Econometrica Supplementary Material

SUPPLEMENT TO "MARKET SELECTION AND THE INFORMATION CONTENT OF PRICES"

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APPENDIX B

B.1. *Proofs of Pooling Calculations*

GIVEN A POOLING BID b_p^n, let the random variables L^n, G^n, and $X^n = L^n + G^n$ denote the number of losers, number of winners (or the number of objects left for the bidders who submit a bid equal to b_p^n), and number of bidders that submit a bid equal to b_p^n, respectively. Let $\bar{L}^n = \mathbb{E}[L^n|P^n = b_p^n, v]$, $\bar{G}^n = \mathbb{E}[G^n|P^n = b_p^n, v]$, and $\bar{X}^n = \bar{L}^n + \bar{G}^n$. Given these definitions, $\Pr[b_p^n \text{ lose}|P^n = b_p^n, v] = \mathbb{E}[L^n/X^n|P^n = b_p^n, v]$ and $\Pr[b_p^n \text{ win}|P^n = b_p^n, v] = \mathbb{E}[G^n/X^n|P^n = b_p^n, v]$. For any type θ that submits the pooling bid, $\Pr(L^n = i|Y_s^n(k_s+1) = \theta, v) = \text{bi}(i; n-1-k_s, \frac{F_s^n([\underline{\theta}_p^n, \theta]|v)}{1-\bar{F}_s^n(\theta|v)})$ and $\Pr(X^n = i|Y_s^n(k_s+1) = \theta, v) = \text{bi}(i; n-1-k_s, \frac{F_s^n([\theta, \theta_p^n]|v)}{\bar{F}_s^n(\theta|v)})$. Therefore, $\mathbb{E}[L^n|Y_s^n(k_s+1) = \theta, v] = n\frac{F_s^n([\underline{\theta}_p^n, \theta]|v)}{1-\bar{F}_s^n(\theta|v)}(1-\kappa_s-\frac{1}{n})$, $\mathbb{E}[X^n|Y_s^n(k_s+1) = \theta, v] = n\kappa_s\frac{F_s^n([\theta, \theta_p^n]|v)}{\bar{F}_s^n(\theta|v)}$, $\bar{L}^n = \int_{\underline{\theta}_p^n}^{\theta_p^n} n\frac{F_s^n([\underline{\theta}_p^n, \theta]|v)}{1-\bar{F}_s^n(\theta|v)}(1-\kappa_s-1/n)\Pr(Y_s^n(k_s+1) = \theta|v)\,d\theta$, and $\bar{X}^n = \int_{\underline{\theta}_p^n}^{\theta_p^n} n\kappa_s\frac{F_s^n([\theta, \theta_p^n]|v)}{\bar{F}_s^n(\theta|v)}\Pr(Y_s^n(k_s+1) = \theta|v)\,d\theta$.

We prove a somewhat stronger version of Lemma A.2 in Lemma B.1 below.

LEMMA B.1: *If* $\lim\Pr(P^n \geq b_p^n|v = 0)$, *then*

$$\lim\Pr\big(b_p^n \text{ lose}|P^n = b_p^n, V = v\big)\Big/\frac{\overline{F}_s^n\big(\underline{\theta}_p^n|v\big)\big(1-\bar{F}_s\big(\underline{\theta}_p^n|v\big)\big)}{nF_s^n\big(\big[\underline{\theta}_p^n, \theta_p^n\big]|v\big)\big(\kappa_s - \bar{F}_s^n\big(\underline{\theta}_p^n|v\big)\big)} = 1.$$

Suppose $\lim\Pr(P^n = b_p^n|v) > 0$.

(i) *If* $\lim F_s^n([\underline{\theta}_p^n, \theta_p^n]|v) > 0$, *then* $\lim\Pr(b_p^n \text{ win}|P^n = b^n, v) = \lim\frac{\kappa_s - \bar{F}_s^n(\theta_p^n|v)}{F_s^n([\underline{\theta}_p^n, \theta_p^n]|v)}$.

(ii) *If* $\sqrt{n}|F_s^n(\theta^n(v)|v) - F_s^n(\underline{\theta}_p^n|v)| \to \infty$, *then* $\lim\frac{F_s^n([\underline{\theta}_p^n, \theta_p^n]|v)}{F_s^n([\underline{\theta}_p^n, \theta^n(v)]|v)}\Pr(b_p^n \text{ lose}|P^n = b_p^n, v) \in (0, \infty)$.

(iii) *If* $\sqrt{n}|F_s^n(\theta^n(v)|v) - F_s^n(\underline{\theta}_p^n|v)| < \infty$, *then* $\lim\sqrt{n}F_s^n([\underline{\theta}_p^n, \theta_p^n]|v)\Pr(b_p^n \text{ lose}|P^n = b_p^n, v) \in (0, \infty)$.

(iv) *If* $\sqrt{n}|F_s^n(\theta^n(v)|v) - F_s^n(\theta_p^n|v)| \to \infty$, *then* $\lim\frac{F_s^n([\underline{\theta}_p^n, \theta_p^n]|v)}{F_s^n([\theta^n(v), \theta_p^n]|v)}\Pr(b_p^n \text{ win}|P^n = b_p^n, v) \in (0, \infty)$.

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(v) *If* $\sqrt{n}|F_s^n(\theta^n(v)|v) - F_s^n(\theta_p^n|v)| < \infty$, *then* $\lim \sqrt{n} F_s^n([\underline{\theta}_p^n, \theta_p^n]|v) \Pr(b_p^n \text{ win}|P^n = b_p^n, v) \in (0, \infty)$.

PROOF: (i) Suppose that $Y_s^n(k_s+1) = \theta^n$, $\theta^n \in [\underline{\theta}_p^n, \theta_p^n]$, and $\bar{F}_s^n(\theta^n|v) \in (\kappa_s - \epsilon_1, \kappa_s + \epsilon_1)$. There are k_s bidders with signals above θ^n and the distribution of G^n is binomial; hence, $\bar{G}_n = \frac{k_s(\bar{F}^n(\theta|v) - \bar{F}_s^n(\theta_p^n|v))}{\bar{F}^n(\theta|v)}$. Also, $\Pr(G^n < (1-\delta)\bar{G}_n | Y_s^n(k_s+1) = \theta^n, v) \leq e^{-\frac{\delta^2}{2}\bar{G}_n}$ for any $\delta \in (0,1)$ by the Chernoff inequality.[18] Similarly, $\bar{L}_n = \frac{(n-1-k_s)(\bar{F}_s^n(\underline{\theta}_p^n|v) - \bar{F}_s^n(\theta^n|v))}{1-\bar{F}_s^n(\theta^n|v)} + 1$ because there are $n-1-k_s$ bidders with signals below θ^n and the distribution of L^n is binomial, and $\Pr(L^n < (1-\delta)\bar{L}_n | Y_s^n(k_s+1) = \theta, v) \leq e^{-\frac{\delta^2}{2}\bar{L}_n}$. The random variables X^n and L^n are independent conditional on $Y_s^n(k_s+1) = \theta^n$. Moreover, $\Pr(b_p^n \text{ win}|Y_s^n(k_s+1) = \theta^n, v) = \mathbb{E}[G^n/(L^n + G^n)|Y_s^n(k_s+1) = \theta^n, v]$. The function $G^n/(L^n + G^n)$ is concave in G^n and convex in L^n. Therefore, using Jensen's inequality and then the Chernoff bound, we obtain

$$\mathbb{E}\left[\frac{G_n}{G_n + \bar{L}_n}\middle| Y_s^n(k_s+1) = \theta^n, v\right] \leq Q_n \leq \mathbb{E}\left[\frac{\bar{G}_n}{\bar{G}_n + L_n}\middle| Y_s^n(k_s+1) = \theta^n, v\right],$$

$$\frac{(1-\delta)\bar{G}_n}{\bar{G}_n(1-\delta) + \bar{L}_n}\left(1 - e^{-\frac{\delta^2}{2}\bar{G}_n}\right) \leq Q_n \leq \frac{\bar{G}_n}{\bar{G}_n + (1-\delta)\bar{L}_n} + e^{-\frac{\delta^2}{2}\bar{L}_n},$$

where $Q_n = \Pr(b_p^n \text{ win}|Y_s^n(k_s+1) = \theta^n, v)$. Our assumption $\lim F_s^n([\underline{\theta}_p^n, \theta_p^n]|v) > 0$ implies either $\bar{G}_n \to \infty$ or $\bar{L}_n \to \infty$ or both. Taking the limits and noting that δ is arbitrary, we obtain $\lim \Pr(b_p^n \text{ win}|Y_s^n(k_s+1) = \theta^n, v) = \lim \frac{\bar{G}_n}{\bar{G}_n + \bar{L}_n}$. Since $\bar{F}_s^n(\theta^n|v) \in (\kappa_s - \epsilon_1, \kappa_s + \epsilon_1)$ by assumption, we have

$$\lim \frac{\kappa_s \dfrac{\left(\kappa_s - \epsilon_1 - \bar{F}_s^n(\theta_p^n|v)\right)}{\kappa_s + \epsilon_1}}{\kappa_s \dfrac{\left(\kappa_s + \epsilon_1 - \bar{F}_s^n(\theta_p^n|v)\right)}{\kappa_s - \epsilon_1} + (1-\kappa_s)\dfrac{\bar{F}_s^n(\underline{\theta}_p^n|v) - \kappa_s + \epsilon_1}{1 - \kappa_s - \epsilon_1}}$$

$$\leq \lim Q_n \leq \lim \frac{\kappa_s \dfrac{\left(\kappa_s + \epsilon_1 - \bar{F}_s^n(\theta_p^n|v)\right)}{\kappa_s - \epsilon_1}}{\kappa_s \dfrac{\left(\kappa_s - \epsilon_1 - \bar{F}_s^n(\theta_p^n|v)\right)}{\kappa_s + \epsilon_1} + (1-\kappa_s)\dfrac{\bar{F}_s^n(\underline{\theta}_p^n|v) - \kappa_s - \epsilon_1}{1 - \kappa_s + \epsilon_1}}.$$

But $\lim \Pr(\bar{F}_s^n(Y_s^n(k_s+1)|v) \in (\kappa_s - \epsilon_1, \kappa_s + \epsilon_1)|v) = 1$ for every $\epsilon_1 > 0$ by the LLN. Hence,

$$\lim \Pr\left(\bar{F}_s^n\left(Y_s^n(k_s+1)|v\right) \in (\kappa_s - \epsilon_1, \kappa_s + \epsilon_1) | Y_s^n(k_s+1) \in \left[\underline{\theta}_p^n, \theta_p^n\right], v\right) = 1.$$

[18] See Janson, Luczak, and Rucinski (2011, Theorem 2.1).

Therefore,

$$\lim \frac{\kappa_s \dfrac{\left(\kappa_s - \epsilon_1 - \bar{F}_s^n\left(\theta_p^n|v\right)\right)}{\kappa_s + \epsilon_1}}{\kappa_s \dfrac{\left(\kappa_s + \epsilon_1 - \bar{F}_s^n\left(\theta_p^n|v\right)\right)}{\kappa_s - \epsilon_1} + (1-\kappa_s)\dfrac{\bar{F}_s^n\left(\underline{\theta}_p^n|v\right) - \kappa_s + \epsilon_1}{1-\kappa_s-\epsilon_1}}$$

$$\leq \lim \Pr\left(b_p^n \text{ wins}|Y_s^n(k_s+1) \in \left[\underline{\theta}_p^n, \theta_p^n\right], v\right)$$

$$\leq \lim \frac{\kappa_s \dfrac{\left(\kappa_s + \epsilon_1 - \bar{F}_s^n\left(\theta_p^n|v\right)\right)}{\kappa_s - \epsilon_1}}{\kappa_s \dfrac{\left(\kappa_s - \epsilon_1 - \bar{F}_s^n\left(\theta_p^n|v\right)\right)}{\kappa_s + \epsilon_1} + (1-\kappa_s)\dfrac{\bar{F}_s^n\left(\underline{\theta}_p^n|v\right) - \kappa_s - \epsilon_1}{1-\kappa_s+\epsilon_1}}.$$

Since this is true for each $\epsilon_1 > 0$, taking $\epsilon_1 \to 0$ shows $\lim \Pr(b_p^n \text{ wins}|P^n = b_p^n, v) = \lim \frac{\kappa_s - \bar{F}_s^n(\theta_p^n|v)}{F_s^n([\underline{\theta}_p^n, \theta_p^n]|v)}$.

(ii)–(v). Further below we argue that the expected number of losers at the pooling bid satisfies $0 < \liminf \frac{\bar{L}^n}{\sqrt{n}} \leq \limsup \frac{\bar{L}^n}{\sqrt{n}} < \infty$ if $\lim \sqrt{n}|F_s^n(\theta^n(v)|v) - F_s^n(\underline{\theta}_p^n|v)| < \infty$, and satisfies $0 < \liminf \frac{\bar{L}^n}{nF_s^n([\underline{\theta}_p^n, \theta^n(v)]|v)} \leq \limsup \frac{\bar{L}^n}{nF_s^n([\underline{\theta}_p^n, \theta^n(v)]|v)} \leq 1$ if $\lim \sqrt{n}|F_s^n(\theta^n(v)|v) - F_s^n(\underline{\theta}_p^n|v)| \to \infty$.

We will prove items (ii) and (iii) using these bounds for $\bar{L}^n$; items (iv) and (v) follow from an identical argument. We begin by proving the lower bounds in items (ii) and (iii). Note that $\Pr(L^n \geq \bar{L}^n - 1|P^n = b_p^n, v) \geq 1/2$ because L^n is a binomial random variable.[19] Therefore,

$$\Pr\left(b_p^n \text{ lose}|P^n = b_p^n, v\right)$$

$$\geq \mathbb{E}\left[\frac{L^n}{X}\Big|L^n \geq \bar{L}^n - 1, P^n = b_p^n, v\right]\Pr\left(L^n \geq \bar{L}^n - 1|P^n = b_p^n, v\right)$$

$$\geq \mathbb{E}\left[\frac{\bar{L}^n - 1}{X}\Big|L^n \geq \bar{L}^n - 1, P^n = b_p^n, v\right]\frac{1}{2}$$

$$\geq \frac{\left(\bar{L}^n - 1\right)/2}{\mathbb{E}\left[X^n|L^n \geq \bar{L}^n - 1, P^n = b_p^n, v\right]} \quad \text{(by Jensen's inequality).}$$

Note that $\mathbb{E}[X^n|L^n \geq \bar{L}^n - 1, P^n = b_p^n, v]\Pr(L^n \geq \bar{L}^n - 1, P^n = b_p^n|v) \leq \mathbf{E}[X^n|v] = nF_s^n([\underline{\theta}_p^n, \theta_p^n]|v)$. Therefore,

$$\Pr\left(b_p^n \text{ lose}|P^n = b_p^n, v\right)$$

$$\geq \frac{\left(\bar{L}^n - 1\right)}{nF_s^n\left(\left[\underline{\theta}_p^n, \theta_p^n\right]|v\right)}\frac{\Pr\left(L^n \geq \bar{L}^n - 1|P^n = b_p^n, v\right)\Pr\left(P^n = b_p^n|v\right)}{2}$$

[19] Conditional on $Y_s^n(k_s+1) = \theta \in [\underline{\theta}_p^n, \theta_p^n]$ and $V = v$, the number of losers L^n is a binomial random variable. The median of the binomial differs from the mean by at most 1. Therefore, $\Pr(L^n \geq \mathbf{E}[L^n|Y_s^n(k_s+1) = \theta, v] - 1|Y_s^n(k_s+1) = \theta, v) \geq 1/2$. In turn, this implies that $\Pr(L^n \geq \bar{L}^n - 1|P^n = b_p^n, v) \geq 1/2$.

and $\Pr(b_p^n \text{ lose}|P^n = b_p^n, v) n F_s^n([\underline{\theta}_p^n, \theta_p^n]|v) \geq (\bar{L}^n - 1)\frac{\Pr(P^n = b_p^n|v)}{4}$. Taking limits and substituting $0 < \liminf \frac{\bar{L}^n - 1}{\sqrt{n}} < \limsup \frac{\bar{L}^n - 1}{\sqrt{n}} < \infty$ if $\lim \sqrt{n}|F_s^n(\theta^n(v)|v) - F_s^n(\underline{\theta}_p^n|v)| < \infty$ and

$$0 < \liminf \frac{\bar{L}^n - 1}{nF_s^n([\underline{\theta}_p^n, \theta^n(v)]|v)} \leq \limsup \frac{\bar{L}^n - 1}{nF_s^n([\underline{\theta}_p^n, \theta^n(v)]|v)} \leq 1$$

if $\lim \sqrt{n}|F_s^n(\theta^n(v)|v) - F_s^n(\underline{\theta}_p^n|v)| \to \infty$ delivers the lower bounds in items (ii) and (iii).

We now establish the upper bounds in items (ii) and (iii). If $\lim \sqrt{n} F_s^n([\underline{\theta}_p^n, \theta_p^n]|v) \in (0, \infty)$, then $\lim \sqrt{n}|F_s^n(\theta^n(v)|V = v) - F_s^n(\underline{\theta}_p^n|V = v)| < \infty$ (because $\lim \Pr(P^n = b_p^n|V = v) > 0$) and the upper bound in item (ii) is trivially satisfied. Suppose $\lim \sqrt{n} F_s^n([\underline{\theta}_p^n, \theta_p^n]|v) = \infty$. Pick $\delta \in (0, 1)$ and let $\bar{Y}^n = nF_s^n([\underline{\theta}_p^n, \theta_p^n]|v)$. Then

$$\begin{aligned} &\Pr[b_p^n \text{ lose}|P^n = b_p^n, v] \\ &\quad\leq \frac{\mathbb{E}[L^n|X^n > (1-\delta)\bar{Y}^n, P^n = b_p^n, v] \Pr(X^n > (1-\delta)\bar{Y}^n|P^n = b_p^n, v)}{(1-\delta)\bar{Y}^n} \\ &\qquad + \Pr(X^n \leq \bar{Y}^n(1-\delta)|P^n = b_p^n, v). \end{aligned}$$

However, $\mathbb{E}[L^n|X^n > (1-\delta)\bar{Y}^n, P^n = b_p^n, v] \Pr(X^n > (1-\delta)\bar{Y}^n|P^n = b_p^n, v) \leq \bar{L}^n$. Therefore,

$$\frac{\Pr[b_p^n \text{ lose}|P^n = b_p^n, v]\bar{Y}^n}{\bar{L}^n} \leq \frac{1}{1-\delta} + \frac{\bar{Y}^n}{\bar{L}^n} \Pr(X^n \leq \bar{Y}^n(1-\delta)|P^n = b_p^n, v).$$

Chernoff's inequality implies that $\lim \Pr(X^n \leq (1-\delta)\bar{Y}^n|v) < \exp(-\frac{\delta^2 \bar{Y}^n}{3})$ and, hence, $\Pr(X^n \leq \bar{Y}^n(1-\delta)|P^n = b_p^n, v) \leq \frac{\exp(-\frac{\delta^2 Y^n}{3})}{\Pr(P^n = b_p^n|v)}$. Therefore,

$$\lim \frac{\Pr[b_p^n \text{ lose}|P^n = b_p^n, v]\bar{Y}^n}{\bar{L}^n} \leq \frac{1}{1-\delta}.$$

Substituting for the number of losers $\bar{L}^n$ now delivers the upper bounds in items (i) and (ii).

We now show that $0 < \liminf \frac{\bar{L}^n}{\sqrt{n}} \leq \limsup \frac{\bar{L}^n}{\sqrt{n}} < \infty$ if $\lim \sqrt{n}|F_s^n(\theta^n(v)|v) - F_s^n(\underline{\theta}_p^n|v)| < \infty$, and $0 < \liminf \frac{\bar{L}^n}{nF_s^n([\underline{\theta}_p^n, \theta^n(v)]|v)} \leq \limsup \frac{\bar{L}^n}{nF_s^n([\underline{\theta}_p^n, \theta^n(v)]|v)} \leq 1$ if $\lim \sqrt{n}|F_s^n(\theta^n(v)|v) - F_s^n(\underline{\theta}_p^n|v)| \to \infty$.

Pick any $\theta^n \in [\underline{\theta}_p^n, \theta_p^n]$ and let $a(\theta^n) := \bar{F}_s^n(\theta^n(v)|v) - \bar{F}_s^n(\theta^n|v) = \kappa_s - \bar{F}_s^n(\theta^n|v)$. Recall that $\Pr(L^n = i|Y_s^n(k_s + 1) = \theta^n, v) = \text{bi}(i; n - 1 - k_s, p^n)$ and $\mathbb{E}[L^n|Y_s^n(k_s + 1) = \theta^n, v] = np(a(\theta^n))(1 - \kappa_s - \frac{1}{n})$, where

$$p(a(\theta^n)) = \frac{\bar{F}_s^n(\underline{\theta}_p^n|v) - \kappa_s + a(\theta^n)}{1 - \kappa_s + a(\theta^n)}.$$

Calculating the number of losers, we find

$$\bar{L}^n = -\left(1 - \kappa_s - \frac{1}{n}\right) \int_{\underline{a}^n}^{\bar{a}^n} np(a)\, d\Lambda(a),$$

where $\underline{a}^n = a(\underline{\theta}_p^n)$, $\bar{a}^n = a(\theta_p^n)$, and

$$\Lambda(a) := \Pr\big(F_s^n\big(Y_s^n(k_s+1)|v\big) - \kappa_s \geq a|P = b_p^n, v\big).$$

Integrating by parts and substituting $p(\underline{a}^n) = 0$, $\Lambda(\bar{a}^n) = 0$, and

$$p'(a) = \frac{\big(1 - \bar{F}_s^n\big(\underline{\theta}_p^n|v\big)\big)}{(1-\kappa_s+a)^2} = \frac{1-\kappa_s+\underline{a}^n}{(1-\kappa_s+a)^2}$$

delivers $\bar{L}^n/n = (1-\kappa_s - \frac{1}{n})(1-\kappa_s+\underline{a}^n)\int_{\underline{a}^n}^{\bar{a}^n} \frac{\Lambda(a)}{(1-\kappa_s+a)^2}\,da$. Hence,

$$C^n \int_{\underline{a}^n}^{\bar{a}^n} \Lambda(a)\,da \leq \bar{L}^n/n \leq \frac{1-\kappa_s}{1-\kappa_s+\underline{a}^n}\int_{\underline{a}^n}^{\bar{a}^n} \Lambda(a)\,da \leq \int_{\underline{a}^n}^{\bar{a}^n} \Lambda(a)\,da,$$

where $C^n = \frac{(1-\kappa_s)(1-\kappa_s+\underline{a}^n)}{(1-\kappa_s+\bar{a}^n)^2}$.

Pick any $\epsilon > 0$ and let a_*^n be such that $\Pr(F_s^n(Y_s^n(k_s+1)|v) - \kappa_s \geq a^n|P^n = b_p^n, v) = 1-\epsilon$. The central limit theorem implies that $\lim \sqrt{n}a_*^n \in (0,\infty)$. Moreover, $\lim \sqrt{n}(a^n - \underline{a}^n) > 0$ because $\Pr(F_s^n(Y_s^n(k_s+1)|v) - \kappa_s \leq a^n|P^n = b_p^n, v) = \epsilon$ for each n. Therefore,

$$\int_{\underline{a}^n}^{a_*^n} \Lambda(a)\,da \leq \int_{\underline{a}^n}^{\bar{a}^n} \Lambda(a)\,da \leq \max\{-\underline{a}^n, 0\} + \int_0^{\bar{a}^n} \Lambda(a)\,da,$$

$$\int_{\underline{a}}^{a_*^n} (1-\epsilon)\,da \leq \int_{\underline{a}^n}^{\bar{a}^n} \Lambda(a)\,da \leq \max\{-\underline{a}^n, 0\} + \frac{\int_0^{\bar{a}^n} e^{-\frac{a^2 n}{2}}\,d\theta}{\Pr\big(P^n = b_p^n|v\big)},$$

$$(1-\epsilon)\big(a_*^n - \underline{a}^n\big) \leq \int_{\underline{a}^n}^{\bar{a}^n} \Lambda(a)\,da = \max\{-\underline{a}^n, 0\} + \frac{\sqrt{2}\,\mathrm{erf}(\sqrt{n}\bar{a})}{\Pr\big(P^n = b_p^n|v\big)\sqrt{\pi n}}$$

$$\leq \max\{-\underline{a}^n, 0\} + \frac{1}{\Pr\big(P^n = b_p^n|v\big)\sqrt{2\pi n}},$$

where $\mathrm{erf}(x) = \frac{1}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt \in [0, 1/2]$ is the error function.

Note that $-\underline{a}^n = F_s^n(\theta^n(v)|V = v) - F_s^n(\underline{\theta}_p^n|V = v)$. Suppose that $-\lim \sqrt{n}\underline{a}^n < \infty$. If $-\lim \sqrt{n}\underline{a}^n = \delta_1 < \infty$, then $\lim \sqrt{n}(a_*^n - \underline{a}^n) = \delta \in (0,\infty)$. The fact that $\frac{\bar{L}^n}{\sqrt{n}} \in (\sqrt{n}C^n \int_{\underline{a}^n}^{\bar{a}^n} \Lambda(a)\,da, \sqrt{n}\int_{\underline{a}^n}^{\bar{a}^n} \Lambda(a)\,da)$ and the bounds for $\int_{\underline{a}^n}^{\bar{a}^n} \Lambda(a)\,da$ together imply that

$$C^n\big((1-\epsilon)\sqrt{n}\big(a_*^n - \underline{a}^n\big)\big) \leq \frac{\bar{L}^n}{\sqrt{n}}$$

$$\leq \max\{\sqrt{n}\underline{a}^n, 0\} + \frac{1}{\Pr\big(P^n = b_p^n|v\big)\sqrt{2\pi}}$$

and

$$0 < (1-\epsilon)C\delta \le \liminf \frac{\bar{L}^n}{\sqrt{n}} \le \limsup \frac{\bar{L}^n}{\sqrt{n}}$$
$$\le \max\{\delta_1, 0\} + \frac{1}{\Pr\big(P^n = b_p^n|v\big)\sqrt{2\pi}} < \infty,$$

where $C = \liminf C^n$.

If $-\lim \sqrt{n}\underline{a}^n = \infty$, then $L^n \in (n\sqrt{n}C^n \int_{\underline{a}^n}^{\bar{a}^n} \Lambda(a)\,da, n\int_{\underline{a}^n}^{\bar{a}^n} \Lambda(a)\,da)$ and the bounds for $\int_{\underline{a}^n}^{\bar{a}^n} \Lambda(a)\,da$ together imply that

$$C^n\left(\frac{(1-\epsilon)n\big(a_*^n - \underline{a}^n\big)}{-n\underline{a}^n}\right) \le \frac{\bar{L}^n}{-n\underline{a}^n} \le 1 - \frac{1}{\Pr\big(P^n = b_p^n|v\big)\underline{a}^n\sqrt{2\pi n}},$$
$$\lim C^n\left((1-\epsilon)\left(\frac{\sqrt{n}a_*^n}{-n\underline{a}^n} + 1\right)\right) \le \liminf \frac{\bar{L}^n}{-n\underline{a}^n} \le \limsup \frac{\bar{L}^n}{-n\underline{a}^n} \le 1,$$
$$0 < C(1-\epsilon) \le \liminf \frac{\bar{L}^n}{-n\underline{a}^n} \le \limsup \frac{\bar{L}^n}{-n\underline{a}^n} \le 1.$$

Q.E.D.

PROOF OF THE CALCULATION FOR THE CASE WHERE $\lim \Pr(P^n \ge b_p^n|v) = 0$ IN LEMMA B.1: As before, let X^n denote the random variable that is equal to the number of bidders in the interval $[\underline{\theta}_p^n, \theta_p^n]$. Redefine L^n to denote the random variable that is equal to the number of losers with signals that exceed $\underline{\theta}_p^n$. Note that $\mathbb{E}[L^n | Y_s^n(k+1) \ge \underline{\theta}_p^n, V = v] = \mathbb{E}[L^n | L^n \ge 1, V = v]$. Pick a $\delta > 0$ and let $d^n = (1-\delta)k_s \frac{F_s^n([\underline{\theta}_p^n, \theta_p^n]|v)}{\bar{F}_s^n(\underline{\theta}_p^n|v)}$, and observe that $\lim \frac{d^n}{\sqrt{n}} > 0$. We will show

$$\lim \frac{\mathbb{E}\left[\frac{L^n}{X^n}\Big|L^n \in [1, d^n], V = 0\right]}{\dfrac{\overline{F}_s^n\big(\underline{\theta}_p^n|V = v\big)\big(1 - \bar{F}_s(\underline{\theta}_p|V = v)\big)}{nF_s^n\big([\underline{\theta}_p^n, \theta_p^n]|V = v\big)\big(\kappa_s - \bar{F}_s(\underline{\theta}_p|V = v)\big)}} = 1$$

and

$$\lim \frac{\Pr\big(b_p^n \text{ loses}|P^n = b_p^n, V = v\big)}{\mathbb{E}\left[\frac{L^n}{X^n}\Big|L^n \in [1, d^n], V = 0\right]} = 1.$$

Step 1: $\lim \frac{\mathbb{E}[L^n|L^n \in [1, d^n], v]}{a^n} = 1$, where $a^n = \frac{\kappa_s(1-\bar{F}_s^n(\underline{\theta}_p^n|V=v))}{\kappa_s - \bar{F}_s^n(\underline{\theta}_p^n|V=v)}$. Note that

$$\lim \mathbb{E}\big[L^n|L^n \in [1, d^n], v\big] = \frac{\sum_{i=1}^{d^n} i bi\big(k_s + i, n; p^n\big)}{\sum_{i=1}^{d^n} \text{bi}\big(k_s + i, n; p^n\big)},$$

where $p^n = \bar{F}_s^n(\underline{\theta}_p^n|v)$. Observe that

$$\frac{\mathrm{bi}(k+i,n;p^n)}{\mathrm{bi}(k+i,n;\kappa_s)}\mathrm{bi}(k+i,n;\kappa_s) = \mathrm{bi}(k+i,n;\kappa_s)\left(\frac{p^n}{\kappa_s}\right)^{k_s}\left(\frac{1-p^n}{1-\kappa_s}\right)^{n-k_s}\left(\frac{p^n(1-\kappa_s)}{\kappa_s(1-p^n)}\right)^i.$$

Therefore

$$\mathbb{E}[L^n|L^n \in [1,d^n],v] = \frac{\sum_{i=1}^{d^n} ir(n)^i \mathrm{bi}(k_s+i,n;\kappa_s)}{\sum_{i=1}^{d^n} r(n)^i \mathrm{bi}(k_s+i,n;\kappa_s)},$$

where $r(n) = \frac{p^n(1-\kappa_s)}{\kappa_s(1-p^n)} < 1$. Pick any $J < d^n$. For each $i < J$,

$$(1-\epsilon^n)\phi\left(\frac{J}{\sqrt{n(1-\kappa_s)\kappa_s}}\right) \le \sqrt{n(1-\kappa_s)\kappa_s}\,\mathrm{bi}(k+i,n;\kappa_s) \le (1+\epsilon^n)\phi(0)$$

by the local limit theorem (Proposition A.1). Hence,

$$(1-\epsilon^n)\frac{\phi\left(\frac{J}{\sqrt{n}}\right)\sum_{i=1}^{J} ir(n)^i}{\phi(0)\sum_{i=1}^{d^n} r(n)^i} \le \frac{\sum_{i=1}^{d^n} ir^i\sqrt{n}\,\mathrm{bi}(k+i,n;\kappa_s)}{\sum_{i=1}^{d^n} r^i\sqrt{n}\,\mathrm{bi}(k+i,n;\kappa_s)} \le \frac{\phi(0)}{\phi\left(\frac{J}{\sqrt{n}}\right)}\frac{\sum_{i=1}^{d^n} ir(n)^i}{\sum_{i=1}^{J} r(n)^i}(1+\epsilon^n).$$

Evaluating the geometric series we find

$$\frac{\phi\left(\frac{J}{\sqrt{n}}\right)(1-\epsilon^n)}{\phi(0)(1-r(n)^{d^n})}\left(\frac{1-r(n)^J}{1-r(n)} - Jr(n)^J\right) \le Q \le \frac{\phi(0)(1+\epsilon^n)}{\phi\left(\frac{J}{\sqrt{n}}\right)(1-r(n)^J)}\left(\frac{1-r(n)^{d^n}}{1-r(n)} - d^n r(n)^{d^n}\right),$$

$$\frac{\phi\left(\frac{J}{\sqrt{n}}\right)(1-\epsilon^n)}{\phi(0)(1-r(n)^{d^n})}\left(\frac{1-r(n)^J}{1-r(n)} - Jr(n)^J\right) \le Q \le \frac{\phi(0)(1+\epsilon^n)}{\phi\left(\frac{J}{\sqrt{n}}\right)(1-r(n)^J)},$$

where $Q = \mathbb{E}[L^n|L^n \in [1,d^n],v]$.

Case 1: $\bar{F}(\underline{\theta}_p|v) < \kappa_s$. In this case, $\lim r(n) = r < 1$. Picking $J = n^{1/4} < d^n$ and taking the limit as $n \to \infty$, we find $\lim \mathbb{E}[L^n|L^n \in [1,d^n],v_i] = \frac{1}{1-r} = \frac{\kappa_s(1-\bar{F}_s(\underline{\theta}_p|V=v))}{\kappa_s - \bar{F}_s(\underline{\theta}_p|V=v)} = \lim a^n$.

Case 2: $\overline{F}_s(\underline{\theta}_p|v_i) = \kappa_s$. In this case, $r(n) < 1$ for all n sufficiently large, but $\lim r(n) = 1$. Note that $\lim \frac{1-r(n)}{1/a^n} = 1$. For any constant m, $ma^n < d^n$ for sufficiently large n because $d^n/a^n \to \infty$. Substituting $1/a^n$ for $1 - r(n)$ and setting $J = ma^n$ for any arbitrary m, we find

$$\frac{\phi(ma^n/\sqrt{n})}{\phi(0)} \frac{a^n\big(1-(1-1/a^n)^{ma^n}\big) - ma^n(1-1/a^n)^{ma^n}}{1-(1-1/a^n)^{d^n}} \frac{1-\epsilon^n}{a^n}$$

$$\leq X \leq \frac{\phi(0)}{\phi(ma^n/\sqrt{n})} a^n \frac{1+\epsilon^n}{a^n},$$

$$\frac{\phi(ma^n/\sqrt{n})}{\phi(0)} \frac{\big(1-(1-1/a^n)^{ma^n}\big) - m(1-1/a^n)^{ma^n}}{1-(1-1/a^n)^{d^n}} (1-\epsilon^n)$$

$$\leq X \leq \frac{\phi(0)}{\phi(ma^n/\sqrt{n})} (1+\epsilon^n),$$

where $X = \frac{\mathbb{E}[L^n|L^n \in [1,d^n], v_i]}{a^n}$. Taking the limit as $n \to \infty$, and noting that $a^n \to \infty$, $a^n/\sqrt{n} \to 0$, and $d^n/a^n \to \infty$, we obtain $(1 - 1/a^n)^{ma^n} \to \exp(-m)$, $\phi(ma^n/\sqrt{n}) \to \phi(0)$, and $(1 - 1/a^n)^{d^n} \to 0$. Therefore,

$$1 - \exp(-m) - \exp(-m)m \leq \lim \mathbb{E}\big[L^n|L^n \in [1, d^n], v_i\big]/a^n \leq 1.$$

As m is arbitrary, taking the limit as $m \to \infty$, we find $\mathbb{E}[L^n|L^n \in [1, d^n], v_1]/a^n \to 1$.

Step 2. We show $\Pr(L^n > d^n|L^n \geq 1, v) \leq A\exp(-d^n/a^n) \to 0$ and $\Pr[Y^n(k+1) > \theta_p^n|Y^n(k+1)] > \underline{\theta}_p^n, v \leq A\exp(-d^n/a^n) \to 0$, where A is an arbitrary positive constant.

Following the procedure from the previous step, we find

$$\Pr(L^n > d^n|L^n \geq 1, v) = \frac{\sum_{i=d^n}^{n-k} \text{bi}(k+i, n; \kappa)}{\sum_{i=1}^{n-k} \text{bi}(k+i, n; \kappa)}$$

$$= \frac{r^{d^n}(1-\epsilon_1^n)\left(a^n + \frac{(1-1/a^n)^n}{a^n} - (1-1/a^n)^n\right)}{(1-\epsilon_2^n)\left(a^n + \frac{(1-1/a^n)^n}{a^n} - (1-1/a^n)^n\right)}$$

$$\leq A\exp(-d^n/a^n),$$

where last inequality is a consequence of the fact that $(1 - 1/a^n)^{d^n}$ is of the order of $\exp(-d^n/a^n)$. Also, we have

$$\Pr\big[Y^n(k+1) > \theta_p^n|Y^n(k+1) > \underline{\theta}_p^n, v\big]$$

$$= \Pr\big(L^n \in [1, d^n]|L^n > 1, v\big)\Pr\big(X^n < L^n|L^n \in [1, d^n], L^n > 1, v\big)$$

$$+ \Pr(L^n > d^n|L^n > 1, v)\Pr(X^n < L^n|L^n > d^n, L^n > 1, v).$$

Consequently,

$$\begin{aligned}\Pr\big[Y^n(k+1) > \theta_p^n | Y^n(k+1) > \theta_p^n, v\big] &\leq \Pr\big(L^n > d^n | L^n > 1, v\big) \\ &\quad + \Pr\big(X^n < L^n | L^n \in [1, d^n], L^n > 1, v\big) \\ &\leq \frac{\sum_{i=d^n}^{n-k} \mathrm{bi}(k+i, n; \kappa)}{\sum_{i=1}^{n-k} \mathrm{bi}(k+i, n; \kappa)} + \exp\big(-\delta^2 d^n/2\big) \\ &\leq A \exp\big(-d^n/a^n\big) + \exp\big(-\delta^2 d^n/2\big) \\ &\leq A \exp\big(-d^n/a^n\big),\end{aligned}$$

where in the last inequality we use the fact that $A \exp(-d^n/a^n) \geq \exp(-\delta^2 d^n/2)$ and redefine the constant A without changing the order of the term.

Step 3. We now show

$$\begin{aligned}&\frac{\overline{F}_s^n\big(\underline{\theta}_p^n | v\big)}{(1+\delta) F^n\big([\underline{\theta}_p^n, \theta_p^n] | v\big)} \frac{\mathbb{E}\big[L^n | L^n \in [1, d^n], v\big]}{k_s} \\ &\quad \leq \Pr\big(b^p \text{ loses} | L^n \geq 1, v\big) \\ &\quad \leq \frac{\overline{F}_s^n\big(\underline{\theta}_p^n | v\big)}{(1-\delta) F^n\big([\underline{\theta}_p^n, \theta_p^n] | v\big)} \frac{\mathbb{E}\big[L^n | L^n \in [1, d^n], v\big]}{k_s} + A \exp\big(-d^n/a^n\big).\end{aligned}$$

We first give a lower bound for the probability of losing:

$$\begin{aligned}&\Pr\big(b^p \text{ loses} | L^n \geq 1, v\big) \\ &\quad \geq \mathbb{E}\left[\min\left[\frac{L^n}{X^n}, 1\right] \middle| L^n \in [1, d^n], v\right] \Pr\big(L^n \in [1, d^n] | L^n \geq 1, v\big).\end{aligned}$$

Note that $\Pr(L^n \in [1, d^n] | L^n \geq 1, v) \to 1$. Thus,

$$\Pr\big(b^p \text{ loses} | L^n \geq 1, v\big) \geq \mathbb{E}\left[\min\left[\frac{L^n}{X^n}, 1\right] \middle| L^n \in [1, d^n], v\right] (1 - \delta_1),$$

where δ_1 is an arbitrarily small constant. The fact that $\min[L^n/X^n, 1]$ is a concave function of X^n and Jensen's inequality together imply that

$$\mathbb{E}\big[\min[L^n/X^n, 1] | L^n \in [1, d^n], v\big] \geq \mathbb{E}\left[\min\left[\frac{L^n}{\mathbb{E}[X^n | L^n, v]}, 1\right] \middle| L^n \in [1, d^n], v\right].$$

By definition $\mathbb{E}[X^n|L^n, v_i] > d^n$. Therefore,

$$\mathbb{E}\left[\min\left[\frac{L^n}{\mathbb{E}[X^n|L^n, v_i]}, 1\right]\Big|L^n \in [1, d^n], v\right] = \mathbb{E}\left[\frac{L^n}{\mathbb{E}[X^n|L^n, v_i]}\Big|L^n \in [1, d^n], v\right]$$
$$= \frac{\overline{F}_s^n(\underline{\theta}_p^n|v)}{F_s^n([\underline{\theta}_p^n, \theta_p^n]|v)}\mathbb{E}\left[\frac{L^n}{L^n + k_s}\Big|L^n \in [1, d^n], v\right].$$

Noticing that $\frac{L^n}{L^n+k}$ is a concave function of L and applying Jensen's inequality implies that

$$\mathbb{E}[\min[L^n/X^n, 1]|L^n \in [1, d^n], v] \geq \frac{\overline{F}_s^n(\underline{\theta}_p^n|v)}{F_s^n([\underline{\theta}_p^n, \theta_p^n]|v)} \frac{\frac{\mathbb{E}[L^n|L^n \in [1, d^n], v]}{k_s}}{\frac{\mathbb{E}[L^n|L^n \in [1, d^n], v]}{k_s} + 1}$$
$$\geq \frac{\overline{F}_s^n(\underline{\theta}_p^n|v)}{F_s^n([\underline{\theta}_p^n, \theta_p^n]|v)} \frac{\frac{\mathbb{E}[L^n|L^n \in [1, d^n], v]}{k_s}}{1 + \delta_2},$$

where $\delta_2 := \mathbb{E}[L^n|L^n \in [1, d^n], v]/k$ is an arbitrary positive constant. Note that $\mathbb{E}[L^n|L^n \in [1, d^n], v]/k \to 0$. Therefore, we can choose δ_2 arbitrarily small for large n. Therefore,

$$\Pr(b^p \text{ loses}|L^n \geq 1, v) \geq (1 - \delta)\frac{\overline{F}_s^n(\underline{\theta}_p^n|v)}{F_s^n([\underline{\theta}_p^n, \theta_p^n]|v)k_s}\mathbb{E}[L^n|L^n \in [1, d^n], v],$$

where $1 - \delta = \min\{1/(1 + \delta_2), 1 - \delta_1\}$.

We now provide an upper bound for the probability of losing:

$$\begin{aligned}
&\Pr(b^p \text{ loses}|L^n \geq 1, v)\\
&\quad \leq \mathbb{E}[\min[L^n/X^n, 1]|L^n \in [1, d^n], v] + \Pr(L^n > d^n|L^n \geq 1, v)\\
&\quad \leq \frac{\overline{F}_s^n(\underline{\theta}_p^n|v)}{(1-\delta)F_s^n([\underline{\theta}_p^n, \theta_p^n]|v)}\mathbb{E}\left[\frac{L^n}{L^n + k_s}\Big|L^n \in [1, d^n], v\right]\\
&\qquad + \Pr(L^n > d^n|L^n \geq 1, v) + \exp(-\delta^2 d^n/3)\\
&\quad \leq \frac{\overline{F}_s^n(\underline{\theta}_p^n|v)}{(1-\delta)F_s^n([\underline{\theta}_p^n, \theta_p^n]|v)} \frac{\mathbb{E}[L^n|L^n \in [1, d^n], v]}{k_s} + A\exp(-d^n/a^n) + \exp(-\delta^2 d^n/3)\\
&\quad \leq \frac{\overline{F}_s^n(\underline{\theta}_p^n|v)}{(1-\delta)F^n([\underline{\theta}_p^n, \theta_p^n]|v)} \frac{\mathbb{E}[L^n|L^n \in [1, d^n], v]}{k_s} + A\exp(-d^n/a^n).
\end{aligned}$$

The first inequality follows because $\mathbb{E}[X^n|L^n = i \in [1, d^n], v] = (k_s + i)\frac{F_s^n([\underline{\theta}_p^n, \theta_p^n]|v)}{\overline{F}_s^n(\underline{\theta}_p^n|v)}$ is less than $(1 - \delta)(k_s + i)\frac{F_s^n([\underline{\theta}_p^n, \theta_p^n]|v)}{\overline{F}_s^n(\underline{\theta}_p^n|v)}$ with probability $\exp(-\delta^2 d^n/3)$ by Chernoff's inequality; the second follows because we showed that $\Pr(L^n > d^n|L^n \geq 1, v) \leq A\exp(-d^n/a^n)$ in Step 2. To obtain the last inequality, we use the fact $A\exp(-d^n/a^n) > \exp(-\delta^2 d^n/3)$ and redefine the constant A without changing the order of the term. The lemma now follows, as $\frac{d^n}{a^n}\exp(-d^n/a^n)\to 0$ because $d^n/a^n\to\infty$ and because the constants δ are arbitrary. *Q.E.D.*

LEMMA B.2: *Fix a sequence of bidding equilibria* $\mathbf{H}$ *and suppose that* $\lim\sqrt{n}|\bar{F}_s^n(\theta_s^n(1)|V=v)-\bar{F}_s^n(\theta_s^n(0)|V=v)|\to\infty$. *If there is pooling by pivotal types, then* $\lim\Pr(P^n\le b_p^n|V=1)=1$ *and* $\lim\Pr(P^n<b_p^n|V=0)=0$.

PROOF: Pooling by pivotal types implies that $\lim\Pr(P^n=b_p^n|V=v)>0$ for $v=0,1$. Suppose $\lim\Pr(P^n<b_p^n|V=0)>0$. Then $\lim\sqrt{n}(F_s^n(\theta_s(0)|0)-F_s^n(\underline{\theta}_p^n|0))\in(-\infty,\infty)$. Moreover, $\lim\Pr(P^n=b_p^n|V=1)>0$ and $\lim\sqrt{n}|\bar{F}_s^n(\theta_s^n(1)|V=1)-\bar{F}_s^n(\theta_s^n(0)|V=1)|\to\infty$ together imply $\lim\sqrt{n}(F_s^n(\theta_s(1)|1)-F_s^n(\underline{\theta}_p^n|1))=\infty$. Along any sequence where the limit in the equation below exists, Lemma A.2 implies that there is a constant C such that

$$\lim\frac{\Pr\big(b^n\text{ lose}|P^n=b_p^n,V=0\big)}{\Pr\big(b^n\text{ lose}|P^n=b_p^n,V=1\big)}\le\frac{1}{\eta}\lim\frac{C}{\sqrt{n}\big(F_s^n\big(\theta_s^n(1)|1\big)-F_s^n\big(\underline{\theta}_p^n|1\big)\big)}=0,$$

showing that pooling is not possible. Therefore, if there is pooling by pivotal types, then $\lim\Pr(P^n<b_p^n|V=0)=0$.

Suppose $\lim\Pr(P^n\le b_p^n|V=1)<1$. Then

$$\lim\sqrt{n}\big(F_s^n\big(\theta_p^n|1\big)-F_s^n\big(\theta_s^n(1)|1\big)\big)\in(-\infty,\infty).$$

Moreover, $\lim\Pr(P^n=b_p^n|V=0)>0$ and $\lim\sqrt{n}|\bar{F}_s^n(\theta_s^n(1)|V=1)-\bar{F}_s^n(\theta_s^n(0)|V=1)|\to\infty$ together imply $\lim\sqrt{n}(F_s^n(\theta_p^n|0)-F_s^n(\theta_s^n(0)|0))=\infty$. Using Lemma A.2, we obtain

$$\lim\frac{\Pr\big(b^n\text{ win}|P^n=b_p^n,V=1\big)}{\Pr\big(b^n\text{ win}|P^n=b_p^n,V=0\big)}$$

$$\le C\lim\frac{F_s^n\big(\big[\underline{\theta}_p^n,\theta_p^n\big]|0\big)}{F_s^n\big(\big[\underline{\theta}_p^n,\theta_p^n\big]|1\big)}\frac{\frac{1}{\sqrt{n}}}{\big(F_s^n\big(\theta_p^n|0\big)-F_s^n\big(\theta_s^n(0)|0\big)\big)}=0,$$

again showing that pooling is not possible. Therefore, if there is pooling by pivotal types, then $\lim\Pr(P^n\le b_p^n|V=1)=1$. *Q.E.D.*

B.2. *Proof of Step 2 of Proposition A.1*

Pick $\theta'\in[1/3,2/3]$ and $\theta''\in[2/3,1]$, and let $\tilde{\boldsymbol{\theta}}=(\theta',\theta'')$. Suppose that $\theta\in[0,\theta')\cup(2/3,\theta'']$ select market s and all others select market r. The expected payoff of a type $\theta\in\mathcal{E}(1)$ that submits a bid equal to $b=1$ in market s or in market r is given by $u_{\tilde{\boldsymbol{\theta}}}(s|\mathcal{E}(1))=G_s(1/3|1)+\int_{1/3}^{\theta'}(1-b_s^n(\theta))\,dG_s(\theta|1)$ and $u_{\tilde{\boldsymbol{\theta}}}(r|\mathcal{E}(1))=G_r(\theta'|1)(1-c)+\int_{\theta'}^{2/3}(1-b_r^n(\theta))\,dG_r(\theta|1)$, where $G_m(\theta|v)=\Pr(Y_m^{n-1}(k_m)\le\theta|V=v)$.[20] The expected

[20]If no types $\theta\in\mathcal{E}(1)\cup\mathcal{E}(1/2)$ bid in market s, then $\mathbb{E}[V|Y_s^{n-1}(k_s)=\theta]$ is not well defined. In this case, any bid $b>0$ is optimal for $\theta\in\mathcal{E}(1/2)$ (and similarly in market r). Although this situation never occurs in equilibrium, for completeness we assume that $\mathbb{E}[V|Y_m^{n-1}(k_m)=\theta]=1/2$ in this case.

payoff of type 1/3 (hence, the payoff for any $\theta \in \mathcal{E}(1/2)$) that submits a bid equal to $b = b_s^n(1/3)$ in market s and the expected payoff of type θ' that submits a bid equal to $b = b_r^n(\theta')$ in market r are given by $u_{\tilde{\boldsymbol{\theta}}}(s|\mathcal{E}(1/2)) = G_s(1/3|1)/2$ and $u_{\tilde{\boldsymbol{\theta}}}(r|\mathcal{E}(1/2)) = G_r(\theta'|1)(1-c)/2 - G_r(\theta'|0)c/2$. Notice that $\Pr(Y_s^{n-1}(k_s) \leq \theta|V = 1)$ and $\Pr(Y_s^{n-1}(k_s) \leq \theta|V = 0)$ are binomial distributions with parameters $\bar{F}_s([\theta, 2/3]|1) + \bar{F}_s([2/3, \theta'']|1)$ and $\bar{F}_s([\theta, 2/3]|0)$. Therefore, the functions $\mathbb{E}[V|Y_m^{n-1}(k_m) = \theta]$, $G_m(\theta|v)$, and $dG_m(\theta|v)$ are continuous in θ' and θ''.

Let $\tilde{\boldsymbol{\theta}} = (\theta', \theta'') \in [1/3, 2/3] \times [2/3, 1]$ and define

$$\Gamma_1(\tilde{\boldsymbol{\theta}}) = \begin{cases} \left[\frac{1}{3}, \frac{2}{3}\right] & \text{if } u_{\tilde{\boldsymbol{\theta}}}\big(s|\mathcal{E}(1/2)\big) = u_{\tilde{\boldsymbol{\theta}}}\big(r|\mathcal{E}(1/2)\big), \\ \frac{2}{3} & \text{if } u_{\tilde{\boldsymbol{\theta}}}\big(s|\mathcal{E}(1/2)\big) > u_{\tilde{\boldsymbol{\theta}}}\big(r|\mathcal{E}(1/2)\big), \\ \frac{1}{3} & \text{if } u_{\tilde{\boldsymbol{\theta}}}\big(s|\mathcal{E}(1/2)\big) < u_{\tilde{\boldsymbol{\theta}}}\big(r|\mathcal{E}(1/2)\big) \end{cases}$$

and

$$\Gamma_2(\tilde{\boldsymbol{\theta}}) = \begin{cases} \left[\frac{2}{3}, 1\right] & \text{if } u_{\tilde{\boldsymbol{\theta}}}\big(r|\mathcal{E}(1)\big) = u_{\tilde{\boldsymbol{\theta}}}\big(r|\mathcal{E}(1)\big), \\ 1 & \text{if } u_{\tilde{\boldsymbol{\theta}}}\big(s|\mathcal{E}(1)\big) > u_{\tilde{\boldsymbol{\theta}}}\big(r|\mathcal{E}(1)\big), \\ 2/3 & \text{if } u_{\tilde{\boldsymbol{\theta}}}\big(s|\mathcal{E}(1)\big) < u_{\tilde{\boldsymbol{\theta}}}\big(r|\mathcal{E}(1)\big). \end{cases}$$

The correspondence $\Gamma = \Gamma_1 \times \Gamma_2$ is upper hemicontinuous, convex valued, and compact valued, and, therefore, has a fixed point (θ_1, θ_2), and this fixed point is an equilibrium. The fixed point is an equilibrium because the correspondence Γ is defined so that all types $\theta \in \mathcal{E}(1/2)$ choose the market that gives them the highest payoff and if $\theta_1 \in (1/3, 2/3)$, then type θ_1 as well as all types $\theta \in \mathcal{E}(1/2)$ are indifferent between the two markets. The situation is similar for types $\theta \in \mathcal{E}(1)$ and all $\theta \in \mathcal{E}(0)$ choose market s by construction. Moreover, conditional on these choices, the bidding function $\mathbb{E}[V|Y_m^{n-1}(k_m) = \theta]$ is a bidding equilibrium in market m, and this bidding function delivers the payoffs used to construct the correspondence Γ. *Q.E.D.*

B.3. *An Equilibrium With Pooling by Pivotal Types*

Pooling by pivotal types is not possible in the illustrative example shown in the paper. Below we construct an equilibrium where there is pooling by pivotal types by altering the signal structure in the illustrative example as

$$f(\theta|V) = \begin{cases} 3\left(1 - g\frac{1-\pi}{\pi}\right)(1-V) & \text{for } \theta \in \mathcal{E}(0) := [0, 1/3), \\ 3\left(gV + (1-V)g\frac{1-\pi}{\pi}\right) & \text{for } \theta \in \mathcal{E}(1/2) := [1/3, 2/3], \\ 3(1-g)V & \text{for } \theta \in \mathcal{E}(1) := (2/3, 1], \end{cases}$$

where $\pi < c < 1/2$ and $g \in [0, 1]$. Types $\theta \in \mathcal{E}(1/2)$ are pessimistic, that is, their belief is $\pi < 1/2$ as opposed to 1/2 as in the illustrative example. The belief of types in $\mathcal{E}(0)$ and $\mathcal{E}(1)$ is equal to 0 and 1, respectively, as in the original illustrative example.

EXAMPLE B.1: Suppose that $\kappa_s < g$ and $\kappa_r > 1 - g$. There exists an $\epsilon > 0$ such that, for all sufficiently large n, there is an equilibrium where all types $\theta \in \mathcal{E}(1)$ select market r and all types $\theta \in \mathcal{E}(1/2)$ bid $b_p = c + \epsilon$ in market s. In this equilibrium, the price in markets s and r is equal to b_p and c, respectively, with probability converging to 1.

We now formally construct the equilibrium described above. Types $\theta \in \mathcal{E}(1/2)$ never opt for market r because $c > \pi$ and all types $\theta \in \mathcal{E}(0)$ submit a bid equal to 0 in market s in any equilibrium. Pick $\epsilon < (1 - 2c)/2$. If all types $\theta \in \mathcal{E}(1/2)$ submit a pooling bid equal to b_p in market s, then their limit payoff at pooling is given by

$$\begin{aligned}&\Pr(V = 1|\theta)(1 - b_p) \lim \Pr(b_p \text{ win}|V = 1) - \Pr(V = 0|\theta) b_p \lim \Pr(b_p \text{ win}|V = 1)\\&\quad = \pi\kappa_s(1 - 2c - 2\epsilon)/g > 0\end{aligned}$$

because the probability of winning conditional on $P = b_p$ converges to κ_s/g and $\kappa_s\pi/g(1 - \pi)$, in states 1 and 0, respectively. Alternatively, if this type instead submits a bid greater than the pooling bid, then the type's limit payoff is $(1 - b_p)\pi - (1 - \pi)b_p = \pi - b_p < 0$ because she wins with probability converging to 1. Therefore, at the limit, each $\theta \in \mathcal{E}(1/2)$ strictly prefers the pooling bid to any higher bid. Also, each $\theta \in \mathcal{E}(1/2)$ strictly prefers the pooling bid to any lower bid because $\kappa_s < g < g(1 - \pi)\pi$ implies that the probability of winning with a lower bid converges to 0. The fact that each $\theta \in \mathcal{E}(1/2)$ strictly prefers the pooling bid to any other bid at the limit implies that these types also prefer the pooling bid for sufficiently large n. Also, types $\theta \in \mathcal{E}(1)$ opt for market r because $b_p > c$.

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