i W_i \| - n^{-1} \sum_{i=1}^{n} \| Z_i W_i \| \right)^2$$

and $K_L = 0.05$ Andrews (2017, p. 34).

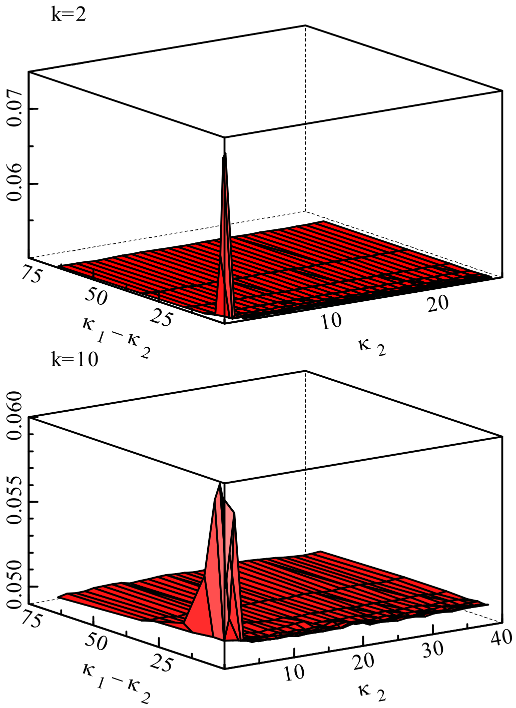

FIGURE 10. Size of the test $\varphi_{\hat{\Lambda}^*}$ in step 5 of EMW's ALFD algorithm for each point under H_1, $\kappa_2 \in [0.1, \bar{\kappa}_2(k)]$, $\bar{\kappa}_2(2) = 25$, $\bar{\kappa}_2(5) = 30$, $\bar{\kappa}_2(10) = 38$, $\bar{\kappa}_2(20) = 46$, discretized into 30 equally spaced points, and $\kappa_1 - \kappa_2 \in \{0, 1, 2, 4, 8, 16, 32, 64\}$. Calculated over 42 points of the nuisance parameter under H_0, using 10,000 Monte Carlo replications with importance sampling.

APPENDIX E: ADDITIONAL NUMERICAL RESULTS

E.1 *Size*

We computed the size of φ_c at significance levels 1%, 5%, and 10% for $k = 2, \ldots, 21$, and $m_W = 1$ using a grid of 42 points in κ_1 equally spaced in log-scale between 0 and 100. The reported size is the maximum of α or the estimated NRPs. The results are reported in Table 23. In all cases, the size of the test is controlled to two decimals, in accordance with Theorem 2.

E.2 *Power*

Here, we report supplementary power comparisons for Section 2.4. The power of the conditional subvector AR test φ_c and the unconditional test φ_{GKMC} are compared to the ALFD estimate of the point-optimal power envelope for $k = 2, 5, 10$, and 20.

Figure 11 gives the difference between the power of 5% level φ_c test and the point-optimal ALFD power bound $\tilde{\pi}$ defined in the step 9 of the algorithm in Section D.3.2, across all alternatives. The power of φ_c is well within 1% of the power bound except for alternatives very close to H_0. The largest deviations from the power bound occur when $\kappa_1 = \kappa_2$.

TABLE 23. Size of the conditional subvector AR test with nominal size α for different k with $m_W = 1$, using critical values given in Tables 3 to 22 and linear interpolation. Computed using 1 million Monte Carlo replications.

	α				α		
k	0.1	0.05	0.01	k	0.1	0.05	0.01
2	0.1000	0.0500	0.0100	12	0.1003	0.0504	0.0101
3	0.1000	0.0504	0.0100	13	0.1004	0.0504	0.0102
4	0.1000	0.0500	0.0100	14	0.1007	0.0506	0.0102
5	0.1000	0.0500	0.0100	15	0.1007	0.0503	0.0102
6	0.1000	0.0500	0.0100	16	0.1013	0.0507	0.0101
7	0.1000	0.0500	0.0100	17	0.1006	0.0509	0.0101
8	0.1000	0.0502	0.0100	18	0.1014	0.0508	0.0101
9	0.1000	0.0500	0.0101	19	0.1017	0.0508	0.0101
10	0.1001	0.0505	0.0101	20	0.1014	0.0511	0.0102
11	0.1005	0.0504	0.0100	21	0.1019	0.0510	0.0102

Figure 12 repeats the comparison of the power of φ_c but with $\tilde{\pi}$ replaced by $\min(\bar{\pi}^{\text{AMS}}, \bar{\pi}^{\text{EMW}})$, where $\bar{\pi}^{\text{AMS}}$, $\bar{\pi}^{\text{EMW}}$ are computed using the algorithms in Sec-

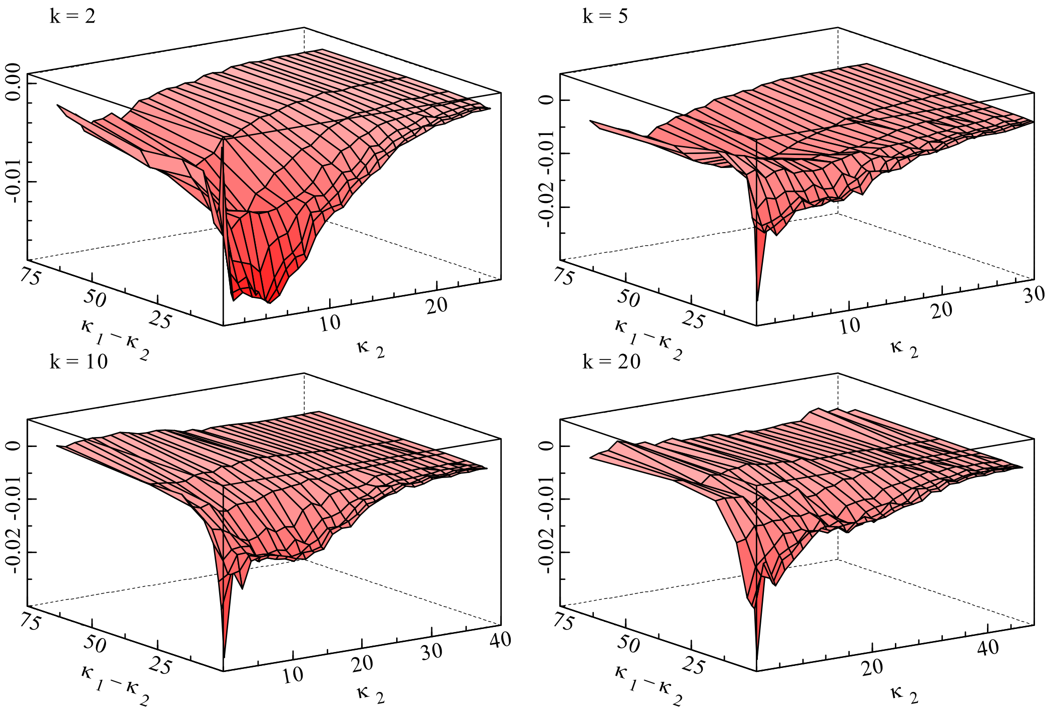

FIGURE 11. Power of 5% level conditional subvector AR test φ_c minus the ALFD power bound $\widetilde{\pi}$ computed by the algorithm in Section D.

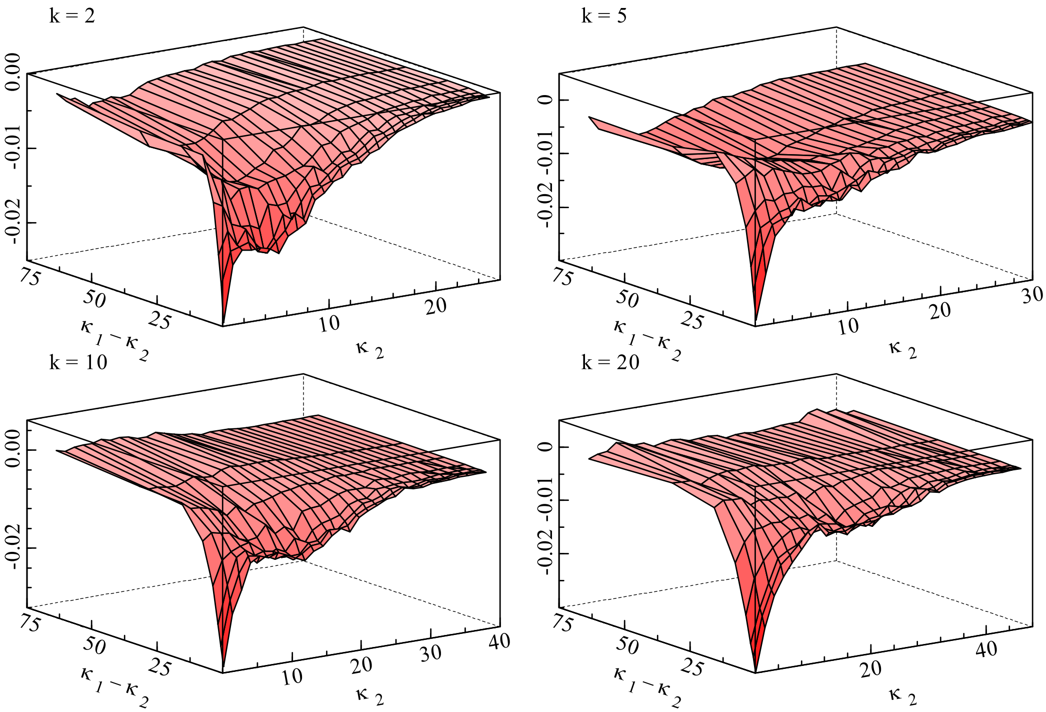

Figure 12. Power of 5% level conditional subvector AR test φ_c minus $\min(\bar{\pi}^{\text{AMS}}, \bar{\pi}^{\text{EMW}})$ power bound, where $\bar{\pi}^{\text{AMS}}$, $\bar{\pi}^{\text{EMW}}$ were computed by the algorithms in Section D.

tions D.3.1 and D.3.2, respectively. Since $\min(\bar{\pi}^{\text{AMS}}, \bar{\pi}^{\text{EMW}}) \geq \tilde{\pi}$, the differences are larger than in Figure 11, but not by much.

Figures 13 through 16 report power comparisons in 2D, where $\kappa_1 - \kappa_2$ is kept fixed in each figure, and the alternative only varies across κ_2. The figures plot the power curves of both test φ_c, φ_{GKMC} at 5% level, and both power bounds, $\min(\bar{\pi}^{\text{AMS}}, \bar{\pi}^{\text{EMW}})$ and $\tilde{\pi}$. We notice that the power of φ_c is very close to both power bounds, which are in turn very close to each other, while the power of the unconditional subvector AR test φ_{GKMC} is noticeably below the power bounds. As $\kappa_1 - \kappa_2$ increases, both power curves get closer to the power bounds, and they essentially collapse on top of each other when $\kappa_1 - \kappa_2 = 64$. This is why we do not consider values higher than that in the simulations. The distance of φ_{GKMC} from φ_c and the power bounds is also somewhat increasing in k.

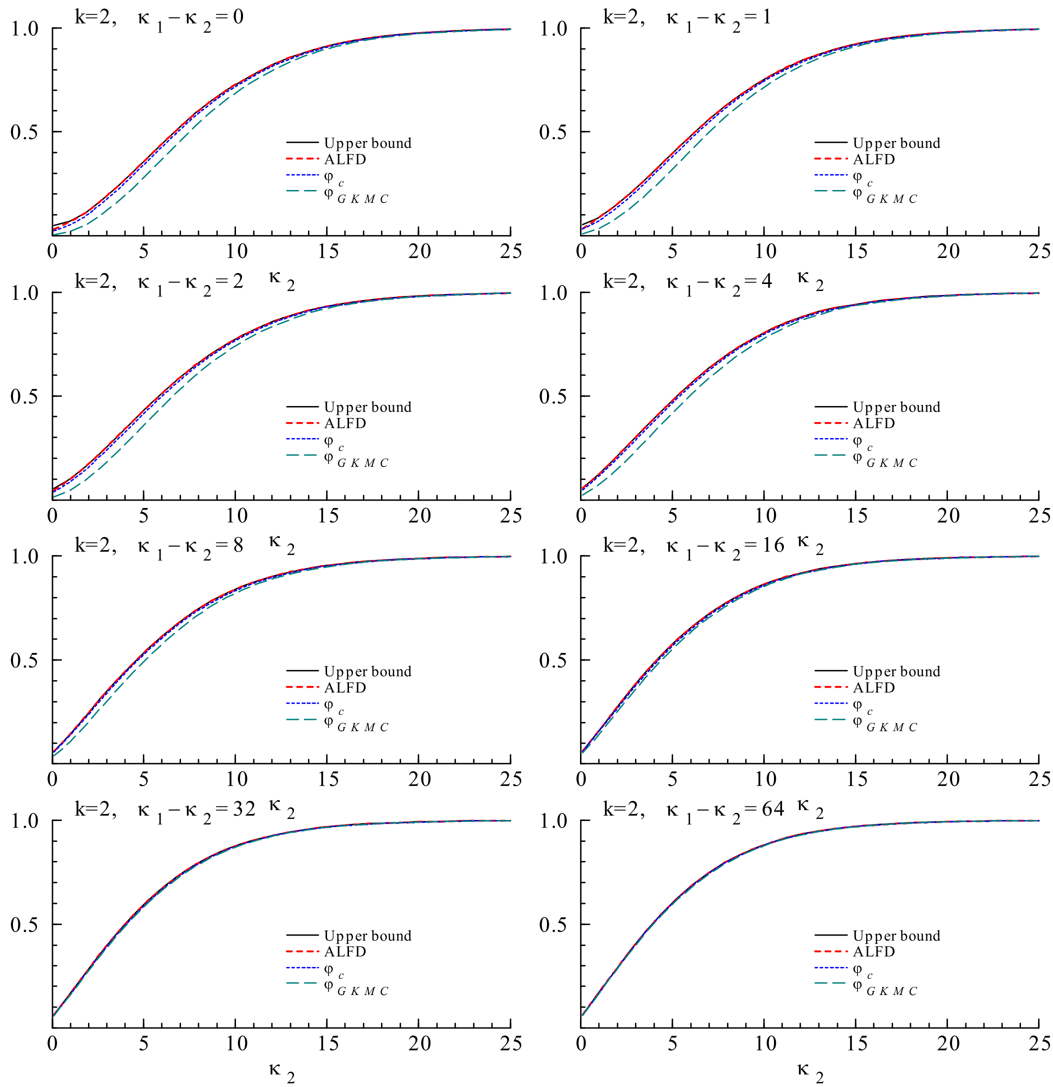

FIGURE 13. Power curves and power bounds of 5% level φ_c and φ_{GKMC} tests as a function of κ_2, at different values of $\kappa_1 - \kappa_2$ when the number of instruments $k = 2$.

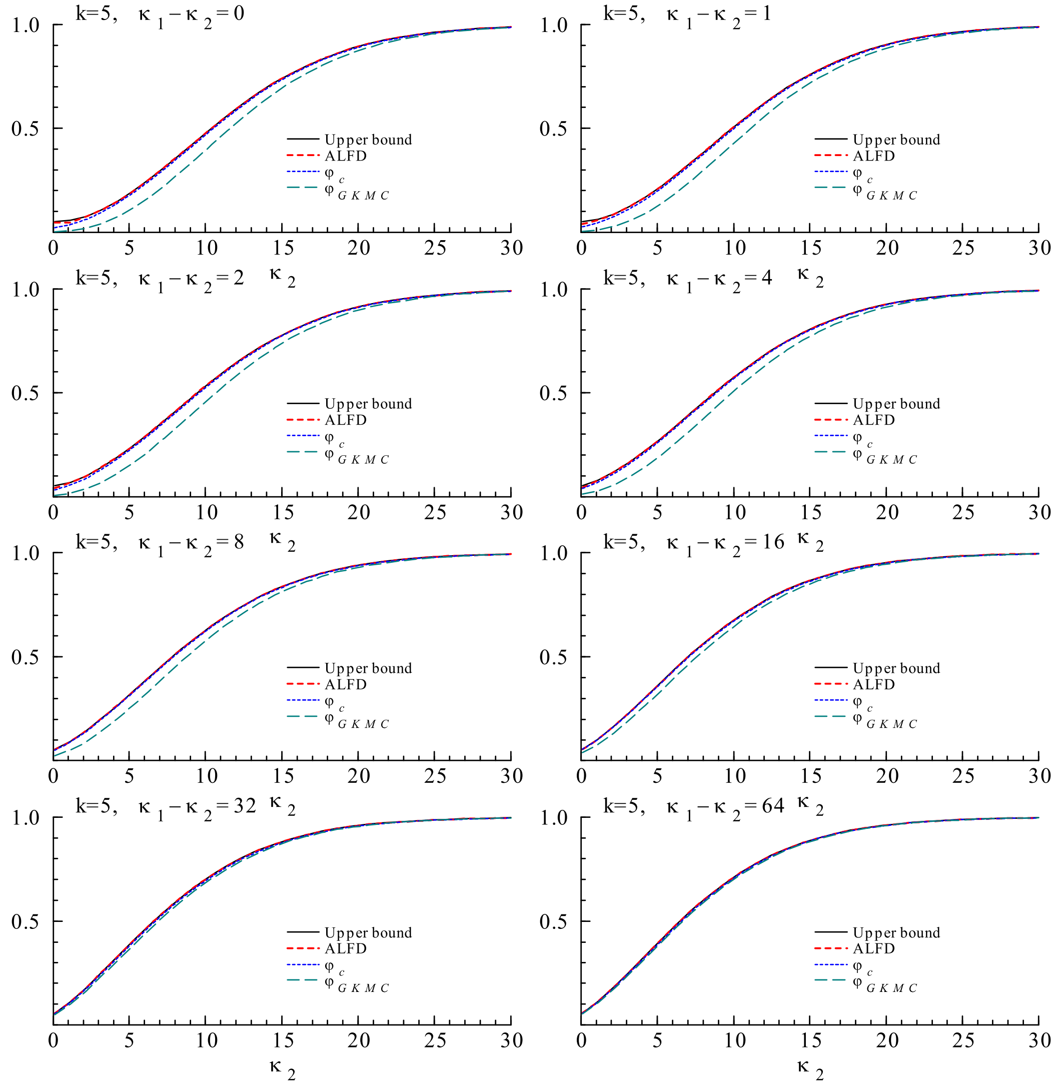

Figure 14. Power curves and power bounds of 5% level φ_c and φ_{GKMC} tests as a function of κ_2, at different values of $\kappa_1 - \kappa_2$ when the number of instruments $k = 5$.

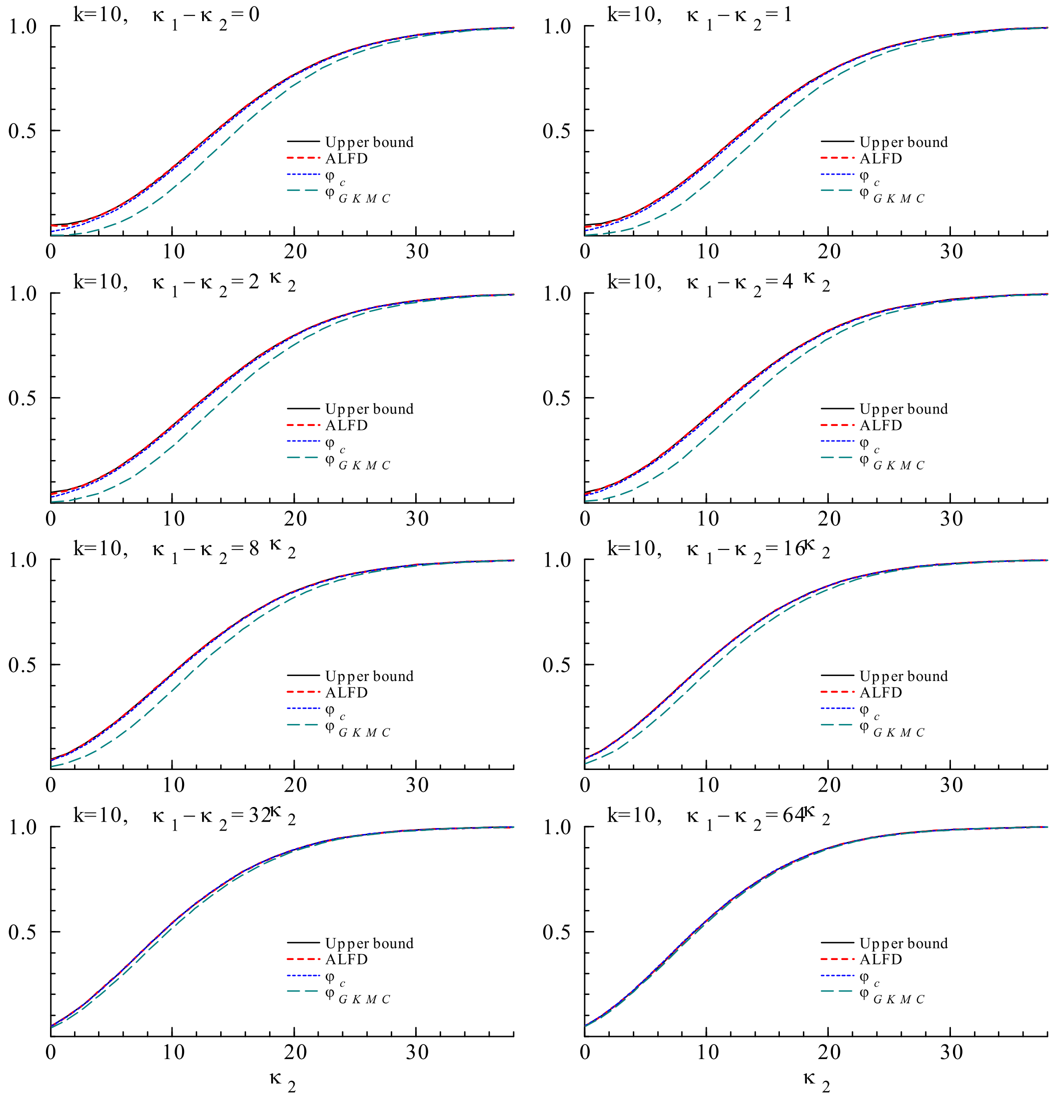

FIGURE 15. Power curves and power bounds of 5% level φ_c and φ_{GKMC} tests as a function of κ_2, at different values of $\kappa_1 - \kappa_2$ when the number of instruments $k = 10$.

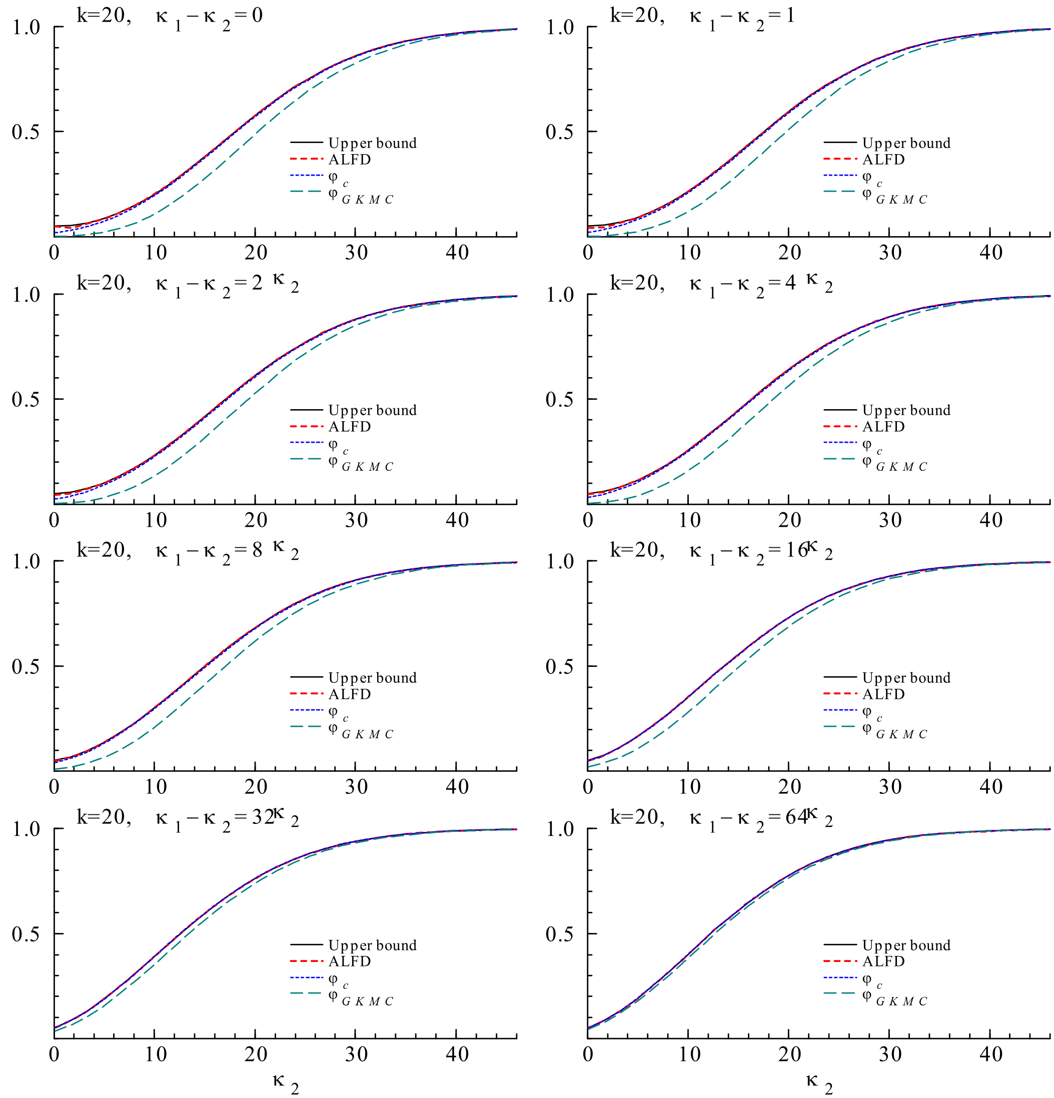

Figure 16. Power curves and power bounds of 5% level φ_c and φ_{GKMC} tests as a function of κ_2, at different values of $\kappa_1 - \kappa_2$ when the number of instruments $k = 20$.

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