Supplement to "Clearinghouses for two-sided matching: An experimental study"

(*Quantitative Economics*, Vol. 7, No. 2, July 2016, 449–482)

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Appendix A: Markets ordinal profiles

Market I. Assortative.

Worker Preferences

$\mathbf{w_1}$: $[f_1] \succ f_2 \succ f_3 \succ f_4 \succ f_5 \succ f_6 \succ f_7 \succ f_8$
$\mathbf{w_2}$: $f_1 \succ [f_2] \succ f_3 \succ f_4 \succ f_5 \succ f_6 \succ f_7 \succ f_8$
$\mathbf{w_3}$: $f_1 \succ f_2 \succ [f_3] \succ f_4 \succ f_5 \succ f_6 \succ f_7 \succ f_8$
$\mathbf{w_4}$: $f_1 \succ f_2 \succ f_3 \succ [f_4] \succ f_5 \succ f_6 \succ f_7 \succ f_8$
$\mathbf{w_5}$: $f_1 \succ f_2 \succ f_3 \succ f_4 \succ [f_5] \succ f_6 \succ f_7 \succ f_8$
$\mathbf{w_6}$: $f_1 \succ f_2 \succ f_3 \succ f_4 \succ f_5 \succ [f_6] \succ f_7 \succ f_8$
$\mathbf{w_7}$: $f_1 \succ f_2 \succ f_3 \succ f_4 \succ f_5 \succ f_6 \succ [f_7] \succ f_8$
$\mathbf{w_8}$: $f_1 \succ f_2 \succ f_3 \succ f_4 \succ f_5 \succ f_6 \succ f_7 \succ [f_8]$

Firm Preferences

$\mathbf{f_1}$: $[w_1] \succ w_2 \succ w_3 \succ w_4 \succ w_5 \succ w_6 \succ w_7 \succ w_8$
$\mathbf{f_2}$: $w_1 \succ [w_2] \succ w_3 \succ w_4 \succ w_5 \succ w_6 \succ w_7 \succ w_8$
$\mathbf{f_3}$: $w_1 \succ w_2 \succ [w_3] \succ w_4 \succ w_5 \succ w_6 \succ w_7 \succ w_8$
$\mathbf{f_4}$: $w_1 \succ w_2 \succ w_3 \succ [w_4] \succ w_5 \succ w_6 \succ w_7 \succ w_8$
$\mathbf{f_5}$: $w_1 \succ w_2 \succ w_3 \succ w_4 \succ [w_5] \succ w_6 \succ w_7 \succ w_8$
$\mathbf{f_6}$: $w_1 \succ w_2 \succ w_3 \succ w_4 \succ w_5 \succ [w_6] \succ w_7 \succ w_8$
$\mathbf{f_7}$: $w_1 \succ w_2 \succ w_3 \succ w_4 \succ w_5 \succ w_6 \succ [w_7] \succ w_8$
$\mathbf{f_8}$: $w_1 \succ w_2 \succ w_3 \succ w_4 \succ w_5 \succ w_6 \succ w_7 \succ [w_8]$

Note: This market was chosen as is.

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DOI: 10.3982/QE496

Market II. One; fully aligned.

Worker Preferences

$\mathbf{w_1}: f_1 \succ f_2 \succ f_3 \succ f_4 \succ f_5 \succ f_6 \succ [f_7] \succ f_8$
$\mathbf{w_2}: f_1 \succ f_2 \succ f_3 \succ f_4 \succ f_5 \succ f_6 \succ f_7 \succ [f_8]$
$\mathbf{w_3}: [f_1] \succ f_2 \succ f_3 \succ f_4 \succ f_5 \succ f_6 \succ f_7 \succ f_8$
$\mathbf{w_4}: f_1 \succ f_2 \succ f_3 \succ f_4 \succ [f_5] \succ f_6 \succ f_7 \succ f_8$
$\mathbf{w_5}: f_1 \succ f_2 \succ f_3 \succ [f_4] \succ f_5 \succ f_6 \succ f_7 \succ f_8$
$\mathbf{w_6}: f_1 \succ [f_2] \succ f_3 \succ f_4 \succ f_5 \succ f_6 \succ f_7 \succ f_8$
$\mathbf{w_7}: f_1 \succ f_2 \succ f_3 \succ f_4 \succ f_5 \succ [f_6] \succ f_7 \succ f_8$
$\mathbf{w_8}: f_1 \succ f_2 \succ [f_3] \succ f_4 \succ f_5 \succ f_6 \succ f_7 \succ f_8$

Firm Preferences

$\mathbf{f_1}: [w_3] \succ w_8 \succ w_7 \succ w_4 \succ w_5 \succ w_6 \succ w_1 \succ w_2$
$\mathbf{f_2}: [w_6] \succ w_4 \succ w_5 \succ w_3 \succ w_2 \succ w_7 \succ w_1 \succ w_8$
$\mathbf{f_3}: [w_8] \succ w_5 \succ w_3 \succ w_6 \succ w_1 \succ w_7 \succ w_2 \succ w_4$
$\mathbf{f_4}: [w_5] \succ w_3 \succ w_4 \succ w_1 \succ w_7 \succ w_8 \succ w_6 \succ w_2$
$\mathbf{f_5}: [w_4] \succ w_1 \succ w_2 \succ w_6 \succ w_7 \succ w_3 \succ w_8 \succ w_5$
$\mathbf{f_6}: w_4 \succ [w_7] \succ w_3 \succ w_8 \succ w_5 \succ w_2 \succ w_6 \succ w_1$
$\mathbf{f_7}: w_5 \succ [w_1] \succ w_3 \succ w_7 \succ w_4 \succ w_8 \succ w_6 \succ w_2$
$\mathbf{f_8}: [w_2] \succ w_1 \succ w_3 \succ w_7 \succ w_6 \succ w_5 \succ w_4 \succ w_8$

Note: This market was constrained to have symmetric preferences on the worker side. Firm preferences were randomly varied to generate a unique stable match.

Market III. Split; two aligned.

Worker Preferences

$\mathbf{w_1}: \lceil f_1 \rceil \succ \lfloor f_2 \rfloor \succ f_3 \succ f_4 \succ f_5 \succ f_6 \succ f_7 \succ f_8$
$\mathbf{w_2}: f_1 \succ f_2 \succ f_3 \succ \lceil f_4 \rceil \succ \lfloor f_5 \rfloor \succ f_6 \succ f_7 \succ f_8$
$\mathbf{w_3}: f_1 \succ f_2 \succ \lceil f_3 \rceil \succ \lfloor f_4 \rfloor \succ f_5 \succ f_6 \succ f_7 \succ f_8$
$\mathbf{w_4}: f_1 \succ \lceil f_2 \rceil \succ \lfloor f_3 \rfloor \succ f_4 \succ f_5 \succ f_6 \succ f_7 \succ f_8$
$\mathbf{w_5}: f_5 \succ f_6 \succ f_7 \succ \lceil f_8 \rceil \succ \lfloor f_1 \rfloor \succ f_2 \succ f_3 \succ f_4$
$\mathbf{w_6}: f_5 \succ \lceil f_6 \rceil \succ \lfloor f_7 \rfloor \succ f_8 \succ f_1 \succ f_2 \succ f_3 \succ f_4$
$\mathbf{w_7}: \lceil f_5 \rceil \succ \lfloor f_6 \rfloor \succ f_7 \succ f_8 \succ f_1 \succ f_2 \succ f_3 \succ f_4$
$\mathbf{w_8}: f_5 \succ f_6 \succ \lceil f_7 \rceil \succ \lfloor f_8 \rfloor \succ f_1 \succ f_2 \succ f_3 \succ f_4$

Firm Preferences

$\mathbf{f_1}: \lceil w_5 \rceil \succ \lfloor w_1 \rfloor \succ w_6 \succ w_2 \succ w_3 \succ w_7 \succ w_4 \succ w_8$
$\mathbf{f_2}: \lceil w_1 \rceil \succ \lfloor w_4 \rfloor \succ w_2 \succ w_3 \succ w_6 \succ w_8 \succ w_7 \succ w_5$
$\mathbf{f_3}: \lceil w_4 \rceil \succ w_1 \succ \lfloor w_3 \rfloor \succ w_2 \succ w_5 \succ w_6 \succ w_8 \succ w_7$
$\mathbf{f_4}: \lceil w_3 \rceil \succ w_1 \succ w_4 \succ \lfloor w_2 \rfloor \succ w_5 \succ w_8 \succ w_7 \succ w_6$
$\mathbf{f_5}: \lceil w_2 \rceil \succ w_1 \succ \lfloor w_7 \rfloor \succ w_5 \succ w_4 \succ w_8 \succ w_3 \succ w_6$
$\mathbf{f_6}: \lceil w_7 \rceil \succ \lfloor w_6 \rfloor \succ w_5 \succ w_8 \succ w_2 \succ w_4 \succ w_1 \succ w_3$
$\mathbf{f_7}: \lceil w_6 \rceil \succ \lfloor w_8 \rfloor \succ w_5 \succ w_7 \succ w_1 \succ w_4 \succ w_2 \succ w_3$
$\mathbf{f_8}: \lceil w_8 \rceil \succ w_7 \succ w_6 \succ \lfloor w_5 \rfloor \succ w_3 \succ w_2 \succ w_4 \succ w_1$

Note: This market was found by constraining one side to have two blocks with symmetric preferences. Firm preferences were randomly varied to generate the two distinct stable matches with moderate truncation required to get to an F-best stable match. We additionally searched for two modifications that changed a single participant's preferences by switching his/her ranked order of two participants from the other side, which would remove one of the two stable matches.

Market IV. Two matches; one very unstable.

Worker Preferences

$\mathbf{w_1}: \lceil f_1 \rceil \succ \lfloor f_7 \rfloor \succ f_8 \succ f_2 \succ f_5 \succ f_4 \succ f_3 \succ f_6$
$\mathbf{w_2}: \lceil f_2 \rceil \succ \lfloor f_8 \rfloor \succ f_3 \succ f_4 \succ f_1 \succ f_6 \succ f_5 \succ f_7$
$\mathbf{w_3}: \lceil f_3 \rceil \succ \lfloor f_4 \rfloor \succ f_1 \succ f_5 \succ f_8 \succ f_2 \succ f_6 \succ f_7$
$\mathbf{w_4}: \lceil f_4 \rceil \succ \lfloor f_1 \rfloor \succ f_5 \succ f_8 \succ f_2 \succ f_3 \succ f_7 \succ f_6$
$\mathbf{w_5}: \lceil f_5 \rceil \succ \lfloor f_2 \rfloor \succ f_1 \succ f_7 \succ f_3 \succ f_8 \succ f_4 \succ f_6$
$\mathbf{w_6}: \lceil f_6 \rceil \succ \lfloor f_5 \rfloor \succ f_8 \succ f_2 \succ f_4 \succ f_7 \succ f_3 \succ f_1$
$\mathbf{w_7}: \lceil f_7 \rceil \succ \lfloor f_6 \rfloor \succ f_2 \succ f_5 \succ f_3 \succ f_1 \succ f_4 \succ f_8$
$\mathbf{w_8}: \lceil f_8 \rceil \succ \lfloor f_3 \rfloor \succ f_1 \succ f_2 \succ f_5 \succ f_6 \succ f_7 \succ f_4$

Firm Preferences

$\mathbf{f_1}: w_2 \succ w_6 \succ w_3 \succ \lceil w_4 \rceil \succ w_5 \succ w_7 \succ w_8 \succ \lfloor w_1 \rfloor$
$\mathbf{f_2}: w_8 \succ w_4 \succ w_6 \succ w_3 \succ \lceil w_5 \rceil \succ w_7 \succ w_1 \succ \lfloor w_2 \rfloor$
$\mathbf{f_3}: \lceil w_8 \rceil \succ w_5 \succ w_1 \succ w_6 \succ w_2 \succ w_4 \succ w_7 \succ \lfloor w_3 \rfloor$
$\mathbf{f_4}: w_2 \succ w_1 \succ w_6 \succ \lceil w_3 \rceil \succ w_7 \succ w_8 \succ w_5 \succ \lfloor w_4 \rfloor$
$\mathbf{f_5}: w_1 \succ w_3 \succ \lceil w_6 \rceil \succ w_7 \succ w_4 \succ w_2 \succ w_8 \succ \lfloor w_5 \rfloor$
$\mathbf{f_6}: w_3 \succ w_1 \succ w_5 \succ \lceil w_7 \rceil \succ w_4 \succ w_8 \succ w_2 \succ \lfloor w_6 \rfloor$
$\mathbf{f_7}: \lceil w_1 \rceil \succ w_4 \succ w_3 \succ w_8 \succ w_5 \succ w_2 \succ w_6 \succ \lfloor w_7 \rfloor$
$\mathbf{f_8}: \lceil w_2 \rceil \succ w_6 \succ w_3 \succ w_1 \succ w_7 \succ w_5 \succ w_4 \succ \lfloor w_8 \rfloor$

Note: This market was found by constraining to two distinct stable matches with maximum truncation required to get to the F-best stable match.

Market V. Two matches; unaligned preferences.

Worker Preferences	*Firm Preferences*
$\mathbf{w_1}: f_6 \succ \lceil f_1 \rceil \succ \lfloor f_8 \rfloor \succ f_4 \succ f_3 \succ f_2 \succ f_5 \succ f_7$	$\mathbf{f_1}: w_3 \succ \lceil w_6 \rceil \succ \lfloor w_1 \rfloor \succ w_7 \succ w_5 \succ w_8 \succ w_2 \succ w_4$
$\mathbf{w_2}: f_3 \succ f_1 \succ \lceil f_2 \rceil \succ \lfloor f_4 \rfloor \succ f_5 \succ f_7 \succ f_8 \succ f_6$	$\mathbf{f_2}: \lceil w_3 \rceil \succ w_8 \succ w_1 \succ w_7 \succ \lfloor w_2 \rfloor \succ w_4 \succ w_6 \succ w_5$
$\mathbf{w_3}: \lceil f_3 \rceil \succ f_6 \succ f_8 \succ \lfloor f_2 \rfloor \succ f_1 \succ f_7 \succ f_5 \succ f_4$	$\mathbf{f_3}: w_1 \succ \lceil w_8 \rceil \succ \lfloor w_3 \rfloor \succ w_4 \succ w_2 \succ w_6 \succ w_5 \succ w_7$
$\mathbf{w_4}: f_3 \succ \lceil f_4 \rceil \succ \lfloor f_7 \rfloor \succ f_5 \succ f_1 \succ f_2 \succ f_6 \succ f_8$	$\mathbf{f_4}: \lceil w_2 \rceil \succ w_1 \succ \lfloor w_4 \rfloor \succ w_5 \succ w_7 \succ w_3 \succ w_8 \succ w_6$
$\mathbf{w_5}: \lceil f_6 \rceil \succ f_1 \succ \lfloor f_5 \rfloor \succ f_3 \succ f_2 \succ f_4 \succ f_8 \succ f_7$	$\mathbf{f_5}: w_2 \succ w_8 \succ w_3 \succ \lceil w_5 \rceil \succ w_1 \succ w_4 \succ \lfloor w_6 \rfloor \succ w_7$
$\mathbf{w_6}: f_6 \succ f_2 \succ f_4 \succ \lceil f_5 \rceil \succ \lfloor f_1 \rfloor \succ f_7 \succ f_8 \succ f_3$	$\mathbf{f_6}: w_2 \succ w_8 \succ \lceil w_7 \rceil \succ \lfloor w_5 \rfloor \succ w_4 \succ w_6 \succ w_3 \succ w_1$
$\mathbf{w_7}: f_8 \succ \lceil f_7 \rceil \succ f_1 \succ \lfloor f_6 \rfloor \succ f_2 \succ f_3 \succ f_5 \succ f_4$	$\mathbf{f_7}: w_1 \succ w_2 \succ w_8 \succ w_6 \succ w_5 \succ \lceil w_4 \rceil \succ w_3 \succ \lfloor w_7 \rfloor$
$\mathbf{w_8}: \lceil f_8 \rceil \succ f_1 \succ f_4 \succ \lfloor f_3 \rfloor \succ f_2 \succ f_7 \succ f_5 \succ f_6$	$\mathbf{f_8}: \lceil w_1 \rceil \succ w_5 \succ \lfloor w_8 \rfloor \succ w_4 \succ w_3 \succ w_7 \succ w_6 \succ w_2$

Note: This market was only constrained by a requirement to have two distinct stable matches. We additionally searched for modifications that changed a single participant's preferences by switching his/her ranked order of two participants from the other side and removed one of the stable matches.

Market VI. Four-by-four market.

Worker Preferences	*Firm Preferences*
$\mathbf{w_1}: \lceil f_2 \rceil \succ f_4 \succ \lfloor f_1 \rfloor \succ f_3 \succ f_8 \succ f_7 \succ f_6 \succ f_5$	$\mathbf{f_1}: \lceil w_1 \rceil \succ \lfloor w_2 \rfloor \succ w_3 \succ w_4 \succ w_5 \succ w_6 \succ w_8 \succ w_7$
$\mathbf{w_2}: f_2 \succ \lceil f_1 \rceil \succ \lfloor f_4 \rfloor \succ f_3 \succ f_7 \succ f_6 \succ f_5 \succ f_8$	$\mathbf{f_2}: \lceil w_3 \rceil \succ \lfloor w_1 \rfloor \succ w_2 \succ w_4 \succ w_8 \succ w_5 \succ w_7 \succ w_6$
$\mathbf{w_3}: \lceil f_4 \rceil \succ \lfloor f_2 \rfloor \succ f_3 \succ f_1 \succ f_6 \succ f_5 \succ f_8 \succ f_7$	$\mathbf{f_3}: w_3 \succ [w_4] \succ w_1 \succ w_2 \succ w_6 \succ w_7 \succ w_5 \succ w_8$
$\mathbf{w_4}: f_1 \succ [f_3] \succ f_4 \succ f_2 \succ f_5 \succ f_8 \succ f_7 \succ f_6$	$\mathbf{f_4}: \lceil w_2 \rceil \succ \lfloor w_3 \rfloor \succ w_4 \succ w_1 \succ w_7 \succ w_8 \succ w_6 \succ w_5$
$\mathbf{w_5}: \lceil f_6 \rceil \succ f_8 \succ \lfloor f_5 \rfloor \succ f_7 \succ f_4 \succ f_3 \succ f_2 \succ f_1$	$\mathbf{f_5}: [w_5] \succ \lfloor w_6 \rfloor \succ w_7 \succ w_8 \succ w_1 \succ w_2 \succ w_4 \succ w_3$
$\mathbf{w_6}: f_6 \succ \lceil f_5 \rceil \succ \lfloor f_8 \rfloor \succ f_7 \succ f_3 \succ f_2 \succ f_1 \succ f_4$	$\mathbf{f_6}: [w_7] \succ \lfloor w_5 \rfloor \succ w_6 \succ w_8 \succ w_4 \succ w_1 \succ w_3 \succ w_2$
$\mathbf{w_7}: \lceil f_8 \rceil \succ \lfloor f_6 \rfloor \succ f_7 \succ f_5 \succ f_2 \succ f_1 \succ f_4 \succ f_3$	$\mathbf{f_7}: w_7 \succ [w_8] \succ w_5 \succ w_6 \succ w_2 \succ w_3 \succ w_1 \succ w_4$
$\mathbf{w_8}: f_5 \succ [f_7] \succ f_8 \succ f_6 \succ f_1 \succ f_4 \succ f_3 \succ f_2$	$\mathbf{f_8}: [w_6] \succ \lfloor w_7 \rfloor \succ w_8 \succ w_5 \succ w_3 \succ w_4 \succ w_2 \succ w_1$

Note: This market was designed to have two four-by-four submarkets. There are two stable (nondistinct) matches within each submarket, for a total of four aggregate stable matches.

Appendix B: Theoretical background

B.1 *The underlying model*

Let W and F be disjoint, finite sets. We call the elements of W proposers and the elements of F receivers. The sets W and F can represent medical residents and hospitals, men and women, parents and schools, etcetera, that are to be matched to one another in the market.[1] A *matching* is a function $\mu : W \cup F \to W \cup F$ such that for all $w \in W$ and $f \in F$,

(i) $\mu(f) \in W \cup \{f\}$,

(ii) $\mu(w) \in F \cup \{w\}$,

(iii) $w = \mu(f)$ if and only if $f = \mu(w)$,

where the notation $\mu(a) = a$ means that participant a is unmatched under μ, and $f = \mu(w)$ denotes that w and f are matched under μ. We denote the set of all possible matchings, given the sets W and F, as $\mathcal{M}$.

[1] Since in many centralized labor market applications, it is the workers (say, the doctors in the NRMP) who serve as proposers and the firms (the hospitals in the NRMP) who serve as receivers, we use the corresponding acronyms W and F to denote the two sets.

A *preference relation* is a linear, transitive, and antisymmetric binary relation (all preferences are strict; no proposer or receiver is indifferent over two distinct partners). A preference relation for a proposer $w \in W$, denoted P_w, is understood to be over the set $F \cup \{w\}$. Similarly, for a receiver $f \in F$, P_f denotes a preference relation over $W \cup \{f\}$. If any participant a prefers remaining unmatched to being matched with another participant a' ($a \, P_a \, a'$), we will say that the match $\mu(a) = a'$ is *not individually rational*, or *unacceptable*, for a. We assume that each proposer (receiver) prefers every receiver (proposer) to remaining unmatched.[2]

A *preference profile* is a list P of preference relations for proposers and receivers:

$$P = \big((P_w)_{w \in W}, (P_f)_{f \in F}\big).$$

As is standard, for $i \in W \cup F$, we denote by P_{-i} the profile of preferences for all agents but i. Let $\mathcal{P}$ be the set of all possible preference profiles, and for an agent $i \in W \cup F$, let $\mathcal{P}_i$ denote the set of possible preferences for i.

We assume that preferences are *strict*. Denote by R_w the weak version of P_w. So $f' \, R_w \, f$ if $f' = f$ or $f' \, P_w \, f$. The definition of R_f is analogous.

Fix a preference profile P. We say that a pair (w, f) *blocks* the matching μ if $f \, P_w \, \mu(w)$ and $w \, P_f \, \mu(f)$. A matching is *stable* if it is individually rational and there is no pair that blocks it.[3] Finally, denote by $S(P)$ the set of all stable matchings.

B.2 *Centralized mechanisms*

A *mechanism* is a function $\phi : \mathcal{P} \to \mathcal{M}$ that assigns a matching to each preference profile. A mechanism is *stable* if $\phi(P) \in S(P)$ for all $P \in \mathcal{P}$.

Gale and Shapley (1962) proved that every preference profile admits a stable matching, and provided the following algorithm to identify one.

ALGORITHM 1 (Deferred Acceptance).

- STEP 0. The set A_0 of *active* proposers consists of all the proposers. All receivers have no tentative partners.
- For $k = 1, 2, \ldots$, repeat the following step until A_k is empty:
- STEP k.

– Each proposer w in A_{k-1} proposes to the highest ranked receiver according to P_w, across all of the receivers w has not proposed to in previous steps of the algorithm.

– Each receiver f chooses the best partner (according to P_f), out of the set of proposers that proposed to f in Step k, and f's tentative match from Step $k-1$. This choice is f's new tentative match; reject all other proposals.

[2]This fits our experimental design where remaining unmatched is the worst outcome.

[3]Since we assume that partners are always acceptable, any matching is individually rational under the true preferences.

– The set A_k is formed from the set of all active proposers rejected in this step: either their proposal to a receiver was rejected or they were tentatively matched in Step $k-1$, and rejected in favor of a new proposal.

- STOP

The tentative matching at the end of the last step is the output matching.

THEOREM 1 (Gale–Shapley Theorem). *The set $S(P)$ is nonempty, and there are two matchings μ_W and μ_F in $S(P)$ such that, for all w, f, and $\mu \in S(P)$,*

$$\mu_W(w)\, R_w\, \mu(w)\, R_w\, \mu_F(w),$$
$$\mu_F(f)\, R_f\, \mu(f)\, R_f\, \mu_W(f).$$

The matching μ_W is called *proposer best* while μ_F is called *receiver best*. Beyond its theoretical role in establishing existence, the DA algorithm is often used in centralized markets. For instance, the National Resident Matching Program uses a close modification of the DA algorithm (where physicians serve as proposers and hospitals serve as receivers).

A mechanism ϕ defines a direct revelation game—the normal-form game where the agents in $W \cup F$ simultaneously report their preferences—so the strategy space of agent i is $\mathcal{P}_i$ and the outcome of a profile P is given by $\phi(P)$. Denote by ϕ_{DA} the mechanism defined by the DA procedure.

For an agent $i \in W \cup F$, truth-telling is a *weakly dominant strategy* if, for any preference profile P'_i different from the true preferences P_i, and any profile $\tilde{P}_{-i}$ of all agents but i, $\alpha_2\mathcal{P}$,

$$\phi(P_i, \tilde{P}_{-i})(i)\, R_i\, \phi\big(P'_i, \tilde{P}_{-i}\big)(i).$$

A mechanism is *strategy proof* if truth-telling is weakly dominant for all agents. As it turns out, we have the following mechanism (see Roth and Sotomayor (1990)).

THEOREM 2 (Strategy Proofness in Stable Mechanisms). *In ϕ_{DA}, truth-telling is weakly dominant for proposers. No stable mechanism is strategy proof.*

Fix a preference profile P, and suppose that all proposers w truthfully choose P_w as their strategy in the direct-revelation game. We consider the *induced* game among receivers, where receivers simultaneously choose a preference profile $\tilde{P}_f \in \mathcal{P}_f$. A *Nash equilibrium* of the induced game is a profile of preferences $(P'_f)_{f\in F}$ such that

$$\phi\big((P_w)_{w\in W}, \big(P'_{f'}\big)_{f'\in F}\big)(f)\, R_f\, \phi\big((P_w)_{w\in W}, \tilde{P}_f, \big(P'_{f'}\big)_{f'\in F\setminus f}\big)(f)$$

for all $f \in F$ and $\tilde{P}_f \in \mathcal{P}_f$.

The following result is well known (again, see Roth and Sotomayor (1990)).

THEOREM 3 (Equilibrium Outcomes in Stable Mechanisms). *Consider any stable mechanism implementing the proposer-optimal stable matching for any reported preferences. In the game induced from truth-telling by the proposers, the set of Nash equilibrium outcomes coincides with the set of stable matchings.*

B.3 *Outcome equivalence*

We present the main intuition behind the equivalence between our game and the DA direct-revelation game. Our game is dynamic, and agents can condition their actions on the outcomes of past decisions. The DA direct-revelation game is static. We impose restrictions on agents' strategies that make the differences between the two games irrelevant. Essentially, our assumptions say that we can think of agents' strategies as being preference relations.

Heuristically, a strategy for a proposer maps any sequence of past proposals (with their corresponding outcomes) into a current proposal. We first restrict strategies to only depend on available proposals. For example, if w_1 can only propose to f_1 or f_2, his choice should be independent of the precise sequence of (rejected) proposals that ended with f_1 and f_2 as the remaining choices. While this restriction seems realistic, it is easy to write down examples that violate it. For example, w_1 may choose f_1 over f_2 when, in a past turn, he proposed to f_3 and was immediately rejected. But he may choose f_2 instead if his proposal to f_3 was initially accepted, and rejected several turns later.

The second restriction is standard in choice theory. A strategy for a proposer is a mapping from sets of available receivers into a proposal; for each set F' of receivers, either some $f \in F'$ is proposed to or no proposal is made. The strategy is then a choice function that can take empty-set values. Under standard conditions from choice theory (such as the congruence axiom of Richter (1966)), we can represent such a strategy with a preference relation.[4]

We make analogous assumptions on receivers' behavior. A receiver's strategy is a decision on which proposal to accept, given any set of proposals made by the active proposers, and any proposer whose proposal the receiver holds. Again, the restrictions we impose are of two types. First, strategies cannot depend on histories per se. Second, strategies obey certain minimal consistency requirements across time, so that they can be represented as preference relations.

We show that a profile of strategies, once represented as a profile of preference relations, generates the same outcome as the profile that would have been generated in the preference-revelation game ϕ_{DA}. Hence, the incentives faced by proposers and receivers in both games are the same.

[4]Namely, we can find a preference ranking P_w such that for any set of available receivers F', if there is some acceptable receiver under P_w, the one that is the most preferred according to P_w in F' is proposed to. The restrictions are reminiscent of the weak axiom of revealed choice, assuring consistency of observed behavior.

B.4 *Static and dynamic deferred-acceptance mechanisms*

The following example, which appeared in Niederle and Yariv (2011), illustrates how weakly dominated strategies on the parts of proposers alone do not lead to the same predictions in the static and dynamic versions of the deferred acceptance mechanism.

Example (Additional Equilibrium Outcomes in the Dynamic Version of Deferred Acceptance). Consider a market consisting of proposers $\{W1, W2, W3\}$ and receivers $\{F1, F2, F3\}$, where all agents prefer to be matched rather than unmatched. Let the induced ordinal preferences $\succsim$ of the three proposers and colors be given by

$$\begin{array}{llll} \mathbf{W1}: & F2 \succ \mathbf{F1} \succ F3 & \mathbf{F1}: & \mathbf{W1} \succ W3 \succ W2 \\ \mathbf{W2}: & F1 \succ \mathbf{F2} \succ F3 & \mathbf{F2}: & \mathbf{W2} \succ W1 \succ W3 \\ \mathbf{W3}: & F1 \succ F2 \succ \mathbf{F3} & \mathbf{F3}: & W1 \succ \mathbf{W3} \succ W2. \end{array}$$

The unique stable matching μ is given below (where we use the convention that each column in the matrix denotes a match between the specified proposer and color), $\mu(Wi) = Fi$ for all i. In particular, the DA mechanism entails a unique equilibrium in weakly undominated strategies yielding μ. Nonetheless, the matching $\tilde{\mu}$ (in which $W1$ and $W2$ swap colors relative to μ) can be induced in our dynamic mechanism:

$$\mu = \begin{pmatrix} W1 & W2 & W3 \\ F1 & F2 & F3 \end{pmatrix}, \qquad \tilde{\mu} = \begin{pmatrix} W1 & W2 & W3 \\ F2 & F1 & F3 \end{pmatrix}.$$

Indeed, the following profile in weakly undominated strategies constitutes part of an equilibrium:

Period 1: Proposer $W3$ makes an offer to $F3$ who accepts.

Period 2: Proposer $W1$ makes an offer to $F2$ and $W2$ makes an offer to $F1$ who accept.

Upon any deviation, offers from agents other than the stable match or the most-preferred match are rejected and all revert to emulating the deferred acceptance strategies (in particular, $F1$ rejects an offer from $W3$).

Notice that time plays an important role in the construction of this equilibrium. Indeed, as highlighted in Niederle and Yariv (2011), the crucial element driving this construction is the ability of some participants to commit and of others to condition their behavior on observed market outcomes (note that once $W3$ is accepted, he cannot escape $F3$).[5]

B.5 *Outcome and strategic equivalence*

In the dynamic setup, at each period t agents monitor only partial activity in the market. We now describe the information each agent has throughout the game. At the beginning

[5] Interestingly, this equilibrium is not robust in that it is not sequential (for instance, $F1$ would need to believe that other agents will deviate as well when observing an offer from $W3$, but the market does not offer enough monitoring for that).

of period t, each proposer w observes a history that consists of the (timed) offers the proposer made and the responses of receivers to those offers, denoted by r for rejection and h for holding (where we use the notational convention that an offer to no receiver is denoted as an offer to $\emptyset$ that is immediately rejected):

$$h^W_{t,w} \in \left((F \cup \emptyset) \times \{r, h\}\right)^{t-1}.$$

The set of all possible histories at time t for proposer w is denoted by $H^W_{t,w}$.

In addition, at each period t, suppose receivers $f_1, \dots, f_{k(t-1)}$ rejected offers from proposer w in periods $1, \dots, t-1$. Denote by $\tilde{F}^t_w = \{f | f \notin \{f_1, \dots, f_{k(t-1)}\}\}$ the set of receivers who have not rejected proposer w yet.

Each receiver acts in the second stage of each period t and observes a history that consists of all (timed) offers she received and a (timed) sequence of offers she held:[6]

$$h^F_{t,f} \in \left(2^W\right)^t \times \left(2^W\right)^t.$$

The set of all possible histories at time t for receiver f is denoted by $H^F_{t,f}$.

In addition, at each period t, suppose proposers $w_1, \dots, w_{k(t-1)}$ made offers to receiver f in periods $1, \dots, t$. Denote by $\tilde{W}^t_f = \{w | w \notin \{w_1, \dots, w_{k(t-1)}\}\}$ the set of proposers who have not made an offer to receiver f.

A strategy for proposer w is a collection of mappings $\{\sigma^W_{t,w}\}$, where $\sigma^W_{t,w} : H^W_{t,w} \to F \cup \emptyset$, and whenever at time t, $\sigma^W_{t,w}(h^W_{t,w}) \neq \emptyset$, then $\sigma^W_{t,w}(h^W_{t,w}) \in \tilde{F}^t_w$. A strategy for receiver f is a collection of mappings $\{\sigma^F_{t,f}\}$, where $\sigma^F_{t,f} : H^F_{t,f} \to (W \cup \emptyset)^{2^W \times (W \cup \emptyset)}$. That is, after each history, the receiver's strategy specifies which proposer (if any) would be held from a menu of proposer offers (when possibly already holding an offer).

Note that for proposers, we could, in fact, describe the strategy as $\sigma^W_{t,w} : H^W_{t,w} \to \{P(w)\}$ (when defining a receiver approached at later periods as less preferred).

If agents condition their behavior on time per se, the dynamic setup may, in principle, lead to very different outcomes than the static one. We make the following assumptions.

Assumption 1 (Stationarity). *Strategies do not depend on sequencing: For any proposer w, there exists $\tau^W_w : 2^F \to F \cup \emptyset$, such that whenever at time t proposer w is not held and under history $h^W_{t,w}$, $\tilde{F}^t_w$ are the receivers he can make an offer to, $\sigma^W_{t,w}(h^W_{t,w}) = \tau^W_w(\tilde{F}^t_w)$.*

For any receiver f, there exists $\tau^F_f : 2^W \times (W \cup \emptyset) \to W \cup \emptyset$, such that whenever at time t receiver f has observed history $h^F_{t,f}$, under which she holds an offer from $f \in F \cup \emptyset$ (where holding an offer from $\emptyset$ is interpreted as not holding an offer), and the set of proposers who made her an offer in t is $\tilde{W}$, then $\sigma^F_{t,f}(h^F_{t,f}) = \tau^F_f(\tilde{W}, w)$.

Assume that proposers make offers whenever they can.

Stationarity in and of itself does not assure a representation through a preference ranking. Indeed, if $\tau^W_w(f_1, f_2) = f_1$, but $\tau^W_w(f_1, f_2, f_3) = f_2$, this would not be consistent

[6] An offer of proposer w to receiver f that is held from period t to t' is recorded as an offer made in periods $t, t+1, \dots, t'$ that is held by the receiver in each of these periods. We use a similar convention for proposers.

with a preference ordering. Namely, a form of independence of irrelevant alternatives is being violated. Furthermore, if $\tau_w^W(f_1, f_2) = f_1$, $\tau_w^W(f_2, f_3) = f_2$, and $\tau_w^W(f_3, f_1) = f_3$, we would obtain a violation of transitivity when trying to explain behavior through a preference ordering. This is in the spirit of violations of the weak axiom of revealed preferences.

The equivalence between the two types of mechanisms rests on a familiar idea from choice theory, effectively a variation of independence of irrelevant alternatives.

Let X be a finite set and let $\mathbb{B} \subseteq 2^X$. A choice function is a function $C : \mathbb{B} \to X$ such that $C(A) \in A$ for all $A \in \mathbb{B}$. We can associate a binary relation $\succ^C$ with C, where $x \succ^C y$ if and only if there is a set $A \in \mathbb{B}$ with $x, y \in A$ and $x = C(A)$. Note that $\succ^C$ is the revealed-preference relation.

The choice function C satisfies the *congruence axiom* if $\succ^C$ is acyclic; that is, if whenever $x_1, \dots, x_K$ is a sequence in X such that

$$x_1 \succ^C x_2 \succ^C \cdots \succ^C x_K,$$

then it is false that $x_K \succ^C x_1$.

In our setup, each proposer w and receiver f is characterized by a choice function, τ_w^W and τ_f^F, respectively. We say that the congruence axiom holds when all agents' choice functions satisfy the congruence axiom.

Proposition 1 (Equivalence). *Whenever stationarity and the congruence axiom hold, equilibria outcomes in weakly undominated strategies of the static DA mechanism coincide with equilibria outcomes in weakly undominated strategies of the dynamic DA mechanism. Furthermore, there is a one-to-one mapping between weakly undominated equilibrium strategy profiles corresponding to the two mechanisms.*

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Co-editor Karl Schmedders handled this manuscript.

Submitted September, 2014. Final version accepted September, 2015.