Supplement to "Identification and estimation of a bidding model for electronic auctions: Appendix"

(*Quantitative Economics*, Vol. 8, No. 2, July 2017, 505–551)

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Appendix

A.1 *A brief primer on B-splines*

As mentioned in our article, B-splines have many attractive qualities as a tool for flexible parametric curve fitting. They can be constructed to fit any continuous curve to arbitrary precision, they are locally low-dimensional, giving nice stability properties, they are convenient to compute, and they can even be adapted to allow for an arbitrary degree of curvature—including discontinuous derivatives—at a point, with finitely many terms. Here, we provide the reader with a brief primer on constructing B-spline functions and their associated bases.

Consider the problem of fitting a continuous nonparametric function $f : [\underline{x}, \overline{x}] \to \mathbb{R}$ with an O-order B-spline, $O \in \mathbb{N}$. We shall compute our B-spline basis functions according to the Cox–de Boor recursion formula (see de Boor (2001, Chapter IX, equations (11)–(15)) with coincident boundary knots. This formula requires us to prespecify a knot vector $\mathbf{k}_K = \{k_1 < k_2 < \cdots < k_K < k_{K+1}\}$ that partitions the domain into K subintervals, with $k_1 = \underline{x}$ and $k_{K+1} = \overline{x}$. Given knot vector $\mathbf{k}_K$ and order O, we wish to compute a set of $(K + O - 1)$ basis functions of degree $D \equiv (O - 1)$, denoted $\mathbb{B}_{\mathbf{k}_K,D} = \{\mathcal{B}_{i,D} : [\underline{x}, \overline{x}] \to \mathbb{R}, i = 1, \dots, K + D\}$. We begin by computing an extended knot vector with D additional copies of the endpoints added in:

$$\overline{\mathbf{k}}_K = \{k_{1-D} = \cdots = k_1 < k_2 < \cdots < k_K < k_{K+1} = \cdots = k_{K+1+D}\}.$$

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DOI: 10.3982/QE233

The Cox–de Boor recursion formula requires that to define basis $\mathbb{B}_{\mathbf{k}_K,D}$, we must first compute all lower-order bases $\mathbb{B}_{\mathbf{k}_K,d}$ for each $d = 0, \ldots, D$. We initiate the recursion by defining the zero-degree basis functions as simply piecewise constant or, for each $i = 1, \ldots, K$,

$$\mathcal{B}_{i,0}(x) = \begin{cases} 1, & \text{if } x \in [k_i, k_{i+1}) \cup \overline{x}, \\ 0, & \text{otherwise.} \end{cases} \tag{A.1}$$

Note from the above formula that there are exactly K of them. With this beginning, we can now define, for each $d = 1, \ldots, D$ and for each $i = 1, \ldots, K + d$,

$$\mathcal{B}_{i,d+1}(x) = \begin{cases} \mathcal{B}_{i,d}(x)\left(\dfrac{x - k_{i-d}}{k_i - k_{i-d}}\right) + \mathcal{B}_{i+1,d}(x)\left(\dfrac{k_{i-d+1} - x}{k_{i+1} - k_{i-d+1}}\right), \\ \quad \text{if } (k_i - k_{i-d})(k_{i+1} - k_{i-d+1}) \neq 0, \\ \mathcal{B}_{i,d}(x)\left(\dfrac{x - k_{i-d}}{k_i - k_{i-d}}\right), \\ \quad \text{if } (k_{i+1} - k_{i-d+1}) = 0, \\ \mathcal{B}_{i+1,d}(x)\left(\dfrac{k_{i-d+1} - x}{k_{i+1} - k_{i-d+1}}\right), \\ \quad \text{if } (k_i - k_{i-d}) = 0. \end{cases} \tag{A.2}$$

It is easy to see from this recurrence relation that the order 2 (degree 1) basis $\mathbb{B}_{\mathbf{k}_K,1}$ results in a set of $K + 1$ piecewise linear functions. Likewise, the order 3 (degree 2) basis $\mathbb{B}_{\mathbf{k}_K,2}$ results in a set of $K + 2$ piecewise quadratic functions, and the order 4 (degree 3) basis $\mathbb{B}_{\mathbf{k}_K,3}$ results in a set of $K + 3$ piecewise cubic functions. Figures A.1–A.4 display examples of a set of bases designed for knot vector $\mathbf{k}_5 = \{-1, -0.6, -0.2, 0.2, 0.6, 1\}$.

Other properties of the basis functions are also worth pointing out. Note that each basis function is globally defined, but any given function in $\mathbb{B}_{\mathbf{k}_K,D}$ is nonzero on at most $O = (D + 1)$ subintervals defined by $\mathbf{k}_K$. Moreover, on each subinterval, exactly O of the $(K + D)$ basis functions are nonzero. This is what gives B-splines their attractive locality properties: for a B-spline function of the form

$$f(x; \alpha) = \sum_{i=1}^{K+D} \alpha_i \mathcal{B}_{i,D}(x),$$

adjusting parameter values relevant to one subinterval will have little or no effect for subintervals sufficiently far away.

Finally, it is easy to see from the above expression that since B-spline functions are linear combinations of B-spline bases, their derivatives are also linear combinations of

B-spline basis functions. Specifically, like global polynomials, the derivative of a Dth-order B-spline is merely a $(D-1)$st-order B-spline, and its integral is an $(D+1)$st-order B-spline: for each $d \geq 1$ and $i = 1, \ldots, (K+d)$ we have

$$\mathcal{B}'_{i,d}(x) = \begin{cases} (d-1)\left(\dfrac{\mathcal{B}_{i,d-1}(x)}{k_i - k_{i-d+1}} + \dfrac{\mathcal{B}_{i+1,d-1}(x)}{k_{i+1} - k_{i-d+2}}\right), \\ \quad \text{if } (k_i - k_{i-d+1})(k_{i+1} - k_{i-d+2}) \neq 0, \\ (d-1)\mathcal{B}_{i,d-1}(x)/(k_i - k_{i-d+1}), \\ \quad \text{if } (k_{i+1} - k_{i-d+2}) = 0, \\ (d-1)\mathcal{B}_{i+1,d-1}(x)/(k_{i+1} - k_{i-d+2}), \\ \quad \text{if } (k_i - k_{i-d+1}) = 0. \end{cases} \tag{A.3}$$

Figure A.5 illustrates $\mathcal{B}_{5,3}(x)$ from Figure A.4, along with its three nontrivial derivatives. This picture illustrates another important property of B-splines: $\mathcal{B}_{i,D}(x)$ has $(D-1)$ continuous derivatives on the interval $[\underline{x}, \overline{x}]$. In fact, the Cox–de Boor recursion formula above was specifically constructed to deliver this property.

A.2 *Extra tables and figures*

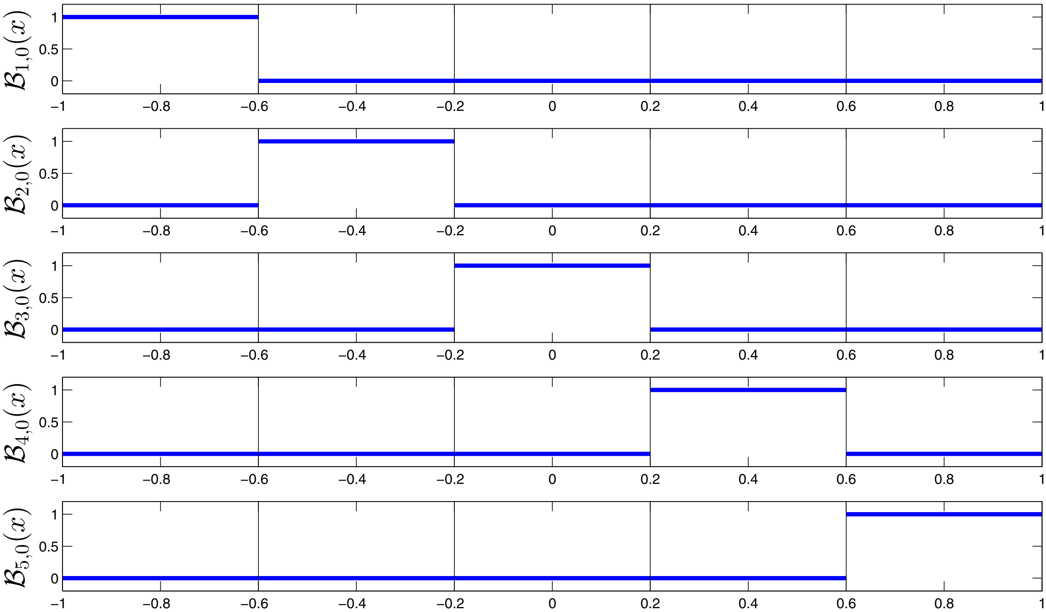

FIGURE A.1. Order 1 (degree 0) B-spline basis functions.

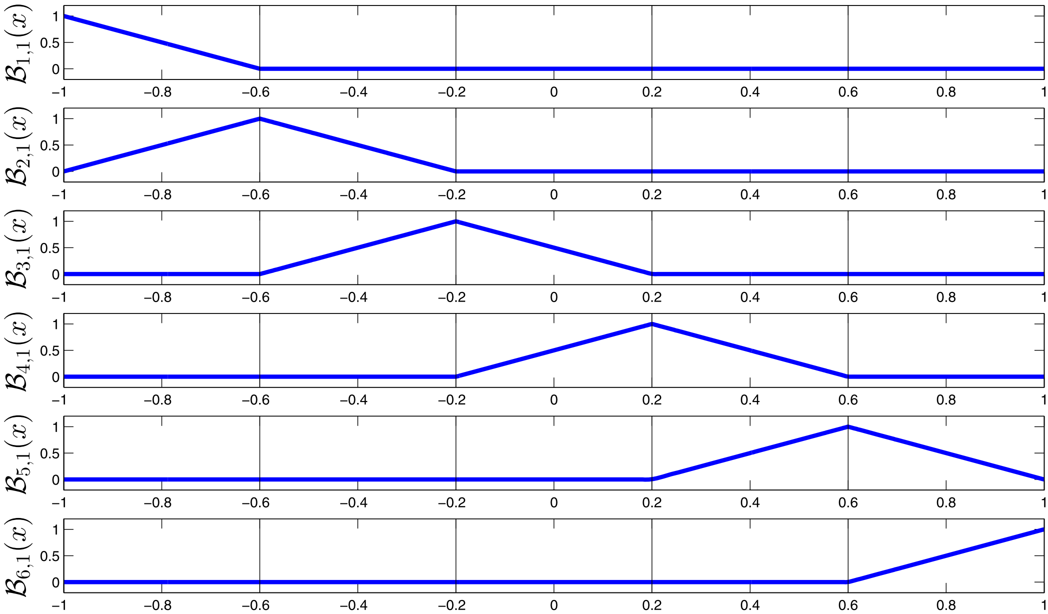

FIGURE A.2. Order 2 (degree 1) B-spline basis functions.

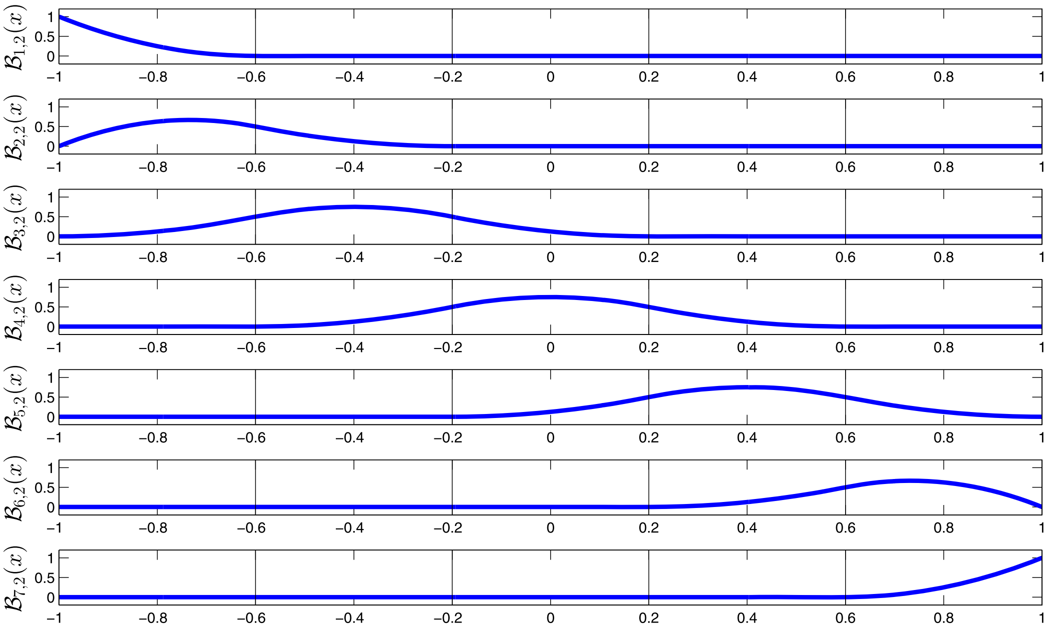

FIGURE A.3. Order 3 (degree 2) B-spline basis functions.

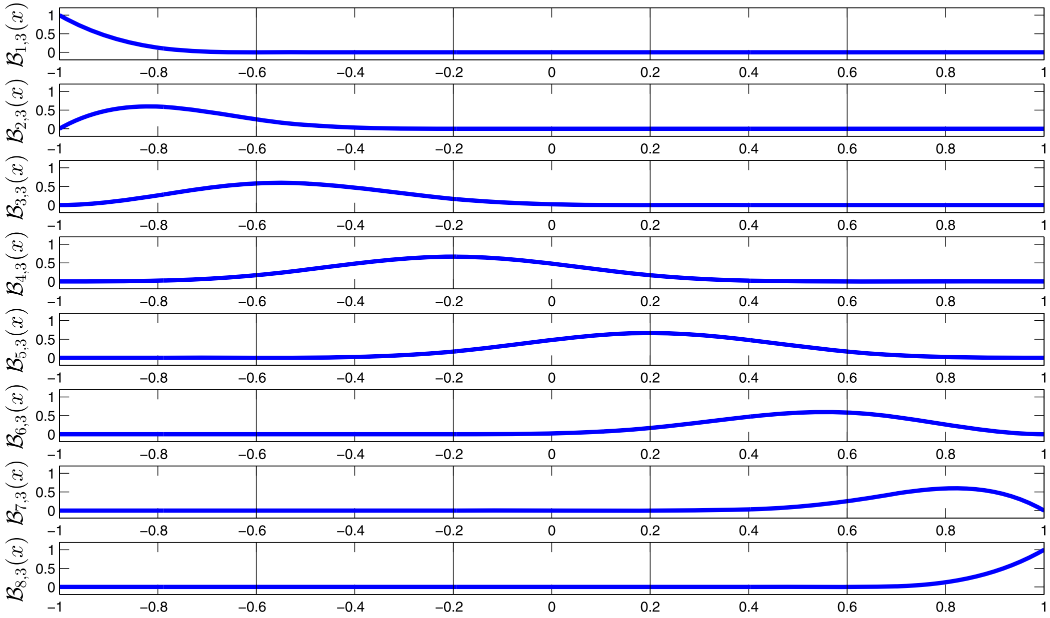

FIGURE A.4. Order 4 (degree 3) B-spline basis functions.

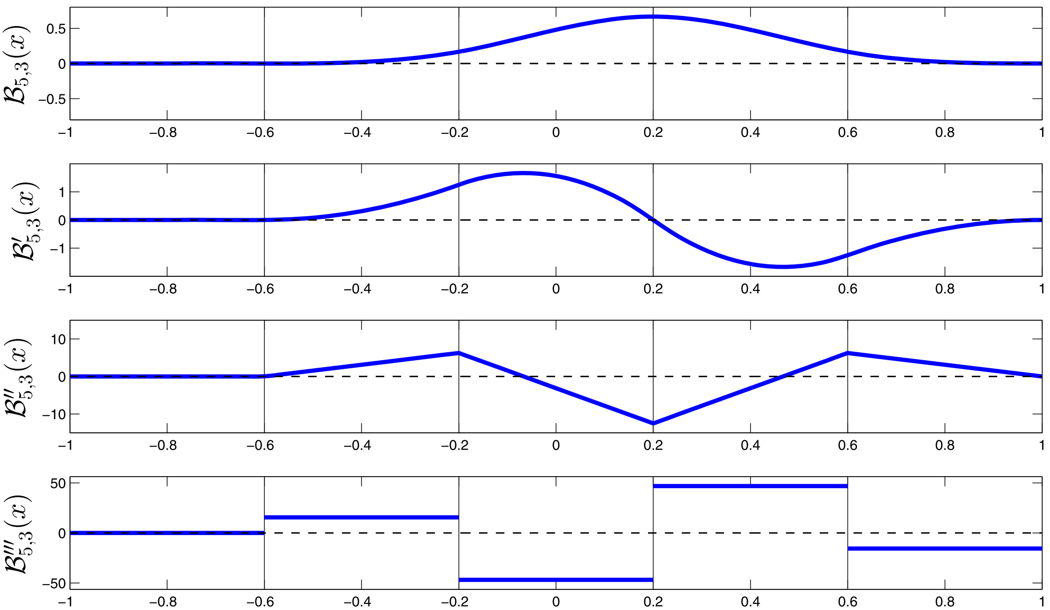

FIGURE A.5. Basis function $\mathcal{B}_{5,3}(x)$ example and its derivatives.

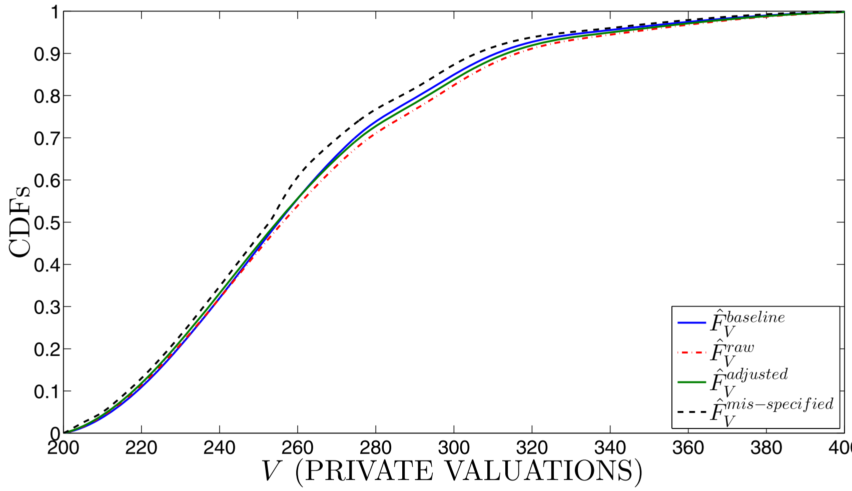

FIGURE A.6. Robustness check: bias resulting from misspecified pricing rule and scale invariance violations.

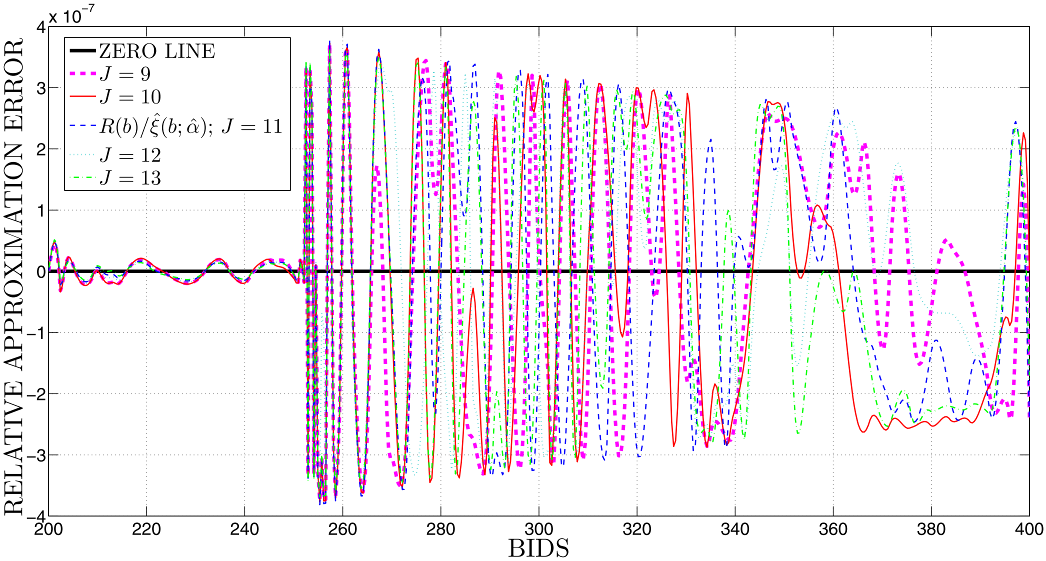

FIGURE A.7. (Sup-norm) approximation error: comparison for various values of J.

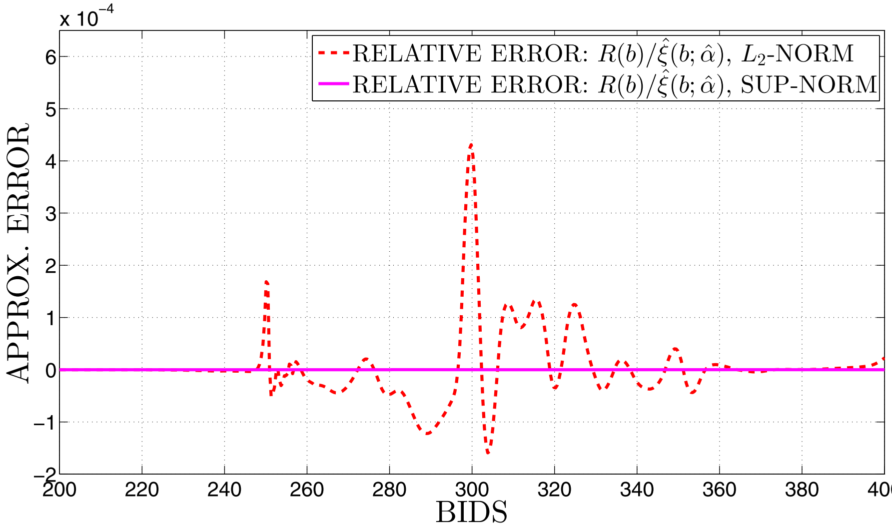

FIGURE A.8. Approximation error comparison: L_∞ $\hat{\xi}$ estimate versus L_2 $\hat{\xi}$ estimate (using the preferred model specification for F_V).

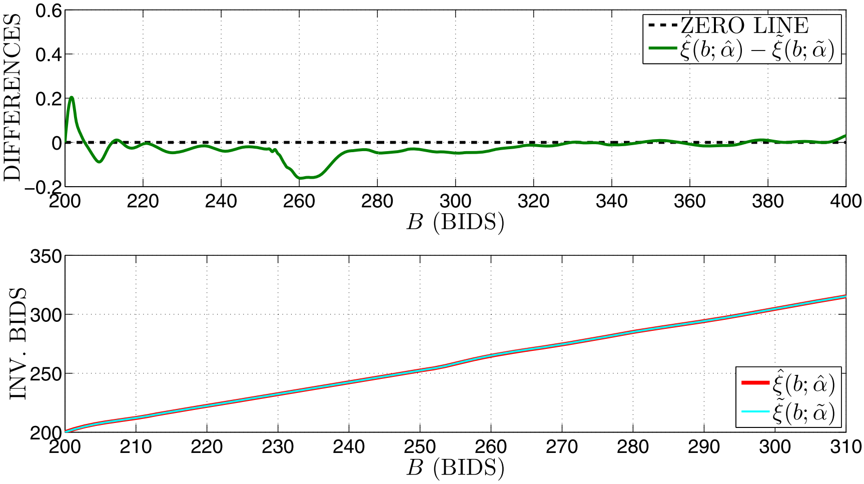

FIGURE A.9. Differential equation solution comparison: runtime Galerkin ODE solution (overfitted) with J knots versus ex post Galerkin ODE solution (square system) with $5J$ knots (holding point estimate $\hat{F}_V$ fixed).

TABLE A.1. Kolmogorov–Smirnov test results: number of null hypotheses rejected and median p-values.

							Power(1.5)			
			Exponential(2.0)		Rayleigh(0.3)		Full Sample		95% Sample	
Δ	T	N	EA	SPA	EA	SPA	EA	SPA	EA	SPA
2%	100	3	0	0	0	0	0	0	0	0
			1.00	0.64	1.00	0.57	0.97	0.78	0.99	0.81
		5	0	0	0	0	0	0	0	0
			1.00	0.60	1.00	0.45	0.98	0.71	1.00	0.68
		10	0	0	0	0	0	0	0	0
			1.00	0.60	1.00	0.33	0.93	0.49	1.00	0.50
	300	3	0	2	0	7	0	63	0	0
			1.00	0.21	1.00	0.15	0.90	0.42	1.00	0.39
		5	0	3	0	108	0	0	0	0
			1.00	0.19	1.00	0.08	0.78	0.24	1.00	0.23
		10	0	2	0	834	0	62	0	63
			1.00	0.15	1.00	0.04	0.56	0.09	0.99	0.09
5%	100	3	0	544	0	919	0	120	0	80
			1.00	0.05	1.00	0.02	0.78	0.11	0.96	0.12
		5	0	261	0	977	0	437	0	417
			1.00	0.08	1.00	0.02	0.71	0.06	0.95	0.05
		10	0	93	0	947	0	895	0	905
			1.00	0.10	1.00	0.02	0.53	0.02	0.96	0.03
	300	3	0	1000	0	1000	1	1000	0	1000
			0.98	0.00	1.00	0.00	0.33	0.00	0.85	0.00
		5	0	1000	0	1000	7	1000	0	1000
			1.00	0.00	1.00	0.00	0.20	0.00	0.81	0.00
		10	0	1000	0	1000	175	1000	0	1000
			1.00	0.00	1.00	0.00	0.08	0.00	0.77	0.00
10%	100	3	0	1000	0	1000	0	1000	0	1000
			1.00	0.00	1.00	0.00	0.44	0.00	0.81	0.00
		5	0	1000	0	1000	0	1000	0	1000
			1.00	0.00	1.00	0.00	0.32	0.00	0.79	0.00
		10	0	1000	0	1000	0	1000	0	1000
			1.00	0.01	1.00	0.00	0.26	0.00	0.80	0.00
	300	3	0	1000	0	1000	211	1000	0	1000
			0.99	0.00	0.99	0.00	0.08	0.00	0.50	0.00
		5	0	1000	0	1000	790	1000	0	1000
			1.00	0.00	1.00	0.00	0.04	0.00	0.41	0.00
		10	0	1000	0	1000	998	1000	0	1000
			1.00	0.00	1.00	0.00	0.01	0.00	0.44	0.00

TABLE A.2. Anderson–Darling test results: number of null hypotheses rejected and median p-values.

							Power(1.5)			
			Exponential(2.0)		Rayleigh(0.3)		Full Sample		95% Sample	
Δ	T	N	EA	SPA	EA	SPA	EA	SPA	EA	SPA
	100	3	0	0	0	0	0	0	0	0
			0.83	0.45	0.83	0.38	0.73	0.42	0.78	0.42
		5	0	0	0	0	0	0	0	0
			0.84	0.42	0.84	0.29	0.71	0.27	0.79	0.27
		10	0	0	0	0	0	1	0	0
			0.85	0.37	0.85	0.20	0.65	0.11	0.78	0.11
	300	3	0	14	0	224	0	63	0	90
			0.83	0.08	0.84	0.06	0.61	0.07	0.75	0.07
		5	0	25	0	1000	0	1000	0	1000
			0.84	0.07	0.85	0.02	0.51	0.02	0.75	0.02
		10	0	988	0	1000	0	1000	0	1000
			0.84	0.04	0.86	0.01	0.34	0.00	0.72	0.00
5%	100	3	0	973	0	1000	0	1000	0	1000
			0.81	0.03	0.82	0.01	0.55	0.02	0.69	0.01
		5	0	987	0	1000	0	1000	0	1000
			0.83	0.03	0.84	0.00	0.48	0.00	0.68	0.00
		10	0	1000	0	1000	0	1000	0	1000
			0.84	0.03	0.86	0.01	0.36	0.00	0.67	0.00
	300	3	0	1000	0	1000	1	1000	0	1000
			0.80	0.00	0.83	0.00	0.25	0.00	0.56	0.00
		5	0	1000	0	1000	3	1000	0	1000
			0.82	0.00	0.85	0.00	0.15	0.00	0.52	0.00
		10	0	1000	0	1000	161	1000	0	1000
			0.81	0.00	0.86	0.00	0.07	0.00	0.49	0.00
10%	100	3	0	1000	0	1000	0	1000	0	1000
			0.81	0.00	0.81	0.00	0.35	0.00	0.57	0.00
		5	0	1000	0	1000	0	1000	0	1000
			0.82	0.00	0.83	0.00	0.28	0.00	0.56	0.00
		10	0	1000	0	1000	0	1000	0	1000
			0.82	0.00	0.85	0.00	0.20	0.00	0.56	0.00
	300	3	0	1000	0	1000	93	1000	0	1000
			0.79	0.00	0.80	0.00	0.08	0.00	0.34	0.00
		5	0	1000	0	1000	748	1000	0	1000
			0.79	0.00	0.82	0.00	0.04	0.00	0.31	0.00
		10	0	1000	0	1000	1000	1000	0	1000
			0.77	0.00	0.84	0.00	0.02	0.00	0.29	0.00

TABLE A.3. Estimates of optimal reserve price for distributions considered.

			Exponential(2.0)		Rayleigh(0.3)		Power(1.5)	
Δ	T	N	EA	SPA	EA	SPA	EA	SPA
2%	100	3	0.36172	0.35386	0.30644	0.29442	0.54552	0.53260
			(0.02834)	(0.02836)	(0.01279)	(0.01332)	(0.01996)	(0.02011)
		5	0.35935	0.35225	0.30576	0.29400	0.54317	0.53115
			(0.02124)	(0.02176)	(0.01006)	(0.01036)	(0.01687)	(0.01722)
		10	0.35962	0.35177	0.30361	0.29226	0.54362	0.53242
			(0.01705)	(0.01741)	(0.00758)	(0.00769)	(0.01197)	(0.01222)
	300	3	0.35960	0.35399	0.30358	0.29228	0.54329	0.53109
			(0.01725)	(0.01729)	(0.00806)	(0.00833)	(0.01339)	(0.01360)
		5	0.35962	0.35386	0.30297	0.29158	0.54390	0.53214
			(0.01392)	(0.01416)	(0.00630)	(0.00651)	(0.01058)	(0.01082)
		10	0.35947	0.35248	0.30201	0.29091	0.54316	0.53204
			(0.01077)	(0.01092)	(0.00465)	(0.00477)	(0.00790)	(0.00807)
5%	100	3	0.36769	0.34272	0.30655	0.27589	0.54482	0.51530
			(0.02548)	(0.02633)	(0.01123)	(0.01287)	(0.01878)	(0.01961)
		5	0.36034	0.34004	0.30529	0.27559	0.54334	0.51565
			(0.02122)	(0.02290)	(0.00939)	(0.01005)	(0.01521)	(0.01636)
		10	0.35812	0.33392	0.30395	0.28233	0.54367	0.52039
			(0.01620)	(0.01766)	(0.00746)	(0.00738)	(0.01162)	(0.01231)
	300	3	0.36602	0.34388	0.30409	0.27421	0.54371	0.51446
			(0.01528)	(0.01808)	(0.00666)	(0.00785)	(0.01147)	(0.01240)
		5	0.36058	0.34134	0.30281	0.27337	0.54308	0.51561
			(0.01296)	(0.01420)	(0.00575)	(0.00637)	(0.00961)	(0.01054)
		10	0.35939	0.33581	0.30164	0.28024	0.54352	0.52021
			(0.00992)	(0.01108)	(0.00453)	(0.00447)	(0.00755)	(0.00814)
10%	100	3	0.34190	0.32344	0.30666	0.24002	0.54478	0.49002
			(0.02410)	(0.04535)	(0.00939)	(0.01042)	(0.01575)	(0.01929)
		5	0.35254	0.29473	0.30449	0.30515	0.54350	0.49393
			(0.01881)	(0.02400)	(0.00893)	(0.00869)	(0.01359)	(0.01633)
		10	0.35725	0.30890	0.30376	0.27877	0.54286	0.50920
			(0.01514)	(0.01474)	(0.00730)	(0.00702)	(0.01085)	(0.01139)
	300	3	0.34444	0.34183	0.30415	0.23794	0.54435	0.48953
			(0.01438)	(0.03260)	(0.00582)	(0.00650)	(0.00918)	(0.01202)
		5	0.35477	0.29694	0.30356	0.25809	0.54352	0.49414
			(0.01170)	(0.01589)	(0.00592)	(0.00564)	(0.00822)	(0.01001)
		10	0.35847	0.30951	0.30190	0.27750	0.54325	0.50965
			(0.01024)	(0.00958)	(0.00459)	(0.00439)	(0.00769)	(0.00812)

TABLE A.4. Expected revenues for an auction with $(N+1)$ bidders.

			Exponential(2.0)		Rayleigh(0.3)		Power(1.5)	
Δ	T	N	EA	SPA	EA	SPA	EA	SPA
2%	100	3	0.40389	0.38176	0.42201	0.40011	0.70419	0.68135
			(0.02161)	(0.02150)	(0.01375)	(0.01368)	(0.01771)	(0.01765)
		5	0.51567	0.49433	0.49567	0.47507	0.79635	0.77464
			(0.01815)	(0.01810)	(0.01208)	(0.01202)	(0.01231)	(0.01215)
		10	0.66294	0.64255	0.58881	0.56937	0.88708	0.86608
			(0.01530)	(0.01524)	(0.01009)	(0.01002)	(0.00723)	(0.00696)
	300	3	0.40179	0.38137	0.42147	0.40093	0.70220	0.68130
			(0.01235)	(0.01230)	(0.00848)	(0.00846)	(0.01123)	(0.01119)
		5	0.51343	0.49339	0.49431	0.47459	0.79561	0.77526
			(0.01212)	(0.01209)	(0.00768)	(0.00765)	(0.00783)	(0.00776)
		10	0.66169	0.64216	0.58837	0.56940	0.88621	0.86636
			(0.00938)	(0.00935)	(0.00625)	(0.00621)	(0.00470)	(0.00457)
5%	100	3	0.40024	0.35122	0.42267	0.37290	0.70459	0.65403
			(0.01990)	(0.01963)	(0.01296)	(0.01286)	(0.01798)	(0.01792)
		5	0.51337	0.46582	0.49418	0.44828	0.79810	0.75091
			(0.01842)	(0.01828)	(0.01166)	(0.01151)	(0.01238)	(0.01227)
		10	0.66170	0.61657	0.58751	0.54517	0.89038	0.84838
			(0.01478)	(0.01469)	(0.00970)	(0.00948)	(0.00708)	(0.00695)
	300	3	0.40078	0.35251	0.42134	0.37261	0.70342	0.65491
			(0.01344)	(0.01331)	(0.00791)	(0.00783)	(0.01050)	(0.01049)
		5	0.51318	0.46627	0.49366	0.44813	0.79679	0.75129
			(0.01166)	(0.01157)	(0.00738)	(0.00723)	(0.00778)	(0.00776)
		10	0.66163	0.61696	0.58750	0.54509	0.88881	0.84861
			(0.00925)	(0.00920)	(0.00584)	(0.00565)	(0.00432)	(0.00429)
10%	100	3	0.39705	0.30496	0.42193	0.33210	0.70585	0.61499
			(0.01986)	(0.01942)	(0.01154)	(0.01110)	(0.01707)	(0.01748)
		5	0.51181	0.42580	0.49312	0.41570	0.80060	0.72162
			(0.01850)	(0.01819)	(0.00984)	(0.00935)	(0.01173)	(0.01221)
		10	0.66087	0.58300	0.58670	0.52055	0.89304	0.83219
			(0.01374)	(0.01375)	(0.00773)	(0.00736)	(0.00617)	(0.00671)
	300	3	0.39808	0.30498	0.42118	0.33185	0.70452	0.61582
			(0.01255)	(0.01229)	(0.00722)	(0.00677)	(0.01021)	(0.01043)
		5	0.51236	0.42598	0.49372	0.41622	0.79869	0.72184
			(0.01156)	(0.01133)	(0.00661)	(0.00610)	(0.00724)	(0.00745)
		10	0.66136	0.58350	0.58746	0.52088	0.89080	0.83247
			(0.00878)	(0.00874)	(0.00535)	(0.00484)	(0.00413)	(0.00439)

Reference

de Boor, C. (2001), *A Practical Guide to Splines*, Revised edition. Springer, New York. [1]

Co-editor Elie Tamer handled this manuscript.

Manuscript received 21 October, 2011; final version accepted 1 August, 2016; available online 14 September, 2016.