Econometrica Supplementary Material

SUPPLEMENT TO “A THEORY OF SIMPLICITY IN GAMES AND MECHANISM DESIGN”

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APPENDIX B: OMITTED PROOFS

THIS SUPPLEMENT contains the proofs of the lemmas used in the proofs of the main theorems in Appendix A, as well as the full proofs of Theorem 6 and Lemma 1 from the main text.

B.1. *Proofs of Lemmas for Theorem 2*

PROOF OF LEMMA A.1: In order to show that there is no OSS mechanism that is equivalent to Γ, suppose, by way of contradiction, that there is such mechanism with game $\tilde{\Gamma}$ and a profile of OSS strategic plans. Let $\tilde{S}$ be the profile of strategies in $\tilde{\Gamma}$ induced by the strategic plans; by Theorem 1, this profile is obviously dominant.

The proof proceeds in a series of steps, which we label 1.1–1.6. (The labeling $k.1 - k.6$ is used because, after proving the result for $k = 1$, we use analogues of these steps to prove Lemma A.2 for arbitrary k.)

Step 1.1. In $\tilde{\Gamma}$, the first mover must be i, and x must be guaranteeable for i. Furthermore, at the empty history, w and z are not guaranteeable for i, but there is a unique action after which w and z are possible. This action is taken by all types of player i that rank either w or z first; we call this action i's focal action.

Proof of Step 1.1. First notice that i must be the first mover. Indeed, in mechanism Γ, agent j receives α_j if and only if agent i prefers x to w and z. Assume that, under $\tilde{\Gamma}$, agent j moves first. Something must be guaranteeable for agent j at this history, say λ.[41] If $\lambda = \alpha_j$, then we have non-equivalence when j prefers α_j the most and agent i does not prefer x to w and z. If $\lambda \neq \alpha_j$, then, we have non-equivalence when j prefers λ the most and i prefers x to w and z. Therefore, the first mover cannot be j. As the same argument works for agent ℓ, the first mover must be i.

Second, note that equivalence implies that i obtains x for any preference profile such that i prefers x the most, and therefore, x is guaranteeable at the first move in $\tilde{\Gamma}$. Analogously, w and z must be possible but not guaranteeable for i at the first move. To see that w cannot be guaranteeable, note that if it were, i would receive w for all preference profiles where she ranked it first, which is not the case in Γ, and so equivalence is violated; the same holds for z. By equivalence, both w and z are possible for i, that is, $w, z \in P_i(h)$.

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[41] That something must be guaranteeable follows because each history has at least two actions, and in any OSP game, there can be at most one action such that there is some payoff that is possible, but not guaranteeable (see the proof of Theorem 5).

Further, there must be a unique action a^* such that $w, z \in P_i((h, a^*))$. If there were two actions a_1, a_2 such that w were possible after both, then any type that prefers w the most would have no obviously dominant action, since w is not guaranteeable; the same holds for z. Therefore, each of w and z are possible after exactly one action, label them a_w and a_z. If $a_w \neq a_z$, then any type that ranks w first and z second would have no obviously dominant action.[42] Therefore, $a_w = a_z$; we call this action i's focal action. Since w and z are possible following only the focal action, all types that rank either w or z first must select it. This completes the proof of Step 1.1.

Step 1.2. In $\tilde{\Gamma}$, at the history following the first focal action by i, agent j moves. At this history, both $\tilde{x}$ and x are guaranteeable for j, while a is not guaranteeable. Further, there is a unique action after which a is possible, and this action is taken by all types of j who rank a first; we call this action j's focal action.

Proof of Step 1.2. Since, per Step 1.1, both w and z are possible for i following the focal action, the focal action cannot lead to a terminal history, and so there must be an agent who moves. We start by showing that the mover must be j. Note that in Γ, agent ℓ receives β_ℓ if and only if agent i prefers either w or z to x, and agent j prefers $\tilde{x}$ the most out of $\{x, \tilde{x}, a\}$. Suppose that i prefers either w or z to x, so that i follows the focal action at the initial history. By the same logic as in Step 1.1, if agent ℓ is the next mover, she must be able to guarantee some payoff, say γ. If $\gamma = \beta_\ell$, this would lead to a non-equivalence when ℓ ranks γ first and j ranks x first. If $\gamma \neq \beta_\ell$, then we have a non-equivalence when ℓ ranks γ first and j ranks $\tilde{x}$ first. Therefore, ℓ cannot be the next mover, and neither can be i (as i just moved) and so it must be j.

The equivalence of Γ and $\tilde{\Gamma}$ implies that for any profile such that i prefers w or z over x and j prefers x the most, j receives x. Because, per Step 1.1, all types of i take the focal action in $\tilde{\Gamma}$, we conclude that following i's focal action, j must be able to guarantee himself x. The same argument applies for $\tilde{x}$. Similarly, equivalence implies that there must be an action for j such that a is possible. Outcome a cannot be guaranteeable for j, because if it were, then j would receive a for all preference profiles where i ranks w or z first and j ranks a first, which is not the case in Γ. By an argument similar to Step 1.1, there cannot be any other actions after which a is possible, and all types of j that rank a first must select this action. We label this action j's focal action.

Step 1.3. In $\tilde{\Gamma}$, following i's focal action and j's focal action, there might be any finite number of consecutive histories at which i and j move. At these histories where i moves, i can clinch x, but neither w nor z is guaranteeable, and there is a unique action (the focal action) after which w and z are possible and that is taken by all types of i that rank w or z first. At these histories where j moves, both $\tilde{x}$ and x are guaranteeable, but a is not guaranteeable, and there is a unique action (the focal action) after which a is possible and is taken by all types of j that rank a first. Following this sequence of focal actions, agent ℓ moves.

Proof of Step 1.3. Since, per Step 1.2, a is possible, but not guaranteeable following j's focal action, the focal action cannot lead to a terminal history, and so must lead to a history at which an agent moves. As j just moved, the next mover must be either i or ℓ. If the next mover is i, as the history is on-path for all types of i who prefer w or z over x, the OSS property of $\tilde{\Gamma}$ implies that either x or else both w and z are clinchable for i. Equivalence implies that neither w nor z can be clinchable for i: if w were clinchable,

[42] Since w is not guaranteeable and z is not possible after a_w, the worst case from any strategy that selects a_w is strictly worse than z, which is possible from a_z. Similarly, since w is not possible following a_z, the worst case is strictly worse than w, which is possible from a_w. Note that an analogous argument would apply to any type that ranks z first and w second.

then i receives w for all profiles such that i prefers w the most and j prefers a the most, which is not the case in Γ; an analogous argument applies for z. Therefore, x must be clinchable. Furthermore, w and z are possible but not guaranteeable for i, and so, as in Step 1.1, OSP implies that there is a unique action after which both w and z are possible, and all types that rank either w or z first take this action (note that these types must have taken the focal action at i's initial move, and so are on-path); we call this action the focal action.

Following the focal action by i, the next mover must be j or ℓ. If it is j, then an analogous argument as for i shows that this agent must have both $x, \tilde{x}$ clinchable, and that there must be a unique action after which a is possible but not guaranteeable; we call it the focal action.

Following j's focal action, the next move is by i or ℓ. If it is by i, then the above argument applies again. We might then have a sequence of moves by i and j to which the above two arguments apply. As the game is finite and at the end of every focal action in the sequence more than one outcome is possible, the focal path of the game must lead to a history at which ℓ is called to play. This proves Step 1.3.

Step 1.4. In $\tilde{\Gamma}$, at ℓ's move following the sequence of focal actions described in Step 1.3, both $\tilde{a}$ and a are guaranteeable for ℓ, while neither c nor x is guaranteeable. There is also a unique action (the focal action) after which c and x are possible for ℓ. This action is taken by all types of ℓ that rank c first.

Proof of Step 1.4. Using arguments similar to Step 1.2, equivalence implies that at ℓ's move, both $\tilde{a}$ and a are guaranteeable for ℓ, while neither c nor x is guaranteeable, but both c and x are possible following a unique action that is taken by all types of agent ℓ that rank c first. Since c is not guaranteeable, this action cannot lead to a terminal history. Since c is possible following only the focal action, all types of ℓ that rank c first must select this action. This proves Step 1.4.

Step 1.5. In $\tilde{\Gamma}$, following the above sequence of focal actions that ends with the first focal action by ℓ, there might be any finite number of consecutive histories at which j and ℓ move. Each of these histories has a unique action (the focal action) after which a is possible for j's moves, and c and x are possible for ℓ's moves. All types of j that rank a first and all types of ℓ that rank c first take their respective focal actions. Following this sequence of focal actions, the next mover is i.

Proof of Step 1.5. Since there are multiple possible outcomes for k following her focal action, the focal action cannot lead to a terminal history. As k just moved, the next mover must be either i or j. First consider the case in which j moves next. The OSS property implies that either both x and $\tilde{x}$ are clinchable for j, or a is clinchable for j. Consider the latter case. If this were true, then under a preference profile where i prefers w most and z second, j prefers a most, and ℓ prefers c most, j would receive a, which is not the case in Γ. Therefore, j must be able to clinch x and $\tilde{x}$. By equivalence, a must be possible for j, but not guaranteeable, and so once again there must be a unique focal action after which a is possible and that is taken by all types of j that prefer a the most (note that all of these types have passed at j's prior moves, and so are on-path). Following the focal action, the next mover is i or ℓ. If it is ℓ, then an analogous argument implies that ℓ must be able to clinch a and $\tilde{a}$, with c possible but not guaranteeable following a unique focal action. There may again be a sequence of moves by j and ℓ for which this argument can be repeated. As the game is finite and at the end of every focal action more than one outcome is possible, the focal path must lead to a history at which i is called to play. This proves step 1.5.

Step 1.6. In $\tilde{\Gamma}$, at i's move following the sequence of focal actions described in Step 1.5, x is not clinchable for i.[43] *At this move, there is a unique action (the focal action) after which w is possible for i; the focal action is also the unique action after which x is possible for i. This focal action is taken by all types of i that rank w first.*

Proof of Step 1.6. By way of contradiction, suppose x is clinchable for i. Then OSP implies that in the continuation game following i's clinching of x, there must be a terminal history at which j receives a: if there were not, then the type of j that prefers a the most and x second would have no obviously dominant action at j's prior moves. At this terminal history, agent ℓ must be assigned something other than x (which was assigned to i) or a (which was assigned to j). But then, the type of ℓ that prefers x the most and a second has no obviously dominant action at ℓ's prior moves, which is a contradiction.[44]

An analogous argument to that which showed that there is a unique action after which w is possible for i in Step 1.1, tells us that there is a unique action (the focal action) after which w is possible for i. By OSP, types of i ranking w first take this action. An analogous argument shows that the focal action is the unique action after which x is possible.

Finishing the proof for $k = 1$.

As the previous step shows that x is not clinchable at the move of i considered there, OSS implies that both w and z must be clinchable for i. This implies that for preference profiles such that i ranks w first and x second, j ranks a first, and k ranks c first, agent i is assigned w. However, under such profiles in Γ, i receives x, which is a contradiction to equivalence. *Q.E.D.*

PROOF OF LEMMA A.2: Take any $k \geq 2$. By way of contradiction, suppose that $\tilde{\Gamma}^{(k)}$ with a profile of strategic plans is a k-step simple mechanism equivalent to $\Gamma^{(k)}$ with greedy strategic plans. The proof begins by repeating steps 1.1–1.6 from the proof of Lemma A.1 above, with the only change being that $\Gamma^{(k)}$ plays the role of Γ and $\tilde{\Gamma}^{(k)}$ plays the role of $\tilde{\Gamma}$. Then, we continue with the addition of steps $k'.3$–$k'.6$ for $k' = 2, 3, \ldots, k$. Each step $k'.3$–$k'.6$ is analogous to the corresponding step 1.3–1.6 from above, except that $a^{(k)}$ plays the role of a, $\tilde{a}^{(k)}$ plays the role of $\tilde{a}$, and $z^{(k)}$ plays the role of z. Finally, the proof for arbitrary k concludes with a final step that is the direct analogue of the finishing step for $k = 1$, except that we apply k-step simplicity instead of OSS. *Q.E.D.*

B.2. *Proofs of Lemmas for Theorem 5*

PROOF OF LEMMA A.3: Let Γ be a millipede game. For a set X of payoffs of agent i and a type $\succ_i$, let $\mathrm{Top}(\succ_i, X)$ be the best payoff in X according to preferences $\succ_i$. Consider some profile of greedy strategies $(S_i(\cdot))_{i\in\mathcal{N}}$. If $\mathrm{Top}(\succ_i, C_i(h)) = \mathrm{Top}(\succ_i, P_i(h))$, then clinching a top payoff is obviously dominant at h. What remains to be shown is if $\mathrm{Top}(\succ_i, C_i(h)) \neq \mathrm{Top}(\succ_i, P_i(h))$, then passing is obviously dominant at h.

Assume that there exists a history h that is on the path of play for type $\succ_i$ when following $S_i(\succ_i)$ such that $\mathrm{Top}(\succ_i, C_i(h)) \neq \mathrm{Top}(\succ_i, P_i(h))$, yet passing is not obviously dominant at h; further, let h be any earliest such history for which this is true. To shorten notation, let $x_P(h) = \mathrm{Top}(\succ_i, P_i(h))$, $x_C(h) = \mathrm{Top}(\succ_i, C_i(h))$, and let $x_W(h)$ be the worst possible payoff from passing and continuing to follow $S_i(\succ_i)$ at all future nodes.

[43] The argument shows that x not only is not clinchable for i but also not guaranteeable.

[44] Note that by equivalence, x must be possible for ℓ at these prior moves, since in Γ, k receives x for type profiles such that i ranks w first, j ranks a first, and ℓ ranks x first.

First, note that $x_W(h) \succsim_i x_W(h')$ for all $h' \subsetneq h$ such that $i_{h'} = i$. Since passing is obviously dominant at all $h' \subsetneq h$, we have $x_W(h') \succsim_i x_C(h')$, and together, these imply that $x_W(h) \succsim_i x_C(h')$ for all such h'. At h, since passing is not obviously dominant and all other actions are clinching actions, we have $x_C(h) \succ_i x_W(h)$; further, since $\mathrm{Top}(\succ_i, C_i(h)) \neq \mathrm{Top}(\succ_i, P_i(h))$, there must be some $x' \in P_i(h) \setminus C_i(h)$ such that $x' \succ_i x_C(h) \succ_i x_W(h)$. The above implies that $x' \succ_i x_C(h) \succ_i x_C(h')$ for all $h' \subsetneq h$ such that $i_{h'} = i$.

Let $X_0 = \{x' : x' \in P_i(h) \text{ and } x' \succ_i x_C(h)\}$; in words, X_0 is a set of payoffs that are possible at all $h' \subseteq h$, and are strictly better than anything that was clinchable at any $h' \subseteq h$ (and therefore have never been clinchable themselves). Order the elements in X_0 according to $\succ_i$, and without loss of generality, let $x_1 \succ_i x_2 \succ_i \cdots \succ_i x_M$.

Consider a path of play starting from h that is consistent with $S_i(\succ_i)$ and ends in a terminal history $\bar{h}$ at which i receives $x_W(h)$. For every $x_m \in X_0$, let h_m denote the earliest history on this path such that $x_m \notin P_i(h_m)$ and either (i) $i_h = i$ or (ii) h_m is terminal. Note that because i is ultimately receiving payoff $x_W(h)$, such a history h_m exists for all $x_m \in X_0$. Let $\hat{h}_{-m}$ be the earliest history at which i moves and at which all payoffs strictly preferred to x_m are no longer possible.

CLAIM: *For all $x_m \in X_0$ and all $h' \subseteq \bar{h}$, we have $x_m \notin C_i(h')$.*

Proof of claim. First, note that $x_m \notin C_i(h')$ for any $h' \subseteq h$ by construction. We show that $x_m \notin C_i(h')$ at any $\bar{h} \supseteq h' \supset h$ as well. Start by considering $m = 1$, and assume $x_1 \in C_i(h')$ for some $\bar{h} \supseteq h' \supset h$. By definition, $x_1 = \mathrm{Top}(\succ_i, P_i(h))$; since $h' \supset h$ implies that $P_i(h') \subseteq P_i(h)$, we have that $x_1 = \mathrm{Top}(\succ_i, P_i(h'))$ as well. Since $x_1 \in C_i(h')$ by supposition, greedy strategies direct i to clinch x_1, which contradicts that she receives $x_W(h)$.[45]

Now, consider an arbitrary m, and assume that for all $m' = 1, \dots, m-1$, payoff $x_{m'}$ is not clinchable at any $h' \subseteq \bar{h}$, but x_m is clinchable at some $h' \subseteq \bar{h}$. Let $x_{m'} \succ_i x_m$ be a payoff that becomes impossible at $\hat{h}_{-m} \subseteq \bar{h}$; if such payoff does not exist, then the argument of the paragraph above applies. There are two cases:

Case (i): $h' \subsetneq \hat{h}_{-m}$. This is the case in which x_m is clinchable while there is some strictly preferred payoff $x_{m'} \succ_i x_m$ that is still possible. By assumption, all $\{x_1, \dots, x_{m-1}\}$ are previously unclinchable at $\hat{h}_{-m}$, and so $x_{m'}$ is previously unclinchable at $\hat{h}_{-m}$. By definition of a millipede game (part 3), we have $x_m \in C_i(\hat{h}_{-m})$. Thus, x_m is the best remaining payoff at $\hat{h}_{-m}$, and is clinchable, and so greedy strategies direct i to clinch x_m at $\hat{h}_{-m}$, which contradicts that she receives $x_W(h)$ (as in footnote 45, the argument still applies if $\hat{h}_{-m}$ is a terminal history).

Case (ii): $h' \supseteq \hat{h}_{-m}$. In this case, x_m becomes clinchable after all strictly preferred payoffs are no longer possible. Thus, again, greedy strategies instruct i to clinch x_m, which contradicts that she is receiving $x_W(h)$. *Q.E.D.*

To finish the proof of Lemma A.3, let $\hat{h} = \max\{h_1, h_2, \dots, h_M\}$ (ordered by $\subset$); in words, $\hat{h}$ is the earliest history on the path to $\bar{h}$ at which no payoffs in X_0 are possible any longer. Let $\hat{x}$ be a payoff in X_0 that becomes impossible at $\hat{h}$. The claim shows that no $x \in X_0$ is clinchable at any $h' \subseteq \hat{h}$, and so we can further conclude that $\hat{x}$ is previously unclinchable

[45] If h' is terminal, then, even though i takes no action at h', according to our notational convention we define $C_i(h') = \{x_1\}$, which also contradicts that she receives payoff $x_W(h)$.

at $\hat{h}$. Therefore, by part 3 in the definition of a millipede game, $x_C(h) \in C_i(\hat{h})$. Since $x_C(h)$ is the best possible remaining payoff at $\hat{h}$, greedy strategies direct i to clinch $x_C(h)$, which contradicts that she receives $x_W(h)$ (as in footnote 45, the argument still applies if $\hat{h}$ is a terminal history).

PROOF OF LEMMA A.4: Ashlagi and Gonczarowski (2018) briefly mentioned this result in a footnote; here, we provide the straightforward proof for completeness. That every OSP game is equivalent to an OSP game with perfect information is implied by our more general Theorem 4. To show that we can furthermore assume that Nature moves at most once, as the first mover, consider a perfect-information game Γ. Let $\mathcal{H}_{\text{nature}}$ be the set of histories h at which Nature moves in Γ. Consider a modified game Γ' in which, at the empty history, Nature chooses actions from $\times_{h\in\mathcal{H}_{\text{nature}}} A(h)$. After each of Nature's initial moves, we replicate the original game, except at each history h at which Nature is called to play, we delete Nature's move and continue with the subgame corresponding to the action Nature chose from $A(h)$ at $\emptyset$. Again, note that for any agent i and history h at which i is called to act, the support of possible outcomes at h in Γ' is a subset of the support of possible outcomes at the corresponding history in Γ (where the corresponding histories are defined by mapping the $A(h)$ component of the action taken at $\emptyset$ by Nature in Γ' as an action made by Nature at h in game Γ). When the support of possible outcomes shrinks, the worst-case outcome from any fixed strategy can only improve, while the best-case can only diminish, and so if a strategy was obviously dominant in Γ, the corresponding strategy will continue to be obviously dominant in Γ', and the two games will be equivalent. *Q.E.D.*

PROOF OF LEMMA A.5: For any history h, let $PnG_i(h) = P_i(h) \setminus G_i(h)$ (where "PnG" is shorthand for "possible but not guaranteeable"). Now, consider any h at which i moves, and assume that at h, there are (at least) two such actions $a_1^*, a_2^* \in A(h)$ as in the statement. We first claim that $PnG_i(h) \cap P_i(h_1^*) \cap P_i(h_2^*) = \emptyset$, where $h_1^* = (h, a_1^*)$ and $h_2^* = (h, a_2^*)$. Indeed, if not, then let x be a payoff in this intersection. By pruning, some type $\succ_i$ is following some strategy such that $S_i(\succ_i)(h) = a_1^*$ that results in a payoff of x at some terminal history $\bar{h} \supseteq (h, a_1^*)$. Note that $\text{Top}(\succ_i, P_i(h)) \neq x$, because otherwise a_1^* would not be obviously dominant for this type (since $x \notin G_i(h)$ and $x \in P_i(h_2^*)$). Thus, let $\text{Top}(\succ_i, P_i(h)) = y$. Note that $y \notin G_i(h)$ (or else it would not be obviously dominant for type $\succ_i$ to play a strategy such that x is a possible payoff). Further, we must have $y \in P_i(h_1^*)$ and $y \notin P_i(h_2^*)$. To see the former, note that if $y \notin P_i(h_1^*)$, then a_1^* is not obviously dominant for type $\succ_i$, which contradicts that $S_i(\succ_i)(h) = a_1^*$; given the former, if $y \in P_i(h_2^*)$, then once again a_1^* would not be obviously dominant for type $\succ_i$. Now, again by pruning, there must be some type $\succ_i'$ such that $S_i(\succ_i')(h) = a_2^*$ that results in payoff x at some terminal history $\bar{h} \supseteq (h, a_2^*)$. By similar reasoning as previously, $\text{Top}(\succ_i', P_i(h)) \neq x$, and so $\text{Top}(\succ_i', P_i(h)) = z$ for some $z \in P_i(h_2^*)$. Since $y \notin P_i(h_2^*)$, we have $z \neq y$, and we can as above conclude that $z \notin G_i(h)$. It is without loss of generality to consider a type $\succ_i'$ such that $\text{Top}(\succ_i', P_i(h) \setminus \{z\}) = y$. Note that, for this type, no action $a \neq a_2^*$ can obviously dominate a_2^* (since $z \notin G_i(h)$). Further, a_2^* itself is not obviously dominant for this type, since the worst case from a_2^* is strictly worse than y (since $y \notin P_i(h_2^*)$ and $z \notin G_i(h)$), while $y \in P_i(h_1^*)$. Therefore, this type has no obviously dominant action at h, which is a contradiction.

Thus, $PnG_i(h) \cap P_i(h_1^*) \cap P_i(h_2^*) = \emptyset$, which means there must be distinct x, y such that (i) $x, y \in PnG_i(h)$, (ii) $x \in P_i(h_1^*)$ but $x \notin P_i(h_2^*)$, and (iii) $y \in P_i(h_2^*)$ but $y \notin P_i(h_1^*)$. Next, for all types of agent i that reach h, it must be that $\text{Top}(\succ_i, P_i(h)) \neq x, y$. To see why, assume there were a type that reaches h such that $\text{Top}(\succ_i, P_i(h)) = x$. Then, by richness, there is a type that reaches h such that $\text{Top}(\succ_i, P_i(h) \setminus \{x\}) = y$. But, note that this type has no obviously dominant action at h. An analogous argument applies switching x with y.

Now, by pruning, there is some type that reaches h that plays a strategy such that $S_i(\succ_i)(h) = a_1^*$ and x is a possible payoff. Let $\text{Top}(\succ_i, P_i(h)) = z$ for this type, where, as just noted, $z \neq x, y$. The fact that $S_i(\succ_i)(h) = a_1^*$ implies that $z \in P_i(h_1^*)$ and $z \notin G_i(h)$; if either of these were false, it would not be obviously dominant for this type to play a strategy such that $S_i(\succ_i)(h) = a_1^*$ and x is a possible payoff. In other words, $z \in PnG(h)$ and $z \in P_i(h_1^*)$. Since we just showed that $PnG_i(h) \cap P_i(h_1^*) \cap P_i(h_2^*) = \emptyset$, we have $z \notin P_i(h_2^*)$. Finally, consider a type $\succ_i$ such that $\text{Top}(\succ_i, P_i(h)) = z$ and $\text{Top}(\succ_i, P_i(h) \setminus \{z\}) = y$. Note that this type has no obviously dominant action at h, which is a contradiction. *Q.E.D.*

PROOF OF LEMMA A.6: Given an OSP mechanism $(\Gamma, S_{\mathcal{N}})$, begin by using Lemma A.4 to construct an equivalent OSP game of perfect information in which Nature moves only at the initial history (if at all). Further, prune this game according to the obviously dominant strategy profile $S_{\mathcal{N}}$. With slight abuse of notation, we denote this pruned, perfect information mechanism by $(\Gamma, S_{\mathcal{N}})$. Consider some history h of Γ at which the mover is $i_h = i$. By Lemma A.5, all but at most one action (denoted a^*) in $A(h)$ satisfy $P_i((h, a)) \subseteq G_i(h)$; this means that any obviously dominant strategy for type $\succ_i$ that does not choose a^* guarantees the best possible outcome in $P_i(h)$ for type $\succ_i$. Define the set

$$\mathcal{S}_i(h) = \big\{S_i : S_i(h) \neq a^* \text{ and at all terminal } \bar{h} \text{ consistent with } S_i, \\ i \text{ receives the same payoff}\big\}.$$

In words, each $S_i \in \mathcal{S}_i(h)$ guarantees a unique payoff for i if she plays strategy S_i starting from history h, no matter what the other agents do.

We create a new game Γ' that is the same as Γ, except we replace the subgame starting from history h with a new subgame defined as follows. If there is an action a^* such that $P_i((h, a^*)) \nsubseteq G_i(h)$ in the original game (of which there can be at most one), then there is an analogous action a^* in the new game, and the subgame following a^* is exactly the same as in the original game Γ. Additionally, there are $M = |\mathcal{S}_i(h)|$ other actions at h, denoted $a_1, \dots, a_M$. Each a_m corresponds to one strategy $S_i^m \in \mathcal{S}_i(h)$, and following each a_m, we replicate the original game, except that at any future history $h' \supseteq h$ at which i is called on to act, all actions (and their subgames) are deleted and replaced with the subgame starting from the history (h', a'), where $a' = S_i^m(h')$ is the action that i would have played at h' in the original game had she followed strategy $S_i^m(\cdot)$. In other words, if i's strategy was to choose some action $a \neq a^*$ at h in the original game, then, in the new game Γ', we ask agent i to "choose" not only her current action, but all future actions that she would have chosen according to $S_i^m(\cdot)$ as well. By doing so, we have created a new game in which every action (except for a^*, if it exists) at h clinches some payoff x, and further, agent i is never called upon to move again.[46]

[46]More precisely, all of i's future moves are trivial moves in which she has only one possible action; hence, these histories may further be removed to create an equivalent game in which i is never called on to move again. Note that this only applies to the actions $a \neq a^*$; it is still possible for i to follow a^* at h and be called upon to make a non-trivial move again later in the game.

We construct strategies in Γ' that are the counterparts of strategies from Γ, so that for all agents $j \neq i$, they continue to follow the same action at every history as they did in the original game, and for i, at history h in the new game, she takes the action a_m that is associated with the strategy S_i^m in the original game. By definition, if all agents follow strategies in the new game analogous to their strategies from the original game, the same outcome is reached, and so Γ and Γ' are equivalent under their respective strategy profiles.

We must also show that if a strategy profile is obviously dominant for Γ, this modified strategy profile is obviously dominant for Γ'. To see why the modified strategy profile is obviously dominant for i, note that if her obviously dominant action in the original game was part of a strategy that guarantees some payoff x, she now is able to clinch x immediately, which is clearly obviously dominant; if her obviously dominant strategy was to follow a strategy that did not guarantee some payoff x at h, this strategy must have directed i to follow a^* at h. However, in Γ', the subgame following a^* is unchanged relative to Γ, and so i is able to perfectly replicate this strategy, which obviously dominates following any of the clinching actions at h in Γ'. In addition, the game is also obviously strategy-proof for all $j \neq i$ because, prior to h, the set of possible payoffs for j is unchanged, while for any history succeeding h where j is to move, having i make all of her choices earlier in the game only shrinks the set of possible outcomes for j, in the set inclusion sense. When the set of possible outcomes shrinks, the best possible payoff from any given strategy only decreases (according to j's preferences) and the worst possible payoff only increases, and so, if a strategy was obviously dominant in the original game, it will continue to be so in the new game. Repeating this process for every history h, we are left with a new game where, at each history, there are only clinching actions plus (possibly) one passing action, and further, every payoff that is guaranteeable at h is also clinchable at h, and i never moves again following a clinching action. This shows parts (i) and (ii). Part (iii) follows immediately from part (ii), due to greedy strategies and the pruning principle. *Q.E.D.*

PROOF OF LEMMA A.7: Let h be any earliest history where some agent i moves such that there is a previously unclinchable payoff z that becomes impossible at h (the case for terminal histories is dealt with separately below). This means that i moves at some strict subhistory $h' \subsetneq h$ and the following are true: (a) $z \notin P_i(h)$; (b) $z \in P_i(h')$ for all $h' \subsetneq h$ such that $i_h = i$; and (c) $z \notin C_i^{\subset}(h)$. Points (b) and (c) imply that z is possible at every $h' \subsetneq h$ at which i is called to move, but it is not clinchable at any of them; thus, for any type of agent i that ranks z first, any obviously dominant strategy has the agent choosing the unique passing action at all $h' \subsetneq h$.

We want to show that $C_i^{\subset}(h) \subseteq C_i(h)$. Towards a contradiction, assume that $C_i^{\subset}(h) \nsubseteq C_i(h)$, and let $x \in C_i^{\subset}(h)$ but $x \notin C_i(h)$. Consider a type $\succ_i$ that ranks z first and x second. By the previous paragraph, this type must be playing some strategy that passes at any $h' \subsetneq h$, and so h is on the path of play for type $\succ_i$. Since $z \notin P_i(h)$ and $x \notin C_i(h)$, by Lemma A.6, part (ii), the worst-case outcome from this strategy is some y that it is strictly worse than both z and x according to $\succ_i$. However, we also have $x \in C_i(h')$ for some $h' \subsetneq h$, and so the best-case outcome from clinching x at h' is x. This implies that passing is not obviously dominant, and thus Γ is not OSP, a contradiction.

Last, consider a terminal history $\bar{h}$. As above, let z be a payoff such that (a), (b), and (c) hold (replacing h with $\bar{h}$). Recall that for terminal histories, we define $C_i(\bar{h}) = \{y\}$, where y is the payoff that obtains at h for i. Towards a contradiction, assume that there is some $x \in C_i(h')$ for some $h' \subsetneq \bar{h}$ but $x \notin C_i(\bar{h})$. Note that (i) $z \neq y$, by (a); (ii) $z \neq x$, by (c); and (iii) $x \neq y$, since $x \notin C_i(\bar{h})$. In other words, x, y, and z are all distinct payoffs for i. Thus, consider the type $\succ_i$ that ranks z first, x second, and y third, followed by all other payoffs.

By (b) and (c), z is possible at every $h'' \subsetneq \bar{h}$ at which i moves, but is not clinchable at any such h''. Thus, any obviously dominant strategy for type $\succ_i$ must have agent i passing at all such histories. This implies that y is possible for this type. However, at h', i could have clinched x, and so the strategy is not obviously dominant, a contradiction. *Q.E.D.*

B.3. *Proof of Theorem 6*

Before proving the theorem, we first formally define a personal clock auction. Given some perfect-information game Γ, define outcome functions g as follows: $g_y(\bar{h}) \subseteq \mathcal{N}$ is the set of agents who are in the allocation $\bar{y}$ that obtains at terminal history $\bar{h}$ (i.e., $i \in g_y(\bar{h})$ if and only if $\bar{y}_i = 1$), and $g_{w,i}(\bar{h}) \in \mathbb{R}$ is the transfer to agent i at $\bar{h}$. The following definition of a personal clock auction is adapted from Li (2017). Note that the game is deterministic, that is, there are no moves by Nature.[47]

Γ is a *personal clock auction* if, for every $i \in \mathcal{N}$, at every earliest history h_i^* at which i moves, either *In-Transfer Falls*: there exists a fixed transfer $\bar{w}_i \in \mathbb{R}$, a going transfer $\tilde{w}_i : \{h_i : h_i^* \subseteq h_i\} \to \mathbb{R}$, and a set of "quitting actions" A^q such that

- For all terminal $\bar{h} \supset h_i^*$, either (i) $i \notin g_y(\bar{h})$ and $g_{w,i}(\bar{h}) = \bar{w}_i$ or (ii) $i \in g_y(\bar{h})$ and $g_{w,i}(\bar{h}) = \inf\{\tilde{w}_i(h_i) : h_i^* \subseteq h_i \subsetneq \bar{h}\}$.
- If $\bar{h} \supsetneq (h, a)$ for some $h \in \mathcal{H}_i$ and $a \in A^q$, then $i \notin g_y(\bar{h})$.
- $A^q \cap A(h_i^*) \neq \emptyset$.
- For all $h_i', h_i'' \in \{h_i \in \mathcal{H}_i : h_i^* \subseteq h_i\}$:
  - If $h_i' \subsetneq h_i''$, then $\tilde{w}_i(h_i') \geq \tilde{w}_i(h_i'')$.
  - If $h_i' \subsetneq h_i''$, $\tilde{w}_i(h_i') > \tilde{w}_i(h_i'')$, and there is no h_i''' such that $h_i' \subsetneq h_i''' \subsetneq h_i''$, then $A^q \cap A(h_i'') \neq \emptyset$.
  - If $h_i' \subsetneq h_i''$ and $\tilde{w}_i(h_i') > \tilde{w}_i(h_i'')$, then $|A(h_i') \setminus A^q| = 1$.
  - If $|A(h_i') \setminus A^q| > 1$, then there exists $a \in A(h_i')$ such that, for all $\bar{h} \supseteq (h_i', a)$, $i \in g_y(\bar{h})$;[48]

or, *Out-Transfer Falls*: as above, replacing every instance of "$i \in g_y(\bar{h})$" with "$i \notin g_y(\bar{h})$" and vice versa.

We now prove Theorem 6. As discussed in the main text, the first part of this theorem follows from our Corollary 1, Li (2017), and the construction of the one-step simple strategic collections for each agent that we now present. This construction also proves the second part of the theorem.

Let Γ be a personal clock auction. We present the construction and argument for in-transfer falls; the case of out-transfer falls is analogous. Consider any $h_i \in \mathcal{H}_i$ and simple-node set $\mathcal{H}_{i,h_i} = \{h' \in \mathcal{H}_i : h_i \subsetneq h'' \subsetneq h' \implies h'' \notin \mathcal{H}_i\}$, and define the strategic plan $S_{i,h_i}(h')$ at $h' \in \mathcal{H}_{i,h_i}$ as follows:

- If $\theta_i + \tilde{w}_i(h_i) > \bar{w}_i$ and $A(h_i) \setminus A^q \neq \emptyset$:

[47] We slightly simplify Definition 15 of Li (2017) by restricting it to perfect-information games: by Theorem 4, for any personal clock auction that satisfies Definition 15 of Li (2017), there is an equivalent mechanism that satisfies the definition we work with. This also applies to the minor correction provided by Li in a corrigendum available on his website; cf. footnote 48 for further details.

[48] The corrigendum issued by Li replaces this statement with one that says if there is more than one non-quitting action at h_i', there is a *continuation strategy* (rather than an action) that guarantees that $i \in g_y(\bar{h})$. The corrigendum also notes, though, that this change does not expand the set of implementable choice rules, because for any newly admissible mechanism, there is always an equivalent mechanism satisfying the original definition in which the agent reports her type at h_i' and does not move again. Thus, our notion of equivalence allows us to work directly with this simpler definition of personal clock auctions.

– *[Action at h_i]* Choose $S_{i,h_i}(h_i) = a \in A(h_i) \setminus A^q$; if it further holds that $|A(h_i) \setminus A^q| > 1$, then choose $S_{i,h_i}(h_i) = a \in A(h_i) \setminus A^q$ such that $i \in g_y(\bar{h})$ for all $\bar{h} \supseteq (h_i, a)$.
– *[Actions at next-histories]* For $h' \in \mathcal{H}_{i,h_i} \setminus \{h_i\}$, if there exists $a \in A(h') \cap A^q$, then $S_{i,h_i}(h') = a$ for some $a \in A(h') \cap A^q$. Else, $S_{i,h_i}(h') = a'$ for some $a' \in A(h')$ such that for all $\bar{h} \supseteq (h, a')$, $i \in g_y(\bar{h})$.

- Else, choose actions such that $S_{i,h_i}(h') \in A^q$ for all $h' \in \mathcal{H}_{i,h_i}$.

To show that this is a one-step simple strategic collection, first consider h_i such that $A(h_i) \setminus A^q = \emptyset$. Then the only actions available at h_i are quitting actions. Thus, the best and worst cases from any action are all $\bar{w}_i$, and one-step dominance holds. Second, consider $\theta_i + \tilde{w}_i(h_i) \leq \bar{w}_i$. Then, the worst case from quitting at h_i is a payoff of $\bar{w}_i$. Since the going transfer can only fall, the best case from playing a non-quitting action at h_i is at most $\theta_i + \tilde{w}_i(h_i) \leq \bar{w}_i$, and so again one-step dominance holds. Third, consider the remaining case in which $\theta_i + \tilde{w}_i(h_i) > \bar{w}_i$ and there exists some $a \in A(h_i) \setminus A^q$. There are two subcases:

First, if $|A(h_i) \setminus A^q| = 1$, then all other actions at h_i are quitting actions, and i's best-case and worst-case payoff from following any such action is $\bar{w}_i$. We must show that the worst case from the perspective of node h_i from following the specified strategic plan gives a weakly greater payoff than $\bar{w}_i$. For any next-history $h'_i \in \mathcal{H}_{i,h_i}$ at which there is a quitting action (i.e., $A(h'_i) \cap A^q \neq \emptyset$), the worst case from the perspective of h_i of following the strategic plan is $\bar{w}_i$. If there is no quitting action at h'_i (i.e., $A(h'_i) \cap A^q = \emptyset$), then, by construction of a personal clock auction, we have that (i) $\tilde{w}_i(h_i) = \tilde{w}_i(h'_i)$, and (ii) there exists an $a' \in A(h'_i)$ such that, for all $\bar{h} \supseteq (h'_i, a')$, we have $i \in g_y(\bar{h})$. Further, for any $h''_i \supsetneq h'_i$, $\tilde{w}_i(h''_i) = \tilde{w}_i(h'_i) = \tilde{w}_i(h_i)$, and so, for any $\bar{h} \supseteq (h'_i, a')$, $g_{w,i}(\bar{h}) = \tilde{w}_i(h_i)$. Therefore, the worst case from following the strategic plan from the perspective of h_i conditional on reaching any such h'_i is $\theta_i + \tilde{w}_i(h_i)$. In either case, the worst case from the strategic plan from the perspective of h_i is weakly better than taking any other action at h_i.

Second, if $|A(h_i) \setminus A^q| > 1$, then the strategic plan instructs i to follow the action $a \in A(h_i)$ such that $i \in g_y(\bar{h})$ for all $\bar{h} \supseteq (h_i, a)$; further, by construction of a personal clock auction, at any $\bar{h} \supseteq (h_i, a)$, we have $g_{w,i}(\bar{h}) = \tilde{w}_i(h_i)$. Since $\theta_i + \tilde{w}_i(h_i) > \bar{w}_i$, this is strictly preferred to the payoff from taking any quitting action at h_i, and since the going transfer cannot rise, it is also weakly preferable to taking any other non-quitting action at h_i.

B.4. *Proof of Lemma for Theorem 7*

PROOF OF LEMMA A.8: By way of contradiction, let $(\Gamma, S_{\mathcal{N},\mathcal{H}})$ be a millipede mechanism that satisfies (i)–(iii) at each history but is not monotonic. The failure of monotonicity implies that there exist an agent i, history $h^* \in \mathcal{H}_i$, history h that follows i's passing move at h^* that is either terminal or in $\mathcal{H}_i$ and such that i does not move between h^* and h, and payoffs x and y such that $x \in (P_i(h^*) \setminus C_i(h^*)) \setminus C_i(h)$ and $y \in C_i(h^*) \setminus C_i(h)$; in particular, $x \neq y$. Without loss of generality, assume that h^* is an earliest history at which monotonicity is violated in this way. This implies that $x \notin C_i(h')$ for any $h' \subseteq h^*$ such that $i_{h'} = i$.[49] In particular, any type $\succ_i$ of agent i that ranks payoff x first passes at any $h' \subseteq h^*$ at which this agent moves.

[49]If $x \in C_i(h')$ for some h', then, by monotonicity, at any next history $h'' \supsetneq h'$ following a pass where i moves, either $x \in C_i(h'')$ or $P_i(h') \setminus C_i(h') \subseteq C_i(h'')$. If the latter holds, then at h'', i has been offered to clinch everything that is possible for her, and so, by greediness, h is not on-path for any type of agent i, and we can construct an equivalent game in which monotonicity is not violated at h^*. Therefore, $x \in C_i(h'')$.

As $x, y \notin C_i(h)$ by the choice of these payoffs, there is some third payoff $z \neq x, y$ such that $z \in C_i(h)$. Let $\succ_i$ be such that $\succ_i$: $x, y, z \ldots$ and $\succ'_i$ be such that $\succ'_i$: $x, z, \ldots$; these types exist by richness, given that we are in a no-transfer environment. Ranking x first, these types are passing at all nodes $h' \subseteq h^*$ at which they move: $S_{i,h'}(\succ_i)(h') = S_{i,h'}(\succ'_i)(h') = a^*(h')$ where $a^*(h')$ denotes the passing action at h'.

We conclude the indirect argument by showing that neither of the following two cases is possible:

Case $y \notin P_i(h)$. If also $x \notin P_i(h)$, then $P_i(h)$ contains some $w \neq x, y$. If $x \in P_i(h)$, then $x \notin C_i(h)$ implies that $x \in P_i((h, a^*(h)))$ and by definition of a passing action, there is some $w \neq x$ such that $w \in P_i((h, a^*))$; furthermore, $w \neq y$ because $y \notin P_i(h)$. In either case, passing at h^* might lead to w, which is worse for $\succ_i$ than y, and i can clinch y at h^*; thus, $S_{i,h^*}(\succ_i)$, which passes at h^*, is not one-step dominant, a contradiction.

Case $y \in P_i(h)$. If $z \in P_i((h, a^*))$, then $x, y \notin C_i(h)$ implies that the worst case for type $\succ_i$ from passing at h^* is at best z, which is worse than clinching y at h^*. Therefore, the passing action $S_{i,h^*}(\succ_i)$ is not one-step dominant at h^* for $\succ_i$, a contradiction. We may thus assume that $z \notin P_i((h, a^*))$. Because $x \notin C_i(h)$, the assumptions of Lemma A.6 imply that x is not guaranteeable at h, and in particular it is not guaranteeable at $(h, a^*(h))$. Thus, the worst case for type $\succ'_i$ from passing at h is strictly worse than z; since $z \in C_i(h)$, this implies that $S_{i,h}(\succ'_i)$ clinches at h. Thus, $x \notin C_i(h)$ allows us to conclude that $x \notin P_i(h)$, as otherwise $S_{i,h}(\succ'_i)$ could not be clinching at h. Since $y \notin C_i(h)$ and $y \in P_i(h)$, we infer that $y \in P_i((h, a^*(h)))$. As at least two payoffs are possible following passing and $x \notin P_i(h)$, there is some $w \neq x, y$ that is possible at $(h, a^*(h))$ and hence also at h. As x is not possible and y is not clinchable at h, the worst case for type $\succ_i$ from the perspective of node h^* from following $S_{i,h^*}(\succ_i)$ is at best w, which is strictly worse than clinching y at h^*. Thus, $S_{i,h^*}(\succ_i)$ is not one-step dominant. *Q.E.D.*

B.5. *Proof of Lemma 1*

Recall that any strongly obviously dominant strategy is greedy. We first note the following lemmas. To state the lemmas, define $\hat{P}_i(h) = \{x \in P_i(h) : \nexists y \in P_i(h) \text{ s.t. } y \rhd_i x\}$ to be the set of possible payoffs for i at h that are undominated in $P_i(h)$.

LEMMA 1: *Let* Γ *be a pruned SOSP game. If a history* h *at which agent* i *moves is payoff-relevant, then* $|\hat{P}_i(h)| \geq 2$.

PROOF OF LEMMA 1: Assume not, and let $\hat{P}_i(h) = \{x\}$, where x is the unique undominated payoff at h.[50] In particular, $x \rhd_i x'$ for all $x' \in P_i(h)$, and $\text{Top}(\succ_i, P_i(h)) = x$ for all types of agent i. Because x is possible at h, there is an action $a \in A(h)$ such that $x \in P_i((h, a))$. Action a does not clinch x; indeed, if $P_i((h, a)) = \{x\}$, then greediness would imply that only actions clinching x are taken, and in a pruned game, h would not be payoff relevant. Thus, there is another $x' \in P_i((h, a))$ such that $x \succ_i x'$ for all types of agent i. Let $a' \neq a$ be an action at $A(h)$. If $x \in P_i((h, a'))$, then, analogously as for a, there is some other $x'' \in P_i((h, a'))$. It is then easy to check that neither a nor a' strongly obviously dominates the other. If $x \notin P_i((h, a'))$, then it would not be strongly obviously dominant (SOD, for shortness) for any type to take action a', which would contradict the game being pruned. *Q.E.D.*

Repeating this argument for every history between h' and h^* at which i moves delivers that $x \in C_i(h^*)$, which is a contradiction.

[50]There must be at least one undominated payoff, since $\trianglerighteq_i$ is transitive and the number of payoffs is finite.

LEMMA 2: *Let* (Γ, S) *be a pruned SOSP mechanism. Let* h_0^i *be any earliest history at which agent i is called to play. Then,* $|\hat{P}_i((h_0^i, a))| \leq 2$ *for all* $a \in A(h_0^i)$, *with equality for at most one* $a \in A(h_0^i)$.

PROOF OF LEMMA 2: Since h_0^i is the first time i is called to move, it is on-path for all types of agent i. We first show that $|\hat{P}_i((h_0^i, a))| \leq 2$ for all $a \in A(h_0^i)$. By way of contradiction, assume that there exists some h_0^i such that $|\hat{P}_i((h_0^i, a))| \geq 3$. Let $x, y, z \in \hat{P}_i((h_0^i, a))$ be three distinct undominated payoffs that are possible following a. As (Γ, S) is pruned, there must be some type, $\succ_i$, for which action a is SOD at h_0^i. Possibly by renaming the outcomes, richness allows us to assume that $\mathrm{Top}(\succ_i, P_i(h_0^i)) = x$ and $x \succ_i y \succ_i z$. For a to be strongly obviously dominant, for all other actions $a' \neq a$ at h_0, the best-case outcome for type $\succ_i$ following a' must be no better than z; in particular, this implies that for all $a' \neq a$ and all $w \in \hat{P}_i((h_0^i, a'))$, $w \not\trianglerighteq_i y$. Let $a'' \neq a$ be an action at h_0. If there is $w \in \hat{P}_i((h_0^i, a''))$ such that $x \not\trianglerighteq_i w$, then there is a type $\succ_i'$ such that $\mathrm{Top}(\succ_i', P_i(h_0^i)) = y$ and $y \succ_i' w \succ_i' x$. For this type, the worst case from a is at best x, while w is possible following a'', so a is not strongly obviously dominant; for any $a' \neq a$, the worst case is strictly worse than y as argued above, while the best case from a is y, and so no $a' \neq a$ is SOD either. Therefore, type $\succ_i'$ has no SOD action, a contradiction showing that no $w \in \hat{P}_i((h_0^i, a''))$ satisfies $x \not\trianglerighteq_i w$. An analogous argument—with z playing the role of x—shows that no $w \in \hat{P}_i((h_0^i, a''))$ satisfies $z \not\trianglerighteq_i w$. Thus, for all a'' and all $w \in \hat{P}_i((h_0^i, a''))$, $x \trianglerighteq w$ and $z \trianglerighteq w$. As x and z are distinct, for any type $\succ_i'$, either $x \succ_i' w$ or $z \succ_i' w$, and in either case, a'' is not a dominant action for a type contrary to (Γ, S) being pruned. This contradiction shows that $|\hat{P}_i((h_0^i, a))| \leq 2$ for all $a \in A(h_0^i)$.

Finally, we show that $|\hat{P}_i((h_0^i, a))| = 2$ for at most one $a \in A(h_0^i)$. Towards a contradiction, let a and a' be two actions such that there are two possible undominated payoffs for i following each, and, for notational purposes, let $\hat{P}_i((h_0^i, a)) = \{x, y\}$ and $\hat{P}_i((h_0^i, a')) = \{w, z\}$, where, a priori, it is possible that $w, z \in \{x, y\}$. As the mechanism is pruned, there is some type $\succ_i$ that selects action a as an SOD action; without loss of generality, let $\mathrm{Top}(\succ_i, P_i(h_0^i)) = x$. Since y is possible following a, in order for a to be SOD, the best case from any $a' \neq a$ must be no better than y, which implies that $w, z \not\trianglerighteq_i x$, and thus $x \neq w, z$. Pruning also implies that some type $\succ_i'$ is selecting action a'' as an SOD action; without loss of generality, let $\mathrm{Top}(\succ_i', P_i(h_0^i)) = z$. Since w is possible following a'', in order for a'' to be SOD, the best case from a must be no better than w for type $\succ_i'$, thus $x, y \not\trianglerighteq_i z$, and so $z \neq x, y$. Thus, we have shown that x, y, z are all distinct, that no outcome in $P_i(h_0^i)$—including z and y—structurally dominates x, and that $y \not\trianglerighteq_i z$. Richness then implies that there is a type $\succ_i$ such that $\mathrm{Top}(\succ_i, P_i(h_0^i)) = x$ and $x \succ_i z \succ_i y$. This type has no SOD action: only a can be SOD because only a makes x possible, but a is not SOD because the worst case from a is at best y, while the best case from a' is z. *Q.E.D.*

Continuing with the proof of Lemma 1, assume that there was a path of the game with two payoff-relevant histories $h_1 \subsetneq h_2$ for some agent i. It is without loss of generality to assume that h_1 and h_2 are the first and second times i is called to play on the path. First, we claim that there are at least two structurally undominated payoffs at h_1, that is, $|\hat{P}_i(h_1)| \geq 2$. To show it by way of contradiction, suppose that $\hat{P}_i(h_1) = \{x\}$, which implies that $x \vartriangleright_i x'$ for all other $x' \in P_i(h_1)$. Then $P_i((h_1, a)) = \{x\}$ for all $a \in A(h_1)$. Indeed, suppose that $x' \neq x$ is possible after some action a at h_1. Then $x, x' \in P_i((h_1, a))$ because otherwise no type of i finds a to be SOD, which is impossible as the game is pruned. If

$x \in P_i((h_1, a'))$ for some action $a' \neq a$ at h_1, then a is not SOD for any type of i, which again is impossible as the game is pruned. Thus, $x \notin P_i((h_1, a'))$ and no type of i finds a' to be SOD, which yet again is impossible in a pruned game. Thus, no $x' \neq x$ is possible after any $a \in A(h_1)$, which contradicts that h_1 is payoff-relevant. This contradiction shows that $\hat{P}_i(h_1)$, being non-empty, has at least two elements.

Let a_1^* be the action such that $h_2 \supseteq (h_1, a_1^*)$. By Lemma 2, one of the below two cases would need to obtain, and to conclude the indirect argument we now show that neither of them obtains.

Case $|\hat{P}_i((h_1, a_1^*))| = 1$. Let z be the unique undominated payoff that is possible after a_1^*; $z \in \hat{P}_i(h_1)$ as otherwise no type of i would find a_1^* to be SOD, which is impossible in a pruned mechanism. Because h_2 is payoff-relevant, Lemma 1 tells us that $|\hat{P}_i(h_2)| \geq 2$, and thus $z \notin \hat{P}_i(h_2)$ as z weakly structurally dominates all outcomes in $P_i(h_2) \subseteq P_i((h_1, a_1^*))$. Let $x \neq z$ be an outcome in $\hat{P}_i(h_1)$ and let $z', z'' \in \hat{P}_i(h_2)$ be distinct undominated payoffs that are possible at h_2, and consider a type $\succ_i$ that ranks the outcomes $z \succ_i x \succ_i z'$. For this type, a_1^* is not SOD at h_1 because z' is possible following a_1^* while $x \notin \{z\} = \hat{P}_i((h_1, a_1^*))$ is possible following some other action at h_1. No action $a \neq a_1^*$ is SOD for $\succ_i$ if $z \notin P_i((h_1, a))$. Hence, $z \in P_i((h_1, a))$, but then a_1^* would not be SOD for any type; impossible as the mechanism is pruned. This contradiction shows that the present case is impossible.

Case $|\hat{P}_i((h_1, a_1^*))| = 2$. Then a_1^* is the unique action with two undominated payoffs from Lemma 2; let us label these payoffs x and y. As the game is pruned, there is some type $\succ_i$ for which a_1^* is strongly obviously dominant; in particular, the payoff $\text{Top}(\succ_i, P_i(h_1))$ is possible following a_1^*, and by renaming payoffs we can set $x = \text{Top}(\succ_i, P_i(h_1))$. For each action $a \neq a_1^*$ at h_1, Lemma 2 implies that $\hat{P}_i((h_1, a)) = \{w_a\}$, for some payoff w_a; action a_1^* being SOD for type $\succ_i$ implies that $w_a \not\succeq_i x$ (and in particular, $w_a \neq x$); and a being SOD for some other type implies that $y \not\triangleright_i w_a$. If $w_a \neq y$, then $y \not\trianglerighteq_i w_a$, and, given that x and y are mutually undominated, richness would give us a type $\succ_i^a$ such that $x \succ_i^a w_a \succ_i^a y$, but for this type, neither a_1^* nor a nor any other action a' at h_1 is SOD because, as shown above, $w_{a'} \neq x$. We conclude that $w_a = y$ for all actions $a \neq a_1^*$ at h_1.

To continue the indirect argument, we now show that $\hat{P}_i(h_2) = \{x, y\}$. The set $\hat{P}_i(h_2)$ has two elements, by Lemma 1, because h_2 is payoff-relevant. Thus, if $\hat{P}_i(h_2) \neq \{x, y\}$, then there would be some $z \neq x, y$ such that $z \in \hat{P}_i(h_2) \subseteq P_i(h_1)$. As x and y are undominated at $(h_1, a_1^*) \subsetneq h_2$, richness would give us type $\succ_i^2$ such that $x \succ_i^2 y \succ_i^2 z$, and for this type a_1^* would not be SOD at h_1 because z would be possible following a_1^* while, as shown above, y would be possible following another action; further, no $a \neq a_1$ would be SOD at h_1 because y would be possible following a while x would be possible following a_1^*. The lack of an SOD action is a contradiction showing that $\hat{P}_i(h_2) = \{x, y\}$. Thus, any $z \in P_i(h_2)$ is structurally dominated by either x or y and, for each type, x or y is the top payoff in $P_i(h_2)$. Since $\hat{P}_i((h_1, a)) = y$ for all $a \neq a_1^*$, strong obvious dominance implies that all and only types $\succ_i^1$ with $x = \text{Top}(\succ_i^1, P_i(h_1))$ select action a_1^* at h_1 and hence these are the types for which h_2 is on path. As y is possible at h_2, there is at least one action $a_2^* \in A(h_2)$ after which y is possible. As at each history agents have at least two actions, there is another action $a_2 \in A(h_2)$, and, as the mechanism is pruned, there are two types $\succ_i^{a_2^*}$ and $\succ_i^{a_2}$ for which h_2 is on path such that $\succ_i^{a_2^*}$ selects a_2^* and $\succ_i^{a_2}$ selects a_2 at h_2. Because we established that x is possible at h_2 and that it is the top possible payoff for both these types, SOSP implies that $x \in P_i((h_2, a_2^*))$ and $x \in P_i((h_2, a_2))$. By construction, $y \in P_i((h_2, a_2^*))$, and hence a_2^* is not SOD for type $\succ_i^{a_2^*}$; a contradiction that concludes the proof of the lemma.

B.6. *Proof of Lemma for Theorem 8*

PROOF OF LEMMA A.9: By way of contradiction, suppose that game Γ', together with greedy strategies, is not a sequential choice mechanism. Let h be an earliest history where the definition of a sequential choice mechanism is violated. As such h is payoff-relevant and Γ' is pruned, Lemma 1 implies that h is a first history at which i moves. Since Γ' is not a sequential choice mechanism, there must be some payoff $x \in P_i(h)$ that i cannot clinch at h. We may assume that x is not dominated, that is, $x \in \hat{P}_i(h)$; indeed, if all $x' \in \hat{P}_i(h)$ were clinchable at h, then greediness would imply that all dominated actions were pruned in Γ'. Since x is not clinchable, for any action $a \in A(h)$ such that $x \in P_i((h, a))$, there is some payoff in $P_i((h, a))$ that is different from x. We fix one such action a.

Case $|P_i(h)| = 2$. Let y be the other payoff in $P_i(h)$. If y were clinchable, then the mechanism would satisfy the definition of sequential choice at h. Since we assumed that the definition is not satisfied at h, neither x nor y is clinchable. Thus, for all $a \in A(h)$, $P_i((h, a)) = \{x, y\}$. As x and y are different payoffs, at least one of $x \succ_i y$ or $y \succ_i x$ holds for some type at h. Because there are at least two actions in $A(h)$, this type does not have a strongly obviously dominant (SOD) action at h, which is a contradiction.

Case $|P_i(h)| \geq 3$ *and* $x \rhd_i y$ *for all* $y \neq x$ *in* $P_i((h, a))$. There is an action $a' \neq a$ at h and, because x is not clinchable at h, there is some $w \neq x$ that belongs to $P_i((h, a'))$. We have $y \trianglerighteq_i w$; indeed, if not, then x being undominated implies that there would exist type $\succ_i$ such that $x \succ_i w \succ_i y$, and, taking into account that x is not clinchable at h, this type would have no SOD action at h. Thus, $x \rhd_i y \trianglerighteq_i w$; but this implies that a' is not SOD for any type, which contradicts the mechanism being pruned.

Case $|P_i(h)| \geq 3$ *and there exists* $y \in P_i((h, a))$ *such that* x *and* y *do not dominate each other.* By Lemma 2, for any $a' \neq a$, the set $\hat{P}_i((h, a'))$ is a singleton. We first claim that for any $a' \neq a$, $\hat{P}_i((h, a')) = \{y\}$. Assume not, that is, there exist some $a' \neq a$ and $w' \neq y$ such that $\hat{P}_i((h, a')) = \{w'\}$. Then also $w' \neq x$; indeed, if $w' = x$, then, as x is both structurally undominated and unclinchable at h, there would be $w \in P_i((h, a'))$ such that $x \rhd_i w$, and—with w possible after a' and x possible after a—no type would find a' to be SOD, contrary to pruning. If $w' \rhd_i x$, then no type would find a to be SOD, which contradicts pruning; we conclude that $w' \ntriangleright_i x$. In particular, $x \notin P_i((h, a'))$. If $y \rhd_i w'$, then $y \notin P_i((h, a'))$ because $w' \in \hat{P}_i((h, a'))$ is undominated at (h, a'). Thus, a' would not be SOD for any type, a contradiction to pruning. We conclude that $y \ntriangleright_i w'$. If $w' \rhd_i y$, then this, and the previously established $w' \ntriangleright_i x$, gives us the existence of type $\succ_i$ such that $x \succ_i w' \succ_i y$. This type has no SOD action at h, a contradiction to pruning. We conclude that $w' \ntriangleright_i y$. If $x \rhd_i w'$, then type $x \succ_i w' \succ_i y$ exists and has no SOD action at h; we conclude that $x \ntriangleright_i w'$. The above four conclusions imply that x, y, w' are mutually undominated at h. Thus, there is a type such that $x \succ_i w' \succ_i y$ and this type has no SOD action at h. This final contradiction shows that $\hat{P}_i((h, a')) = \{y\}$ for all $a' \neq a$.

We further claim that $P_i((h, a')) = \{y\}$ for all $a' \neq a$; indeed, if this were not the case, then there is some a' and some $w' \in P_i((h, a'))$ such that $y \rhd_i w'$. As the mechanism is pruned, some type $\succ_i'$ takes action a'; but, the worst case from a', for all types, is at best w', while y is possible following a; thus, a' is not SOD for type $\succ_i'$. This contradiction shows that $P_i((h, a')) = \{y\}$ for all $a' \neq a$.

Finally, let $z \neq x, y$ be some third payoff that is possible at h. In light of the previous paragraph, $z \in P_i((h, a))$, and $z \notin P_i((h, a'))$ for all other $a' \neq a$. As $\hat{P}_i(h) = \{x, y\}$, z dominates neither x nor y, and richness gives us a type such that $x \succ_i y \succ_i z$. This type has no SOD action at h; this contradicts the mechanism being SOSP and establishes the theorem. *Q.E.D.*

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