Supplementary Material

Supplement to "Evaluating factor pricing models using high-frequency panels"

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This supplement contains some useful lemmas and their proofs and the proofs of the theorems presented in the main paper.

Useful lemmas and their proofs

Let (A_t) be an Ito process given by

$$\frac{dA_t}{A_t} = f_t\,dt + g_t\,dB_t,$$

where (B_t) is a Brownian motion with respect to a filtration $(\mathcal{F}_t)$, to which (f_t) and (g_t) are adapted. We assume

Assumption A.1. *For all* $0 \le s \le t \le T$, $a_T(t-s) \le \int_s^t g_u^2\,du \le b_T(t-s)$, *where* a_T *and* b_T *are some constants depending only on* T.

Assumption A.2. *We have* $\sup_{t\ge 0}|f_t| = O_p(1)$.

Assumption A.3. *We have* $\inf_{t\ge 0} A_t > 0$ *and* $\sup_{0\le t\le T} A_t = O_p(c_T)$, *with* (c_T) *depending only on* T.

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In the subsequent development of our theory, we assume that the Ito process A satisfies Assumptions A.1–A.3.

LEMMA A.1. *We have*

$$\sup_{|s-t|\le\delta}|A_t-A_s| = O_p\big(\delta^{1/2-\varepsilon}b_T^{1/2}c_T\big)$$

for any $\varepsilon>0$, *uniformly in* $0\le s,t\le T$.

PROOF. Write

$$A_t-A_s=\int_s^t f_uA_u\,du+\int_s^t g_uA_u\,dB_u$$

for $0\le s\le t\le T$. We may easily deduce that

$$\left|\int_s^t f_uA_u\,du\right|\le\Big(\sup_{0\le t\le T}A_t\Big)\int_s^t|f_u|\,du=O_p(\delta c_T) \tag{A.1}$$

uniformly in $0\le s,t\le T$, due to Assumptions A.2 and A.3. Moreover, if we let $C_t=\int_0^t g_sA_s\,dB_s$, then C is a continuous martingale with

$$\begin{aligned}[C]_t-[C]_s&=\int_s^t g_u^2A_u^2\,du\\ &\le\Big(\sup_{0\le t\le T}A_t^2\Big)\int_s^t g_u^2\,du=O_p\big(\delta b_Tc_T^2\big)\end{aligned} \tag{A.2}$$

uniformly in $0\le s,t\le T$. Since C is a continuous martingale, we may represent it as

$$C_t=\big(D\circ[C]\big)_t \tag{A.3}$$

with the DDS Brownian motion D of C, due to the celebrated theorem by Dambis, Dubins, and Schwarz in, for example, Revuz and Yor (1994, Theorem 5.1.6, p. 173). Now we may deduce from (A.3), together with the modulus of continuity of Brownian motion and (A.2), that

$$\begin{aligned}\sup_{|t-s|\le\delta}|C_t-C_s|&\le\sup_{|t-s|\le\delta}\left|\big(D\circ[C]\big)_t-\big(D\circ[C]\big)_s\right|\\ &\le\sup_{|t-s|\le\delta}\left|[C]_t-[C]_s\right|^{1/2-\varepsilon}=O_p\big(\delta^{1/2-\varepsilon}b_T^{1/2}c_T\big)\end{aligned} \tag{A.4}$$

for any $\varepsilon>0$, uniformly in $0\le s,t\le T$. Upon noticing that $c_T\delta=o(\delta^{1/2-\varepsilon}b_Tc_T)$ for any $\varepsilon>0$, the stated result follows immediately from (A.1) and (A.4). The proof is therefore complete. □

LEMMA A.2. *We have*

$$\max_{1\le m\le M}\left|\int_{(m-1)\delta}^{m\delta}\frac{dA_t}{A_t}-\frac{A_{m\delta}-A_{(m-1)\delta}}{A_{(m-1)\delta}}\right|=O_p\big(\delta^{1-\varepsilon}b_Tc_T\big)$$

for any $\varepsilon>0$.

PROOF. Define

$$
\begin{aligned}
R_m &= \int_{(m-1)\delta}^{m\delta} \frac{dA_t}{A_t} - \frac{A_{m\delta} - A_{(m-1)\delta}}{A_{(m-1)\delta}} \\
&= \int_{(m-1)\delta}^{m\delta} \left(1 - \frac{A_t}{A_{(m-1)\delta}}\right)\frac{dA_t}{A_t} = \int_{(m-1)\delta}^{m\delta} \frac{A_t - A_{(m-1)\delta}}{A_{(m-1)\delta}}(f_t\,dt + g_t\,dB_t).
\end{aligned} \tag{A.5}
$$

We have

$$
\begin{aligned}
\int_{(m-1)\delta}^{m\delta} \frac{A_t - A_{(m-1)\delta}}{A_{(m-1)\delta}} f_t\,dt &\le \frac{1}{\inf_t A_t}\Big(\sup_{(m-1)\delta\le t\le m\delta} |A_t - A_{(m-1)\delta}|\Big) \int_{(m-1)\delta}^{m\delta} f_t\,dt \\
&= O_p\big(\delta^{3/2-\varepsilon} b_T^{1/2} c_T\big) = O_p\big(\delta^{1-\varepsilon} b_T c_T\big)
\end{aligned} \tag{A.6}
$$

uniformly in $m = 1, \ldots, M$, due in particular to Lemma A.1. Moreover,

$$
\int_{(m-1)\delta}^{t} \frac{A_s - A_{(m-1)\delta}}{A_{(m-1)\delta}} g_s\,dB_s
$$

is a continuous martingale, whose increment in quadratic variation over interval $[(m-1)\delta, m\delta]$ is bounded by

$$
\begin{aligned}
&\int_{(m-1)\delta}^{m\delta} \left(\frac{A_t - A_{(m-1)\delta}}{A_{(m-1)\delta}}\right) g_t^2\,dt \\
&\le \frac{1}{\inf_t A_t^2}\Big(\sup_{(m-1)\delta\le t\le m\delta} |A_t - A_{(m-1)\delta}|^2\Big) \int_{(m-1)\delta}^{m\delta} g_t^2\,dt \\
&= O_p\big(\delta^{2-\varepsilon} b_T^2 c_T^2\big).
\end{aligned}
$$

Consequently, we may show that

$$
\int_{(m-1)\delta}^{m\delta} \frac{A_t - A_{(m-1)\delta}}{A_{(m-1)\delta}} g_t\,dB_t = O_p\big(\delta^{1-\varepsilon} b_T c_T\big) \tag{A.7}
$$

uniformly in $m = 1, \ldots, M$, using the same argument as in the proof of Lemma A.2. The stated result now follows immediately from (A.5), (A.6), and (A.7). □

Subsequently, we let

$$
dF_t = \frac{dA_t}{A_t} \quad \text{and} \quad dG_t = g_t dB_t,
$$

and define

$$
\begin{aligned}
[F]_t^\delta &= \sum_{m\delta\le t} \left(\frac{A_{m\delta} - A_{(m-1)\delta}}{A_{(m-1)\delta}}\right)^2, \\
[G]_t^\delta &= \sum_{m\delta\le t} (G_{m\delta} - G_{(m-1)\delta})^2.
\end{aligned}
$$

LEMMA A.3. *We have*

$$\sup_{0\le t\le T}\big|[G]_t^\delta-[G]_t\big|=O_p\big((\delta T)^{1/2}b_T\big).$$

PROOF. Under Assumption A.1, the stated result follows immediately from Lemma A3.1 of Park (2009). □

LEMMA A.4. *We have*

$$\sup_{0\le t\le T}\big|[F]_t^\delta-[G]_t\big|=O_p\big(\delta^{1/2-\varepsilon}Tb_T^{3/2}c_T^2\big)$$

for any $\varepsilon>0$.

PROOF. Define

$$\big[F^\delta\big]_t=\sum_{m\delta\le t}(F_{m\delta}-F_{(m-1)\delta})^2=\sum_{m\delta\le t}\left(\int_{(m-1)\delta}^{m\delta}\frac{dA_t}{A_t}\right)^2,$$

and note that

$$\big|[F]_t^\delta-[G]_t^\delta\big|\le\big|[F]_t^\delta-\big[F^\delta\big]_t\big|+\big|\big[F^\delta\big]_t-[G]_t^\delta\big|. \tag{A.8}$$

We may readily deduce from Lemmas A.1 and A.2 that

$$\begin{aligned}\big|[F]_t^\delta-\big[F^\delta\big]_t\big|&=\sum_{m\delta\le t}\left[\left(\frac{A_{m\delta}-A_{(m-1)\delta}}{A_{(m-1)\delta}}\right)^2-\left(\int_{(m-1)\delta}^{m\delta}\frac{dA_t}{A_t}\right)^2\right]\\&\le\frac{2}{\inf_t A_t}\Big(\max_{1\le m\le M}|A_{m\delta}-A_{(m-1)\delta}|\Big)\\&\quad\times M\left(\max_{1\le m\le M}\left|\int_{(m-1)\delta}^{m\delta}\frac{dA_t}{A_t}-\frac{A_{m\delta}-A_{(m-1)\delta}}{A_{(m-1)\delta}}\right|\right)\\&=(T/\delta)O_p\big(\delta^{1/2-\varepsilon}b_T^{1/2}c_T\big)O_p\big(\delta^{1-\varepsilon}b_Tc_T\big)=O_p\big(\delta^{1/2-\varepsilon}Tb_T^{3/2}c_T^2\big)\end{aligned} \tag{A.9}$$

for all $0\le t\le T$.

Moreover, it follows that

$$\big[F^\delta\big]_t=[G]_t^\delta+2\sum_{m\delta\le t}\left(\int_{(m-1)\delta}^{m\delta}f_t\,dt\right)(G_{m\delta}-G_{(m-1)\delta})+\sum_{m\delta\le t}\left(\int_{(m-1)\delta}^{m\delta}f_t\,dt\right)^2,$$

where we have

$$\sum_{m\delta\le t}\left(\int_{(m-1)\delta}^{m\delta}f_t\,dt\right)^2\le MO_p\big(\delta^2\big)=O_p(\delta T)$$

and

$$
\begin{aligned}
&\Big|\sum_{m\delta\le t}\Big(\int_{(m-1)\delta}^{m\delta} f_t\,dt\Big)(G_{m\delta}-G_{(m-1)\delta})\Big| \\
&\quad\le \Big[\sum_{m\delta\le t}\Big(\int_{(m-1)\delta}^{m\delta} f_t\,dt\Big)^2\Big]^{1/2}\Big[\sum_{m\delta\le t}(G_{m\delta}-G_{(m-1)\delta})\Big]^{1/2} \\
&\quad= O_p\big((\delta T)^{1/2}\big)O_p\big((Tb_T)^{1/2}\big)=O_p\big(\delta^{1/2}Tb_T^{1/2}\big)
\end{aligned}
$$

uniformly in $0\le t\le T$. Note that $\delta T=o(\delta^{1/2}Tb_T^{1/2})$. Consequently, we have

$$
\big|[F^\delta]_t-[G]_t^\delta\big|=O_p\big(\delta^{1/2}Tb_T^{1/2}\big) \tag{A.10}
$$

uniformly in $0\le t\le T$. The stated result follows from Lemma A.3, and (A.8), (A.9), and (A.10). Note that

$$
\delta^{1/2}Tb_T^{1/2},\ (\delta T)^{1/2}b_T=o\big(\delta^{1/2-\varepsilon}Tb_T^{3/2}c_T^2\big),
$$

and, therefore, the terms we consider in Lemma A.3 and (A.10) become negligible. □

In what follows, we let

$$
H_t=\inf_{s>0}\big\{[G]_s>t\big\}
$$

and analogously define

$$
H_t^\delta=\inf_{s>0}\big\{[F]_s^\delta>t\big\}
$$

for $0\le t\le [G]_T$.

Lemma A.5. *We have*

$$
\sup_{0\le t\le [G]_T}\big|H_t^\delta-H_t\big|=O_p\big(\delta^{1/2-\varepsilon}Ta_T^{-1}b_T^{3/2}c_T^2\big)
$$

for any $\varepsilon>0$.

Proof. The proof is virtually identical to that of Corollary 3.3 of Park (2009) and, therefore, it is omitted. □

In the following lemma, we define M_n by $\delta M_n=H_{n\Delta}^\delta$ for $n=1,\dots,N$.

Lemma A.6. *We have*

$$
\max_{1\le n\le N}\Bigg|\int_{H_{(n-1)\Delta}}^{H_{n\Delta}}\frac{dA_t}{A_t}-\sum_{m=M_{n-1}+1}^{M_n}\frac{A_{m\delta}-A_{(m-1)\delta}}{A_{(m-1)\delta}}\Bigg|=O_p\big(\delta^{1/4-\varepsilon}T^{1/2}a_T^{-1/2}b_T^{5/4}c_T\big)
$$

for any $\varepsilon>0$.

PROOF. We let

$$R_n = \int_{H_{(n-1)\Delta}}^{H_{n\Delta}} \frac{dA_t}{A_t} - \sum_{m=M_{n-1}+1}^{M_n} \frac{A_{m\delta} - A_{(m-1)\delta}}{A_{(m-1)\delta}}$$

and write

$$|R_n| \le |R_n^a| + |R_n^b|, \tag{A.11}$$

where

$$R_n^a = \int_{H_{(n-1)\Delta}}^{H_{n\Delta}} \frac{dA_t}{A_t} - \int_{H_{(n-1)\Delta}^\delta}^{H_{n\Delta}^\delta} \frac{dA_t}{A_t},$$

$$R_n^b = \int_{H_{(n-1)\Delta}^\delta}^{H_{n\Delta}^\delta} \frac{dA_t}{A_t} - \sum_{m=M_{n-1}+1}^{M_n} \frac{A_{m\delta} - A_{(m-1)\delta}}{A_{(m-1)\delta}}.$$

Moreover, we define

$$I_n = \min\big(H_{n\Delta}, H_{n\Delta}^\delta\big) \quad \text{and} \quad J_n = \max\big(H_{n\Delta}, H_{n\Delta}^\delta\big)$$

for $n = 1, \ldots, N$.

We have

$$|R_n^a| \le \left| \int_{H_{(n-1)\Delta}}^{H_{n\Delta}} f_t\, dt - \int_{H_{(n-1)\Delta}^\delta}^{H_{n\Delta}^\delta} f_t\, dt \right| + \left| \int_{H_{(n-1)\Delta}}^{H_{n\Delta}} g_t\, dB_t - \int_{H_{(n-1)\Delta}^\delta}^{H_{n\Delta}^\delta} g_t\, dB_t \right|.$$

The first term is bounded by

$$2 \max_{1\le n\le N} \int_{I_n}^{J_n} f_t\, dt \le 2\Big(\sup_{0\le t\le T} |f_t|\Big) \max_{1\le n\le N} \big|H_{n\Delta} - H_{n\Delta}^\delta\big|$$

for all $n = 1, \ldots, N$, and the quadratic variation of the second term is bounded by

$$2 \max_{1\le n\le N} \int_{I_n}^{J_n} g_t^2\, dt \le 2b_T \max_{1\le n\le N} \big|H_{n\Delta} - H_{n\Delta}^\delta\big|$$

for all $n = 1, \ldots, N$. Clearly, the first term is of order smaller than that of the second term. Therefore, it follows from Lemma A.5 that

$$R_n^a = O_p\big(\delta^{1/4-\varepsilon} T^{1/2} a_T^{-1/2} b_T^{5/4} c_T\big) \tag{A.12}$$

uniformly in $n = 1, \ldots, N$.

Furthermore, we have

$$\begin{aligned} |R_n^b| &\le \sum_{m=M_{n-1}+1}^{M_n} \left| \int_{(m-1)\delta}^{m\delta} \frac{dA_t}{A_t} - \frac{A_{m\delta} - A_{(m-1)\delta}}{A_{(m-1)\delta}} \right| \\ &\le \max_{1\le n\le N} \big|H_{n\Delta}^\delta - H_{(n-1)\Delta}^\delta\big| \max_{1\le m\le M} \left| \int_{(m-1)\delta}^{m\delta} \frac{dA_t}{A_t} - \frac{A_{m\delta} - A_{(m-1)\delta}}{A_{(m-1)\delta}} \right| \end{aligned}$$

for all $n = 1, \ldots, N$. However, we may readily deduce that

$$\max_{1\le n\le N}\left|H^\delta_{n\Delta} - H^\delta_{(n-1)\Delta}\right| \le \max_{1\le n\le N}|H_{n\Delta} - H_{(n-1)\Delta}| + 2\max_{1\le n\le N}\left|H_{n\Delta} - H^\delta_{n\Delta}\right|$$

and

$$\max_{1\le n\le N}|H_{n\Delta} - H_{(n-1)\Delta}| \le \frac{\Delta}{a_T} = O_p\left(a_T^{-1}\right).$$

Consequently, it follows from Lemma A.2 that

$$R^b_n = O_p\left(\delta^{1-\varepsilon}a_T^{-1}b_T c_T\right) \tag{A.13}$$

for any $\varepsilon > 0$, uniformly in $n = 1, \ldots, N$. Note that

$$\max_{1\le n\le N}\left|H_{n\Delta} - H^\delta_{n\Delta}\right| = O_p\left(\delta^{1/2-\varepsilon}Ta_T^{-1}b_T^{3/2}c_T^2\right) = o_p\left(a_T^{-1}\right)$$

due to Lemma A.5. The stated result now follows immediately from (A.11), (A.12), and (A.13). Note that R^b_n is of order smaller than that of the first term of R^a_n. □

The proofs of theorems

Proof of Theorem 3.1. Throughout the proof, we set $T_n = H_{n\Delta}$, where H is introduced above Lemma A.5. Note that $(Tb_T)^{-1/2} = O(N^{-1/2})$, since $N\Delta \le Tb_T$ and Δ is constant. The result for (c_n) may easily be obtained if we let $X_1 = A$ and apply Lemma A.5. It follows that

$$\begin{aligned}\max_{1\le n\le N}\left|c^\delta_n - c_n\right| &= \max_{1\le n\le N}\left|\left(T^\delta_n - T^\delta_{n-1}\right) - (T_n - T_{n-1})\right|\\ &\le 2\max_{1\le n\le N}\left|H_{n\Delta} - H^\delta_{n\Delta}\right|\\ &= O_p\left(\delta^{1/2-\varepsilon}Ta_T^{-1}b_T^{3/2}c_T^2\right) = o_p\left((Tb_T)^{-1/2}\right) = o_p\left(N^{-1/2}\right).\end{aligned}$$

Similarly, we may simply apply Lemma A.6 with $X_j = A$ and note that

$$\frac{\delta^{1/4-\varepsilon}T^{1/2}b_T^{5/4}c_T}{a_T^{1/2}} = o\left(\frac{1}{T^{1/2}b_T^{1/2}}\right) = o\left(N^{-1/2}\right)$$

to deduce the stated result for (x_{nj}).

The proof for (u_{in}) is slightly more involved. Note that

$$\max_{1\le n\le N}\left|u^\delta_{ni} - u_{ni}\right| \le 2\max_{1\le n\le N}|U_{iT^\delta_n} - U_{iT_n}|. \tag{A.14}$$

However, we have

$$U_{iT^\delta_n} - U_{iT_n} = \int_0^{T^\delta_n}\omega_{it}\,dZ_{it} - \int_0^{T_n}\omega_{it}\,dZ_{it},$$

whose quadratic variation is bounded by

$$\max_{1\leq n\leq N}\left|T_n^\delta - T_n\right| = \max_{1\leq n\leq N}\left|H_{n\Delta}^\delta - H_{n\Delta}\right| = O_p\left(\delta^{1/2-\varepsilon}Ta_T^{-1}b_T^{3/2}c_T^2\right)$$

uniformly in $n=1,\ldots,N$, due to Lemma A.5. It follows that

$$\max_{1\leq n\leq N}|U_{iT_n^\delta} - U_{iT_n}| = O_p\left(\left(\delta^{1/2-\varepsilon}Ta_T^{-1}b_T^{3/2}c_T^2\right)^{1/2}\right) \tag{A.15}$$

and, therefore,

$$\max_{1\leq n\leq N}\left|u_{ni}^\delta - u_{ni}\right| = o\left(N^{-1/2}\right),$$

due to (A.14) and (A.15), and $(\delta^{1/2-\varepsilon}Ta_T^{-1}b_T^{3/2}c_T^2)^{1/2} = o((Tb_T)^{-1/2}) = o(N^{-1/2})$.

To finish the proof, we note that

$$\left|y_{ni}^\delta - y_{ni}\right| \leq |\alpha_i|\left|c_n^\delta - c_n\right| + \sum_{j=1}^J |\beta_{ij}|\left|x_{nj}^\delta - x_{nj}\right| + \left|u_{ni}^\delta - u_{ni}\right|$$

uniformly in $i=1,\ldots,I$, from which, along with our previous results, we may easily deduce the stated result for (y_{ni}). □

Proof of Corollary 3.2. We may readily deduce the stated result for $\hat{\Sigma}$ from

$$\begin{aligned}\frac{1}{N}\sum_{n=1}^N \hat{u}_n^\delta\hat{u}_n^{\delta\prime} &= \frac{1}{N}\sum_{n=1}^N u_n^\delta u_n^{\delta\prime} + O_p\left(N^{-1/2}\right)\\ &= \frac{1}{N}\sum_{n=1}^N u_n u_n' + O_p\left(N^{-1/2}\right),\end{aligned}$$

due to the well known regression asymptotics and Theorem 3.1.

For the proof of our result for $\tilde{\Sigma}$, we assume that $I=J=1$, and suppress the subscripts i and j for notational simplicity. The proof for the general case is essentially the same and can easily be established as in the simple case we consider here. We write

$$\hat{U}_{m\delta} - \hat{U}_{(m-1)\delta} = (U_{m\delta} - U_{(m-1)\delta}) - R_{m\delta}$$

with

$$R_{m\delta} = (\hat{\alpha}-\alpha)\delta + (\hat{\beta}-\beta)\frac{X_{m\delta} - X_{(m-1)\delta}}{X_{(m-1)\delta}},$$

so that

$$(\hat{U}_{m\delta} - \hat{U}_{(m-1)\delta})^2 = (U_{m\delta} - U_{(m-1)\delta})^2 - 2(U_{m\delta} - U_{(m-1)\delta})R_{m\delta} \tag{A.16}$$

for $m = 1, \ldots, M$. However, we have

$$\begin{aligned}\frac{1}{N}\sum_{m=1}^{M} R_{m\delta}^2 &\leq 2(\hat{\alpha}-\alpha)^2\frac{\delta^2 M}{N} + 2(\hat{\beta}-\beta)^2\frac{1}{N}\sum_{m=1}^{M}\left(\frac{X_{m\delta}-X_{(m-1)\delta}}{X_{(m-1)\delta}}\right)^2 \\ &= o(N^{-2}) + O(N^{-1}) = O(N^{-1})\end{aligned} \tag{A.17}$$

and

$$\begin{aligned}&\left|\frac{1}{N}\sum_{m=1}^{M}(U_{m\delta}-U_{(m-1)\delta})R_{m\delta}\right| \\ &\quad\leq \left[\frac{1}{N}\sum_{m=1}^{M}(U_{m\delta}-U_{(m-1)\delta})^2\right]^{1/2}\left[\frac{1}{N}\sum_{m=1}^{M}R_{m\delta}^2\right]^{1/2} = O(N^{-1/2}).\end{aligned} \tag{A.18}$$

Now it follows immediately from (A.16), (A.17), and (A.18) that $\tilde{\Sigma}^\delta = \tilde{\Sigma} + O_p(N^{-1/2})$, and the proof is complete. □

Proof of Corollary 3.3. Let $\tau(\alpha)$ and $\tau(\beta_j)$ be the continuous time versions of the Wald statistics $\tau^\delta(\alpha)$ and $\tau^\delta(\beta_j)$ introduced in (29) and (31) of the main paper, that is,

$$\tau(\alpha) = \left(c'c - c'X\left(X'X\right)^{-1}X'c\right)\hat{\alpha}'\bar{\Sigma}^{-1}\hat{\alpha},$$

$$\tau(\beta_j) = \left(x_j'x_j - x_j'X_j\left(X_j'X_j\right)^{-1}X_j'x_j\right)\hat{\beta}_j'\bar{\Sigma}^{-1}\hat{\beta}_j,$$

where c, X, $\hat{\alpha}$, x_j, X_j, $\hat{\beta}_j$, and $\bar{\Sigma}$ are defined from regression (21) correspondingly as c^δ, X^δ, $\hat{\alpha}^\delta$, x_j^δ, X_j^δ, $\hat{\beta}_j^\delta$, and $\bar{\Sigma}^\delta$ that are defined from regression (24). Furthermore, let $\hat{\gamma} = (\hat{\alpha}', \hat{\beta}')'$, where $\hat{\alpha}$ and $\hat{\beta} = (\hat{\beta}_1', \ldots, \hat{\beta}_J')'$, which are, respectively, I and IJ dimensional, are the OLS estimators of α and $\beta = (\beta_1', \ldots, \beta_J')'$.

Define $Z = (c, X)$ and let R be an $I \times I(J+1)$-dimensional matrix given by $R = (I_I, 0_{I\times IJ})$ so that we may represent the null hypothesis $\mathbb{H}_0 : \alpha_1 = \cdots = \alpha_I = 0$ as $R\gamma = 0$ with $\gamma = (\alpha', \beta')'$. Then we may write

$$\tau(\alpha) = (\hat{\gamma}-\gamma)'R'\left(R\left[\left(Z'Z\right)^{-1}\otimes\bar{\Sigma}\right]R'\right)^{-1}R(\hat{\gamma}-\gamma). \tag{A.19}$$

However, due to Assumption 3.1, we have $Z'Z/N \to_p \Lambda$ and

$$\sqrt{N}(\hat{\gamma}-\gamma) \to_d \mathbb{N}\left(0, \Lambda^{-1}\otimes\Sigma\right),$$

and, therefore, it follows that

$$\tau(\alpha) = \left[R\sqrt{N}(\hat{\gamma}-\gamma)\right]'\left(R\left[\left(\frac{Z'Z}{N}\right)^{-1}\otimes\bar{\Sigma}\right]R'\right)^{-1}\left[R\sqrt{N}(\hat{\gamma}-\gamma)\right] \to_d \chi_I^2 \tag{A.20}$$

as $N \to \infty$.

Now we write

$$\tau^\delta(\alpha) = \left(\hat{\gamma}^\delta-\gamma\right)'R'\left(R\left[\left(Z^{\delta\prime}Z^\delta\right)^{-1}\otimes\bar{\Sigma}^\delta\right]R'\right)^{-1}R\left(\hat{\gamma}^\delta-\gamma\right)$$

analogously as in (A.19), where $\hat{\gamma}^{\delta}$ is defined similarly as $\hat{\gamma}$ from $\hat{\alpha}^{\delta}$, and $\hat{\beta}^{\delta} = (\hat{\beta}_1^{\delta\prime}, \ldots, \hat{\beta}_J^{\delta\prime})'$ and $Z^{\delta} = (c^{\delta}, X^{\delta})$. Therefore, we may easily deduce from Theorem 3.1 and Corollaries 3.2 and 3.3 that

$$\tau^{\delta}(\alpha) = \tau(\alpha) + o_p(1),$$

from which and (A.20) it follows that

$$\tau^{\delta}(\alpha) \to_d \chi_I^2$$

as $N \to \infty$. This was to be shown. The proof for $\tau^{\delta}(\beta_j)$ is entirely analogous and is omitted to save space. □

References

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