Supplement to "Jump factor models in large cross-sections"

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Jia Li
Department of Economics, Duke University

Viktor Todorov
Department of Finance, Kellogg School of Management, Northwestern University

George Tauchen
Department of Economics, Duke University

This Online Supplementary Material contains all proofs for the results in the main text.

Appendix SA: Proofs

Throughout the proofs, we use K to denote a generic constant that may change from line to line. For a sub σ-field $\mathcal{G} \subseteq \mathcal{F}$ and a sequence X_n of random variables, we write $X_n \xrightarrow{\mathcal{L}|\mathcal{G}} X$ if the $\mathcal{G}$-conditional law of X_n converges in probability to that of X under a metric that is associated with the weak convergence of probability measures. By a standard localization procedure, we can strengthen Assumption 3 as the following without loss of generality:

Assumption SA1. *Suppose Assumption* 3 *holds with* $T_1 = \infty$. *Moreover, the processes* α_j, λ_j, λ_Z, b_f, σ_f, $\tilde{\sigma}_j^2$ *and* $\tilde{J}_{Y,j}$ *are bounded, uniformly in* j.

SA.1 *Preliminary results*

In this subsection, we introduce some notation and preliminary estimates that are used in the sequel. We consider a sequence Ω_n of random events defined by $\Omega_n \equiv \{$distinct jump times of the Poisson process $t \mapsto \mu([0,t], E)$ are at least $2k_n\Delta_n$ apart$\}$. Since $k_n\Delta_n \to 0$ and the jumps of Z is of finite activity, $\mathbb{P}(\Omega_n) \to 1$. Therefore, we can restrict our calculations to Ω_n without loss of generality. It is (notationally) convenient to extend the definition of the spot jump beta to all $t \in [0, T]$ such that, on each path, $\beta_{j,t} = \beta_{j,\tau}$ for $t \in [\tau - k_n\Delta_n, \tau + k_n\Delta_n]$. This extension is well behaved on Ω_n and our analysis only concerns the behavior of $\beta_{j,t}$ around shrinking neighborhoods around the jump times. (It should be noted that $\beta_{j,t}$ defined as such is not adapted to $\mathcal{F}_t$.)

Jia Li: jl410@duke.edu
Viktor Todorov: v-todorov@northwestern.edu
George Tauchen: george.tauchen@duke.edu

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We also consider the following sequence of events:

$$\Omega'_n \equiv \left\{ \sum_{j=1}^{N_n} 1_{\{\Delta^n_{i(n,\tau)} \tilde{J}_{Y,j} \neq 0\}} \leq \lfloor N_n q^w_n \rfloor / 2 \text{ for some } \tau \in \mathcal{T} \right\}. \tag{SA.1}$$

By Markov's inequality,

$$\mathbb{P}\left(\sum_{j=1}^{N_n} 1_{\{\Delta^n_{i(n,\tau)} \tilde{J}_{Y,j} \neq 0\}} > \lfloor N_n q^w_n \rfloor / 2 \right) \leq \frac{2}{\lfloor N_n q^w_n \rfloor} \sum_{j=1}^{N_n} \mathbb{P}\big(\Delta^n_{i(n,\tau)} \tilde{J}_{Y,j} \neq 0\big) \leq K\Delta_n / q^w_n \to 0.$$

Since $\mathcal{T}$ is finite, we have $\mathbb{P}(\Omega'_n) \to 1$.

We denote the continuous part of Y_j and Z as, respectively,

$$Y'_{j,t} \equiv \int_0^t \alpha_{j,u}\, du + \int_0^t \lambda^\top_{j,u}\, df_u + \epsilon_{j,t}, \qquad Z'_t \equiv \int_0^t \lambda^\top_{Z,u}\, df_u. \tag{SA.2}$$

The diffusive residual process is then defined as

$$\widetilde{Y}'_{j,t} \equiv Y'_{j,t} - \beta_{j,t} Z'_t = \int_0^t \alpha_{j,u}\, du + \left(\int_0^t \lambda^\top_{j,u}\, df_u - \beta_{j,t} \int_0^t \lambda^\top_{Z,u}\, df_u \right) + \epsilon_{j,t}. \tag{SA.3}$$

We denote

$$\xi_{n,j,s} \equiv \frac{\Delta^n_{i(n,s)} \widetilde{Y}'_j}{\Delta^n_{i(n,s)} Z}, \tag{SA.4}$$

which can be decomposed as

$$\xi_{n,j,s} = \xi'_{n,j,s} + \xi''_{n,j,s}, \tag{SA.5}$$

where

$$\begin{cases} \xi'_{n,j,s} \equiv \dfrac{1}{\Delta Z_s} \big(\tilde{\lambda}^\top_{j,s-} (f_s - f_{(i(n,s)-1)\Delta_n}) + \tilde{\lambda}^\top_{j,s} (f_{i(n,s)\Delta_n} - f_s) \\ \qquad + \epsilon_{i(n,s)\Delta_n} - \epsilon_{(i(n,s)-1)\Delta_n} \big), \\ \xi''_{n,j,s} \equiv \dfrac{1}{\Delta Z_s} \displaystyle\int_{(i(n,s)-1)\Delta_n}^{i(n,s)\Delta_n} \alpha_{j,u}\, du + \Delta^n_{i(n,s)} \widetilde{Y}'_j \left(\frac{1}{\Delta^n_{i(n,s)} Z} - \frac{1}{\Delta Z_s} \right) \\ \qquad + \dfrac{1}{\Delta Z_s} \displaystyle\int_{(i(n,s)-1)\Delta_n}^{s} (\lambda_{j,u} - \lambda_{j,s-})^\top\, df_u \\ \qquad - \dfrac{\beta_{j,s}}{\Delta Z_s} \displaystyle\int_{(i(n,s)-1)\Delta_n}^{s} (\lambda_{Z,u} - \lambda_{Z,s-})^\top\, df_u \\ \qquad + \dfrac{1}{\Delta Z_s} \displaystyle\int_s^{i(n,s)\Delta_n} (\lambda_{j,u} - \lambda_{j,s})^\top\, df_u - \frac{\beta_{j,s}}{\Delta Z_s} \int_s^{i(n,s)\Delta_n} (\lambda_{Z,u} - \lambda_{Z,s})^\top\, df_u. \end{cases} \tag{SA.6}$$

We further rewrite $\xi'_{n,j,s}$ as

$$\xi'_{n,j,s} = \Delta_n^{1/2} \sum_{q \in \{s-, s+\}} w_{n,q} \big(\tilde{\lambda}^\top_{j,q} \Sigma^{1/2}_{f,q} \zeta_{n,q} + R_{n,j,q} \big), \tag{SA.7}$$

where we define

$$\begin{cases} \zeta_{n,s-} \equiv \Sigma_{f,s-}^{-1/2} \dfrac{f_s - f_{(i(n,s)-1)\Delta_n}}{\sqrt{s-\big(i(n,s)-1\big)\Delta_n}}, & \zeta_{n,s+} \equiv \Sigma_{f,s}^{-1/2} \dfrac{f_{i(n,s)\Delta_n} - f_s}{\sqrt{i(n,s)\Delta_n - s}}, \\ R_{n,j,s-} \equiv \dfrac{\epsilon_{j,s} - \epsilon_{j,(i(n,s)-1)\Delta_n}}{\sqrt{s-\big(i(n,s)-1\big)\Delta_n}}, & R_{n,j,s+} \equiv \dfrac{\epsilon_{j,i(n,s)\Delta_n} - \epsilon_{j,s}}{\sqrt{i(n,s)\Delta_n - s}}, \\ w_{n,s-} \equiv \dfrac{1}{\Delta Z_s}\sqrt{\dfrac{s-\big(i(n,s)-1\big)\Delta_n}{\Delta_n}}, & w_{n,s+} \equiv \dfrac{1}{\Delta Z_s}\sqrt{\dfrac{i(n,s)\Delta_n - s}{\Delta_n}}. \end{cases} \tag{SA.8}$$

LEMMA SA1. *Under Assumptions* 3 *and* 4, *we have for* $p, q \in \{\tau-, \tau+, \eta-, \eta+\}$:

(a) $N_n^{-1}\sum_{j=1}^{N_n} R_{n,j,p}R_{n,j,q} = O_p(N_n^{-1/2})$ *when* $p \neq q$;

(b) $N_n^{-1}\sum_{j=1}^{N_n} R_{n,j,q}^2 \xrightarrow{\mathbb{P}} M_\epsilon(q)$;

(c) $N_n^{-1}\sum_{j=1}^{N_n} R_{n,j,p}\tilde{\lambda}_{j,q} = O_p(N_n^{-1/2})$;

(d) $N_n^{-1}\Delta_n^{-1}\sum_{j=1}^{N_n}(\xi''_{n,j,\tau} - \xi''_{n,j,\eta})^2 = O_p(\Delta_n)$.

PROOF OF LEMMA SA1(A). We prove the case with $p = \tau-$ and $q = \tau+$ in detail, while noting that the other cases can be proved in exactly the same way. Note that the jump times of the Poisson measure μ are necessarily independent of the Brownian motions $\widetilde{W}_j$, $1 \leq j \leq N_n$. Let $\mathcal{G}_t$ be the smallest filtration such that $\mathcal{F}_t \subseteq \mathcal{G}_t$ and the jump times of μ are $\mathcal{G}_t$-measurable. The processes $(\widetilde{W}_j)_{1\leq j\leq N_n}$ remain to be Brownian motions with respect to $(\mathcal{G}_t)_{t\geq 0}$. Consequently, ϵ_j is a $(\mathcal{G}_t)_{t\geq 0}$-martingale, and hence,

$$\mathbb{E}[R_{n,j,\tau-}R_{n,j,\tau+}] = 0. \tag{SA.9}$$

Moreover, for $j \neq m$,

$$\mathbb{E}[R_{n,j,\tau-}R_{n,j,\tau+}R_{n,m,\tau-}R_{n,m,\tau+}] = \mathbb{E}\big[R_{n,j,\tau-}R_{n,m,\tau-}\mathbb{E}[R_{n,j,\tau+}R_{n,m,\tau+}|\mathcal{G}_\tau]\big] = 0, \tag{SA.10}$$

where the first equality holds because $R_{n,j,\tau-}R_{n,m,\tau-}$ is $\mathcal{G}_\tau$-measurable and the second equality holds because $\widetilde{W}_j$ and $\widetilde{W}_m$ are orthogonal. Since the processes $\tilde{\sigma}_j^2$ are uniformly bounded, $\mathbb{E}[R_{n,j,\tau\pm}^4] \leq K$ holds due to a standard estimate for continuous Itô processes. By the Cauchy–Schwarz inequality, this further implies that

$$\mathbb{E}\big[R_{n,j,\tau-}^2 R_{n,j,\tau+}^2\big] \leq K. \tag{SA.11}$$

From (SA.9), (SA.10), and (SA.11), we deduce

$$\mathbb{E}\left[\left(N_n^{-1}\sum_{j=1}^{N_n} R_{n,j,p}R_{n,j,q}\right)^2\right] \leq K N_n^{-1}.$$

The assertion of part (a) then readily follows.

(b) We consider the case $q=\tau-$ in detail while noting that the other cases can be proved in the same way. By using Itô's formula, we can decompose

$$R^2_{n,j,\tau-}=U_{n,j}+U'_{n,j},\quad \text{where}$$
$$U_{n,j}\equiv\frac{1}{\tau-\big(i(n,\tau)-1\big)\Delta_n}\int_{(i(n,\tau)-1)\Delta_n}^{\tau}\tilde{\sigma}^2_{j,u}\,du,$$
$$U'_{n,j}\equiv\frac{2}{\tau-\big(i(n,\tau)-1\big)\Delta_n}\int_{(i(n,\tau)-1)\Delta_n}^{\tau}(\epsilon_{j,u}-\epsilon_{j,(i(n,\tau)-1)\Delta_n})\,d\widetilde{W}_{j,u}.$$

We note that $\mathbb{E}[U'_{n,j}]=0$ for each j and $\mathbb{E}[U'_{n,j}U'_{n,m}]=0$ for $j\neq m$. In addition, $\mathbb{E}|U'_{n,j}|^2\leq K$. From these estimates, it readily follows that

$$\frac{1}{N_n}\sum_{j=1}^{N_n}U'_{n,j}=O_p\big(N_n^{-1/2}\big). \tag{SA.12}$$

Next, note that by Assumption 3(v),

$$\mathbb{E}\big|U_{n,j}-\tilde{\sigma}^2_{j,\tau-}\big|\leq\mathbb{E}\Big[\sup_{s,t,|s-t|\leq\Delta_n}\big|\tilde{\sigma}^2_{j,s}-\tilde{\sigma}^2_{j,t}\big|\Big]\leq K\Delta_n^{1/2}.$$

From this estimate and Assumption 4, we deduce

$$\frac{1}{N_n}\sum_{j=1}^{N_n}U_{n,j}=\frac{1}{N_n}\sum_{j=1}^{N_n}\tilde{\sigma}^2_{j,\tau-}+o_p(1)=M_\epsilon(\tau-)+o_p(1). \tag{SA.13}$$

The assertion of part (b) then follows from (SA.12) and (SA.13).

(c) By Assumption 4, $\tilde{\lambda}_{j,q}$ is conditionally independent of $R_{n,j,p}$, and hence, $\mathbb{E}[R_{n,j,p}\tilde{\lambda}_{j,q}|\mathcal{G}_0]=\mathbb{E}[R_{n,j,p}|\mathcal{G}_0]\mathbb{E}[\tilde{\lambda}_{j,q}|\mathcal{G}_0]=0$. In addition, for $j\neq m$,

$$\mathbb{E}\big[R_{n,j,p}R_{n,m,p}\tilde{\lambda}_{j,q}\tilde{\lambda}_{m,q}^\top|\mathcal{G}_0\big]=\mathbb{E}[R_{n,j,p}R_{n,m,p}|\mathcal{G}_0]\mathbb{E}\big[\tilde{\lambda}_{j,q}\tilde{\lambda}_{m,q}^\top|\mathcal{G}_0\big]=0,$$

where the second equality follows from the orthogonality between $\widetilde{W}_j$ and $\widetilde{W}_m$. Since $\tilde{\lambda}_{j,q}$ is bounded, $R_{n,j,p}\tilde{\lambda}_{j,q}$ has bounded second moment. The assertion of part (c) readily follows from these facts.

(d) First, since the α_j's are uniformly bounded, it is easy to see that $(\Delta Z_s)^{-1}\times\int_{(i(n,s)-1)\Delta_n}^{i(n,s)\Delta_n}\alpha_{j,u}\,du=O_p(\Delta_n)$ uniformly in j. Hence,

$$\frac{1}{N_n\Delta_n}\sum_{j=1}^{N_n}\bigg(\frac{1}{\Delta Z_s}\int_{(i(n,s)-1)\Delta_n}^{i(n,s)\Delta_n}\alpha_{j,u}\,du\bigg)^2=O_p(\Delta_n). \tag{SA.14}$$

Further note that, uniformly in j, we have $\mathbb{E}|\Delta^n_{i(n,s)}\widetilde{Y}'_j|^2\leq K\Delta_n$, and hence,

$$\frac{1}{N_n\Delta_n}\sum_{j=1}^{N_n}\big(\Delta^n_{i(n,s)}\widetilde{Y}'_j\big)^2=O_p(1). \tag{SA.15}$$

It is easy to see that

$$\frac{1}{\Delta_{i(n,s)}^n Z} - \frac{1}{\Delta Z_s} = O_p\big(\Delta_n^{1/2}\big). \tag{SA.16}$$

From (SA.15) and (SA.16), we deduce

$$\frac{1}{N_n \Delta_n} \sum_{j=1}^{N_n} \left(\Delta_{i(n,s)}^n \widetilde{Y}_j' \left(\frac{1}{\Delta_{i(n,s)}^n Z} - \frac{1}{\Delta Z_s} \right) \right)^2 = O_p(\Delta_n). \tag{SA.17}$$

We then note that, since the processes λ_j's are $(1/2)$-Hölder continuous under L_2-norm uniformly in j (Assumption 3(v)), the following estimate also holds uniformly:

$$\mathbb{E}\left[\left(\int_{(i(n,s)-1)\Delta_n}^{s} (\lambda_{j,u} - \lambda_{j,s-})^\top \, df_u \right)^2 \right] \leq K \Delta_n^2.$$

Hence,

$$\frac{1}{N_n \Delta_n} \sum_{j=1}^{N_n} \left(\frac{1}{\Delta Z_s} \int_{(i(n,s)-1)\Delta_n}^{s} (\lambda_{j,u} - \lambda_{j,s-})^\top \, df_u \right)^2 = O_p(\Delta_n). \tag{SA.18}$$

Similarly,

$$\begin{cases} \dfrac{1}{N_n \Delta_n} \displaystyle\sum_{j=1}^{N_n} \left(\dfrac{\beta_{j,s}}{\Delta Z_s} \displaystyle\int_{(i(n,s)-1)\Delta_n}^{s} (\lambda_{Z,u} - \lambda_{Z,s-})^\top \, df_u \right)^2 = O_p(\Delta_n), \\ \dfrac{1}{N_n \Delta_n} \displaystyle\sum_{j=1}^{N_n} \left(\dfrac{1}{\Delta Z_s} \displaystyle\int_{s}^{i(n,s)\Delta_n} (\lambda_{j,u} - \lambda_{j,s})^\top \, df_u \right)^2 = O_p(\Delta_n), \\ \dfrac{1}{N_n \Delta_n} \displaystyle\sum_{j=1}^{N_n} \left(\dfrac{\beta_{j,s}}{\Delta Z_s} \displaystyle\int_{s}^{i(n,s)\Delta_n} (\lambda_{Z,u} - \lambda_{Z,s})^\top \, df_u \right)^2 = O_p(\Delta_n). \end{cases} \tag{SA.19}$$

With an appeal to the Cauchy–Schwarz inequality, the assertion of part (d) then follows from (SA.14), (SA.17), (SA.18), and (SA.19). □

Next, we set

$$\begin{cases} A_n(s) \equiv \dfrac{1}{N_n \Delta_n} \displaystyle\sum_{j=1}^{N_n} \big(\xi'_{n,j,s}\big)^2, \quad s \in \{\eta, \tau\}, \\ B_n(\eta, \tau) \equiv \dfrac{1}{N_n \Delta_n} \displaystyle\sum_{j=1}^{N_n} \xi'_{n,j,\eta} \xi'_{n,j,\tau}. \end{cases} \tag{SA.20}$$

The following lemma collects some convergence results that we use for deriving limiting distributions.

Lemma SA2. *Suppose that Assumptions* 3 *and* 4 *hold. Then*

$$\big(A_n(\eta), A_n(\tau), B_n(\eta, \tau)\big) \xrightarrow{\mathcal{L}\text{-}s} \big(\mathcal{A}(\eta), \mathcal{A}(\tau), \mathcal{B}(\eta, \tau)\big),$$

where $\xrightarrow{\mathcal{L}\text{-}s}$ *denotes* $\mathcal{F}$*-stable convergence in law.*

Proof of Lemma SA2. By Theorem 4.3.1 in Jacod and Protter (2012),

$$(w_{n,q}, \zeta_{n,q})_{q\in\{\eta-,\eta+,\tau-,\tau+\}} \xrightarrow{\mathcal{L}\text{-}s} (w_q, \zeta_q)_{q\in\{\eta-,\eta+,\tau-,\tau+\}}. \tag{SA.21}$$

Recall the definitions in (SA.7) and (SA.20). We have, for $s \in \{\eta, \tau\}$,

$$\begin{aligned} A_n(s) &= \frac{1}{N_n}\sum_{j=1}^{N_n}\Bigg(\sum_{q\in\{s-,s+\}} w_{n,q}\big(\tilde{\lambda}_{j,q}^{\top}\Sigma_{f,q}^{1/2}\zeta_{n,q} + R_{n,j,q}\big)\Bigg)^2 \\ &= \sum_{p,q\in\{s-,s+\}} w_{n,p}w_{n,q}\zeta_{n,p}^{\top}\Sigma_{f,p}^{1/2}\Bigg(\frac{1}{N_n}\sum_{j=1}^{N_n}\tilde{\lambda}_{j,p}\tilde{\lambda}_{j,q}^{\top}\Bigg)\Sigma_{f,q}^{1/2}\zeta_{n,q} \\ &\quad + \sum_{q\in\{s-,s+\}} w_{n,q}^2\Bigg(\frac{1}{N_n}\sum_{j=1}^{N_n}R_{n,j,q}^2\Bigg) + O_p\big(N_n^{-1/2}\big), \end{aligned} \tag{SA.22}$$

where the rate for the $O_p(N_n^{-1})$ term in the last line is obtained using Lemma SA1(a), (c). Similarly,

$$\begin{aligned} &B_n(\eta,\tau) \\ &= \frac{1}{N_n}\sum_{j=1}^{N_n}\Bigg(\sum_{p\in\{\tau-,\tau+\}} w_{n,p}\big(\tilde{\lambda}_{j,p}^{\top}\Sigma_{f,p}^{1/2}\zeta_{n,p} + R_{n,j,p}\big)\Bigg) \\ &\quad \times\Bigg(\sum_{q\in\{\eta-,\eta+\}} w_{n,q}\big(\tilde{\lambda}_{j,q}^{\top}\Sigma_{f,q}^{1/2}\zeta_{n,q} + R_{n,j,q}\big)\Bigg) \\ &= \sum_{p\in\{\tau-,\tau+\}}\sum_{q\in\{\eta-,\eta+\}} w_{n,p}w_{n,q}\zeta_{n,p}^{\top}\Sigma_{f,p}^{1/2}\Bigg(\frac{1}{N_n}\sum_{j=1}^{N_n}\tilde{\lambda}_{j,p}\tilde{\lambda}_{j,q}^{\top}\Bigg)\Sigma_{f,q}^{1/2}\zeta_{n,q} \\ &\quad + O_p\big(N_n^{-1/2}\big). \end{aligned} \tag{SA.23}$$

By Assumption 4 and Lemma SA1(b),

$$\Sigma_{f,p}^{1/2}\Bigg(\frac{1}{N_n}\sum_{j=1}^{N_n}\tilde{\lambda}_{j,p}\tilde{\lambda}_{j,q}^{\top}\Bigg)\Sigma_{f,q}^{1/2} \xrightarrow{\mathbb{P}} M_C(p,q), \qquad \frac{1}{N_n}\sum_{j=1}^{N_n}R_{n,j,q}^2 \xrightarrow{\mathbb{P}} M_\epsilon(q). \tag{SA.24}$$

We further note that the limiting variables $M_C(p,q)$ and $M_\epsilon(q)$ are $\mathcal{F}$-measurable. Hence, by the property of stable convergence in law, we can deduce the assertion of Lemma SA2 from (SA.21), (SA.22), (SA.23), and (SA.24). □

Finally, we show in Lemma SA3 some consistency results for the spot jump beta estimates.

Lemma SA3. *Under Assumptions* 3 *and* 5, *the following holds for* $s \in \mathcal{T}$:

(a) $\sup_{1\le j\le N_n} |\hat{\beta}_{n,j,s} - \beta_{j,s}| 1_{\{\Delta_{i(n,s)}^n \tilde{J}_{Y,j}=0\}} = o_p(1)$;

(b) $N_n^{-1}\sum_{j=1}^{N_n}|\hat{\beta}_{n,j,\tau}-\beta_{j,\tau}|^2=o_p(1)$.

PROOF OF LEMMA SA3(A). Note that

$$\hat{\beta}_{n,j,s}-\beta_{j,s}=\frac{\Delta_{i(n,s)}^n Y_j-\beta_{j,s}\Delta_{i(n,s)}^n Z}{\Delta_{i(n,s)}^n Z}. \tag{SA.25}$$

By localization, we can assume that $\tilde{\sigma}_j^2$, Σ_f, λ_j and β_j are bounded. By a standard estimate for continuous Itô semimartingales (applied to the continuous parts of Y_j and Z), we have for any $p\geq 1$,

$$\mathbb{E}\big[\big|\Delta_{i(n,s)}^n Y_j-\beta_{j,s}\Delta_{i(n,s)}^n Z\big|^p 1_{\{\Delta_{i(n,s)}^n \tilde{J}_{Y,j}=0\}}\big]\leq K_p\Delta_n^{p/2},$$

for some constant K_p. By using a maximal inequality, we deduce that

$$\sup_{1\leq j\leq N_n}\big|\Delta_{i(n,s)}^n Y_j-\beta_{j,s}\Delta_{i(n,s)}^n Z\big|1_{\{\Delta_{i(n,s)}^n \tilde{J}_{Y,j}=0\}}=O_p\big(\Delta_n^{1/2}N_n^{\iota}\big) \tag{SA.26}$$

for some arbitrarily small (but fixed) constant $\iota>0$. Then, by Assumption 5,

$$\sup_{1\leq j\leq N_n}\big|\Delta_{i(n,s)}^n Y_j-\beta_{j,s}\Delta_{i(n,s)}^n Z\big|1_{\{\Delta_{i(n,s)}^n \tilde{J}_{Y,j}=0\}}=o_p(1).$$

Note that $1/\Delta_{i(n,s)}^n Z=O_p(1)$. The assertion of the lemma then readily follows from the above estimate and equation (SA.25).

(b) It is easy to see that $\hat{\beta}_{n,j,\tau}$, $1\leq j\leq N_n$, are uniformly bounded with probability approaching one. We then note that

$$\begin{aligned}
&\frac{1}{N_n}\sum_{j=1}^{N_n}|\hat{\beta}_{n,j,\tau}-\beta_{j,\tau}|^2\\
&=\frac{1}{N_n}\sum_{j=1}^{N_n}|\hat{\beta}_{n,j,\tau}-\beta_{j,\tau}|^2 1_{\{\Delta_{i(n,s)}^n \tilde{J}_{Y,j}=0\}}+\frac{1}{N_n}\sum_{j=1}^{N_n}|\hat{\beta}_{n,j,\tau}-\beta_{j,\tau}|^2 1_{\{\Delta_{i(n,s)}^n \tilde{J}_{Y,j}\neq 0\}}\\
&\leq\Big(\sup_{1\leq j\leq N_n}|\hat{\beta}_{n,j,s}-\beta_{j,s}|1_{\{\Delta_{i(n,s)}^n \tilde{J}_{Y,j}=0\}}\Big)^2+\frac{K}{N_n}\sum_{j=1}^{N_n}1_{\{\Delta_{i(n,s)}^n \tilde{J}_{Y,j}\neq 0\}}\\
&=o_p(1),
\end{aligned}$$

as claimed in part (b). □

SA.2 *Proof of Proposition 1*

PROOF OF PROPOSITION 1. Recall that the spot jump betas $\beta_{j,s}$ are bounded by assumption. By Lemma SA3 and the boundedness of $\tilde{J}_{Y,j}$, we further deduce that the beta estimates $\hat{\beta}_{n,j,s}$ are uniformly (in j) bounded with probability approaching one. Since the loss function $L(\cdot)$ is Lipschitz on bounded sets (Assumption 1), we can now assume that $L(\cdot)$ is globally Lipschitz without loss of generality. Hence, by Lemma SA3,

$$\begin{aligned}
&\frac{1}{N_n}\sum_{j=1}^{N_n}\big|L(\hat{\beta}_{n,j,\tau}-\hat{\beta}_{n,j,\eta})-L(\chi_{j,\eta,\tau})\big| \\
&\leq \frac{1}{N_n}\sum_{j=1}^{N_n}\big|L(\hat{\beta}_{n,j,\tau}-\hat{\beta}_{n,j,\eta})-L(\chi_{j,\eta,\tau})\big|1_{\{|\Delta^n_{i(n,\tau)}\tilde{J}_{Y,j}|+|\Delta^n_{i(n,\eta)}\tilde{J}_{Y,j}|=0\}} \\
&\quad+\frac{K}{N_n}\sum_{j=1}^{N_n}1_{\{|\Delta^n_{i(n,\tau)}\tilde{J}_{Y,j}|+|\Delta^n_{i(n,\eta)}\tilde{J}_{Y,j}|>0\}} \\
&\leq K\max_{s\in\{\eta,\tau\},1\leq j\leq N_n}|\hat{\beta}_{n,j,s}-\beta_{j,s}|1_{\{\Delta^n_{i(n,s)}\tilde{J}_{Y,j}=0\}}+O_p(\Delta_n) \\
&=o_p(1).
\end{aligned}\tag{SA.27}$$

Next, we set

$$\xi_n=\frac{1}{N_n}\sum_{j=1}^{N_n}\big(L(\chi_{j,\eta,\tau})-\mathbb{E}\big[L(\chi_{j,\eta,\tau})|\mathcal{F}_{\eta-}\big]\big).$$

Under Assumption 6, ξ_n is the average of $\mathcal{F}_{\eta-}$-conditionally independent variables with zero conditional mean. Hence,

$$\begin{aligned}
\mathbb{E}\big[\xi_n^2|\mathcal{F}_{\eta-}\big]&=\frac{1}{N_n^2}\sum_{j=1}^{N_n}\mathbb{E}\big[\big(L(\chi_{j,\eta,\tau})-\mathbb{E}\big[L(\chi_{j,\eta,\tau})|\mathcal{F}_{\eta-}\big]\big)^2|\mathcal{F}_{\eta-}\big] \\
&\leq\frac{1}{N_n^2}\sum_{j=1}^{N_n}\mathbb{E}\big[L(\chi_{j,\eta,\tau})^2|\mathcal{F}_{\eta-}\big]=O_p\big(N_n^{-1}\big)=o_p(1).
\end{aligned}$$

In particular, this implies that $\mathbb{E}[|\xi_n|\wedge 1|\mathcal{F}_{\eta-}]=o_p(1)$. By the bounded convergence theorem, we further deduce $\mathbb{E}[|\xi_n|\wedge 1]\to 0$. But this is equivalent to $\xi_n=o_p(1)$. This, together with (SA.27), implies that

$$\frac{1}{N_n}\sum_{j=1}^{N_n}L(\hat{\beta}_{n,j,\tau}-\hat{\beta}_{n,j,\eta})=\frac{1}{N_n}\sum_{j=1}^{N_n}\mathbb{E}\big[L(\chi_{j,\eta,\tau})|\mathcal{F}_{\eta-}\big]+o_p(1).\tag{SA.28}$$

Since $q_n^w\to 0$, the winsorized estimator $\hat{V}_n$ differs from $N_n^{-1}\sum_{j=1}^{N_n}L(\hat{\beta}_{n,j,\tau}-\hat{\beta}_{n,j,\eta})$ by an $o_p(1)$ term. The assertion of the proposition then follows from (SA.28). □

SA.3 *Proof of Theorem 1*

Proof of Theorem 1. Step 1. The proof proceeds in two steps. Recall Ω'_n from (SA.1). Since $\mathbb{P}(\Omega'_n)\to 1$, we can restrict our calculations to Ω'_n without loss of generality. In this step, we show that

$$\Delta_n^{-1}\hat{V}_n = \frac{1}{N_n\Delta_n}\sum_{j=1}^{N_n} L(\xi_{n,j,\tau} - \xi_{n,j,\eta}) + o_p(1). \tag{SA.29}$$

From (SA.26), we see that

$$\sup_{1\le j\le N_n} |\hat{\beta}_{n,j,s} - \beta_{j,s}| 1_{\{\Delta^n_{i(n,s)}\tilde{J}_{Y,j}=0\}} = O_p\big(\Delta_n^{1/2}N_n^{\iota}\big) \tag{SA.30}$$

for some fixed but arbitrarily small constant $\iota > 0$. In restriction to Ω_n' and the null hypothesis, $\bar{B}_{n,\eta,\tau}$ is bounded by two times of the left-hand of the above display. Hence,

$$\bar{B}_{n,\eta,\tau} = O_p\big(\Delta_n^{1/2}N_n^{\iota}\big). \tag{SA.31}$$

We note that

$$\begin{aligned}
&\bigg|\frac{1}{\Delta_n N_n}\sum_{j=1}^{N_n}\bigg| L\big(|\hat{\beta}_{n,j,\tau} - \hat{\beta}_{n,j,\eta}| \wedge \bar{B}_{n,\eta,\tau}\big) \\
&\quad - L\big(|\hat{\beta}_{n,j,\tau} - \hat{\beta}_{n,j,\eta}|\big)|1_{\{|\Delta^n_{i(n,\tau)}\tilde{J}_{Y,j}|+|\Delta^n_{i(n,\eta)}\tilde{J}_{Y,j}|=0\}}| \\
&\le \frac{\lceil q_n^w N_n\rceil}{\Delta_n N_n}\sup_{1\le j\le N_n} L\big(|\hat{\beta}_{n,j,\tau} - \hat{\beta}_{n,j,\eta}|\big)1_{\{|\Delta^n_{i(n,\tau)}\tilde{J}_{Y,j}|+|\Delta^n_{i(n,\eta)}\tilde{J}_{Y,j}|=0\}} \\
&= O_p\big(q_n^w N_n^{2\iota}\big) = o_p(1),
\end{aligned} \tag{SA.32}$$

where the inequality follows from the fact that the winsorization is active for at most $\lceil q_n^w N_n\rceil$ terms ($\lceil\cdot\rceil$ denotes the ceiling function); the first equality follows from (SA.30); the second equality follows from Assumptions 2 and 5 with ι chosen sufficiently small. Note that in restriction to $\{|\Delta^n_{i(n,\tau)}\tilde{J}_{Y,j}| + |\Delta^n_{i(n,\eta)}\tilde{J}_{Y,j}| = 0\}$ and the null hypothesis, $\hat{\beta}_{n,j,\tau} - \hat{\beta}_{n,j,\eta} = \xi_{n,j,\tau} - \xi_{n,j,\eta}$. Hence,

$$\begin{aligned}
&\frac{1}{\Delta_n N_n}\sum_{j=1}^{N_n} L\big(|\hat{\beta}_{n,j,\tau} - \hat{\beta}_{n,j,\eta}| \wedge \bar{B}_{n,\eta,\tau}\big)1_{\{|\Delta^n_{i(n,\tau)}\tilde{J}_{Y,j}|+|\Delta^n_{i(n,\eta)}\tilde{J}_{Y,j}|=0\}} \\
&\quad = \frac{1}{\Delta_n N_n}\sum_{j=1}^{N_n} L(\xi_{n,j,\tau} - \xi_{n,j,\eta})1_{\{|\Delta^n_{i(n,\tau)}\tilde{J}_{Y,j}|+|\Delta^n_{i(n,\eta)}\tilde{J}_{Y,j}|=0\}} \\
&\qquad + o_p(1).
\end{aligned} \tag{SA.33}$$

Next, we note that

$$\frac{1}{N_n\Delta_n}\sum_{j=1}^{N_n} L\big(|\hat{\beta}_{n,j,\tau} - \hat{\beta}_{n,j,\eta}| \wedge \bar{B}_{n,\eta,\tau}\big)1_{\{|\Delta^n_{i(n,\tau)}\tilde{J}_{Y,j}|+|\Delta^n_{i(n,\eta)}\tilde{J}_{Y,j}|>0\}}$$

$$\leq \frac{L(\bar{B}_{n,\eta,\tau})}{N_n\Delta_n}\sum_{j=1}^{N_n} 1_{\{|\Delta^n_{i(n,\tau)}\tilde{J}_{Y,j}|+|\Delta^n_{i(n,\eta)}\tilde{J}_{Y,j}|>0\}}$$

$$= O_p\big(\Delta_n N_n^{2\iota}\big) = o_p(1), \tag{SA.34}$$

where the inequality follows from the monotonicity of $L(\cdot)$ and the last line follows from (SA.31) and the fact that $\mathbb{P}(\Delta^n_{i(n,s)}\tilde{J}_{Y,j} \neq 0) \leq K\Delta_n$. Similarly, we can show that

$$\frac{1}{N_n\Delta_n}\sum_{j=1}^{N_n} L(\xi_{n,j,\tau} - \xi_{n,j,\eta}) 1_{\{|\Delta^n_{i(n,\tau)}\tilde{J}_{Y,j}|+|\Delta^n_{i(n,\eta)}\tilde{J}_{Y,j}|>0\}} = o_p(1). \tag{SA.35}$$

From (SA.33), (SA.34), and (SA.35), we deduce (SA.29) as wanted.

Step 2. It remains to derive the convergence of $(N_n\Delta_n)^{-1}\sum_{j=1}^{N_n} L(\xi_{n,j,\tau} - \xi_{n,j,\eta})$. Recall the definition of $\xi'_{n,j,s}$ from (SA.6). Let L_n be defined as

$$L_n \equiv \frac{1}{N_n\Delta_n}\sum_{j=1}^{N_n}\big(\xi'_{n,j,\tau} - \xi'_{n,j,\eta}\big)^2. \tag{SA.36}$$

Recalling the definitions in (SA.20), we can rewrite L_n as

$$L_n = A_n(\eta) + A_n(\tau) - 2B_n(\eta,\tau). \tag{SA.37}$$

Then, by Lemma SA2,

$$L_n \xrightarrow{\mathcal{L}\text{-}s} \mathcal{L}(\eta,\tau) \equiv \mathcal{A}(\eta) + \mathcal{A}(\tau) - 2\mathcal{B}(\eta,\tau). \tag{SA.38}$$

From (SA.5), we further see that

$$\begin{aligned}
&\frac{1}{N_n\Delta_n}\sum_{j=1}^{N_n} L(\xi_{n,j,\tau} - \xi_{n,j,\eta}) \\
&\quad = L_n + \frac{2}{N_n\Delta_n}\sum_{j=1}^{N_n}\big(\xi'_{n,j,\tau} - \xi'_{n,j,\eta}\big)\big(\xi''_{n,j,\tau} - \xi''_{n,j,\eta}\big) \\
&\qquad + \frac{1}{N_n\Delta_n}\sum_{j=1}^{N_n}\big(\xi''_{n,j,\tau} - \xi''_{n,j,\eta}\big)^2.
\end{aligned} \tag{SA.39}$$

By Lemma SA1(d), the last term in (SA.39) is $O_p(\Delta_n)$. By the Cauchy–Schwarz inequality, this estimate and (SA.38) further imply that the second term on the right-hand side of (SA.39) is $o_p(1)$. Therefore,

$$\frac{1}{N_n\Delta_n}\sum_{j=1}^{N_n} L(\xi_{n,j,\tau} - \xi_{n,j,\eta}) = L_n + o_p(1).$$

The assertion of the theorem then follows from (SA.29) and (SA.38). □

SA.4 *Proof of Theorem 2*

We start with the proof of part (a) and part (b). We provide details for the case with $q = \tau-$, while noting that the case with $q = \tau+$ only requires a change of notation. Hence, we suppress (in most cases) the dependence on q in our notations for simplicity. More specifically, we write $\hat{X}_n$, $\hat{F}_n$, $\hat{\Lambda}_n$, Λ_n^*, $\mathcal{E}_n$, H, Σ_f, M_Λ^* and M_C^* in place of $\hat{X}_n(q)$, $\hat{F}_n(q)$, $\hat{\Lambda}_n(q)$, $\Lambda_n^*(q)$, $\mathcal{E}_n(q)$, H_q, $\Sigma_{f,q}$, $M_\Lambda^*(q,q)$, and $M_C^*(q,q)$, respectively. We denote the jth column of a generic matrix A by $A_{\cdot j}$. Recall the sequence Ω_n of events defined as in Section SA.1. Since $\mathbb{P}(\Omega_n) \to 1$, we can restrict our calculations below in Ω_n without loss of generality.

Below, we denote $\Gamma_n \equiv \{\gamma \in \mathbb{R}^{k_n} : \gamma^\top \gamma = k_n\}$. Note that each column of $\hat{F}_n$ is an element of Γ_n. We collect some useful estimates in Lemma SA4, where we denote

$$\tilde{\Lambda}_n^* \equiv (\lambda_{1,\tau-} - \tilde{\beta}_{n,1,\tau}\lambda_{Z,\tau-}, \ldots, \lambda_{N_n,\tau-} - \tilde{\beta}_{n,N_n,\tau}\lambda_{Z,\tau-})^\top. \tag{SA.40}$$

We also consider an $N_n \times k_n$ matrix $\mathcal{E}_n' = [e_{j,l}']_{1\le j\le N_n, 1\le l\le k_n}$ defined as

$$\begin{aligned} e_{j,l}' \equiv{}& \Delta_n^{-1/2} \int_{(i(n,\tau-)+l-1)\Delta_n}^{(i(n,\tau-)+l)\Delta_n} \alpha_{j,s}\, ds + \Delta_n^{-1/2} \int_{(i(n,\tau-)+l-1)\Delta_n}^{(i(n,\tau-)+l)\Delta_n} (\lambda_{j,u} - \lambda_{j,\tau-})^\top df_u \\ &- \tilde{\beta}_{n,j,\tau} \Delta_n^{-1/2} \int_{(i(n,\tau-)+l-1)\Delta_n}^{(i(n,\tau-)+l)\Delta_n} (\lambda_{Z,u} - \lambda_{Z,\tau-})^\top df_u. \end{aligned} \tag{SA.41}$$

LEMMA SA4. *Under the conditions of Theorem* 2, *the following statements hold*:

(a) $\sup_{\gamma\in\Gamma_n} k_n^{-2} N_n^{-1} \gamma^\top \mathcal{E}_n^\top \mathcal{E}_n \gamma = o_p(1)$;

(b) $\sup_{\gamma\in\Gamma_n} k_n^{-2} N_n^{-1} \gamma^\top \mathcal{E}_n'^\top \mathcal{E}_n' \gamma = o_p(1)$;

(c) $\sup_{\gamma\in\Gamma_n} k_n^{-1} N_n^{-1} |\gamma^\top \mathcal{E}_n^\top \Lambda_n^*| = o_p(1)$;

(d) $\sup_{\gamma\in\Gamma_n} k_n^{-1} N_n^{-1} |\gamma^\top \mathcal{E}_n'^\top \Lambda_n^*| = o_p(1)$;

(e) $N_n^{-1} \tilde{\Lambda}_n^{*\top} \Lambda_n^* = M_\Lambda^* + o_p(1)$ *and* $N_n^{-1} \tilde{\Lambda}_n^{*\top} \tilde{\Lambda}_n^* = M_\Lambda^* + o_p(1)$.

PROOF OF LEMMA SA4. (a) Recall that the (j,l) element of $\mathcal{E}_n$ is given by $e_{j,l} \equiv \Delta_{i(n,\tau-)+l}^n \epsilon_j / \Delta_n^{1/2}$. We observe

$$\begin{aligned} &\frac{1}{k_n^2 N_n} \gamma^\top \mathcal{E}_n^\top \mathcal{E}_n \gamma \\ &\quad = \frac{1}{k_n^2 N_n} \sum_{l=1}^{k_n} \sum_{m=1}^{k_n} \gamma_l \gamma_m \sum_{j=1}^{N_n} e_{j,l} e_{j,m} \\ &\quad \le \left(\frac{1}{k_n^2} \sum_{l=1}^{k_n} \sum_{m=1}^{k_n} \gamma_l^2 \gamma_m^2 \right)^{1/2} \left(\frac{1}{k_n^2} \sum_{l=1}^{k_n} \sum_{m=1}^{k_n} \left(\frac{1}{N_n} \sum_{j=1}^{N_n} e_{j,l} e_{j,m} \right)^2 \right)^{1/2} \\ &\quad = \left(\frac{1}{k_n^2} \sum_{l=1}^{k_n} \sum_{m=1}^{k_n} \left(\frac{1}{N_n} \sum_{j=1}^{N_n} e_{j,l} e_{j,m} \right)^2 \right)^{1/2}, \end{aligned} \tag{SA.42}$$

where the first equality is by definition, the inequality is by the Cauchy–Schwarz inequality, and the last line follows from $\gamma^\top\gamma = k_n$.

We decompose the majorant side of (SA.42) as

$$\frac{1}{k_n^2}\sum_{l=1}^{k_n}\sum_{m=1}^{k_n}\left(\frac{1}{N_n}\sum_{j=1}^{N_n} e_{j,l}e_{j,m}\right)^2$$
$$= \frac{1}{k_n^2}\sum_{l=1}^{k_n}\left(\frac{1}{N_n}\sum_{j=1}^{N_n} e_{j,l}^2\right)^2 + \frac{1}{k_n^2}\sum_{l,m,l\neq m}\left(\frac{1}{N_n}\sum_{j=1}^{N_n} e_{j,l}e_{j,m}\right)^2. \qquad \text{(SA.43)}$$

By a standard estimate for continuous Itô semimartingales, $\mathbb{E}[e_{j,l}^4] \leq K$; this holds uniformly in $j \in \{1, \ldots, N_n\}$ because the idiosyncratic variances $\tilde{\sigma}_j^2$ are uniformly (locally) bounded under Assumption 3(iii). Hence, by Jensen's inequality,

$$\mathbb{E}\left[\left(\frac{1}{N_n}\sum_{j=1}^{N_n} e_{j,l}^2\right)^2\right] \leq \mathbb{E}\left[\frac{1}{N_n}\sum_{j=1}^{N_n} e_{j,l}^4\right] \leq K.$$

From here, it follows that the first term on the right-hand side of (SA.43) is $o_p(1)$. In view of (SA.42) and (SA.43), it remains to show that the second term on the right-hand side of (SA.43) is also $o_p(1)$.

To this end, we observe the following for $l \neq m$: (i) $\mathbb{E}[e_{j,l}e_{j,m}] = 0$ because the process $\boldsymbol{\epsilon}_j$ is a martingale; (ii) $\mathbb{E}[e_{j,l}^2 e_{j,m}^2] \leq K$; and (iii) the variables $(e_{j,l}e_{j,m})_{1\leq j\leq N_n}$ are uncorrelated, which can be shown by using repeated conditioning and the orthogonality among the Brownian motions $(\widetilde{W}_j)_{j\geq 1}$. Hence,

$$\mathbb{E}\left[\left(\frac{1}{N_n}\sum_{j=1}^{N_n} e_{j,l}e_{j,m}\right)^2\right] \leq KN_n^{-1} \to 0,$$

which implies, as wanted,

$$\frac{1}{k_n^2}\sum_{l,m,l\neq m}\left(\frac{1}{N_n}\sum_{j=1}^{N_n} e_{j,l}e_{j,m}\right)^2 = O_p\big(N_n^{-1}\big) = o_p(1).$$

This finishes the proof of part (a).

(b) Similar to (SA.42), we can derive

$$\sup_{\gamma\in\Gamma_n}\frac{1}{k_n^2 N_n}\gamma^\top\mathcal{E}_n'^\top\mathcal{E}_n'\gamma \leq \left(\frac{1}{k_n^2}\sum_{l=1}^{k_n}\sum_{m=1}^{k_n}\left(\frac{1}{N_n}\sum_{j=1}^{N_n} e'_{j,l}e'_{j,m}\right)^2\right)^{1/2}. \qquad \text{(SA.44)}$$

In addition, we observe

$$\mathbb{E}\left|\Delta_n^{-1/2}\int_{(i(n,\tau-)+l-1)\Delta_n}^{(i(n,\tau-)+l)\Delta_n}(\lambda_{j,u}-\lambda_{j,\tau-})^\top\, df_u\right|^4$$

$$\leq K\Delta_n^{-2}\mathbb{E}\left[\left(\int_{(i(n,\tau-)+l-1)\Delta_n}^{(i(n,\tau-)+l)\Delta_n}\|\lambda_{j,u}-\lambda_{j,\tau-}\|^2\,du\right)^2\right]$$
$$\leq K\Delta_n^{-1}\mathbb{E}\left[\int_{(i(n,\tau-)+l-1)\Delta_n}^{(i(n,\tau-)+l)\Delta_n}\|\lambda_{j,u}-\lambda_{j,\tau-}\|^4\,du\right]$$
$$\leq K\Delta_n^{-1}\mathbb{E}\left[\int_{(i(n,\tau-)+l-1)\Delta_n}^{(i(n,\tau-)+l)\Delta_n}\|\lambda_{j,u}-\lambda_{j,\tau-}\|^2\,du\right]\leq K\Delta_n, \tag{SA.45}$$

where the first inequality is by the Burkholder–Davis–Gundy inequality, the second inequality is by Jensen's inequality, and the last line holds because $\lambda_{j,u}$ is bounded and $(1/2)$-Hölder continuous under L_2-norm uniformly in j. Similarly,

$$\mathbb{E}\left|\Delta_n^{-1/2}\int_{(i(n,\tau-)+l-1)\Delta_n}^{(i(n,\tau-)+l)\Delta_n}(\lambda_{Z,u}-\lambda_{Z,\tau-})^\top\,df_u\right|^4\leq K\Delta_n. \tag{SA.46}$$

Under Assumption 7, $(\tilde{\beta}_{j,n,\tau})_{1\leq j\leq N_n}$ are uniformly bounded with probability approaching one, so we can assume that these variables are bounded without loss of generality. Hence, from (SA.45) and (SA.46), we deduce that

$$\mathbb{E}\left|e'_{j,l}\right|^4\leq K\Delta_n. \tag{SA.47}$$

Hence, by the Cauchy–Schwarz inequality, we further have

$$\mathbb{E}\left[\left(\frac{1}{N_n}\sum_{j=1}^{N_n}e'_{j,l}e'_{j,m}\right)^2\right]\leq K\Delta_n. \tag{SA.48}$$

The assertion of part (b) then follows from (SA.44) and (SA.48).

(c) We denote the (j,k) element of Λ_n^* by $\lambda_{j,k}^*$. We note that for each $k\in\{1,\ldots,r\}$ (recalling that $\Lambda_{n,\cdot k}^*$ denotes the kth column of Λ_n^*),

$$\frac{1}{k_nN_n}\left|\gamma^\top\mathcal{E}_n^\top\Lambda_{n,\cdot k}^*\right|=\left|\frac{1}{k_n}\sum_{l=1}^{k_n}\gamma_l\left(\frac{1}{N_n}\sum_{j=1}^{N_n}e_{j,l}\lambda_{j,k}^*\right)\right|$$
$$\leq\left(\frac{1}{k_n}\sum_{l=1}^{k_n}\gamma_l^2\right)^{1/2}\left(\frac{1}{k_n}\sum_{l=1}^{k_n}\left(\frac{1}{N_n}\sum_{j=1}^{N_n}e_{j,l}\lambda_{j,k}^*\right)^2\right)^{1/2}$$
$$=\left(\frac{1}{k_n}\sum_{l=1}^{k_n}\left(\frac{1}{N_n}\sum_{j=1}^{N_n}e_{j,l}\lambda_{j,k}^*\right)^2\right)^{1/2}, \tag{SA.49}$$

where the first line is by definition, the second line is by the Cauchy–Schwarz inequality and the last line follows from $\gamma\in\Gamma_n$. Under Assumption 8, $e_{j,l}$ is independent of $\lambda_{j,k}^*$; hence, the variables $(e_{j,l}\lambda_{j,k}^*)_{1\leq j\leq N_n}$ are uncorrelated and have zero mean and bounded second moment. It is then easy to see that

$$\mathbb{E}\left[\frac{1}{k_n}\sum_{l=1}^{k_n}\left(\frac{1}{N_n}\sum_{j=1}^{N_n} e_{j,l}\lambda_{j,k}^*\right)^2\right] \leq K/N_n.$$

Therefore,

$$\frac{1}{k_n}\sum_{l=1}^{k_n}\left(\frac{1}{N_n}\sum_{j=1}^{N_n} e_{j,l}\lambda_{j,k}^*\right)^2 = O_p\left(N_n^{-1}\right) = o_p(1). \tag{SA.50}$$

The assertion of part (c) then follows from (SA.49) and (SA.50).

(d) Like (SA.49), we can derive

$$\frac{1}{k_n N_n}\left|\gamma^\top \mathcal{E}_n'^\top \Lambda_{n,\cdot k}^*\right| \leq \left(\frac{1}{k_n}\sum_{l=1}^{k_n}\left(\frac{1}{N_n}\sum_{j=1}^{N_n} e_{j,l}'\lambda_{j,k}^*\right)^2\right)^{1/2}. \tag{SA.51}$$

We further note that

$$\begin{aligned}\mathbb{E}\left[\left(\frac{1}{N_n}\sum_{j=1}^{N_n} e_{j,l}'\lambda_{j,k}^*\right)^2\right] &\leq \mathbb{E}\left[\frac{1}{N_n}\sum_{j=1}^{N_n}\left(e_{j,l}'\lambda_{j,k}^*\right)^2\right] \\ &\leq \frac{K}{N_n}\sum_{j=1}^{N_n}\mathbb{E}\left[\left(e_{j,l}'\right)^2\right] \leq K\Delta_n,\end{aligned}$$

where the first inequality is by Jensen's inequality, the second inequality holds because $\lambda_{j,k}^*$ is bounded and the last inequality can be derived similarly as (SA.47). In view of (SA.51), the assertion of part (d) readily follows.

(e) From the definitions of Λ_n^* and $\tilde{\Lambda}_n^*$ respectively from (3.9) and (SA.40), we see that (recall $q = \tau-$)

$$\tilde{\Lambda}_n^* - \Lambda_n^* = \left(\left(\beta_{1,\tau}^* - \tilde{\beta}_{n,1,\tau}\right)\lambda_{Z,\tau-}, \ldots, \left(\beta_{n,N_n,\tau}^* - \tilde{\beta}_{n,N_n,\tau}\right)\lambda_{Z,\tau-}\right)^\top.$$

Therefore, by Assumption 7,

$$\frac{1}{N_n}\left(\tilde{\Lambda}_n^* - \Lambda_n^*\right)^\top\left(\tilde{\Lambda}_n^* - \Lambda_n^*\right) = o_p(1). \tag{SA.52}$$

That is, $N_n^{-1}\|\tilde{\Lambda}_n^* - \Lambda_n^*\|^2 = o_p(1)$. Since $N_n^{-1}\Lambda_n^{*\top}\Lambda_n^* \xrightarrow{\mathbb{P}} M_\Lambda^*$ by Assumption 8, the estimate above readily implies the assertions in part (e). □

We are now ready to prove part (a) and part (b) of Theorem 2. We remind the reader that we fix $q = \tau-$ for proving these parts.

Proof of Theorem 2(a). Step 1. We prove part (a) of Theorem 2 in several steps. In this step, we show that

$$\sup_{\gamma\in\Gamma_n}\left|\Xi_n(\gamma) - \Xi_n^*(\gamma)\right| = o_p(1), \tag{SA.53}$$

where $\Xi_n(\cdot)$ and $\Xi_n^*(\cdot)$ are defined as

$$\Xi_n(\gamma) \equiv \frac{1}{k_n^2 N_n} \gamma^\top \hat{X}_n^\top \hat{X}_n \gamma, \qquad \Xi_n^*(\gamma) \equiv \frac{1}{k_n^2 N_n} \gamma^\top F_n \Lambda_n^{*\top} \Lambda_n^* F_n^\top \gamma. \tag{SA.54}$$

Below, we denote the (j,l) element of $\hat{X}_n$ by

$$\xi_{n,j,l} \equiv \frac{\Delta_{i(n,\tau-)+l}^n Y_j \wedge u_n \vee (-u_n) - \tilde{\beta}_{n,j,\tau} \Delta_{i(n,\tau-)+l}^n Z}{\sqrt{\Delta_n}}.$$

We set

$$\begin{aligned}\xi'_{n,j,l} \equiv \Delta_n^{-1/2} &\int_{(i(n,\tau-)+l-1)\Delta_n}^{(i(n,\tau-)+l)\Delta_n} \alpha_{j,s}\, ds \\ &+ \Delta_n^{-1/2} \int_{(i(n,\tau-)+l-1)\Delta_n}^{(i(n,\tau-)+l)\Delta_n} (\lambda_{j,s} - \tilde{\beta}_{n,j,\tau} \lambda_{Z,s})^\top df_s + \Delta_n^{-1/2} \Delta_{i(n,\tau-)+l}^n \epsilon_j.\end{aligned}$$

Note that

$$\mathbb{E}\left|\xi_{n,j,l} - \xi'_{n,j,l}\right|^2 \leq K\Delta_n. \tag{SA.55}$$

We now define $\hat{X}'_n$ as a $N_n \times k_n$ matrix whose (j,l) element is given by $\xi'_{n,j,l}$ and let

$$\Xi'_n(\gamma) = \frac{1}{k_n^2 N_n} \gamma^\top \hat{X}'^\top_n \hat{X}'_n \gamma.$$

By (SA.55),

$$\frac{1}{k_n N_n} \left\|\hat{X}_n - \hat{X}'_n\right\|^2 = \frac{1}{k_n N_n} \sum_{j=1}^{N_n} \sum_{l=1}^{k_n} \left|\xi_{n,j,l} - \xi'_{n,j,l}\right|^2 = o_p(1). \tag{SA.56}$$

By the Cauchy–Schwarz inequality and the triangle inequality,

$$\begin{aligned}\sup_{\gamma \in \Gamma_n} \left|\Xi_n(\gamma) - \Xi'_n(\gamma)\right| &= \frac{1}{k_n^2 N_n} \sup_{\gamma \in \Gamma_n} \left|\gamma^\top \left(\hat{X}_n^\top \hat{X}_n - \hat{X}'^\top_n \hat{X}'_n\right)\gamma\right| \\ &\leq \frac{1}{k_n^2 N_n} \sup_{\gamma \in \Gamma_n} \|\gamma\|^2 \left\|\hat{X}_n^\top \hat{X}_n - \hat{X}'^\top_n \hat{X}'_n\right\| \\ &= \frac{1}{k_n N_n} \left\|\hat{X}_n^\top \hat{X}_n - \hat{X}'^\top_n \hat{X}'_n\right\| \\ &\leq \frac{2}{k_n N_n} \left\|\hat{X}'_n\right\| \left\|\hat{X}_n - \hat{X}'_n\right\| + \frac{1}{k_n N_n} \left\|\hat{X}_n - \hat{X}'_n\right\|^2.\end{aligned}$$

It is easy to see that $\|\hat{X}'_n\| = O_p(\sqrt{k_n N_n})$. Hence, by (SA.56),

$$\sup_{\gamma \in \Gamma_n} \left|\Xi_n(\gamma) - \Xi'_n(\gamma)\right| = o_p(1). \tag{SA.57}$$

To show (SA.53), it remains to show that $\sup_{\gamma\in\Gamma_n}|\Xi'_n(\gamma)-\Xi^*_n(\gamma)|=o_p(1)$. We note that, by a standard result for spot covariance estimation

$$F_n^\top F_n/k_n \xrightarrow{\mathbb{P}} \Sigma_f. \tag{SA.58}$$

In particular, $\|F_n\|=O_p(k_n^{1/2})$. Hence,

$$\sup_{\gamma\in\Gamma_n}\left\|\gamma^\top F_n/k_n\right\| \leq \sup_{\gamma\in\Gamma_n}\|\gamma\|\|F_n\|/k_n = O_p(1). \tag{SA.59}$$

Under Assumption 8, $\Lambda_n^{*\top}\Lambda_n^* = O_p(N_n)$. It then follows that

$$\sup_{\gamma\in\Gamma_n}\Xi^*_n(\gamma)=O_p(1). \tag{SA.60}$$

Recall the definitions in (3.9), (SA.40), and (SA.41). We can decompose $\hat{X}'_n$ as

$$\hat{X}'_n = \tilde{\Lambda}^*_n F_n^\top + \mathcal{E}_n + \mathcal{E}'_n. \tag{SA.61}$$

Hence,

$$\hat{X}'_n - \Lambda^*_n F_n^\top = \left(\tilde{\Lambda}^*_n - \Lambda^*_n\right)F_n^\top + \mathcal{E}_n + \mathcal{E}'_n. \tag{SA.62}$$

We can then decompose

$$\begin{aligned}\Xi'_n(\gamma)-\Xi^*_n(\gamma) = {} & \frac{2}{k_n^2 N_n}\gamma^\top F_n \Lambda_n^{*\top}\left[\left(\tilde{\Lambda}^*_n - \Lambda^*_n\right)F_n^\top + \mathcal{E}_n + \mathcal{E}'_n\right]\gamma \\ & + \frac{1}{k_n^2 N_n}\gamma^\top\left(\left(\tilde{\Lambda}^*_n - \Lambda^*_n\right)F_n^\top + \mathcal{E}_n + \mathcal{E}'_n\right)^\top\left(\left(\tilde{\Lambda}^*_n - \Lambda^*_n\right)F_n^\top + \mathcal{E}_n + \mathcal{E}'_n\right)\gamma.\end{aligned} \tag{SA.63}$$

By Lemma SA4(a), (b),

$$\frac{1}{k_n^2 N_n}\sup_{\gamma\in\Gamma_n}\gamma^\top\mathcal{E}_n^\top\mathcal{E}_n\gamma = o_p(1), \qquad \frac{1}{k_n^2 N_n}\sup_{\gamma\in\Gamma_n}\gamma^\top\mathcal{E}_n'^\top\mathcal{E}'_n\gamma = o_p(1). \tag{SA.64}$$

Further using the Cauchy–Schwarz inequality, we can deduce that $\sup_{\gamma\in\Gamma_n}|\gamma^\top\mathcal{E}_n^\top\mathcal{E}'_n\gamma| = o_p(1)$; hence,

$$\frac{1}{k_n^2 N_n}\sup_{\gamma\in\Gamma_n}\gamma^\top\left(\mathcal{E}_n+\mathcal{E}'_n\right)^\top\left(\mathcal{E}_n+\mathcal{E}'_n\right)\gamma = o_p(1). \tag{SA.65}$$

In addition, by Lemma SA4(e) and (SA.59),

$$\sup_{\gamma\in\Gamma_n}\frac{1}{k_n^2 N_n}\gamma^\top F_n^\top\left(\tilde{\Lambda}^*_n - \Lambda^*_n\right)^\top\left(\tilde{\Lambda}^*_n - \Lambda^*_n\right)F_n^\top\gamma = o_p(1). \tag{SA.66}$$

By (SA.65) and (SA.66), as well as the Cauchy–Schwarz inequality, we deduce

$$\frac{1}{k_n^2 N_n}\sup_{\gamma\in\Gamma_n}\gamma^\top\left(\left(\tilde{\Lambda}^*_n - \Lambda^*_n\right)F_n^\top + \mathcal{E}_n + \mathcal{E}'_n\right)^\top\left(\left(\tilde{\Lambda}^*_n - \Lambda^*_n\right)F_n^\top + \mathcal{E}_n + \mathcal{E}'_n\right)\gamma = o_p(1). \tag{SA.67}$$

By (SA.60) and the Cauchy–Schwarz inequality, (SA.67) further implies that

$$\frac{2}{k_n^2 N_n} \sup_{\gamma\in\Gamma_n} \left|\gamma^\top F_n \Lambda_n^{*\top}\left[(\tilde{\Lambda}_n^* - \Lambda_n^*)F_n^\top + \mathcal{E}_n + \mathcal{E}_n'\right]\gamma\right| = o_p(1). \tag{SA.68}$$

By (SA.63), (SA.67), and (SA.68), we deduce $\sup_{\gamma\in\Gamma_n} |\Xi_n'(\gamma) - \Xi_n^*(\gamma)| = o_p(1)$, and hence, (SA.53) as wanted.

Step 2. In this step, we show that

$$S_n^*\left(\hat{F}_n^{*\top} F_n/k_n\right)\Sigma_f^{-1/2} H \xrightarrow{\mathbb{P}} I_r, \tag{SA.69}$$

where we recall that $S_n^* = \mathrm{diag}(\mathrm{sign}(\hat{F}_n^{*\top} F_n (F_n^\top F_n/k_n)^{-1/2} H))$ and H is the ordered eigenvector matrix of M_C^*. Below, we denote by D_j the jth largest eigenvalue of M_C^* and write $D = \mathrm{diag}(D_1, \ldots, D_r)$.

We first show that

$$\sup_{\gamma\in\Gamma_n} \Xi_n^*(\gamma) \xrightarrow{\mathbb{P}} D_1. \tag{SA.70}$$

To see this, we note that we can represent $\gamma \in \Gamma_n$ as

$$\gamma = F_n\left(F_n^\top F_n/k_n\right)^{-1/2} H\delta + \tilde{\gamma}, \tag{SA.71}$$

where $\tilde{\gamma}$ is the projection error of γ onto the column space of F_n such that $F_n^\top \tilde{\gamma} = 0$. We can then rewrite

$$\sup_{\gamma\in\Gamma_n} \Xi_n^*(\gamma) = \sup_{\|\delta\|\le 1} \delta^\top H^\top M_{C,n}^* H\delta, \quad \text{where}$$

$$M_{C,n}^* \equiv \left(\frac{F_n^\top F_n}{k_n}\right)^{1/2} \left(\frac{\Lambda_n^{*\top}\Lambda_n^*}{N_n}\right)\left(\frac{F_n^\top F_n}{k_n}\right)^{1/2}.$$

Hence, $\sup_{\gamma\in\Gamma_n} \Xi_n^*(\gamma)$ is the largest eigenvalue of $M_{C,n}^*$. By (SA.58) and Assumption 8,

$$M_{C,n}^* \xrightarrow{\mathbb{P}} \Sigma_f^{1/2} M_\Lambda^* \Sigma_f^{1/2} \equiv M_C^*.$$

Since the mapping for calculating the unique largest eigenvalue is continuous, we deduce (SA.70) by using the continuous mapping theorem.

By the construction of $\hat{F}_n$, its first column $\hat{F}_{n,\cdot 1}$ satisfies

$$\Xi_n(\hat{F}_{n,\cdot 1}) = \sup_{\gamma\in\Gamma_n} \Xi_n(\gamma).$$

By (SA.53), $\sup_{\gamma\in\Gamma_n} \Xi_n(\gamma) = \sup_{\gamma\in\Gamma_n} \Xi_n^*(\gamma) + o_p(1)$, which implies $\Xi_n(\hat{F}_{n,\cdot 1}) \xrightarrow{\mathbb{P}} D_1$ because of (SA.70). Using the uniform convergence result in (SA.53), we further deduce

$$\Xi_n^*(\hat{F}_{n,\cdot 1}) \xrightarrow{\mathbb{P}} D_1. \tag{SA.72}$$

We now represent $\hat{F}_{n,\cdot 1}$ in the format of (SA.71), that is,

$$\hat{F}_{n,\cdot 1} = F_n\left(F_n^\top F_n/k_n\right)^{-1/2} H\hat{\delta}_1 + \tilde{\gamma}_1, \tag{SA.73}$$

such that $F_n^\top \tilde{\gamma}_1 = 0$. From (SA.72) and (SA.73), we see

$$\begin{aligned} o_p(1) &= \Xi_n^*(\hat{F}_{n,\cdot 1}) - D_1 \\ &= \hat{\delta}_1^\top H^\top M_{C,n}^* H \hat{\delta}_1 - D_1 \\ &= \hat{\delta}_1^\top H^\top \big(M_{C,n}^* - M_C^*\big) H \hat{\delta}_1 + \hat{\delta}_1^\top H^\top M_C^* H \hat{\delta}_1 - D_1 \\ &= \hat{\delta}_1^\top H^\top \big(M_{C,n}^* - M_C^*\big) H \hat{\delta}_1 + \hat{\delta}_1^\top D \hat{\delta}_1 - D_1, \end{aligned}$$

where the last line follows from the eigenvalue decomposition $M_C^* = HDH^\top$. Since $\|\hat{\delta}_1\| \leq 1$ and $M_{C,n}^* - M_C^* = o_p(1)$, the above display implies that

$$\hat{\delta}_1^\top D \hat{\delta}_1 - D_1 = o_p(1).$$

Since D_1 is the unique largest eigenvalue, this further implies that $\hat{\delta}_{11}^2 \xrightarrow{\mathbb{P}} 1$ and $\hat{\delta}_{1j}^2 \xrightarrow{\mathbb{P}} 0$ for $j \geq 2$. In particular, $\|\hat{\delta}_1\| \xrightarrow{\mathbb{P}} 1$ which implies that $\tilde{\gamma}_1^\top \tilde{\gamma}_1 / k_n \xrightarrow{\mathbb{P}} 0$.

Let $S_{n,j}^*$ denote the jth diagonal element of S_n^*. Note that by (SA.73),

$$\hat{F}_{n,\cdot 1}^\top F_n / k_n = \hat{\delta}_1^\top H^\top \big(F_n^\top F_n / k_n\big)^{1/2}.$$

Hence,

$$\hat{\delta}_1^\top = \big(\hat{F}_{n,\cdot 1}^\top F_n / k_n\big)\big(F_n^\top F_n / k_n\big)^{-1/2} H.$$

By the definition of $S_{n,1}^*$, the first element of $S_{n,1}^*(\hat{F}_{n,\cdot 1}^\top F_n / k_n)(F_n^\top F_n / k_n)^{-1/2} H$ is nonnegative. Hence,

$$\begin{aligned} &S_{n,1}^* \big(\hat{F}_{n,\cdot 1}^\top F_n / k_n\big)\big(F_n^\top F_n / k_n\big)^{-1/2} H \\ &\quad = \big(|\hat{\delta}_{11}|, S_{n,1}^* \hat{\delta}_{12}, \ldots, S_{n,1}^* \hat{\delta}_{1r}\big) \xrightarrow{\mathbb{P}} (1, 0, \ldots, 0). \end{aligned}$$

By (SA.58), we further deduce that

$$S_{n,1}^* \big(\hat{F}_{n,\cdot 1}^\top F_n / k_n\big) \Sigma_f^{-1/2} H \xrightarrow{\mathbb{P}} (1, 0, \ldots, 0),$$

which shows the convergence in (SA.69) for the first row.

By repeating the same argument (by setting Γ_n as the subspace orthogonal to previous eigenvectors), we can prove the convergence in (SA.69) for the jth row, $2 \leq j \leq r$.

Step 3. In this step, we finish the proof for part (a) of Theorem 2. We denote

$$\tilde{D}_n = N_n^{-1} \big(\hat{\Lambda}_n^* - \Lambda_n^* \Sigma_f^{1/2} H S_n^*\big)^\top \big(\hat{\Lambda}_n^* - \Lambda_n^* \Sigma_f^{1/2} H S_n^*\big).$$

The assertion of part (a) can be rewritten as $\text{Trace}[\tilde{D}_n] = o_p(1)$.

We decompose

$$\tilde{D}_n = \tilde{D}_{n,1} - \tilde{D}_{n,2} - \tilde{D}_{n,2}^\top + \tilde{D}_{n,3},$$

where

$$\begin{cases} \tilde{D}_{n,1} \equiv N_n^{-1} \hat{\Lambda}_n^{*\top} \hat{\Lambda}_n^*, & \tilde{D}_{n,2} \equiv N_n^{-1} \hat{\Lambda}_n^{*\top} \Lambda_n^* \Sigma_f^{1/2} H S_n^*, \\ \tilde{D}_{n,3} \equiv N_n^{-1} S_n^* H^\top \Sigma_f^{1/2} \Lambda_n^{*\top} \Lambda_n^* \Sigma_f^{1/2} H S_n^*. \end{cases}$$

To prove Trace$[\tilde{D}_n] = o_p(1)$, it suffices to show that

$$\tilde{D}_{n,k} \xrightarrow{\mathbb{P}} D, \quad k = 1, 2, 3, \tag{SA.74}$$

where we recall that D is the diagonal matrix that collects the ordered eigenvalues of M_C^*. Below, we prove (SA.74) for each case.

Case $k = 1$: Recall that we partition $\hat{F}_n = [\hat{F}_n^* \vdots \hat{F}_n^0]$, where $\hat{F}_n^*$ collects the first r columns of $\hat{F}_n$. We set

$$\hat{\Lambda}_n'^* = \frac{1}{k_n} \hat{X}_n' \hat{F}_n^* = \frac{1}{k_n} \big(\tilde{\Lambda}_n^* F_n^\top + \mathcal{E}_n + \mathcal{E}_n' \big) \hat{F}_n^*. \tag{SA.75}$$

Note that

$$\begin{aligned} \big\| \tilde{D}_{n,1} - N_n^{-1} \hat{\Lambda}_n'^{*\top} \hat{\Lambda}_n'^* \big\| &= N_n^{-1} \big\| \hat{\Lambda}_n^{*\top} \hat{\Lambda}_n^* - \hat{\Lambda}_n'^{*\top} \hat{\Lambda}_n'^* \big\| \\ &= k_n^{-2} N_n^{-1} \big\| \hat{F}_n^{*\top} \hat{X}_n^\top \hat{X}_n \hat{F}_n^* - \hat{F}_n^{*\top} \hat{X}_n'^\top \hat{X}_n' \hat{F}_n^* \big\| = o_p(1), \end{aligned} \tag{SA.76}$$

where the first two equalities are by definition and the last one is by (SA.57). Subsequently, by (SA.75), we can decompose $\tilde{D}_{n,1}$ as

$$\begin{aligned} \tilde{D}_{n,1} &= \frac{1}{k_n^2 N_n} \hat{F}_n^{*\top} \big(\tilde{\Lambda}_n^* F_n^\top + \mathcal{E}_n + \mathcal{E}_n' \big)^\top \big(\tilde{\Lambda}_n^* F_n^\top + \mathcal{E}_n + \mathcal{E}_n' \big) \hat{F}_n^* + o_p(1) \\ &= \tilde{D}_{n,1,1} + \tilde{D}_{n,1,2} + \tilde{D}_{n,1,2}^\top + \tilde{D}_{n,1,3} + o_p(1), \end{aligned}$$

where

$$\begin{cases} \tilde{D}_{n,1,1} \equiv \big(\hat{F}_n^{*\top} F_n / k_n \big) \big(\tilde{\Lambda}_n^{*\top} \tilde{\Lambda}_n^* / N_n \big) \big(F_n^\top \hat{F}_n^* / k_n \big), \\ \tilde{D}_{n,1,2} \equiv \big(k_n^{-1} N_n^{-1} \hat{F}_n^{*\top} (\mathcal{E}_n + \mathcal{E}_n')^\top \tilde{\Lambda}_n^* \big) \big(F_n^\top \hat{F}_n^* / k_n \big), \\ \tilde{D}_{n,1,3} \equiv k_n^{-2} N_n^{-1} \hat{F}_n^{*\top} (\mathcal{E}_n + \mathcal{E}_n')^\top (\mathcal{E}_n + \mathcal{E}_n') \hat{F}_n^*. \end{cases}$$

From (SA.69),

$$\frac{1}{k_n} \hat{F}_n^{*\top} F_n - S_n^* H^\top \Sigma_f^{1/2} = o_p(1), \qquad \frac{1}{k_n} \hat{F}_n^{*\top} F_n = O_p(1). \tag{SA.77}$$

Hence, recalling that H is the eigenvector matrix of $M_C^* = \Sigma_f^{1/2} M_\Lambda^* \Sigma_f^{1/2}$ and S_n^* is a diagonal matrix with ± 1 on its diagonal, we deduce

$$\tilde{D}_{n,1,1} = S_n^* H^\top \Sigma_f^{1/2} M_\Lambda^* \Sigma_f^{1/2} H S_n^* + o_p(1) = D + o_p(1).$$

By Lemma SA4, we see that $\tilde{D}_{n,1,2}$ and $\tilde{D}_{n,1,3}$ are both $o_p(1)$. From these estimates, (SA.74) for the case $k = 1$ readily follows.

Case $k=2$: By (SA.76) and the Cauchy–Schwarz inequality,

$$\tilde{D}_{n,2} \equiv N_n^{-1}\hat{\Lambda}_n'^{*\top}\Lambda_n^*\Sigma_f^{1/2}HS_n^* + o_p(1).$$

By (SA.75), we can thus decompose $\tilde{D}_{n,2}$ as $\tilde{D}_{n,2} = \tilde{D}_{n,2,1} + \tilde{D}_{n,2,2} + o_p(1)$ where

$$\begin{cases} \tilde{D}_{n,2,1} \equiv \big(\hat{F}_n^{*\top}F_n/k_n\big)\big(\tilde{\Lambda}_n^{*\top}\Lambda_n^*/N_n\big)\Sigma_f^{1/2}HS_n^*, \\ \tilde{D}_{n,2,2} \equiv \big(k_n^{-1}N_n^{-1}\hat{F}_n^{*\top}\big(\mathcal{E}_n + \mathcal{E}_n'\big)^\top\Lambda_n^*\big)\Sigma_f^{1/2}HS_n^*. \end{cases}$$

By (SA.77) and Lemma SA4(e), we deduce

$$\tilde{D}_{n,2,1} = S_n^*H^\top\Sigma_f^{1/2}M_\Lambda^*\Sigma_f^{1/2}HS_n^* + o_p(1) = D + o_p(1).$$

By Lemma SA4(c), (d), $\tilde{D}_{n,2,2} = o_p(1)$. This proves (SA.74) for the case $k=2$.

Case $k=3$: By Assumption 8, it is obvious that

$$\tilde{D}_{n,3} = S_n^*H^\top\Sigma_f^{1/2}M_\Lambda^*\Sigma_f^{1/2}HS_n^* + o_p(1) = D + o_p(1).$$

This finishes the proof of (SA.74), and hence, part (a) of Theorem 2. □

Proof of Theorem 2(b). We fix $j \in \{r+1,\ldots,\bar{r}\}$. Recall that $\hat{\Lambda}_{n,\cdot j}$ denote the jth column of $\hat{\Lambda}_n$. By the definitions of $\hat{\Lambda}_n$ and $\hat{F}_n$,

$$\frac{1}{N_n}\hat{\Lambda}_{n,\cdot j}^\top\hat{\Lambda}_{n,\cdot j} = \Xi_n(\hat{F}_{n,\cdot j}). \tag{SA.78}$$

Like in (SA.73), for each $k \in \{1,\ldots,r\}$, we can represent

$$\hat{F}_{n,\cdot k} = F_n\big(F_n^\top F_n/k_n\big)^{-1/2}H\hat{\delta}_k + \tilde{\gamma}_k, \tag{SA.79}$$

where $F_n^\top\tilde{\gamma}_k = 0$. Following a similar argument as in Step 2 of the proof of Theorem 2(a), we can show that, for each $k, k' \in \{1,\ldots,r\}$ with $k \neq k'$,

$$\hat{\delta}_{kk}^2 \xrightarrow{\mathbb{P}} 1, \qquad \hat{\delta}_{kk'} \xrightarrow{\mathbb{P}} 0, \qquad \tilde{\gamma}_k^\top\tilde{\gamma}_k/k_n \xrightarrow{\mathbb{P}} 0. \tag{SA.80}$$

We also represent

$$\hat{F}_{n,\cdot j} = F_n\big(F_n^\top F_n/k_n\big)^{-1/2}H\hat{\delta}_j + \tilde{\gamma}_j, \tag{SA.81}$$

where $F_n^\top\tilde{\gamma}_j = 0$. Since $\hat{F}_{n,\cdot j}^\top\hat{F}_{n,\cdot k}/k_n = 0$ for $1 \le k \le r$ (because $\hat{F}_n$ collects the eigenvectors of $\hat{X}_n^\top\hat{X}_n$), we have

$$\hat{\delta}_j^\top\hat{\delta}_k + \tilde{\gamma}_j^\top\tilde{\gamma}_k/k_n = 0. \tag{SA.82}$$

Since $\tilde{\gamma}_k^\top\tilde{\gamma}_k/k_n \xrightarrow{\mathbb{P}} 0$ and $\tilde{\gamma}_j^\top\tilde{\gamma}_j/k_n \le 1$, we have $\tilde{\gamma}_j^\top\tilde{\gamma}_k/k_n = o_p(1)$ by the Cauchy–Schwarz inequality. Therefore, $\hat{\delta}_j^\top\hat{\delta}_k = o_p(1)$ for $1 \le k \le r$. By (SA.80) above, this implies $\hat{\delta}_j = o_p(1)$. Hence,

$$\Xi_n^*(\hat{F}_{n,\cdot j}) = \hat{\delta}_j^\top HM_{C,n}^*H^\top\hat{\delta}_j = o_p(1). \tag{SA.83}$$

By (SA.53), $\Xi_n(\hat{F}_{n,\cdot j}) = o_p(1)$. The assertion of part (b) readily follows from (SA.78). $\square$

Proof of Theorem 2(c). By Assumption 8,

$$H_p^\top \Sigma_{f,p}^{1/2} \frac{\Lambda_n^*(p)^\top \Lambda_n^*(q)}{N_n} \Sigma_{f,q}^{1/2} H_q \xrightarrow{\mathbb{P}} H_p^\top M_C^*(p,q) H_q. \tag{SA.84}$$

We observe

$$\begin{aligned}
&\frac{1}{N_n}\left\| \hat{\Lambda}_n^*(p)^\top \hat{\Lambda}_n^*(q) - S_n^*(p) H_p^\top \Sigma_{f,p}^{1/2} \Lambda_n^*(p)^\top \Lambda_n^*(q) \Sigma_{f,q}^{1/2} H_q S_n^*(q) \right\| \\
&\quad \leq \frac{1}{N_n}\left\| \left(\hat{\Lambda}_n^*(p) - \Lambda_n^*(p)\Sigma_{f,p}^{1/2} H_p S_n^*(p)\right)^\top \Lambda_n^*(q) \Sigma_{f,q}^{1/2} H_q \right\| \\
&\qquad + \frac{1}{N_n}\left\| H_p^\top \Sigma_{f,p}^{1/2} \Lambda_n^*(p)^\top \left(\hat{\Lambda}_n^*(q) - \Lambda_n^*(q)\Sigma_{f,q}^{1/2} H_q S_n^*(q)\right) \right\| \\
&\qquad + \frac{1}{N_n}\left\| \left(\hat{\Lambda}_n^*(p) - \Lambda_n^*(p)\Sigma_{f,p}^{1/2} H_p S_n^*(p)\right)^\top \left(\hat{\Lambda}_n^*(q) - \Lambda_n^*(q)\Sigma_{f,q}^{1/2} H_q S_n^*(q)\right) \right\|.
\end{aligned}$$

By the Cauchy–Schwarz inequality and Theorem 2(a), we deduce that the terms on the majorant side of the above display are all $o_p(1)$. Hence, by (SA.84),

$$\frac{1}{N_n} \hat{\Lambda}_n^*(p)^\top \hat{\Lambda}_n^*(q) - S_n^*(p) H_p^\top M_C^*(p,q) H_q S_n^*(q) = o_p(1). \tag{SA.85}$$

In particular,

$$\frac{1}{N_n} \hat{\Lambda}_n^*(p)^\top \hat{\Lambda}_n^*(q) = O_p(1). \tag{SA.86}$$

By Theorem 2(b),

$$\frac{1}{N_n} \hat{\Lambda}_n^0(p)^\top \hat{\Lambda}_n^0(q) = o_p(1). \tag{SA.87}$$

By the Cauchy–Schwarz inequality, (SA.86), and (SA.87), we deduce

$$\frac{1}{N_n} \hat{\Lambda}_n^*(p)^\top \hat{\Lambda}_n^0(q) = o_p(1). \tag{SA.88}$$

The assertion of part (c) then follows from (SA.85), (SA.87), and (SA.88). $\square$

Proof of Theorem 2(d). By part (c) of Theorem 2,

$$\begin{aligned}
\mathrm{Trace}\left[\widehat{M}_{C,n}(q,q)\right] &= \mathrm{Trace}\left[S_n^*(q) H_q^\top M_C^*(q,q) H_q S_n^*(q)\right] + o_p(1) \\
&= \mathrm{Trace}\left[M_C^*(q,q)\right] + o_p(1) \\
&= \mathrm{Trace}\left[M_\Lambda^*(q,q)\Sigma_{f,q}\right] + o_p(1),
\end{aligned}$$

where the second inequality follows from the orthogonality of $H_q S_n^*(q)$ and the last line holds because $M_C^*(q,q) = \Sigma_{f,q}^{1/2} M_\Lambda^*(q,q) \Sigma_{f,q}^{1/2}$. We also note from (SA.56) that

$$\frac{1}{k_n N_n}\left\|\hat{X}_n(q)\right\|^2 = \frac{1}{k_n N_n}\left\|\hat{X}_n'(q)\right\|^2 + o_p(1).$$

Hence, it remains to show that

$$\frac{1}{k_n N_n}\left\|\hat{X}_n'(q)\right\|^2 \xrightarrow{\mathbb{P}} \operatorname{Trace}\left[M_\Lambda^*(q,q)\Sigma_{f,q}\right] + M_\epsilon(q). \tag{SA.89}$$

To show (SA.89), we consider the following decomposition:

$$\begin{aligned}\left\|\hat{X}_n'(q)\right\|^2 &= \operatorname{Trace}\left[\hat{X}_n'(q)^\top \hat{X}_n'(q)\right] \\ &= \operatorname{Trace}\left[\tilde{\Lambda}_n^*(q)^\top \tilde{\Lambda}_n^*(q) F_n(q)^\top F_n(q)\right] \\ &\quad + \operatorname{Trace}\left[\left(\mathcal{E}_n(q)+\mathcal{E}_n'(q)\right)^\top \left(\mathcal{E}_n(q)+\mathcal{E}_n'(q)\right)\right] \\ &\quad + 2\operatorname{Trace}\left[F_n(q)\tilde{\Lambda}_n^*(q)^\top \left(\mathcal{E}_n(q)+\mathcal{E}_n'(q)\right)\right].\end{aligned} \tag{SA.90}$$

By Lemma SA4(e) and (SA.58),

$$\frac{1}{k_n N_n}\operatorname{Trace}\left[\tilde{\Lambda}_n^*(q)^\top \tilde{\Lambda}_n^*(q) F_n(q)^\top F_n(q)\right] \xrightarrow{\mathbb{P}} \operatorname{Trace}\left[M_\Lambda^*(q,q)\Sigma_{f,q}\right]. \tag{SA.91}$$

In the proof of Lemma SA4(c), (d), we have shown that

$$\frac{1}{k_n}\left\|\frac{1}{N_n}\left(\mathcal{E}_n(q)+\mathcal{E}_n'(q)\right)^\top \Lambda_n^*(q)\right\|^2 = o_p(1).$$

In addition, by (SA.52),

$$\begin{aligned}&\frac{1}{k_n}\left\|\frac{1}{N_n}\left(\mathcal{E}_n(q)+\mathcal{E}_n'(q)\right)^\top \left(\tilde{\Lambda}_n^*(q)-\Lambda_n^*(q)\right)\right\|^2 \\ &\quad \leq \frac{\left\|\mathcal{E}_n(q)+\mathcal{E}_n'(q)\right\|^2}{k_n N_n}\cdot\frac{\left\|\tilde{\Lambda}_n^*(q)-\Lambda_n^*(q)\right\|^2}{N_n} = o_p(1).\end{aligned}$$

Hence, $\|(\mathcal{E}_n(q)+\mathcal{E}_n'(q))^\top \tilde{\Lambda}_n^*(q)\| = o_p(N_n k_n^{1/2})$. Also note that $\|F_n(q)\| = O_p(k_n^{1/2})$. Therefore, by the Cauchy–Schwarz inequality,

$$\left\|\frac{1}{k_n N_n}\left(\mathcal{E}_n(q)+\mathcal{E}_n'(q)\right)^\top \tilde{\Lambda}_n^*(q) F_n(q)^\top\right\| \leq \frac{1}{k_n N_n}\left\|F_n(q)\right\|\left\|\left(\mathcal{E}_n(q)+\mathcal{E}_n'(q)\right)^\top \tilde{\Lambda}_n^*(q)\right\| = o_p(1).$$

Consequently,

$$\frac{1}{k_n N_n}\operatorname{Trace}\left[F_n(q)\tilde{\Lambda}_n^*(q)^\top \left(\mathcal{E}_n(q)+\mathcal{E}_n'(q)\right)\right] = o_p(1). \tag{SA.92}$$

In view of (SA.90), (SA.91), and (SA.92), (SA.89) will be implied by

$$\frac{1}{k_n N_n}\operatorname{Trace}\left[\left(\mathcal{E}_n(q)+\mathcal{E}_n'(q)\right)^\top \left(\mathcal{E}_n(q)+\mathcal{E}_n'(q)\right)\right] \xrightarrow{\mathbb{P}} M_\epsilon(q). \tag{SA.93}$$

Finally, we show (SA.93). For each j, we denote

$$\xi_{n,j} \equiv \frac{1}{k_n}\sum_{l=1}^{k_n}\left(\frac{\Delta_{i(n,q)+l}^n \epsilon_j}{\sqrt{\Delta_n}}\right)^2,$$

$$\xi'_{n,j} \equiv \frac{1}{k_n\Delta_n}\int_{i(n,q)\Delta_n}^{i(n,q)\Delta_n+k_n\Delta_n} \tilde{\sigma}_{j,u}^2\,du, \qquad \xi''_{n,j} \equiv \xi_{n,j} - \xi'_{n,j}.$$

Then we can decompose

$$\begin{aligned}\frac{1}{k_nN_n}\operatorname{Trace}\big[\mathcal{E}_n(q)^\top\mathcal{E}_n(q)\big] &= \frac{1}{k_nN_n}\sum_{j=1}^{N_n}\sum_{l=1}^{k_n}\left(\frac{\Delta^n_{i(n,q)+l}\epsilon_j}{\sqrt{\Delta_n}}\right)^2 \\ &= \frac{1}{N_n}\sum_{j=1}^{N_n}\xi'_{n,j} + \frac{1}{N_n}\sum_{j=1}^{N_n}\xi''_{n,j}.\end{aligned}$$

We note that conditional on $\mathcal{F}_{i(n,q)\Delta_n}$, the variables $(\xi''_{n,j})_{1\le j\le N_n}$ are uncorrelated with zero mean and bounded variances. Hence,

$$\frac{1}{N_n}\sum_{j=1}^{N_n}\xi''_{n,j} = o_p(1). \tag{SA.94}$$

In addition, we note that

$$\begin{aligned}\frac{1}{N_n}\sum_{j=1}^{N_n}\xi'_{n,j} - \frac{1}{N_n}\sum_{j=1}^{N_n}\tilde{\sigma}^2_{j,q} &= \frac{1}{N_n}\sum_{j=1}^{N_n}\frac{1}{k_n\Delta_n}\int_{i(n,q)\Delta_n}^{i(n,q)\Delta_n+k_n\Delta_n}\big(\tilde{\sigma}^2_{j,u}-\tilde{\sigma}^2_{j,q}\big)\,du \\ &= O_p\big(k_n^{1/2}\Delta_n^{1/2}\big) = o_p(1).\end{aligned}$$

It readily follows that

$$\frac{1}{N_n}\sum_{j=1}^{N_n}\xi'_{n,j} \xrightarrow{\mathbb{P}} M_\epsilon(q). \tag{SA.95}$$

By (SA.94) and (SA.95),

$$\frac{1}{k_nN_n}\operatorname{Trace}\big[\mathcal{E}_n(q)^\top\mathcal{E}_n(q)\big] \xrightarrow{\mathbb{P}} M_\epsilon(q). \tag{SA.96}$$

We further note that

$$\frac{1}{k_nN_n}\operatorname{Trace}\big[\mathcal{E}'_n(q)^\top\mathcal{E}'_n(q)\big] = \frac{1}{k_nN_n}\sum_{j=1}^{N_n}\sum_{l=1}^{k_n}\big(e'_{j,l}\big)^2 = O_p(\Delta_n). \tag{SA.97}$$

With an appeal to the Cauchy–Schwarz inequality, we deduce (SA.93) from (SA.96) and (SA.97). This finishes the proof of part (d) of Theorem 2. □

SA.5 *Proof of Theorem 3*

(a) First, by Theorem 2(c), (d), it is obvious that $\tilde{L}_n(\eta,\tau) = O_p(1)$. Hence, the quantile $cv_{n,\alpha} = O_p(1)$. Next, we consider the case under the null hypothesis, so $M^*_C(p,q)$ coincides with $M_C(p,q)$.

We partition $\tilde{\zeta}_q^\top = (\tilde{\zeta}_q^{*\top}, \tilde{\zeta}_q^{0\top})$, where $\tilde{\zeta}_q^*$ is r-dimensional. By Theorem 2(c), (d), we have, for $s \in \{\eta, \tau\}$,

$$\begin{cases} \tilde{A}_n(s) = \displaystyle\sum_{p,q\in\{s-,s+\}} \tilde{w}_{n,p}\tilde{w}_{n,q}\tilde{\zeta}_p^{*\top} S_n^*(p) H_p^\top M_C(p,q) H_q S_n^*(q)\tilde{\zeta}_q^* \\ \qquad\qquad + \displaystyle\sum_{q\in\{s-,s+\}} \tilde{w}_{n,q}^2 M_\epsilon(q) + o_p(1), \\ \tilde{B}_n(\eta,\tau) = \displaystyle\sum_{p\in\{\tau-,\tau+\}}\sum_{q\in\{\eta-,\eta+\}} \tilde{w}_{n,p}\tilde{w}_{n,q}\tilde{\zeta}_p^{*\top} S_n^*(p) H_p^\top M_C(p,q) H_q S_n^*(q)\tilde{\zeta}_q^* + o_p(1). \end{cases}$$

We note that the r-dimensional vectors $H_q S_n^*(q)\tilde{\zeta}_q^*$ are, conditionally on $\mathcal{F}$, standard normal and mutually independent across $q \in \{\tau-, \tau+, \eta-, \eta+\}$. We also observe that for $s \in \{\eta, \tau\}$, $\Delta_{i(n,s)}^n Z \xrightarrow{\mathbb{P}} \Delta Z_s$. Hence,

$$\big(H_q S_n^*(q)\tilde{\zeta}_q^*, \tilde{w}_{n,q}\big)_{q\in\{\tau-,\tau+,\eta-,\eta+\}} \xrightarrow{\mathcal{L}|\mathcal{F}} (\zeta_q, w_q)_{q\in\{\tau-,\tau+,\eta-,\eta+\}}, \tag{SA.98}$$

where $\xrightarrow{\mathcal{L}|\mathcal{F}}$ denotes the convergence of conditional law in probability. It follows that

$$\big(\tilde{A}_n(\eta), \tilde{A}_n(\tau), \tilde{B}_n(\eta,\tau)\big) \xrightarrow{\mathcal{L}|\mathcal{F}} \big(\mathcal{A}(\eta), \mathcal{A}(\tau), \mathcal{B}(\eta,\tau)\big).$$

Consequently, $\tilde{L}_n(\eta,\tau) \xrightarrow{\mathcal{L}|\mathcal{F}} \mathcal{L}(\eta,\tau)$. We further note that the $\mathcal{F}$-conditional distribution function of $\mathcal{L}(\eta,\tau)$ is continuous and strictly increasing. Hence, $cv_{n,\alpha} \xrightarrow{\mathbb{P}} cv_\alpha$.

(b) The assertion on the asymptotic level follows from part (a) and Theorem 1. Under the alternative, $\Delta_n^{-1}\hat{V}_n$ diverges to $+\infty$ in probability by Proposition 1. The power property then follows from $cv_{n,\alpha} = O_p(1)$.

References

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